The Mathematics Enthusiast

Volume 8
Number 1 *Numbers 1 & 2*

Article 17

1-2011

Disrupting gifted teenager’s mathematical identity with epistemological messiness

Paul Betts

Laura McMaster

Disrupting gifted teenager's mathematical identity with epistemological messiness

Paul Betts, University of Winnipeg, Canada

Laura McMaster, Miles Macdonell Collegiate, Canada

Abstract. Mathematics is widely perceived as a universal and uncontested discipline, contrary to the philosophy of mathematics literature. Other researchers have considered the potential role of philosophy in school, but there is little work with gifted students engaged with issues concerning the nature of mathematics. We developed a philosophy of mathematics unit intended to enlarge gifted students' perceptions of the nature of mathematics by exposing the uncritical and tidy rendering of mathematics within school math. Using a narrative methodology, we attended to gifted student's students' stories of relationship with mathematics, based on the premise that a person's relationship with mathematics is inextricably woven together with their identity. In this paper, we will focus on the experiences of three gifted teenagers during our philosophy of mathematics unit. We found that these students were disrupted and compartmentalized their school math and philosophy of mathematics experiences and beliefs. We conclude that substantive experiences with the nature of mathematics should be a regular component of school math.

Key words: philosophy of mathematics, gifted high school students, mathematical identity, narrative

INTRODUCTION

> For me, yeah I don't like it – grayness in math. I think of math as right or wrong (Dorothy, a high school student in the IB program).

It is well known that mathematics is perceived as a universal and unquestioned body of knowledge. This positioning of the nature of mathematics in wider society is reflected in curricular documents and in the teaching of mathematics. Curriculum recommendations do not, to our knowledge, ever refer to possible philosophically-based goals or learning outcomes (see, for example, National Council of Teachers of Mathematics, 2000; Western and Northern Canadian Protocol, 2006; Manitoba Education, 2008). Teachers, en masse, believe that mathematics is absolute (at a superficial level), and reproduce these beliefs among their students (Philipp, 2007).

The Montana Mathematics Enthusiast, **ISSN 1551-3440, Vol. 8, nos.1&2**, pp.329-354

Contrary to the popular positioning of mathematics in school math and the wider society, philosophers have debated the epistemological status of mathematics at least as far back as Socrates. These debates revolve around the questioning of mathematics as an absolute body of knowledge and are far from resolved. Various fallibilist positions have been developed by mathematicians (e.g., Davis & Hersh, 1981), mathematics educators (e.g., Ernest, 1998), philosophers (e.g., Lakatos, 1976), and cognitive scientists (e.g., George Lakoff & Nunez, 2000).

Given the premise that school math is presented as a neat, tidy and undisputable collection of facts, we developed a "messy" conception of the nature of mathematics, and then developed activities intended to explore this messiness with gifted teenagers. Our goal was to use messiness to expand gifted high school students' conceptions of the nature of mathematics. Not surprisingly, given many years of exposure to a narrow and tidy vision of mathematics in school, the gifted students we worked with struggled to make sense of mathematics as messy.

Philosophy based programs of study for children and young adults are not a new idea. For example, the Philosophy for Children (P4C) program was initiated in the seventies by Lipman, premised by the idea that children and young adults can think philosophically, and so philosophy should not be relegated to college-level study (Lipman, Sharp, & Oscanyan, 1980). These programs also tend to share the following qualities: (1) pedagogy is rooted in open dialogue, where a context, such as a story (e.g., Lipman, 1988;), or a story beginning (e.g., Matthews, 1984) is used to trigger a teacher-facilitated discussion of a philosophical issue; and (2) content is usually focused on general philosophical issues such morals, ethics, truth, and rarely considers discipline-based issues. The effects of these programs have been well documented. In general, these programs improve the thinking (e.g., Naji & Ghazinezhad, 2008) and other curriculum-based skills of students (e.g., Trickey & Topping, 2004).

Specific issues arise when considering exposing children and young adults to ideas from the philosophy of mathematics. Given that mathematics is perceived as (superficially) absolute within school math and by the wider society, what is there to discuss philosophically? Daniel et.al. used the P4C model to develop a philosophy of mathematics program (called P4CM) for children and young adults (Daniel, Lafortune, Pallascio, & Schleifer, 1999). Stories with mathematical content were used to trigger open dialogue concerning philosophy of mathematics. They found various kinds of evidence that participation in P4CM is beneficial; for example, negative attitudes toward mathematics are reduced (Lafortune, Daniel, Pallascio, & Schleifer, 1999). Others have successfully implemented variations on P4CM. For example, while working with junior high students, Martin (2008) used a story that raises the issue of making a perfect cube to trigger a conversation about whether a perfect cube actually exists; the ideas within the conversation of these students were consistent with ontological views of Aristotle and Plato.

For all of these philosophical programs, questions remain concerning the process of students' development of enhanced thinking. In particular, while participating in open philosophical dialogue, students would be individually making sense of philosophical issues. There would likely be changes in their informal, implicit personal philosophies. What sorts of changes might occur and how do they occur? These questions apply equally to gifted and the general student population. In particular, it is not clear in what ways gifted students would respond to a unit of activities focusing on issues concerning the philosophy of mathematics. For example, we found various positioning of gifted students toward a philosophy of mathematics unit, including confusion, resistance and engagement (McMaster & Betts, 2007).

We wondered if the gifted students we work with would be more or less open to entertaining alternative visions of mathematics. Would they be responsive? Would their gifted abilities contribute to or hinder their responsiveness? We considered this research to be a first foray into these questions, and therefore deliberately decided to take an exploratory approach. Because we hoped to expand students' perspectives of mathematics, because we believed gifted students would be able to handle ideas that would appear foreign to their past experiences with mathematics, and because our activities come near the end of a course on philosophy, we did not expect to be disruptive of their relationship with mathematics. Our efforts disrupted student relationships with mathematics, but it is unclear whether their perspectives were enlarged in a stable way. In this paper, we will look closely at how three gifted students adapted to the disruptions triggered by a "messy" rendering of the nature of mathematics. We will suggest that these students navigated the disruption by compartmentalizing their experiences, which likely allowed them to protect their identities in relation to mathematics.

RESEARCH METHODS

In this research project, our goal was to expand our IB student's appreciation of mathematics. To detect this goal, we used a narrative methodology (Clandinin & Connelly, 2000) – we sought to detect student's stories of identity in relation to mathematics. Narrative assumes that we use story to make sense of experience and that experience is storied (Clandinin & Connelly, 2000). Hence, we looked for student stories that suggested how they were positioning themselves in relation to the ideas presented during our philosophy of mathematics unit.

A narrative approach is suitable to the nature of our research questions. This paper represents results from an initial project concerning gifted student's positioning of self and others that are triggered by experiences with the nature of mathematics. We are less concerned with what students know or learned about mathematics or the nature of mathematics. Rather, we sought to understand the role of their identities (as learners and gifted students) as they struggled with novel ideas concerning the nature of mathematics. Our focus is on experience and identity; hence the uses of a narrative approach. In what follows, we describe the participants and their context for this study, methods of data collection and analysis, and our philosophy of mathematics unit based on epistemological messiness.

Participants and context

The students we worked with were enrolled full time in the International Baccalaureate (IB) program, and in their final year of high school. We consider these students to be gifted because they are high performing academically and highly motivated to be successful. The IB program is a "demanding two-year curriculum that meets the needs of highly motivated students" (International Baccalaureate Organization, 2005-2009c), and so it is considered an advanced placement program of study, attracting students with the highest grades in regular studies. At the very least, all the students were academically precocious, based on grades. The three students we focus on in this paper are among the best in the IB program at this school. Dorothy is a multi-sport athlete with high marks in all subjects, and scored in the top 15% in English among all IB programs in the world. Mary consistently receives the highest grades in all subjects among all students in the IB program in her school. John scores high marks in all courses, and is considered brilliant in math and science by his teachers.

The IB program is implemented by high schools around the world, all following an academically advanced and standardized curriculum (International Baccalaureate Organization, 2005-2009b). The IB curriculum includes a course called Theory of Knowledge (ToK), which focuses on ways of knowing, including epistemological issues specific to major disciplines (International Baccalaureate Organization, 2005-2009a). The philosophy of math unit within ToK is an ideal location to introduce the notion of epistemological messiness. All students who participated in this project were enrolled in the Theory of Knowledge course, as well as other IB courses that would be considered advanced versions of standard high school courses, such as Math, Science, and English. The philosophy of mathematics unit came near the end of the ToK course, so general ideas (e.g., Plato's Forms, aesthetics) where available to apply to the particular case of the discipline of mathematics.

<u>Data collection and analysis</u>

Stories of student identity in relation to mathematics were constructed from data collected before, during and after the philosophy of mathematics unit. Before beginning the unit, we interviewed each participant, seeking to establish their appreciation of and attitudes about mathematics. These interviews revealed what we expected: mathematics is absolute and so why would there be a need to consider philosophical aspects of mathematics. Thus, we knew at the start of the unit that students would tend to story the experience with narratives such as "math is inaccessible" and "math is black and white." We speculated that these stories would be intimately tied to how they made sense of ideas. For example, Plato's Forms may prop up a student's identity of "yes, math is inaccessible." We also suspected that students would need to negotiate tensions between ideas developed during the unit and their life of experiences with mathematics dominated by just one vision of mathematics, namely, math is a perfect and uncontested body of knowledge.

During the unit, we asked students to write a reflective journal at the end of each class. These served a pedagogical purpose: they provided us with insight on student thinking for assessment purposes, allowed us to provide feedback to students for the purpose of encouraging further elaboration of their ideas, and were used to showcase student ideas in subsequent classes. The journals were also used for research purposes. They became a source of data for detecting student's stories of identity in relation to mathematics. We also kept field notes of interesting conversations that occurred during the classes, which also served as a source of data.

After the unit was complete, we selected ten students to participate in an in-depth interview. We used two criteria to select candidates. First, we sought candidates that seemed to display exceptional giftedness. Although this tended to correlate with grades, we looked for students who displayed exceptional thinking during classes, such as an ability to develop an idea or the soundness of their ideas. Second, based on the journals and field notes, we tried to select candidates with differing reactions toward the unit. For example, John aggressively accepted Formalism throughout although began to consider Embodiment at the end of the unit, whereas Mary quietly embraced Platonism throughout, whereas Dorothy seemed undecided throughout but tentatively considered Proofs and Refutations (see next subsection for descriptions of these philosophical positions). The final interview lasted about an hour, was open ended, and focused on encouraging and challenging students to describe and develop their views concerning the nature of mathematics.

We decided to describe the stories of three students, Dorothy, Mary and John. We believe that each of these students are quite different, and taken as a whole are reflective of the diversity of the class. They are also among the most gifted of the students who participated. And yet, despite the diversity suggestive of our small sample, we found a common theme in their stories, namely, a navigation of disruption of their identity in relation to mathematics. In the next sections, we will try to illustrate these stories of navigation of disruption.

Data analysis proceeded in two phases. First, we focused solely on developing the story of each of the three students (and the other students who participated in the post-interview), without comparison. We each developed a story and then, through a process of dialogue and reexamination of the data, we came to an agreement on each story. Our differing perspectives as regular-teacher-of-these-students and researcher-from-outside-the school were complementary, and, we believe, adds to the trustworthiness of our interpretations. From these stories, we selected three for further analysis. We then looked for themes across the stories of the three case participants. It was during this second phase that we came to agree on the theme of disruption, and when we began orienting their stories as one of navigating disruption.

An epistemologically messy philosophy of mathematics unit

Numerous mathematicians have described the work they do using journey or process metaphors, which belie the neat and tidy presentations of mathematics found in most expository texts, including school math text books. For example,

> *When asked what it was like to set about proving something, the mathematician likened proving a theorem to seeing the peak of a mountain and trying to climb to the top. One establishes a base camp and begins scaling the mountain's sheer face, encountering obstacles at every turn, often retracing one's steps and struggling every foot of the journey. Finally when the top is reached, one stands examining the peak, taking in the view of the surrounding countryside and then noting the automobile road up the other side! (Kleinhenz, 2007)*

What is described in math textbooks and taught in school math classes is the "automobile road up the other side," which clearly hides most of what it means to do math. And yet, if students are to appreciate math (a goal found in all curriculum documents we are aware of!), then they should experience the doing of mathematics. The quote above begins to question a tidy rendering of doing mathematics – there are frustrations, false starts, back tracking, and numerous other accomplishments and setbacks along the way. At the very least, problem solving is more than a linear sequence of steps, and there is always more to do even after a problem is solved. This is a starting point for recognizing that school math experiences hide philosophical issues. It is with this rejection of the tidiness in the representations of school math that we use as a starting point for the nature of mathematics as messy.

We wish to be critical of a tidy vision of the nature of mathematics that seems to be universally propagated by school math. Mathematics as a perfect and uncontested body of knowledge is a tidy position – there is no uncertainty and hence no messiness. But numerous philosophers have questioned the certainty of mathematics. For example, Davis and Hersh (1981), who are mathematicians, suggest that mathematics is a human endeavor, and hence subject to the same fallibilism as any human endeavor. Ernest (1998) developed a fallibilist position by drawing on a social constructivist perspective. Although school math does not explicitly present a philosophy of mathematics, its tidy enactment commonly engenders superficial absolutist positions among student's personal working philosophies.

We developed a philosophy of math unit based on exploring four distinct philosophical positions concerning the discipline of mathematics. Each of these positions were given credence as viable philosophies, where deeper explorations of each were intended to invite students to attend to and critique the tidy renditions of school math common to their mathematical experiences. First, we broadly distinguish between Absolutism (math is universal and infallible) and Humanism (math is fallible). We then developed two example positions for each broad category: Platonism and Formalism for Absolutism, and Proofs and Refutations and Embodiment for Humanism.

We gradually developed each of these positions through a series of activities, each activity usually built from a specific high school math context but examined from a philosophical perspective and with minimal attention to teaching the mathematics involved (we ensured that the mathematical concepts explored were familiar to students). A messy rendition for the nature of mathematics emerges in two ways: each philosophical position by itself carries numerous opportunities for critique of the neatness of school math, and the availability of multiple positions for the nature of mathematics is an opportunity to perceive the philosophy of mathematics as a contested body of knowledge. In what follows, we provide a brief description of each position, and of one example activity, to illustrate the content of our philosophy of mathematics unit [see Betts (2007) and McMaster & Betts (2007) for more detailed descriptions of our philosophy of mathematics unit].

Platonism is an absolutist position based on Plato's "allegory of the cave" and Plato's "Forms" (Govier, 1997). Forms are the ideal – a universal representation of the particulars accessible to humans. The cave allegory suggests that humans perceive only shadows of perfection – being the Forms – and are chained down, unable to be free of the cave to experience perfection. So, in particular, mathematical concepts, such as the fraction ½ and a drawing of a line, are but imperfect representations of the Form for a concept. School math pretends to present ideas as if they are ideal. A line is drawn as if it is a "perfect" line, rather than as a representation of a line that is good enough for the purposes of the current mathematical argument. Platonism can trigger critique of the neatness of school math because humans cannot access the ideal – the Forms – and hence must account for the imperfection of a human representation of a mathematical idea. Erdos, one of the most prolific mathematicians ever, was a proponent of Platonism (Hersh, 1997).

Formalism is also an Absolutist position, and is based on the premise that error enters into mathematics when its ideas are operationalized in human contexts (Hersh, 1997). For example, Russell's paradox arises because it is represented using language, and so is subject to the fallibility of language. Mathematicians such as Hilbert set out to formalize mathematics as a symbolic system independent of language (Mancosu, 1998). In essence, mathematics is a set of symbols and rules for manipulating these symbols, which have no meaning in the real world. According to the mathematician Hardy, only pure mathematics, mathematics that is unconcerned with application in the real world, is real mathematics (Hardy, 1992). Students can engage with the idea that mathematics does not come in a perfect package; rather, mathematicians have worked hard to remove error from mathematics. Hardy would argue that school math is not real mathematics, which can lead students to question the tidy renditions of school math as misleading.

One of the Humanist positions is based solely on the ideas developed by Lakatos in his book Proofs and Refutations. Lakatos used the historical development of Euler's formula to describe various iterations of the following process: conjecture, proof of conjecture, refutation of conjecture (e.g., by counter example; critique of proof, definitions, and/or axioms), leading to a new conjecture (by modifying definitions, axioms, and/or the actual conjecture) (Lakatos, 1976). This position is messy in two ways. First, a result is sanctioned by the mathematical community not just by proof, but by a process of error detection and adjustment to account for the error. Mathematical decisions can be based on criteria other than logic, such as aesthetics. Second, a theorem can always come under scrutiny, even if it has been sanctioned as true by the mathematics community – in other words, we can never be 100% sure that a conjecture and its proof is true because another refutation may arise in the future.

The other humanist position, which we call Embodiment, is based on the ideas of Lakoff and Nunez. The reader should consult other writers for more detailed descriptions of embodied cognition in general (e.g., G. Lakoff & Johnson, 1999) and as it relates to mathematics (e.g., George Lakoff & Nunez, 2000). A key principle is that mathematical ideas start from our experience as humans and are built up via a series of metaphorical mappings. For example, the notion of continuity of the real number line comes from our embodied experience of motion. The real numbers is a discrete and infinite collection of numbers, but is also represented as a continuous line. We can manage these realizations of real numbers because we can experience continuous motion between two points, which is also a travelling of an infinite number of discrete points (e.g., the halfway point). The embodied experience of motion from A to B is metaphorically mapped onto the notion of an interval of the real number line, such as all real numbers from 0 to 1. An embodied vision of mathematics is messy because the idea that mathematics is universal and independent of humanity is completely rejected.

One of the activities we used near the beginning of the unit involved the circle. We asked the students to come up with more than one answer to the following question, and to be able to justify their answers: How many sides does a circle have? We know of 5 distinct and mathematically viable answers to this question, of which, we will describe three: (1) no sides because sides are straight and a circle is curved; (2) one side, which is the edge going all the way around the circle; and (3) infinitely many, because a circle is the limiting case of a regular n-sided shape as n approaches infinity (in the limit, there is an infinite number of sides, each of length 0). After generating a list of answers that seemed mathematically correct and trying to justify which answer could/should be the correct answer, we asked students to reflect and discuss what this situation means for the nature of mathematics.

This activity allowed us to develop several philosophical issues, based on the ideas of students. If, for example, we pick one answer, how do we know for certain it is correct, which allows us to point to a broad distinction between Absolutism and Humanism. The distinction arises because of the potential for opting for an answer that later turns out to be rejected – does this mean that mathematicians can eventually remove all error with careful analysis, or is mathematics a human endeavor so that it must be a fallible body of knowledge. Another issue arises concerning the inaccuracy that must arise in drawing a circle, which leads to the idea of a perfect circle and Plato's Forms. Finally, in the debate about which answers to accept, the issue of agreeing on the definition of a side arises, leading to a discussion of Lakatos' heuristic, where we consider the refutation of an idea through the contesting of a definition.

The circle activity is not immediately used to illustrate Embodiment. The Embodied position is difficult to develop because it is based on attending to subtle and taken-for-granted aspects of human experience. After an initial encounter with Embodiment that is not grounded in a mathematical context, we revisit previous examples for evidence of this position. For the circle example above, we wonder how to imagine how n-sided shapes of increasing n approach a circle but the circle doesn't disappear in the limiting case. How do we do this? We can't draw a circle as an infinite number of sides of length zero. But we can experience a circle as a continuous curving line that loops back onto itself, which is metaphorically mapped onto the limiting case definition of circle.

The example above also illustrates the pedagogical principles used to implement our philosophy of mathematics unit. We followed teaching ideas used in the P4C model. In particular, we sought to establish an environment where open dialogue concerning the nature of mathematics was facilitated. We encouraged students to state and defend philosophical positions. We resisted the urge to tell students about the philosophical positions of others or to suggest a "best" position. We considered it tantamount that students not consider us, as teachers, to be the final arbiters of a correct philosophical position or argument. Rather, we sought opportunities to validate student thinking by labeling their ideas as following a specific philosophical position. So, for example, when a student argued that the circle example above suggests that mathematical results change over time, we suggested that their position was similar to that of Lakatos.

Students were encouraged to and did begin to develop their own personal philosophical positions concerning the nature of mathematics, rather than merely reproducing ideas from us. Our pedagogical emphasis on dialogue and refusal to sanction one philosophical position as correct led students to think deeply about the philosophical implications of the mathematical contexts we explored and about their own experiences with mathematics. Students had no difficulty applying to the case of mathematics ideas previously developed in their philosophy course. We noticed students suggesting that aesthetics is an important consideration, leading to an interrogation of proof as the only arbiter of mathematical truth. Students also were able to critique mathematical aspects of the contexts we presented. The idea that we must decide on the meaning of a side during the circle activity above was brought forward by the students without prompting from us. With some scaffolding we were able to help students notice superficial uses of Humanism and Absolutism. They, for example, began to recognize that falliblism is not the same as solipsism. Most significantly was the critical thinking inherent in the questioning of students: How do we gain knowledge of the Forms if we can only access the imperfect – if we are trapped in the cave? Why isn't it possible for mathematical results to be certain even though they emerge from human experience? These questions respectively represent a critique of Platonism and a synthesis of Absolutism and Embodiment. The students, in general, did engage with philosophical ideas, critique the neatness of school math, and begin to appreciate the philosophy of mathematics as a contested body of knowledge [see McMaster & Betts (2007) for further details].

RESULTS - DESCRIBING EACH STUDENT'S STORY OF NAVIGATING DISRUPTION

Our focus in this paper is not what students learned. We saw significant evidence that the students engaged with philosophical ideas, and take it as a given that they learned about philosophies of mathematics. Our focus on student identity leads us to notice how their thinking about the nature of mathematics was intricately woven together with their relationship with mathematics and their identity in general. Philosophical ideas concerning the nature of mathematics are not evident within these students' prior experience, but their identities do matter as they take-in and work with these ideas, independent of their common experiences of math as a neat collection of facts and rules to follow. We found that many student's identities in relation to mathematics were disrupted. In previous work, we developed a general description and characterization of the disruption for all students who participated in our philosophy of mathematics unit (see McMaster & Betts, 2007). In this paper, we focus on a deeper description of how three students navigated the disruption of their identity in relation to mathematics. Each student started the unit believing mathematics was tidy and uncontested, and this belief was implicitly challenged by the activities during the unit. For each case, we try to establish a chronology for each story, based on their identity before the philosophy of math unit (as per pre-interview), during the unit (journals and in-class observations), and after the unit (as per post-interview).

DOROTHY

Before the philosophy of mathematics unit began, Dorothy expressed a joy for learning in general and math in particular. She expressed a real satisfaction in obtaining the right answer in math, which is a feeling that she has valued since early elementary school. She prefers certain branches of math, such as algebra and trigonometry, over others such as probability, because she doesn't like having what she calls "options" in probability. She also likes the process of working through a precise sequence of steps, where that process is clear and linear, a process she describes as "exactly how things fall into place." She wants to know how it works, but only wants it to work in one way. One right answer is what she wants. At the conclusion of this pre-interview, she states, though not rudely, that she just wants to "stop talking about math." Dorothy is interested only in the business of doing math that involves arriving at the right answer, and finds it uncomfortable and disconcerting to delve into the philosophical issues that accompany it. Dorothy carries a tidy rendering of the nature of mathematics: results are either right or wrong, there is one method for solving each problem, and each method is essentially an algorithm. Dorothy's implicit personal philosophy of mathematics is a superficial form of absolutism, propped up by her success with doing school math.

During the philosophy of mathematics unit, Dorothy freely expresses feelings of confusion. She is clearly not comfortable with this feeling, and tends to seek simplistic resolutions to the issues presented in class. For example, in an early journal she wrote: "It is just simpler to accept what we are told than to dispute it." She does not want to enter this debate at all, but since she is required to, she advocates for math that is simple and useful. In a later journal, she agrees with the Humanist position because it is simple, not just from a mathematical point of view, but from a human point of view. She wrote: "We must make adjustments to mathematical concepts for sheer simplicity in life." She objects to discussing the issues. This discussion frustrates her because she does not see its purpose. We believe this is because she has been indoctrinated to be very goal oriented, rather than to see the value of the messy discussion that we undertook in class.

After the philosophy of math unit, Dorothy was still reluctant to talk about the nature of math, and seeks to keep issues of messiness in math from threatening her prior experiences with a right/wrong dichotomy approach to math that she has been trained to value. She is willing to discuss various elements of math as long as those discussions do not threaten what she sees as math's fundamental operations, such as how formulas work, or whether they work. She is so comfortable with the right/wrong approach to math that she is only willing to discuss the issues underpinning math if she can consider them like separate issues that don't threaten what she feels she actually does in math class - different philosophical positions can call different issues into question but that doesn't mean that there is more than one answer to problems she is asked to solve in math class. Dorothy seems to be able to compartmentalize math to make these discussions feel safe to her; that is, we can talk about the philosophy of math as long as it doesn't prevent her from being able to seek the right answer to a math problem.

Throughout her post-interview, the interviewer challenged Dorothy to consider her position more critically, especially her tendency to agree with both Absolutism and Humanism as acceptable philosophies of mathematics. For example, she wants math to always produce one right answer (Absolutism), but she also wants math to be personal, under the control of the person performing mathematics, and describes the evolution of mathematical knowledge in humanist terms. She becomes aware that her position is untenable, but this is not enough for her to change her position. We believe this illustrates just how deeply seated her ideas about math are.

When challenged further to defend her position, Dorothy's sense of security, sourced in following rules, procedures, using formulas, and getting to the right answer, is threatened. Throughout the interview, she repeatedly changes the topic, laughs, flirts, indicates she doesn't care about the issues raised, and tries to brush off the interviewer. For example:

> Interviewer: So there is no interpretation or opinion in mathematics?
>
> Dorothy: No [laughs].
>
> Interviewer: And yet you did talk about grayness coming into our philosophy of math?
>
> Dorothy: Ahhh okay [laughs].
>
> Interviewer: Your turn.
>
> Dorothy: Noooo [laughs] it shouldn't be my turn!

Given the frequency of these exchanges, we don't believe these comments are random – Dorothy is profoundly uncomfortable. At one point, she even makes a borderline inappropriate comment (i.e., "Men!") which targets the interviewer. We see this as additional evidence of her attempts to get out of the tight spot in which she finds herself. In addition, she sees the discussion itself as combative, even saying at one point to the interviewer "You win". The student is uncomfortable, defensive, and almost rude. A process that she has found satisfying and which has fed her self-concept regarding mathematics, the process of targeting and then obtaining a right answer, is being seriously challenged, and she is seeking ways to bail out. Her desire to avoid the issues altogether is closely connected to Dorothy's need for control. If the philosophy of math is integrated into her mathematical experience, she feels a loss of control, and literally doesn't know what to do. A huge source of her feelings of academic success becomes threatened, and her self-concept along with it. She tries to avoid issues from the philosophy of mathematics to keep it separate from her experience of school math.

MARY

Based on the initial interview, Mary believes that she does well in math because she works hard. She doesn't believe she is good at mathematics, although she does enjoy doing mathematics and that enjoyment seems to be strongly tied to her experiences of success with math due to hard work. Mary does what she is supposed to do in math class (and in all courses). She accepts the knowledge of the instructor at face value and without question – the math that is taught in school was developed by mathematicians in the past and is true without question. There is no need to question the results of mathematics. Mary is happy with this state of affairs because it is easy to figure out what responses are correct, so that she can be successful in terms of grades and feel good about her hard work. Mary perceives mathematical results as either right or wrong, which are sanctioned by teachers as the communicators of the work of mathematicians. This is a tidy rendering of the nature of math because of the simple relationship between teachers and mathematicians and the unquestioned acceptance of the ultimate and universal truth of the mathematics learned in school. For Mary, these beliefs about school math extend to all of mathematics.

Mary's conception of mathematics was challenged during the philosophy of mathematics unit. We presented the idea that mathematics might not be absolute and that humans might be inextricably implicated with what is considered true in mathematics. Now, Mary must face the possibility that the canons of mathematics, which she is so successful at reproducing on math tests, might not be so certain. She faces the possibility that the nature of mathematics involves uncertainty, which causes a problem for her desire to detect and reproduce right answers.

Mary adapts to the discomfort caused by epistemological messiness in two ways. First, she keeps mathematics at arm's length. For example, during the final interview, she said:

> I also agree with the fact that math has always existed and is not created by human beings or anyone else. When we, as humans, find out some new mathematical concept, we are really just discovering something that was always there.

This quote is representative of Mary's position in two ways. She rarely used "I" to state her position, and when she did, she reverted to "we" (which was much more common), as if to distance herself from the position. Further, the quote represents Mary's belief that math is ubiquitous – it is "everywhere." Mary deliberately places mathematics outside her personal experience, and the only reason she experiences math is because "we" cannot help bumping into it – it is needed for "us" to "survive." Keeping mathematics at arm's length is comfortable for Mary. It allows her to keep mathematics as objective and separate from us, which protects her comfortable acceptance of the absolutism of mathematics.

The second way Mary adapts to her discomfort during the unit is to be slow to commit to an answer or to sit on the fence. For example, in the first journal she wrote:

> It can be argued whether math is independent and can act alone or if it needs language to exist.

In the last journal, when asked to pick one of the four camps, she wrote:

> My philosophy of math is Platonism, as it is the philosophy of math that makes the most sense to me. I feel that there is not really one philosophy of math that is completely right.

In the first quote, she states a contentious issue, but will not take a position. In the second quote, she selects a position but makes a qualification. Throughout the final interview, she was slow to answer, tried to give short and non-committal answers, and would qualify with phrases such as "I'm not sure." The only idea that Mary would commit to was that "we" can never be "sure." She uses uncertainty in general to protect her belief that mathematics can be certain. She qualifies or doesn't commit because she is looking for the school sanctioned right answer to reproduce. The Theory of Knowledge course reinforced the idea that knowledge is never certain. Mary is doing what school has taught her to do, namely, to reproduce the right answer.

When Mary does commit to a position it is because there is a strong emotional connection to her zone of comfort with mathematics. In the final journal, when reflecting on whether school math has influenced her beliefs, she wrote:

> Although high school math has been a major influence on my beliefs of Platonism, I think my personal traits and the way in which I think also contribute

> to my Platonist views. I like things that are black and white that give me definite answers. I do not want to be caught in a no man's land, as I will not know what to do because I will not know what the right thing to do is. Platonism tells me that concepts have definite answers. This is what makes me happy because I will know what I am doing, and can tell if I am doing the right thing.

A Platonist view of mathematics is a security blanket for Mary. During the final interview, when pressed on this issue, she admitted as much. Formalism is rejected because math loses its real life ubiquitous nature (she is perhaps worried that formal math is so abstract that she will no longer be able to understand it) – this is the safety of keeping math at arm's length. Embodiment is rejected because math is not separate from humans, and so she cannot maintain an impersonal relationship with mathematics. Proofs and Refutations is hedged by the possibility of finding absolute answers or the surety that "we" can never be sure. These are strongly emotional positions, in the sense that she feels strongly about keeping mathematics impersonal and separate. She selects Platonism because she feels strongly about wanting to feel happy about knowing there are right answers that she can correctly reproduce.

JOHN

John is extremely good at achieving 100% on math tests and exams. He was also one of the few participants who expressed a genuine love of mathematics. During his pre-interview, he stated, "I like the fact that in math you can derive an answer and be certain of it…" He admires the work of mathematicians, and feels a sense of pleasure when his ability to be the only one in a math class who can solve a challenging problem positions him as the "mathematician" of the class. A key word used in his pre-interview is "comfortable." He likes math because it makes him feel comfortable. He knows what to do, he's good at doing it, and he experiences satisfaction at the achievement of the one, unique answer. Math is at the center of his self-concept; he in fact claims it to be at the center of "everything." John has simplified the nature of mathematics by conflating what he does in school math with the work of a mathematician. He sees himself as a problem solver, and his success on math tests props up this perception of mathematics – he is comfortable with his perceptions. His comfort with school math generates a blind spot in recognizing the potential differences between how math is rendered in school and how mathematicians experience mathematics.

During the philosophy of math unit, John's discomfort was minimal at first but slowly increased. After the first class he wrote: "I think about math today in another light, one in which I am not used to thinking." John is just a little bit worried because the first class has triggered the thought that the math he is comfortable with might not be the math of mathematicians. In subsequent journals, we find evidence of an increase in his concerns about mathematics. Later in the unit he wrote:

> I learned about the idea of embodiment today. I, however, don't buy it. I believe that we, as humans, despite our given restrictions within the reality by which we live, are capable of extrapolating our knowledge into areas and dimensions unprovable by our current capabilities. I still believe firmly in absolutism.

The words "however" in sentence two and "firmly" in the last sentence are not needed by John to express his ideas. Their presence suggests how important John felt it was to emphasize his position, and hence his increasing concern with the ideas presented.

Part of his discomfort is rooted in his respect for mathematics and mathematicians. During class discussions, we learned that John read about mathematicians and mathematics out of interest (not as required school reading). In one of his journals he wrote: "We <u>have</u> a problem. And a mathematician <u>must</u> [emphasis not added] be able to accept it." He admires mathematicians, but is discovering an element of being a mathematician that is outside his comfort zone. His discomfort with the ideas presented during class activities has increased. We believe this is because he has available to him increasing evidence that the mathematics he is comfortable with is not so neat and tidy. The one right answer he is certain exists for every problem and takes pleasure in finding has been challenged.

John also values critical thinking – he is curious about ideas but is also skeptical. We found this evident in his questioning and challenging disposition during class discussions. We also found this evident in journal entries. For example, from two of his journals:

> I feel that this is an extremely deep topic, in need of further explanation, and look forward to further exploring it.

And:

> There are always abstract exceptions to mathematics. It is indeed fascinating to wonder about it and analyze, realizing we might never truly achieve an answer. It is rather the thought process that makes it all worthwhile.

John values thinking for the sake of thinking. John believes that an idea must stand up to critique before it is accepted, and he wants to engage in such critical thought processes. He finds pleasure in engaging with ideas. Thus, when he found his ideas about mathematics to be challenged, he took this challenge seriously because he values critical thinking.

He must find a way to navigate the disruption in his comfort with the math he has experienced in school. He does this, with pleasure, through critical thinking. Although he adamantly agreed with Absolutism, through his skeptical challenging and questioning, he eventually found problems with both Platonism and Formalism. His initial reactions to both Proofs/Refutations and Embodiment was rejection because they represented a rejection of Absolutism. Now his skeptical disposition was to question and critique in order to find reasons to also reject these positions. But Embodiment, in his perception, was difficult to reject. We spoke several times after class about Embodiment – it was clear that his valuing of pure thinking was the essence of his curiosity and questioning. He wanted to make sure he understood in order to make sure the ideas could withstand critical evaluation. So, although he rejected Embodiment in his second last journal, his last journal started to describe a philosophy of math that he labeled "Embodied Absolutism." John is finding a way to protect his absolute vision of and experience with mathematics through his pure joy with pure thinking.

During the final interview, John's explication of his ideas continued, predicated on his joy of engaging with ideas. He sees Embodied Absolutism as a philosophical project – a thought experiment – in which the problem is deciding what is absolute and what is embodied. For example, John argued that although error may arise due to perception, and this is because of our embodiment, the concept that is perceived is still absolute. When challenged on this idea, he acknowledged that he might be wrong about "where Absolutism stops and embodiment starts." His post interview is singularly focused on his recognition that there is still thinking to be done on his philosophy, and he is willing and happy to do that thinking.

John's comfort/pleasure with math is inextricably tied up with his comfort/pleasure with thinking, where the thinking he values is oriented by both curiosity about pure thought and by skepticism of all ideas. But absolutism is the one idea, at least at the ontological level, which cannot be challenged – there must be some absolutes. For example, he noted that the "…fact that we are embodied…is absolute." At the beginning of the post-interview, he described the ideas as fascinating (a word used several times in his journal as well). The philosophy of math has been a cerebral game, but he loves playing this kind of thinking game, and so his comfort with thinking about ideas "for the fun of it" protects him from the disruption of his comfort with school math. School math becomes compartmentalized – his experiences with school math remain separate and protected from his thinking about the nature of mathematics.

CONCLUSIONS

In summary, Dorothy starts from her comfort with the absolutism of mathematics, and then the messiness of philosophy of mathematics disrupts this comfort. To deal with this discomfort, she seeks simplistic answers. For example, she tries to simultaneously agree and disagree with a Humanist position. She tries to avoid philosophy of mathematics issues altogether. Epistemological messiness thwarts what she really likes about math, what she describes as its lack of "greyness." Dorothy experiences profound discomfort, and to protect her sense of identity in relation to mathematics, she compartmentalizes philosophy of mathematics to keep it separate from her experiences with school math.

Mary wants to maintain her identity with mathematics as an objective and separate body of knowledge with which she need not think or feel personally about. When the ubiquitous, objective and absolute mathematics that she is happy with (because she can successfully reproduce it for her teachers) is challenged, she feels discomfort. She does not want to face the prospect that a nature of mathematics, which she is happy with, might not be representative of mathematics. She protects her sense of identity by being non-committal, qualifying her answers, or keeping ideas at arm's length from her personal beliefs. This allows her to maintain a sense of success – if she doesn't commit, she doesn't need to face being wrong. When she does commit, it is to maintain and protect a strong emotional connection to mathematics – that is, that she is happy with math and that would allow her to maintain an impersonal relationship with math.

John genuinely loves mathematics and pure thinking and these are intricately tied together. But it is skepticism that ultimately protects his identity in relation to mathematics. Although his comfort with school math is disrupted, he is critical of all ideas except the idea that there must be some absolutes. That there must be some absolutes and his joy of pure thinking leads him to synthesize absolutism with embodiment. This allows him to ultimately protect his identity in relation to school math because, in the end, it is pure thought that matters and is valued. Mathematics is based on pure thought. Descartes would be proud.

We would like to highlight several features of these three stories of navigating of disruption. First, a story of identity in relation to mathematics is intrinsically and fundamentally bound up with a story of identity in general. Dorothy's outgoing nature was the story of her sense of indecision in terms of the philosophies of mathematics that we presented. Mary's quietness is her way of seeking the answer that she will quietly embrace and, given the opportunity, reproduce on tests if her answer is the curriculum-sanctioned correct answer. John's skepticism is fundamental to both his continued rejection of ideas but also his eventual acknowledgement of the skepticism of the Embodiment position.

Second, we believe that these students compartmentalize their disruption. The Theory of Knowledge (ToK) course is a mental game. For Dorothy, the game doesn't really matter. For Mary, she quietly plays the game by looking for the sanctioned right answer, which is absolute in math class and "there is no right answer" in ToK. John enjoys playing the mental game of debating ideas, but when in math class, he understands the procedure presented and gets mad at himself when he makes a "stupid" mistake on a test – the skepticism of TofK does not carry over into math class. We believe this compartmentalizing is important for maintaining a sense of coherent identity in relation to mathematics for these students. If they did not compartmentalize their experience of our messy philosophy of mathematics unit, their experience outside the unit would also be disrupted, which could potentially undermine their status as the "smart" (math) students as sanctioned by their teachers

From the results of this project, we make several recommendations. It may seem that our philosophy of mathematics unit failed to enrich these gifted student's beliefs concerning the nature of mathematics – their beliefs were only disrupted leading to a compartmentalization of school math and philosophy of mathematics. But our philosophy of mathematics unit was only a two-week intervention compared to 12 years of enculturation into a narrow and tidy vision of math. These gifted students are focused on maintaining their success (read grades) in the IB program. In particular, their IB mathematics teacher was resistant to the ideas explored in our unit, so the "geography" of the school math course and ToK course may have contributed to the compartmentalization we observed. Given the social milieu of our project, perhaps mathematics as messy is too foreign for these gifted students to occasion change in their relationship with or perceptions of mathematics. Other renderings of mathematics could be used to enrich our messy framing of the nature of mathematics, such as by Byers (2007), who uses concepts such as mystery and ambiguity to describe key processes in the development of mathematical ideas. It may be that activities can be created based on mystery/ambiguity that resonate, rather than disrupt, while still occasioning richer conceptions of mathematics among these gifted students.

We also believe that the disruption and compartmentalization experienced by these gifted students is a curricular issue. All mathematics curricula, to our knowledge, state a major goal is for students to appreciate the products and processes of mathematics. And yet, a richer exploration of the nature of mathematics with gifted high school students is disruptive of their personal identity in relation to mathematics. This is because mathematics curriculum, as enacted in math classrooms, is singularly narrow in its tidy vision of mathematics. Most curricula try to point to the richness of mathematics through a list of mathematical processes (e.g., problem solving, reasoning) that should be infused throughout the teaching of all skills and concepts. But this list is easily framed by a narrow vision of mathematics. We believe that curriculum documents should endorse a "critical engagement" mathematical process, which signals teachers concerning some of the messiness of mathematics. The goal would be opportunities for students to experience some of the messiness of mathematics as regularly as problem solving and throughout their K-12 school math program. A critical engagement mathematical process could be enacted throughout K-12, so that children/teachers are not enculturated/enculturating a narrow and tidy vision of mathematics.

The curricular recommendations above have significant implications for teacher professional learning. The expansion of mathematics curriculum to include messiness places considerable demands on all K-12 teachers, especially given the common belief among teachers and wider society that the nature of mathematics is uncontested and uncontestable. The professional learning of teachers is a significant concern because it effects all kinds and levels of teacher education, at a time when it is not clear how to effectively invite teachers into the current agenda (e.g., National Council of Teachers of Mathematics, 2000) to reform mathematics education; and because current calls for reform are effectively silent concerning philosophically-based goals or learning outcomes, so are insufficient based on our curricular recommendations. We consider these implications as a call for collective and action-based research that raises the status of philosophy of mathematics among all educational stakeholders.

Finally, this project raises questions for further research. Our research questions for this initial project were exploratory in nature and focused on gifted students. Subsequent research could consider more closely how student identity is related to student relationships with mathematics. For example, what is the relationship between expanded or disrupted perceptions of the nature of mathematics and success in school math? Would gifted abilities contribute to or hinder a student's responsiveness to issues concerning the nature of mathematics? More precise research questions are needed to expand the literature, and subsequent research could consider the general student population as compared to gifted students.

References

Betts, P. (2007). Using messiness to appreciate the nature of mathematics as a discipline: A call to enhance school math. *Learning and Teaching Mathematics, 5*, 26-30.

Byers, W. (2007). *How mathematicians think*: Using ambiguity, contradiction, and paradox to create mathematics. Princeton: Princeton University Press.

Clandinin, J., & Connelly, F. M. (2000). *Narrative inquiry: Experience and story in qualitative research*. San Francisco: Josey-Bass.

Daniel, M. F., Lafortune, L., Pallascio, R., & Schleifer, M. (1999). Philosophical reflection and cooperative practices in an elementary school mathematics classroom. *Canadian Journal of Education/Revue canadienne de l'education, 24*(4), 426-440.

Davis, P. J., & Hersh, R. (1981). *The mathematical experience*. Cambridge: Birkhauser.

Ernest, P. (1998). *Social constructivism as a philosophy of mathematics*. Albany, NY: State University of New York.

Govier, T. (1997). *Socrates' children: Thinking and knowing in the western tradition*. Peterborough, Canada: Broadview Press.

Hardy, G. H. (1992). *A mathematician's apology*. Cambridge: Canto.

Hersh, R. (1997). *What is mathematics, really?* New York: Oxford University Press.

International Baccalaureate Organization. (2005-2009a). *Diploma programme curriculum core requirements: Theory of knowledge* Retrieved July 17, 2009, from http://www.ibo.org/diploma/curriculum/core/knowledge/index.cfm

International Baccalaureate Organization. (2005-2009b). *IB Home Page*. Retrieved July 17, 2009, from http://www.ibo.org/

International Baccalaureate Organization. (2005-2009c). *Three programmes at a glance*. Retrieved July 17, 2009, from http://www.ibo.org/programmes/index.cfm

Kleinhenz, R. J. (2007). *When asked what it was like to set about (...)*. Retrieved July 17, 2009, from http://www.math.univ-montp2.fr/SPIP/-Kleinhenz-Robert-J-

Lafortune, L., Daniel, M. F., Pallascio, R., & Schleifer, M. (1999). Evolution of pupils' attitudes to mathematics when using a philosophical approach. *Analytic Teaching, 20*(1), 33-44.

Lakatos. (1976). *Proofs and refutations : the logic of mathematical discovery*. Cambridge: Cambridge University Press.

Lakoff, G., & Johnson, C. M. (1999). *Philosophy in the flesh*. New York: Basic Books.

Lakoff, G., & Nunez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books Inc.

Lipman, M. (1988). *Philosophy goes to school*. Philadelphia: Temple University Press.

Lipman, M., Sharp, A. M., & Oscanyan, F. S. (1980). *Philosophy in the classroom* (2nd ed.). Philadelphia: Temple University Press.

Mancosu, P. (1998). *From Brouwer to Hilbert: The debate on the foundations of mathematics in the 1920's*. New York: Oxford University Press.

Manitoba Education, C. a. Y. (2008). *Kindergarten to grade 8 mathematics: Manitoba curriculum framework of outcomes*. Winnipeg, Canada: Manitoba Education, Citizenship and Youth.

Martin, C. (2008). Philosophy in the Mathematics Classroom. *Mathematics Teaching Incorporating Micromath, 210*, 16-17.

Matthews, G. B. (1984). *Dialogues with children*. Cambridge: Harvard University Press.

McMaster, L., & Betts, P. (2007). Making school math messy: Deepening mathematical appreciation in gifted high school students. *Gifted and Talented International, 22*(2), 89-95.

Naji, S., & Ghazinezhad, P. (2008). The impact of philosophy for children courses on thinking skills and behavioral performance in children. *Journal Of Curriculum Studies, 2*(7), 123-150.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Philipp, R. A. (2007). Mathematics teachers' belief and affect. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (Vol. 1, pp. 257-315). Charlotte, NC: Information Age Publishing.

Trickey, S., & Topping, K. J. (2004). 'Philosophy for children': a systematic review. *Research Papers in Education, 19*(3), 365-380.

Western and Northern Canadian Protocol. (2006). *The common curriculum framework for K-9 mathematics*. Alberta, Canada: Western and Northern Canadian Protocol.