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Seeking More Than Nothing: Two Elementary Teachers' Conceptions of Zero

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Abstract: *Zero is a complex and important concept within mathematics, yet prior research has demonstrated that students, pre-service teachers, and teachers all have misconceptions about and/or lack of knowledge of zero. Using a hermeneutic approach based upon Gadamer's philosophy, this study examined how two elementary mathematics teachers understand zero and how and when zero enters into their teaching of mathematics. The results of this study add new insights into the understandings of teachers and students related to zero and the origins, relationships between, and consequences of those understandings. Significant gaps and misconceptions within both teachers' understandings of zero suggest the need for pre-service education programs to bring attention to the development of a more complete and meaningful understanding of zero.*

Key words: Zero; In-service teachers; Elementary teachers; Mathematics teachers; Prospective teacher development; Teacher research

What is zero? When asked, many teachers and students will tell you that zero is "nothing" (Wheeler, 1987; Leeb-Lundberg, 1977; Wilcox, 2008; Crespo & Nicol, 2006; Wheeler & Feghali, 1983). For being nothing, though, it has gotten a lot of attention in mathematics education research. Since the 1960s, various researchers have explored how students understand zero (Inhelder & Piaget, 1964, Pasternack, 2003; Evans, 1983; Baroody, Gannon, Berent, & Ginsburg, 1983; Wheeler, 1987; Neuwirth Beal, 1983; Reys & Grouws, 1975; Allinger, 1980; Leeb-Lundberg, 1977; Whitelaw, 1984; Kamii, 1981; Crespo & Nicol, 2006) with findings including that students confuse zero with the letter "O", believe that zero is not a number , believe zero is just a part of the symbol for the 'digit' ten (believe that zero is 'nothing' and therefore can be ignored, and have difficulties within arithmetic calculations (including, but not limited to, division by zero) when zero is involved. Much research has also focused on teacher's and prospective teacher's understanding of division by zero (Crespo & Nicol, 2006; Ball, 1990; Evan, 1993; Wheeler & Feghali, 1983; Even & Tirosh, 1995). What is remarkable and disheartening is that the same errors in thinking and understanding about division by zero are noted in Crespo & Nicol's (2006) work as were noted originally by Wheeler & Feghali (1983) in their research. Given the extensiveness of student and teacher misunderstanding about zero, it is also notable that, other than Wheeler & Feghali's (1983) research, no research designed to explore specifically teachers understandings of zero, other than those studies related to division by zero, has been done. The research upon which this article is based was intended to help to begin filling in some of that void.

This article reports about a qualitative research study designed to explore how teachers conceive and preconceive of zero, both personally and within their classrooms. Through the use of Gadamer's (1989) hermeneutic philosophy of understanding, two teachers were engaged though dialogue and explorations in the consideration of the questions: "How do you understand zero," and "When and why does zero become a part of teaching and learning in your classroom". Many of the results of this study parallel those found by Wheeler & Feghali, but because of the study's qualitative design, the results also help to paint a picture of the thinking and reasoning behind those results for these two teachers. These insights could be used to begin developing aspects within teacher education programs that would help prospective teachers to learn both the "that" and "why" knowledge (Shulman, 1986). related to zero and how to share that knowledge with future students

1. BACKGROUND

This study is informed by the history of the development and evolution of zero as a mathematical concept, research into student understanding of zero, and research related to teacher understanding of zero. These topics are discussed in the following sections.

1.1 Evolution of zero

The history of zero is one that has been well documented. Around the turn of the last century, and the millennium, three authors, Barrow, Kaplan, and Seife, each wrote detailed accounts of that history, and it is upon those three authors' works that this section is based. The development of the mathematical concept of zero (including its roles as a place holder, a number, and a symbol) is one that happened quickly in some societies, such as India and the Mayan culture. In Greece, however, the acceptance of zero took more than 1000 years (Barrow, 2000). These variances in timing were the result of differences in the religious and philosophical beliefs of the societies themselves. Within India's Hindu religion, there were many gods that represented different dualisms in life, one being that of the void and the infinite. Thus, when zero as a placeholder reached India, the extension of the concept to a quantity was natural because a related idea (the void) already existed in the religion. In Greece, however, the notion of zero as a number contradicted a mathematical proof that God existed (Kaplan, 1999). In addition, Greek philosophers were unwilling to accept that a symbol could be used to represent a void, since a void is nothing and one cannot represent something that does not exist. Even the Greek mathematicians found zero challenging because it could not be represented by a shape as Pythagoras' philosophy said that all numbers could (Seife, 2000).

Once zero found its place in different societies, however, it quickly became integrated into mathematical thought and logic, and soon opened doors to formalized arithmetic, algebra, and calculus, to name just a few areas. Even mathematicians' understanding and conceptualization of numbers began to evolve through their explorations of zero and its meaning, leading Newton, for one, to conclude that "…mathematical quantities [are]… not … consisting of very small parts, but as described by a continuous motion" (Kaplan, 1999, p. 156). Thus, zero had evolved from being an arbitrary symbol used to denote a blank space to a quantity of such complexity and depth that it allowed mathematics to move into a realm that embraced asymptotic and limiting behaviors.

Despite the discrepancies related to how zero was welcomed into the number and mathematical systems of different societies, zero has become a foundational, complex, and multi-functional concept in modern mathematics (Seife, 2000; Barrow, 2000; Kaplan, 1999). As such, it can be argued, a robust understanding of zero is a necessary part of students and teachers' abilities to think and work mathematically with confidence and competence.

1.2 Student understanding

Given the revolutionary impact of zero on mathematics, it is surprising, yet understandable, to realize the limited understanding and many misconceptions that students have related to zero. Inhelder & Piaget (1964) found that children under the age of 10 or 11 do not recognize the null set (i.e., a set whose defining characteristic is the lack of a characteristic, such as the set of pictures with no birds versus the set of pictures with one bird) when sorting.

Research also reveals that students do not recognize zero as a number; they see it as only a part of the symbol for ten (Pasternack, 2003; Evans, 1983; Baroody et. al., 1983). In fact, many students believe that "[Zero] isn't really a number … it is just nothing" (Wheeler, 1987, p. 42), with the implication that it can be ignored whenever it occurrs. As an alternative, some students recognize zero as a number that "develops and exists separately from other number rules" (Evans, 1983, p. 96). Similar results are also found in the research of Neuwirth Beal (1983) and Reys & Grouws (1975). All of these notions of zero were found to support students' misunderstandings related to computations involving zero (Wheeler, 1987; Neuwirth Beal, 1983; Anthony & Walshaw, 2004; Evans, 1983).

Students have also been shown to struggle with the mathematical concept of zero because of the inconsistent use of oral and written language related to zero within society. Baroody, et. al. (1983), Allinger (1980), and Whitelaw (1984) all report that students frequently confuse the the number "0" and the letter "O". Society's frequent use of "oh" when stating area codes, license plates, phone numbers, and room numbers is cited as a common source for the students equating of zero and the letter "O" by these researchers.

Even the naming of numbers in the English language, zero is not mentioned within the name (e.g., 203 is read as two hundred three, and not two hundred zero three). This convention of mathematics can cause students to misunderstand and misuse zero. Students try to "spell" numbers in the same way they "spell" words, thus one hundred twenty is often written as 10020 (Kamii, 1981). This "ignoring of zero" convention in the naming of numbers supports the student belief that zero is "nothing" and thus it can be ignored (Wheeler, 1987).

Two researchers (Leeb-Luneberg, 1977; Wilcox, 2008), did demonstrate, however, that young students can develop an understanding of zero. In Leeb-Luneberg's (1977) research, the students concluded that zero must be "… nothing - of something!" (25), while Wilcox's grade 1 daughter told her mother: "No [zero] means something. It means you don't have anything" (204). In both studies, the children were able to resolve the Greek dilemma that zero cannot exist because it is nothing by refining their understanding of zero to be that it represents none of some item. Cockburn and Littler's (2008) *Mathematical misconceptions: A guide for primary teachers* includes an opening chapter (Chapter 0) that specifically addresses students' misunderstandings about zero and how to correct and prevent them, with a number of the activities suggested mimicking those used by Leeb-Luneberg and Wilcox .

1.3 Teacher understanding

Consider first the area of research into teachers' understandings of zero that is most prominent in the literature – that of division by zero. Even & Tiroch (1995), found that when asked what 4 divided by zero was, most answered "undefined", however, when the same teachers were asked to explain why "most could not supply any appropriate explanation" (9) beyond stating that it was a mathematics rule. A number of researchers, however, did not find the same results (Crespo & Nicol, 2006; Ball, 1990; Wheeler & Feghali, 1983). Instead, these researchers found that most pre-service teachers did not even know that the answer should be undefined, let alone why. In some cases, the pre-service teachers recalled learning that anything divided by zero was zero, and in some others they reasoned out the answer of zero by thinking of zero as "nothing".

Wheeler & Feghali's (1983) study of pre-service teachers' understandings of zero, as noted earlier, is the only research that has explored this topic in breadth (beyond division by zero). The study revealed that pre-service teachers have many of the same misunderstandings and lack of knowledge as the above-mentioned research indicated for students. One such similarity between teacher and student understandings of zero is in relation to the exchange of the word "oh" for the number "zero" (Baroody, et. al., 1983; Allinger, 1980; Whitelaw, 1984). Also like the students, many of the pre-service teachers that Wheeler and Feghali (1983) worked with believed that zero is not a number (Wheeler & Feghali, 1983), referring to it as 'nothing'. The pre-service teachers in this research explained that it did not meet their criteria for what a number is, namely that a number represents 'something' and therefore it must be countable. This interplay between zero and numbers, and 'nothing' and 'something' is remarkably similar to some of the arguments made by Greek philosophers almost 2000 years ago (Kaplan, 1999), , and plays a large role in both Leeb Luneberg's (1977) Wilcox's (2008) studies of how young children can come to understand zero. .

Wheeler & Feghali (1983) also repeated Inhelder &Piaget's (1964) earlier testing involving the null set with the pre-service teachers and found that, like the children decades before, the participants were not inclined to sort cards into sets that included a set of cards *without* particular characteristics. In following up with the pre-service teachers after the test, Wheeler & Feghali (1983) also found that some of the participants would not accept a null set even when it was presented as a possible solution to consider.

The only other place that demonstrations of teachers' understandings of zero (other than those related to division by zero), can be found is hidden within other research topics. One example of such research is Ma's (1999) study, which compared the mathematical understandings of teachers in the US and Singapore. Ma demonstrated through this research that the teachers from Singapore possessed a much higher level of "profound understanding" of mathematics than did the US teachers. When considered through the lens of "what's happening with zero", her research also revealed that, in particular, the US teachers had misconceptions about zero within the concepts of place value and number decomposition. These misconceptions become evident through Ma's study of the teachers' understanding of two-digit subtraction and multi-digit multiplication. With respect to subtraction, the data collected revealed that the teachers did not understand the role of the decomposition of numbers into different groupings of tens and ones within the subtraction algorithm that they taught. As a result, these teachers also lacked understanding of place value and the role of zero within place value and the decomposition and subtraction of numbers. With respect to multiplication, the teachers were shown a students' solution to a multi-digit multiplication question which included a common error made by students – that of failing to account for the place value of the individual digits in the multiplicand and multiplier (or "forgetting to put zeros at the end" of each partial product). Ma asked the teachers how they would correct the student. Although the teachers noted that the student "did not understand the rationale of the algorithm" (p. 29), the teachers' own explanations of the student's errors revealed that the teachers did not understand place value and its role in multiplication themselves. Some of the teachers even suggested that rather than putting zeros in the partial products, the student should be encouraged to use Xs to hold the places. These teachers argued that by using zeros as the place holders, the students would be led to believe that the partial products were actually larger than what they really were (e.g., 4920 versus 492 which comes from multiplying 123 by the 4 in 645). Thus, in both subtraction and multiplication, the US teachers lacked understanding of number decomposition, place value, and zero.

1.4 Understanding of zero

Whether the understanding of zero be considered from the perspective of students or teachers, there is clear evidence within mathematics education research that there is cause for concern, not only in the upper middle-level grades of mathematics, when teachers are introducing students to the complexities of dividing by zero for the first time, but even within the earliest elementary grades, and with respect to the understanding of teachers in general. This study was designed to explore the understandings of zero that elementary teachers have, how they came to have those understandings, and how they engage their students in understanding zero.

2. METHODOLOGY

Given the nature of the research questions for this study, a qualitative approach was required in order to explore the nuances and contextualizations for the different participants' understandings of zero. In this section, the methodology, method of data collection, and methods for analysis of the data are described.

2.1 Gadamer's hermeneutic philosophy

In order to explore the understandings, often hidden, that teachers have of zero, and how these ideas developed, were supported by experiences, and the issues those ideas raised for the teachers, a qualitative methodology based upon Gadamer's (1989) hermeneutic philosophy was used to frame the collection of data. In this hermeneutic approach, what and how one knows about an idea or concept is defined by one's past and present horizons of understanding related to the idea. In Gadamer's theory, the past horizon is a cognitive construct that contains the historical knowledge and resulting traditions that define a concept. As well, every person also has a present horizon of understanding that encompasses everything that one believes and understands about the concept at a particular point in time. Gadamer argues that the past horizon easily influences one's present horizon (with or without intention or recognition) and thus needs to be exposed so that the present horizon can be better understood and evaluated. Through dialogue, the two horizons fuse together, with the past horizon remaining fixed, but the present horizon "continually … being formed because we are continually having to test all our prejudices" (Gadamer, 1989, p. 306). By engaging in a hermeneutic dialogue, Gadamer maintains that these prejudices are exposed and evaluated individually as the participants in the dialogue ask questions and seek to clarify their own understanding.

Gadamer (1989) proposes that one cannot develop a rich understanding by one's self. Rather, it is through dialogue with others that one becomes not only aware of one's own horizons of understanding, but also of the understandings of others. Anyone involved such a dialogue is not expected to outright reject their own horizons, nor reject those of others. Instead, the dialogue is intended to help each person understand the horizons of others and, as a result, their own horizons expand with this understanding. The goal is not to seek an ultimate truth, which Gadamer argues cannot be attained, but rather to play with the possibilities of understanding.

Experiences play a large role in the defining of one's horizons of understanding, so within dialogue it is important for those involved to share their own experiences, and to engage in new experiences. It is through openness to experience and consideration of other's horizons of understanding that meaning can be clarified, sought, and expanded (Gadamer, 1989). The environment within which dialogue and experiences occur must be structured to be open and non-judgmental (Silverman, 1991).

Gadamer (1989) also emphasizes that at any given moment, and in relation to any given context, participants in dialogue access only a limited portion of their horizons of understanding. Thus, dialogue plays a final role to expand the portion of the horizons of understanding being engaged by the participants so that both speaker and listeners are engaging in the discussion with broader, better defined, and more closely aligned horizons of understanding which include awareness of the beliefs and knowledge of everyone involved (Silverman, 1991).

2.2 Collection and analysis of data

With Gadamer's hermeuntic philosophy as the guiding methodology for this research, it was important for data to be collected dialogues and experiences that the participants were engaged in. These interactions were designed to allow the researcher to explore and come to understand the past and present horizons of understanding of zero held by the participants. In addition, the data collection also had to allow for the two teachers to help in giving direction to the next dialogues and experiences as a response to their own changing awareness and curiousity about their understandings of zero.

Working within the above noted framework for the data collection, the study was comprised of three three-hour meetings of the two teachers and the researcher, an in-class teaching session in each of the teachers' classrooms for one hour (during which the researcher explored the teachers' students' understandings of zero while the teacher observed and kept anecdotal records), and an interview with the teachers immediately following the in-class session with her students. Each of these interactions focused on questions and activities that sought to reveal more of and challenge each of the teachers' present horizons of understanding zero and the relationship between those horizons to their past horizons of understanding. To help expose some of the nuances of their past horizons, the two teachers were also encouraged to keep a journal of their memories of learning about zero, as well as of any experiences or dialogues that they had outside of our scheduled meeting times with colleagues, friends, or students regarding zero. These journals contributed greatly to helping direct the study's conversations and explorations. Although the researcher was explicitly involved in the directing and redirecting of the conversations and activities, great care was taken to not reveal aspects of her own present or past horizons. It must be acknowledged however, that the understandings of the researcher very likely influenced choices regarding activities and directions taken with the participants.

The specific research questions for this study were: “How do you understand zero” and “When and why does zero become a part of teaching and learning in your classroom”? These questions provided the initial direction for the group meetings, but with the acknowledgement that the questions might be modified or replaced in order to be responsive to the dialogue, and hence the horizons of understandings of the two teachers.

Each of the three meetings was tape-recorded, transcribed, and the transcripts verified by the two teachers. In addition, the one post-classroom session interview (only one was done due to time restraints for the one participant) was also tape-recorded, transcribed, and verified by the teacher. The classroom visits were not recorded, but rather both the teachers and researcher made anecdotal notes of the experiences and these were discussed and reflected upon during the follow-up interview and the third group meeting.

Prior to the first group meeting, the two teachers were asked to reflect upon what they knew about zero, when they learned about it, and what language was used with respect to zero. These memories were a source of much of the dialogue during the first meeting, as was an exploration of works of children’s literature and their inclusion/exclusion of zero. Although much of the children’s literature used was familiar to the two teachers, the exploration of zero within the literature was a new experience for them and was a rich source for the ongoing dialogue. The dialogue that resulted from the teachers’ memories of learning about zero and from their analysis of the role in the children’s literature both sought to expose not only parts of their present horizon of understanding, but also to have the teachers consider what some of society's past horizon understandings of zero are and to reflect upon the validity and relevance of those understandings to their individual situations.

The second meeting began with the sharing of experiences and recalled memories the teachers had had related to zero since the previous meeting. This meeting also focused on what the teachers believed about students and the number zero, including what students know about zero and what they should know about zero and why. The question of why students should learn particular facts or ideas about zero again helped to expose some of the past horizons of understanding for zero held within our society, while also engaging the teachers in revealing more of their present horizon of understanding zero. This information helped to inform the choice of activities that the researcher used in each of the classroom sessions with the participants' students.

The final group meeting involved the teachers sharing new ideas, recalled memories, and reflections on their observations of the in-class sessions, as well as a discussion of what they felt to be most important for elementary students and teachers to understand about zero and how those understandings might be developed. It was an opportunity to bring together many of the different facets of their present and past horizons of understandings of zero to develop a broader perspective of what each teacher knew and believed about zero, while also allowing for new ideas and connections to be made. The meeting ended with the researcher sharing some of her present horizon understandings of zero and teaching and learning about it, as well as how the dialogues and experiences had helped to reveal, clarify, change, and expand the researcher's own horizons of understanding zero.

The in-class teaching sessions were included in this study to provide a new experience for the teachers that could engage them in further reflection, discussion, and exploration of their horizons of understanding. The activities used in each session were different, and were designed to engage with the students within the contexts that their current mathematics study was focused. In one classroom, the students were focusing on place value and its role in addition and subtraction, while in the other classroom, students were learning about two-digit multiplication. The activities in each session were designed to probe how the students in each classroom understood zero and to explore how the teachers interpreted and understood the students' engagement and responses to the activities. The interview that was held immediately following one of the in-class session provided the researcher with an opportunity explore how that teacher placed the experience within her present horizon of understanding zero as it related to the understandings of both herself and her students. For the second session, this dialogue occurred in the final group meeting and involved both teachers.

The analysis of the data involved the recognizing and codifying of common themes of agreement and/or disagreement within the two teachers' horizons of understanding zero. Once the themes were defined, the researcher then compared those understandings with the historical development of zero and to the prior research findings related to student, pre-service teacher, and teacher understandings of zero.

3. RESULTS

The analysis of the collected data revealed a number of links between the history of zero, past research findings related to student, pre-service teachers, and teachers understanding of zero and the two teachers horizons of understanding zero. However, there were ideas generated by the two teachers that had not previously been referenced in the literature. The main six themes that emerged from the data analysis are described in this section. In this discussion of the results, the pseudonyms Elaine and Nora will be used for the two teachers.

3.1 The start of knowing

When first questioned about when and how they first learned about zero, both Nora and Elaine had very few memories, which resulted in much speculation and misgivings about what and how they had been taught. Elaine spoke of learning that zero was the starting point of numbers, but not a number itself. Then, in middle school, Elaine was told that zero was the "middle of the integers", like a type of physical divider. These two meanings were irresolvable for Elaine as a student. Without knowing of the social construction and evolution of zero and the integers, as well as not knowing that zero defines a quantity, Elaine's learning about zero, as well as about whole numbers and integers, was relegated to points of trivia to be remembered, but not, necessarily, understood. Elaine's view of zero as not being a number correlates with findings from Wheeler & Feghali's (1983) study of pre-service teachers and with research involving students (Evans, 1983; Anthony & Walshaw, 2004; Wheeler, 1987), however, the emphasis on zero being viewed as a starting point is one that was not put forward in prior research.

Nora quickly came to the conclusion that her memories of learning about zero were in fact memories of not learning about zero. As had been the case for many students in Wheeler's (1987) study and for many of the pre-service teachers in Wheeler and Feghali's (1983) research, all that Nora could recall was being told that "zero is nothing". Nora struggled with this definition of zero. She frequently spoke of how it gave rise to her belief that zero was not important and as a result could be ignored. Throughout our discussions, Nora regularly returned to this definition and trying to remedy it was a major motivation for her seeking of a philosophical and theoretical understanding of the concept of zero. Eventually, Nora expanded her definition to "nothing of *some thing"*. This modification to her definition parallels the conclusion by the students in Leeb-Lundberg's (1977) research: "Zero is nothing – of something!" (25) and that of Wilcox's (2008) daughter's statement that: "… [zero] means something. It means you don't have anything" (204). This modification to her definition allowed Nora to later create a philosophical understanding of zero that made sense of the technical roles and rules for zero that she had learned as a student.

The students in both teachers' classes also perceived zero as "nothing" and said it could be ignored. This concerned both Nora and Elaine, and Nora was particularly troubled to hear her declare zero was not a number. The students reasoned that if zero was a number, then they would have been taught it when they were taught about one to ten and that it would be said in number names (e.g., we say "twenty" and "twenty-one", but if zero was a number, we would say "twenty-zero"). Although some of the past research had demonstrated students' confusion over the naming of numbers containing zeros (Kamii, 1981; Baroody et. al., 1983), these explanations about their thinking about number naming revealed that the students were generalizing patterns and ideas from the mathematics that they had learned which were causing them to come to invalid conclusions (zero is not a number).

3.2 Memories related to computations

With respect to zero within computations and computational procedures, Nora's memories again focused on the lack of inclusion of zero. Her memories were primarily about addition and subtraction, and specifically being taught about "carrying" and "borrowing" the "1" (see Figure 1).

Figure 1. Carrying the one

As a student, Nora was perplexed by this procedure because she did not understand where the "1" came from. As an adult, Nora realized that the "1" actually represented "10", but she questioned why the procedure had been described as "carry the 1" and not "carry the 10". She argued that by dropping the zero off the ten, this procedure had confirmed her belief that zero could be ignored. Nora questioned what she really understood about the role of zero in any computation, and if she was ignoring it everywhere.

Similarly, Elaine spoke of learning to divide questions such as $20\overline{)340}$ and being told to "just knock off the zeros". Elaine did not know why she could do this, only that ignoring the zeros made things easier.

Nora and Elaine's descriptions of their limited and misconstrued understandings of computations involving zero reflect many of those by Ma (1999) with the teachers from the United States' understandings of subtraction and multiplication and of those identified in Neuwirth Beal's (1983), Evans' (1983), Anthony & Walshaw's (2004), and Wheeler's (1987) research involving students. Both teachers had learned to do computations involving zero as procedures without any basis of understanding. Over time, they had come to theoretically understand some of the computational procedures they had been taught, but many of the procedures had remained unquestioned until our meetings.

3.3 Zero Outside the Classroom.

Elaine and Nora both spoke of how much of their understanding of zero developed outside the classroom. Elaine recalled having an aunt who used the word 'aught' in place of zero, and both teachers spoke of how 'oh' was frequently used when saying the number zero in different contexts, such as postal codes. Such uses of "oh" was also frequently noted in other research (Wheeler & Feghali, 1983; Allinger, 1980; Baroody et. al., 1983; Whitelaw, 1984). This avoidance of the word zero had reinforced many of Nora and Elaine's mathematically incorrect beliefs about zero, including that it can be ignored.

Elaine and Nora also talked about how, while regularly avoided in numerical contexts, the word "zero" was frequently used in non-numerical contexts. Elaine remembered zero being associated with people in ways that indicated they were defective or substandard, and Nora provided the example of the novel *Holes* (Sachar, 2000) and its central character, Zero, as verification. Both teachers agreed that these experiences gave them the belief that "zero" somehow indicated a deficiency or disapproval and that they did not have such a pessimistic (or emotional) attitude towards other individual numbers. Although Allinger (1980) mentions the occurrence of this type of use of the word zero, Nora and Elaine proposed that this non-mathematical contextualization of zero influenced how they view and think about the mathematical concept of zero. Whether it be the casual replacing of the word "zero" with "oh" or the use of zero as a qualitative descriptor, their everyday encounters with the quantity zero and the word "zero" presented the two teachers with incomplete and often misleading facts that contributed to their lack of theoretical understanding of zero within the context of mathematics.

3.4 Marginalization and legitimatization of zero

Throughout this study, Elaine and Nora struggled with their understanding of zero in terms of what they had been taught as a student, in making sense of what they teach to their question that emerged through our dialogues: "why do I think I was taught about zero in the way that I was," led Nora and Elaine to seek for and identify ways in which zero had been marginalized, not only for themselves, but within society as a whole. The natural consequence was that they both then sought ways in which this marginalization could be corrected and zero could become a legitimate and valued number. In both cases, Elaine and Nora returned to their earliest memories of knowing zero to define where and how this legitimatization could take place.

Nora began by looking for where zero was ignored, with her first concern emerging from students' learning of oral and symbolic representation of numbers and letters. She spoke of how, "in public it's acceptable to switch 'zero' and 'oh'," and that "when kids are starting to read, they get confused with "1" and "l" … but it's not viewed acceptable to confuse them." Nora said that she felt that little or no attention was given to students not confusing the letter "O" and the numeral "0". This inconsistent use of words and the failure to give equal status and time to the number zero as was given to other numbers and letters bothered Nora. This concern regarding equity for all numbers in the classroom became a frequent perspective that she brought to the discussion. In order to legitimatize zero within the context of numbers, Nora felt that it was important that a deliberate attempt be made at using the correct vocabulary when speaking in contexts involving zero, and to put an emphasis on distinguishing between the written letter "O" and the numeral "0".

During the in-class session with Nora's students, another representational confusion emerged that surprised both Nora and the researcher. When asked to identify objects that they had zero of, Nora's students pointed to the objects as they said "clock", "fan", and "my earrings", all three of which were circular in shape and were present in the room, if not in the direct possession of the student identifying the object.. After some discussion, it became evident that the students thought that "zero" meant "having the shape of a circle". For Nora, this confusion of the shape of a circle with the meaning of zero added an additional source of marginalization of zero in that zero. She argued that this too was an aspect of teaching and learning about zero that must be prevented. Although there were references to the shape of the letter "O" and the digit "0" in some of the research related to students (Baroody, et. al., 1983; Allinger, 1980), Nora's students identified a more extensive misunderstanding and point of confusion, that of extending beyond the letter "O" to any object that is circular, which has not been discussed prior in the research.

When considering children's literature related to numbers, Nora was concerned that in many of the works we explored, the only inclusion of zero was through the numeral "10" and not as a number in its own right. Nora also felt that in those works where zero was presented, the ways in which it was shown and described was inconsistent with that of other numbers. As an example, in *Zero is not enough* (Zimmermann, 1990), Nora was very concerned because although zero was defined in words, it was not on any of the placards held up by the monsters in the illustrations, whereas, all of the numbers from 1 to 10 were on placards. Nora argued that because the book did not present zero in exactly the same way it did all other numbers that the book wasn't doing zero justice. In order to correct this injustice, and give zero equal standing and importance with the other numbers in the book, Nora suggested the addition of a monster holding a placard with zero on it.

Nora raised the concern that teachers marginalize zero in their teaching of mathematics. Nora began the discussion by admitting that, "honestly – I [have] never taught zero," adding that she felt that "teachers just aren't… aware of incorporating zero into lessons". Nora went on to reflect that, "I think maybe I don't understand zero the way I should… If I can't explain it myself, how am I explaining it to the children?" This marginalization of zero through the teaching and learning process was one that Nora felt was continuing from when she had been a student. She argued for more emphasis on zero in university education methods classes and in resources as ways to legitimatize and bring zero into the classroom.

Nora noted that zero was not included as a number in the hundreds charts used in elementary classrooms. Although there do exist commercially produced charts that include the numbers of 0 to 99 rather than 1 to 100, Nora proposed a unique change to the chart in which zero would be added above the column containing multiples of tens. Nora argued that by placing the 0 above the 10, the students would see the connection between the two quantities in terms of both having zero-ones. In this way, zero was being placed in its equitable position in relation to the other numbers.

Nora's focus on where zero is ignored also brought her to consider many situations and contexts that sit outside the realm of theoretical mathematics. For example, Nora was concerned that in naming streets and avenues, "you have First Avenue, Second Avenue, Third Avenue, but no Zero Avenue". Again, Nora felt that this was an error that could and should be corrected.

Nora also struggled with recording time. She noted that between 12:59 and 1:00 there should be a time called 0 to signify starting the cycle over again: "zero would be the millisecond between 12:59 and 1:00." Similarly, Nora proposed that there should be a year 0 between 1 BC and 1 AD. In both cases, Nora was suggesting that these infused zeros were more of a movement to 1 than a quantity. Although fundamentally different, it is interesting to note how this new conceptualization that Nora had for zero was similar in some respects to the understanding of zero as a process, motion, or change that emerged during the development in calculus (Kaplan, 1999). In her suggested inclusion of zero into time on clocks and the calendar years, Nora did not recognize that her approaches were not giving zero the same status or role as other numbers and thus did not help in her search for that particular type of legitimatization of zero.

For Elaine, the marginalization of zero occurred in situations in which there was a starting point, but no zero. Her attempts at legitimatizing zero began with her proposal of a solution to the issue of zero not being a starting point for integers. By viewing the integers as two separate sets of numbers, positive and negative, she argued that zero was in fact the starting point for both sets. Elaine was not aware that by doing this, she was taking the stance that the integers can only be viewed as two distinct sets, and not as one cohesive set of numbers. There is some question as to whether she viewed the zero she defined (as the starting point for positive and negative numbers) as a single entity or two distinct ones.

Elaine then moved the conversation to changes outside of the specific realm of mathematics that needed to be made in order to incorporate zero in meaningful contexts as a starting point. First, Elaine spoke of baseball, and of changing "home" base to "zeroth" base: "Its home base, but you're going to first – so where are you starting from? … They're not calling it zero, but it's your staring point". Elaine did not recognize that, by the same logic, the home base would then have to become "fourth" base at the desired completion of a batter's run.

Elaine also argued that timers should always count up from zero, rather than down from a specified time because zero is where timing starts. Elaine explained that she tended to interpret the passage of time “from a moment in the future” back to the present time rather than from the present to a point in the future. The history of the Mayan society includes a similar notion of counting back in time to zero (Seife, 2000).

Next, Elaine asked “does zero gravity exist?” to which Nora replied that zero gravity was “buoyancy – it’s when you float”. Together, the two teachers concluded that zero must be the lowest value for a measurement of gravity. The discussion of the two teachers around gravity demonstrated that their understanding and conceptualization of zero had a definite link and impact on other understandings outside the context of mathematics.

Elaine’s final recommendation was to change the role of zero with respect to temperature, She argued that zero should be the starting point of, or lowest possible, temperature. Nora disagreed with this proposal, because she valued 0º C as the freezing point of water. Nora’s argument can be seen as supporting her goal to make zero something that cannot be ignored, while Elaine’s argument supported her belief that zero should always be a starting point. Interestingly, Elaine did not argue for zero being the starting point of both the positive and negative temperatures as she had done for the role of zero in integers. Both teachers were unaware of the Kelvin system of measuring temperature.

In all these cases, Nora and Elaine were attempting to bring zero to the attention of the public eye, to make it “something” rather than “nothing”, and to bring consistency to the “world” of zero. Their arguments were almost exclusively based upon and in reaction to the technicalities that they remembered learning about zero and were rarely supported by an understanding of the theoretical and socially constructed aspects of zero.

3.5 Representing zero

As Elaine and Nora formed a sense of importance for zero, they began to argue for purposeful and meaningful teaching about zero to students. Both teachers felt that students should learn about zero as soon as they started learning about whole numbers, but Elaine struggled with finding a way that students so young could understand a concept as abstract as zero. Nora, on the other hand, proposed three different types of activities that young children could engage in to begin their understanding of zero: through the absence of specified objects, as an empty set that proceeds one, and as the result of subtraction.

One of Nora's suggestions was to highlight the absence of a specific type of object on a page in a literature book. For example, in the book *Ten friends* (Goldstone, 2001) the description of an illustration as showing "2 teachers, 2 trolls, and 2 tycoons…" could be modified to read "2 teachers, 2 trolls, 0 frogs…". Nora explained that she specifically chose frogs because, although they did not appear on the current page, they were found on other pages in the book and thus had been a possibility for the situation and had connections to the students understandings of quantity. As an extension activity, Nora suggested that students could create their own pictures with statements describing the quantity of objects or things present; including what the students noticed that there was zero of. With respect to the historical development of the concept of zero, Nora's suggested activities intended to engage students in understanding zero as a quantity, much in the same way that the mathematicians of India came to understand zero (Seife, 2000).

A second type of representation for zero that Nora introduced was intended to help students understand zero as the quantity before one. She described how she could use manipulatives to represent the narrative of, "first I have zero, now I have one, now I have two…". She explained that, by having no manipulatives present for the first part of the narrative, students would come to understand that zero represents a set of objects that is empty. Although she did not refer to this notion as the "empty set", her activity was an attempt to have student construct an understanding of zero as the "null set" as emerged through the history of zero (Kaplan, 1999). Based upon the research of Inhelder and Piaget (1964), as well as Wheeler and Feghali (1983), the understanding that Nora sought to have students gain was one that students and pre-service teachers would have great difficulty in attaining; however, according to Leeb-Luneberg's (1973) and Wilcox's (2008) research would indicate that in fact young students could understand zero in this way.

Nora also extended this notion of the empty set to understanding place value. In *Anno's counting book* (Anno, 1977)*,* Nora noted how the inclusion of an empty set on the page about 10 could be used to support the students' learning of place value. In the book, 10 x 1 grids are shown on each of the pages associated with a number between 0 and 11. For example, on the zero page, none of the grid is coloured in, while on the page for 11, two grids are shown, one completely coloured in and one with only one square coloured. On the page for 10 Nora noticed that only one grid was shown and it was completely coloured in. To connect the students' understanding of zero being a null set to its role in the place value system, Nora argued that the page for 10 should in fact have two grids on it – one completely coloured in and one with no colour. In this example, Nora was brining together the roles of zero as a quantity and as a place holder, just as the development of the Hindu-Arabic number system historically did (Seife, 2000).

Finally, Nora reasoned that students could represent and understand zero in relation to subtraction. In this case, Nora described an activity she would use in which students would tell the subtraction story that could have resulted in a particular picture that she shows to them. For example, the story for a picture of trees without birds might be that “five birds were there and then flew away”. This understanding of zero is one that also emerged in India, and became very important centuries later in the advent of the two-column bookkeeping system (Kaplan, 1999). As well, Leeb-Lundberg’s (1977) research demonstrated that elementary students were capable of understanding zero as the subtraction of equal quantities.

Elaine liked Nora’s suggestions as activities that students could do, but questioned whether the students would actually understand zero from these experiences. Elaine spoke of young students being at a concrete stage, yet all of Nora’s forms of representation required students to abstract the notion of zero from concrete representations of quantities that are not zero. Elaine wondered how students could learn about zero concretely, in which they were able to “see” and “touch” zero. This quandary was one of the root causes for Greek society’s struggle with accepting zero as a quantity for hundreds of years (Barrow, 2000; Kaplan, 1999; Seife, 2000). Leeb-Luneberg’s (1977) and Wilcox’s (2008) research demonstrates instances in which young students could understand the abstract nature of zero, and Cockburn’s and Littler’s (1977) chapter 0 includes a number of activities and ideas for teachers to use that are similar in intention to those proposed by Nora.

3.6 Zero and student learning

When contemplating the current role of zero in elementary students’ mathematical learnings, Nora and Elaine focused on zero within place value and number compositions, and within computations. With respect to place value and number composition, Nora reflected that if she gave her students the numbers 45, 54, and 37, “they’ll see the 7 and think that the 37 is the bigger number”. Nora explained that this showed her that the students did not understand place value. This discussion prompted Elaine to add that when showing students numbers on a place value mat "If [it was] a number in which there was something in every place value [the students had] no problem. But as soon as I removed something off the matt," then the students believed that nothing existed there. Elaine explained that it was because the students do not understand zero as a quantity that they don't know how to deal with zero in place value in numbers. Nora agreed and said that this was the root of her students not being able to order the numbers correctly. Pasternack (2003), Evans (1983), and Baroody et. al. (1983) all provided similar evidence related to, but gained through different tasks, students' misunderstanding of zero in place value.

Nora and Elaine hypothesized that their students' lack of understanding about the role of zero in place value was also the source of their problems with naming numbers. They stated that the students were merely relying on procedures that they had memorized to name numbers. Elaine spoke of how many of her students "just lose [the zero]" when naming numbers, such as 204 being called twenty-four. Interestingly, Nora's students had said in the in-class session that zero was not a number, and one of the arguments that they provided was "you don't say it when you read numbers, but you say all the other number". One student then provided the example of 20 (twenty) and 21 (twenty-one). This student argued that if zero was a number, then it would be called 20 "twenty-zero". Previous research, such as that of Baroody et. al. (1983) and Kamii (1981), spoke of students ignoring the zero in naming numbers as Elaine had noticed, but the previous research did not indicate that students' reasoning in this regard might actually be a result of our naming conventions for numbers and the way that it treats zero differently.

Elaine and Nora generated a number of suggestions for what they believed to be important in students' development of an understanding of zero. First, Elaine stated that students must learn that although zero can act as a place-holder, it cannot be ignored because it indicates something about the size of the quantity. Nora agreed and added that place value should receive more emphasis in grade one. Both teachers felt that there was too much emphasis in the earlier grades on addition with not enough emphasis on number decomposition. Elaine commented “9 + 1 = 10 or ten plus one more is 11 … we’re not teaching [place value] – we’re teaching addition”. Elaine continued to provide examples, circling the addition sign (+) in every statement she wrote, explaining that the emphasis was on that sign and not on knowing the numbers.

In discussing the four operations on whole numbers, both Elaine and Nora argued for the standard procedures and algorithms to be deemphasized and left until later in the students learning. Instead, the two teachers felt that it was important that the students use their understanding of place value and number decomposition, as well as of the operations, to develop strategies for performing different types of calculation. Their concerns about student and teacher misconceptions were reflective of the findings in Ma’s (1999) study involving US and Shanghai teachers. Repeatedly throughout their discussions about the teaching and learning of the operations, the two teachers kept revisiting the importance of students being flexible in their understanding of the decomposition of numbers (e.g., recognizing 204 as 20 tens and 4 ones, 19 tens and 14 ones, etc) and the role of zero in place value. Both teachers also emphasized the importance of never giving the students the impression that zeros were being ignored or dropped off.

4. DISCUSSION

The past and present horizons of understanding that emerged from the dialogues and experiences that Nora and Elaine engaged in during this study had many parallels as well as some departures from what previous literature had noted for students, pre-service teachers and teachers. This section summarizes those similarities and differences and then proposes a framework that could be applied to future research that is analyzing one's understanding of zero.

4.1 Results summary

Through the dialogues and experiences the uncovering of the past and present horizons of understanding of zero for both Nora and Elaine highlighted understandings that were both correct and incorrect. Just as had been previously noticed for students (Baroody, et. al., 1983; Allinger, 1980; Whitelaw, 1984) and pre-service teachers (Wheeler & Fegahli, 1983) Nora struggled with her understanding of zero as "nothing", but she was able to adjust her definition to "nothing of something" as had also been done by Leeb-Lunburg's (1977) students. Alternatively, Elaine sought to make sense of zero being a starting point in all situations, resulting in zero not necessarily being a number, just as pre-service teachers argued in Wheeler & Feghali's (1983) research.

For both teachers, zero in computations was not well understood and was a source of frustration in trying to teach students. Many of these same issues had been raised by Ma (1999), Neuwirth Beal (1983), Evans (1983), Anthony & Walshaw (2004), and Wheeler (1987) with respect to teachers and students involved in their research. Nora and Elaine's emphasis on the importance of understanding zero as part of place value and number decomposition was also a finding of Pasternack (2003), Evans (1983), Baroody et. al (1983), and Kamii (1981).

Elaine and Nora brought forward the use of the word "oh" in the place of "zero" and the resulting confusion between the two concepts by students which was a finding in the research of Wheeler and Feghali, (1983) for pre-service teachers as well as for students in the research of

Allinger (1980), Baroody et. al. (1983) and Whitelaw (1984). The negativity associated with the word zero that Nora and Elaine noted can also be found within the research of Allinger (1980).

The two teachers engagement in the dialogues and experiences also revealed insights and ways of perceiving zero that did not emerge in previous reported research. The use of a hermeneutic approach allowed the teachers to explore the reasons why they held the beliefs that they did about zero, and as a result caused them to also question the validity of what they knew and had been told. The result was that the teachers sought to identify cases where zero had not been included, both within their learning as a student and in society in general, and to propose ways to rectify the situation. In many of these instances, the teachers were not aware of the cultural history that had led to the marginalizations of zero that they perceived. Elaine and Nora also grappled with how zero could be represented and understood by elementary students, taking them into an exploration of their beliefs about cognitive readiness and pedagogy with relation to zero.

Finally, Nora's students brought forward an understanding of zero which had not been reported previously – that zero is the same thing as any circle. Although some research (Allinger, 1980' Baroody et. al., 1983; Whitelaw, 1984) mentions the exchange of the word "oh" for "zero", there is no discussion of students assuming that because the symbol for zero was circular in nature that zero must be in itself a circle.

4.2 A way to conceptualize the understandings

As the group meetings and in-class sessions proceeded, it became evident that the teachers were exploring and struggling with knowledge that had evolved over time and that had resulted in two different, yet related categories of understandings: procedural and technical understandings of zero, and philosophical and theoretical understandings of zero (see Figure 2).

The philosophical and theoretical understandings of zero are the result of both societal conventions related to zero as well as the theories and axioms of meaning given to zero by past and contemporary academic mathematicians. On the other hand, the technical and procedural understandings of zero are the routines carried out in mathematical situations that involve zero. These two categories are related to each other in that the technical and procedural understandings of zero can be directly explained by the philosophical and theoretical understandings of zero. For example, why one puts zeros in when doing multi-digit multiplication is a direct consequence of the theoretical definition of the place value of the quantities being multiplied.

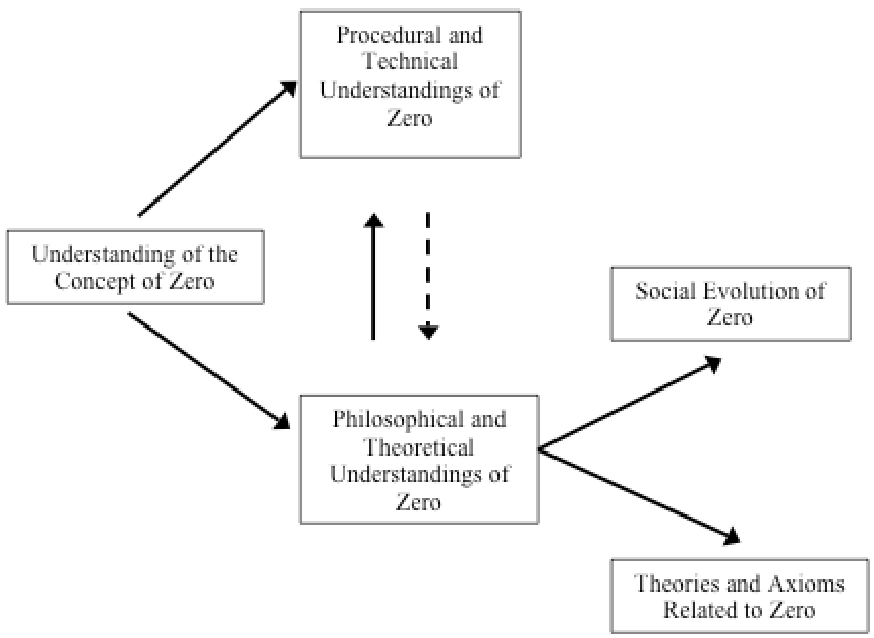

Figure 2. A Conceptual Framework for Viewing Teachers' Conceptions of Zero

Initially, it was the researcher's view that one's procedural and technical understandings would be supported by one's philosophical and theoretical understandings of zero, however, Nora and Elaine tended to have procedures and techniques that they used, but they did not have the understanding of why they should use those procedures and techniques. Consider even where the two teachers began in their memories of learning about zero. Nora believed it was nothing (something to be ignored) and Elaine believed it was the starting point. Both conceptions of zero were based on nothing more than a single "fact" that had been stated without any evidence of reasoning or support.

As adults, they had begun to construct possible philosophical and theoretical underpinnings for their procedures and techniques. In many cases, such as Nora's desire to write "10" rather than just "carry the 1" in addition, and Elaine's emphasis on number decomposition before starting in on operations, can be seen to be developing conceptualizations of the theoretical and philosophical foundations of understanding zero. However, there are many cases where the two teachers' attempts at building theoretical and philosophical reasoning for their technical and procedural understandings conflicted with the true theoretical and philosophical understandings of zero within mathematics. This often caused the two teachers to venture into "reforms" to the way zero is used and known that, although interesting, are neither practical nor informative from a mathematical perspective. Elaine, believing that zero must be a starting point, which it need not be, becomes trapped in an exploration to "convert" the world around her, while Nora, believing that zero should be everywhere where any other number is, seems ready to begin a crusade to bring zero to its rightful place of respect. Neither teacher is aware of the conventions, defined by society over time, that have brought rise to these situations that they desire to change. As a result, their efforts in trying to get to understand zero better become derailed by their imaginations and misconceptions. Thus, it would seem that the interplay, or lack thereof, between the two categories within understanding of zero, philosophical and theoretical, and technical and procedural, can impact the depth and accuracy of understanding that one has about zero..

5. CONCLUSION

The concept of zero is a complex mix of social evolution, theoretical mathematics, and procedures. As a result, the interplay between these aspects of the understanding of zero is of great importance in developing a cohesive and mathematically correct understanding of the concept. Reflecting once more upon the research related to students' and teachers' understanding of zero done prior to this study, it appears that Nora and Elaine's abundance of procedural and technical understandings, without strong philosophical and theoretical understandings of zero, may be the status quo for students and teachers alike.

Understanding zero is foundational to understanding mathematics. Whether it be place value and whole number computational situations as Nora and Elaine discussed, or other topics such as locating fractions on a number line or understanding division by zero and asymptotes of functions, misconceptions about zero can lead to students failing to learn key ideas in mathematics and teachers struggling to try to correct the situation without understanding it themselves. Research such as Leeb-Luneberg (2003) and Wilcox (2006), and this study have shown that given interactive, contextualized, and meaningful learning experiences, both students and teachers can learn to better understand zero. Zero needs to become more than "nothing" within the classroom and in pre-service education programs.

The research that this article is based upon considered a very limited sample, only two teachers; however, it does reveal some findings that are new, promising, and as such, warrant further investigation. Such future research, involving a larger and similar population, may very well provide insights into teachers' philosophical and theoretical as well as procedural and technical understandings of zero, which in turn could help to better inform pre-service teacher education programs and in-service teacher professional development in relation to the creation of robust understandings of zero by pre-service and in-service teachers.

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