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The promise of interconnecting problems for enriching students' experiences in mathematics

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Abstract: The *interconnecting problem approach* suggests that often one and the same mathematical problem can be used to teach various mathematical topics at different grade levels. How is this approach useful for the development of mathematical ability and the enrichment of mathematical experiences of all students including the gifted ones? What are the benefits for teachers' and what would teachers need to implement this approach? What directions would further research on these issues take? The paper discusses these and closely related questions.

I propose that a long-term study of a progression of mathematical ideas revolved around one interconnecting problem is useful for developing a perception of mathematics as a connected subject for all learners. Having a natural appreciation for linking learned material, mathematically-able students exposed to this approach could develop more comprehensive thinking, applicable in many other problem solving situations, such as multiple-solution tasks. Because the problem's solutions vary in levels of difficulty, as well as conceptual richness, the approach allows teachers to form a strategic vision through a systematic review of various mathematical topics in connection with one problem.

General pedagogical ideas outlined in this paper are supported by discussions of concrete mathematical examples and classroom applications. While individual successful practices of using this approach are known to be taking place, the need for more data collection and interpretation is highlighted.
Key words: multiple-solution problems, connectedness of mathematics, constructions in geometry, teaching support of mathematically inclined students.

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1. Interconnecting problems and giftedness in mathematics

Mathematically gifted learners differ from average learners in their ability to perceive and retain mathematical information (Krutetskii, 1976). Apparently, they possess a well-organized interconnected web of mathematical knowledge (Noss&Hoyles, 1996) which manifests itself in flexibility of handling data, originality of interpretations, ability to transfer and generalize mathematical ideas (Greenes, 1981), and creativity of approaches taken when problem solving. According to Polya (1973), besides extracting relevant information from the memory, "in solving a mathematical problem we have to construct an argument connecting the material recollected to a well-adapted whole" (Polya, p.157). This ability to logically organize and process mathematical information is yet another distinguishing characteristic of mathematical talent (Krutetskii, 1976).

A learner could be a *good exercise doer* but still be incapable of adjusting standard techniques for answering unfamiliar questions (see e.g. discussion in Greenes, 1981). In teachers' words, "some of them [students] who solve standard problems quickly and easily meet an impasse when solving problems requiring independent thoughts" (Krutetskii, p. 176). This observation implies that the goal of the teacher consists of helping a dedicated learner go beyond *instrumental* understanding secured by knowing mathematical procedures, and achieve *relational* understanding (Skemp, 1987) between different mathematical topics, which assumes connections of various mathematical ideas. "An ability to establish and use a wide range of connections offers students alternative paths to the solution. … with a formulation of each new connection … the likelihood of discovering a solution in enhanced" (Hodgson, 1995, p.19). The emphasis on making connections is important not only for the teaching of mathematically gifted learners but is becoming one of the core didactical principles of the modern mathematical curricula (NCTM, 2000).

Researchers distinguish several ways of manifesting students' higher ability: in quality of the product, in characteristics of the process, and as a subjective experience. There also exists a variety of possibilities to describe and study the phenomenon of creativity (see e.g. Sriraman (2004a) for a review of this topic). As for the driving force of mathematical creativity, interaction of ideas in the mind of the thinker is considered as one of the most important factors in this process (Ervynck, 1991). Consequently, some authors proposed to measure flexibility of thinking and creativity in mathematics by the number of produced solutions to a given problem as well as the ability of the solver to switch between different representations of the problem (Krutetskii, 1976; Laycock, 1970, Silver, 1997). From this perspective, problems which allow multiple solutions present a promising tool for nurturing of giftedness and enhancement of the quality of teaching in general (Stigler & Hiebert, 1999; Fennema & Romberg, 1999). Leikin and her collaborators extensively studied *multiple-solution connecting tasks* which they define as "tasks that contain an explicit requirement for solving the problem in multiple ways" (Leikin & Levav-Waynberg, 2008, p.234). They view these tasks as a valuable tool for the examination of mathematical creativity (Leikin & Lev, 2007).

The approach considered in this paper also focuses on problems with multiple solutions but those problems are used with a different pedagogical emphasis. The idea is not to solve the problem in many different ways at once. Instead, one problem is used throughout a learner's development over a long period of time. Each problem's solution is considered from different perspectives as the learner builds his mathematical confidence over several years of schooling. In particular, problems connecting elementary and advanced solutions as well as various methods and techniques are valuable for this purpose. The intuition developed through elementary approaches to the problem may be used by the learner for a better understanding of more advanced methods and at the same time for making connections between the various approaches. While learners at different stages of their growth "may be able to solve a particular problem, the manner of solution and the consequences of long-term development of learning can be very different, moving from rigid use of a single procedure through increasing flexibility to symbolic operations on thinkable concepts" (Tall, 2006, p.200). Multiple-solution problems used to specifically support the progression of the learner are the subject of this paper.

I call a problem *interconnecting* if it possesses the following characteristics:

(1) allows simple formulation (without specialized mathematical terms and notions);

(2) allows various solutions at both elementary and advanced levels;

(3) may be solved by various mathematical tools from distinct mathematical branches, which leads to finding multiple solutions, and

(4) is used in different grades and courses and can be understood in various contexts.

Due to the wide range of difficulty levels of its solutions, the same interconnecting problem may appear at the elementary school level, and then in progressive grades until the advanced level. The students, familiar with the problem from their prior hands-on experience, will use their intuition to support the more elaborated techniques presented symbolically in the upper grades. This would allow students to see their old problem in a new light and interpret new methods in terms of an old and familiar example, and thus linking the new concept with the existing schemata. Rephrasing Watson and Mason's description of reference examples, an interconnecting problem is "the one that becomes extremely familiar and is used to test out conjectures, to illustrate the meaning of theorems" (Watson & Mason, 2005, p.7).

From a learner's standpoint, a problem is *interconnecting* if its solution has been understood by the learner from several conceptual perspectives after working on the problem over an extended period of time. This definition of interconnectedness does not only characterize a problem but also demands a continuous engagement and certain cognitive effort from a learner, suggesting that same problem can be interconnecting for one student but not yet for another. Thus, the possibility of identifying and developing mathematically gifted students is embedded in the definition of interconnecting problems. Once understood, an interconnecting problem may be used by the solver as a model of flexible thinking in another problem context. The possibility for creative solutions arises from the learner's familiarity with other interconnecting problems because this familiarity allows the learner to have a comprehensive grasp of the new problem. In the next section I discuss interconnecting problems in comparison with various types of other mathematical activities and teaching approaches.

2. The place of interconnecting problems among other teaching approaches

There are various types of mathematical activities students face during their lessons. Different activities have different learning objectives. For instance, mathematical exercises help students to develop proficiency with various standard techniques and rules. In contrast, recreational problems appeal to students' common sense and intuition. There are also problems which combine some features of both the exercise and recreational types. These problems, on the one hand, are very intuitive and on the other hand incorporate special knowledge in a natural fashion. Their elementary solutions may not be immediately apparent but when found they demonstrate how several basic facts can be useful in a non-routine situation. They help to activate and connect basic knowledge and allow the student to discover new relations and properties. According to Polya (1945) and Schoenfeld (1985), this type of problem plays a very important role in the development of a strong mathematical background of a learner.

Careful and meaningful construction of appropriate learning environments for gifted students is a difficult pedagogical issue. First, according to Diezmann & Watters (2002) in order to have a cognitive value for a learner, the mathematical task must have a level of difficulty appropriate for the learner, that is, it must be at the psychological edge between his/her comfort and risk-taking zones (Vygotski, 1978).In addition, if suitable learning-stimulating tasks are not given "at the right moment, then some intellectual abilities may not have the chance to develop"(Sierpinska, 1994, p.140). Students need to be challenged during all years of education because "when the student comes to study mathematics at the university level, the propitious moment [in his/her development] would have passed, and it may be too late for the teaching intervention to have any effect" (Sierpinska, 1994, p.140).

Tasks which require finding multiple solutions present a challenge not only for students but also for their teachers. Besides a general direction to employ different representations of the same mathematical concept (NCTM, 2000), teachers are insufficiently advised how to incorporate multiple-solution tasks in their lessons and how to assess their students' progress in solving them (Leikin&Levav-Waynberg, 2007). I suggest that familiarity of students with interconnecting problems during their entire educational process creates a culture of mathematical thinking that makes solving multiple-solution tasks more accessible. Through interconnecting problem, students may acquire the habit of analyzing a given problem in multiple ways as a systematic approach to problem solving and learning mathematics.

In a way, the interconnecting problem approach complements the *strand of problems* approach (Weber et al, 2006; Powell et al, 2009). The strand of problems approach uses *isomorphic* problems (English, 1993; Hung, 2000; Maher & Martino, 1996; Sriraman, 2004b), which appear to be different but employ the same underlying mathematical structure, and allows students to develop "problem-solving schemas within a specific mathematical domain" (Powell et al, p.139). Both approaches employ Bruner's proposal of spiral curriculum, the view that curriculum should revisit basic topics and ideas learned over an extended period of time. This proposal correlates with the phenomenon of the spacing effect found in studies of memory: learning of fewer items in a longer period of time is more effective than repeated studies in a short period of time (Crowder, 1976). Thus reinforcement and revisiting is necessary in order to achieve fluency in understanding and comprehension of some material. But the revisiting can happen in different ways. In the *strand of problems* approach, the learner returns to the same mathematical idea or technique by solving a number of different problems. Here the challenge is to recognize that different problems have the same mathematical structure and thus the same method can be employed to solve all of them.

In contrast, in the *interconnecting problem* approach the learner always deals with the same problem but employs different mathematical ideas and consequently, methods to solve it. This leads to establishing links between different topics learned in mathematics curriculum. In sum, the two complementary approaches are based on different paradigms: *one problem linked with multiple ideas (or concepts)* and *many problems linked with one idea (or concept)*, which allows building a network of knowledge, especially if the approaches are used in a combination. This view is schematically presented in Figure 1.

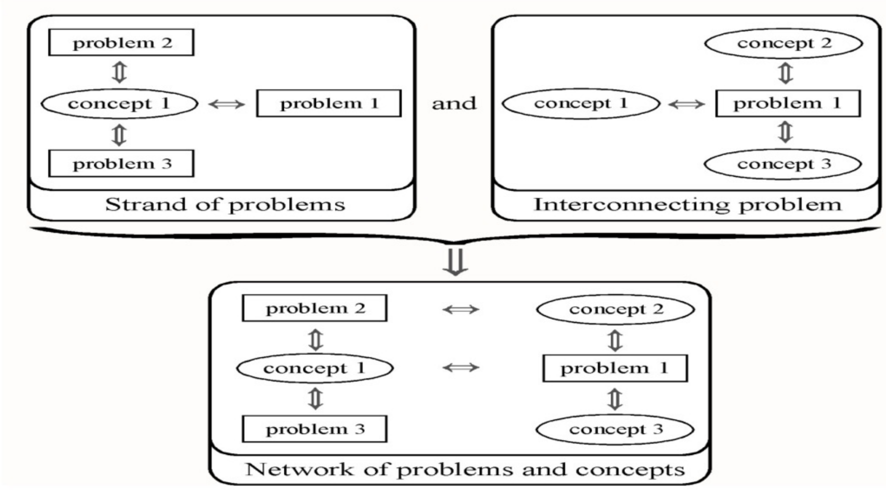

Figure 1: Strand of problems and interconnecting problems generate a network of concepts and problems.

In this respect, the interconnecting problem approach becomes an integral part of a teaching strategy aimed at creating a learning environment fostering mathematical intellectual growth and giftedness in particular.In the next section I give an example of interconnecting problem and examine its potential for learner's development.

3. An example of an interconnecting problem

As many other good mathematical questions, this problem arose from practical needs in an engineering design project. It was conveyed to me in a conversation with my friend, who also mentioned that the majority of his colleagues, former university graduates, could not find a reasonable solution to it. I took it as a challenge to illustrate that the problem can be solved at different levels of grade school education and thus serve as an interconnecting problem for a learner of mathematics.

Problem: *Start with an arbitrary angle ABC and point E inside the angle. The problem is to draw a circle tangent to the sides of the angle and passing through the point E (that is we need to construct the center and the radius of the circle).*

In this section I will consider four possible approaches to this problem that can be applicable at different stages of learner's cognitive development and related to different mathematical tools and representations of the question. The first approach is very intuitive and can be demonstrated with manipulatives. This corresponds to *enactive* stage of problem representation (Bruner, 1966). Two other approaches, similarity-based and parabola-based, are geometrical approaches. They can be classified in Bruner's terminology as *iconic* because they involve reasoning based on the properties of the drawn objects. The third method develops further the idea of parabola-based approach by moving it towards algebraic formalization and rigorous description of the solutions in terms of their coordinates. The local network of knowledge build around this problem over time can be schematically shown in the following figure.

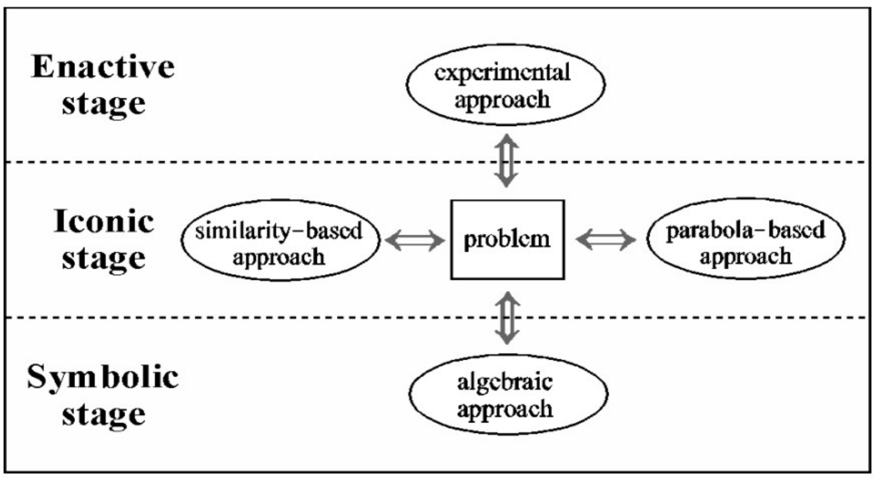

Figure 2: Approaches to the problem appropriate during several developmental stages.

Below I present mathematical details pertinent to each of the approaches. In this section I give a more algorithmic, step-by-step description of each method. The next section discusses ideas and concepts underlying these methods.

A. Experimental approach:

We bring into play a 3D model to help students understand that the solution to the problem exists. Consider a conical basket and imagine putting your finger on a point located inside the basket. Keeping the basket and the finger in the static position, ask if it is possible to find a ball or spherical balloon such that when it is placed in the basket the finger will touch the surface of the balloon. It is clear that if the balloon is too small, then the finger will be far from its surface, while if the balloon is too big, the finger will deform or break the surface. Is it possible to get a balloon of the right size? The solution then is very intuitive: we place a small balloon and inflate it until it touches the finger. This experiment can convince students that the problem has a solution no matter what the size of the cone is and where the finger points. It does not define the radius and position of the center yet, but shows that it can be determined mechanically, doing the experiment with real manipulatives. Note that our original problem is a plane section of this 3D model.

The next two approaches are purely geometrical. They can be discussed with a child who starts to notice and understand properties of drawn objects such as circles, triangles, tangent lines, perpendicular segments, etc.

B. Similarity-based approach:

For this approach I refer to Figure 3.

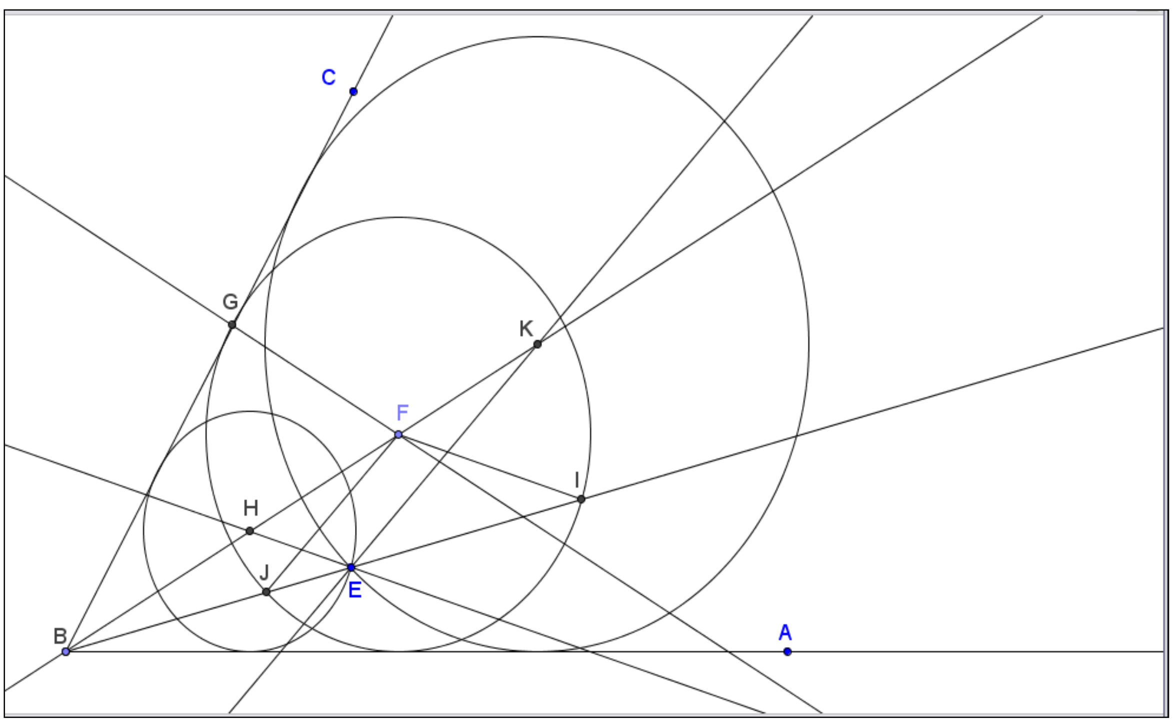

Figure 3: Pure geometrical similarity-based approach.

I. First we draw an arbitrary *auxiliary circle* tangent to the sides of the angle but not passing through the point E. We do it by the following steps:

1. Draw an angular bisector of ABC; we know that all circles tangent to the sides of the angle have their centers on this bisector.
2. We pick an arbitrary point F on the bisector as the center of the auxiliary circle.
3. We drop a perpendicular from the point F to one of the sides of the angle, BC.
4. The intersection point of the perpendicular and the side is called by G, and FG is the radius of the auxiliary circle.

II. Our second step is to connect the vertex B of the angle and the given point E by a ray BE. Since point E lies inside the angle, the ray BE intersects our auxiliary circle in two points, called J and I. The segments FJ and FI are radii of the auxiliary circle.

III. Our last step is to draw two lines through point E: one line is parallel to segment FJ and another is parallel to segment FI. These two lines intersect with the angular bisector BF at points K and H respectively.

We claim that points K and H are the centers of the required circles; their radii are segments KE and HE respectively.

This method is not applicable if E lies on the bisector BF or on one of the sides of the angle. The latter case is discussed in (Jones, 1998) along with an analysis of students' approaches to solve the problem. In the special case when E lies on the bisector BF we follow another approach, which is in fact easier (see Figure 3a).

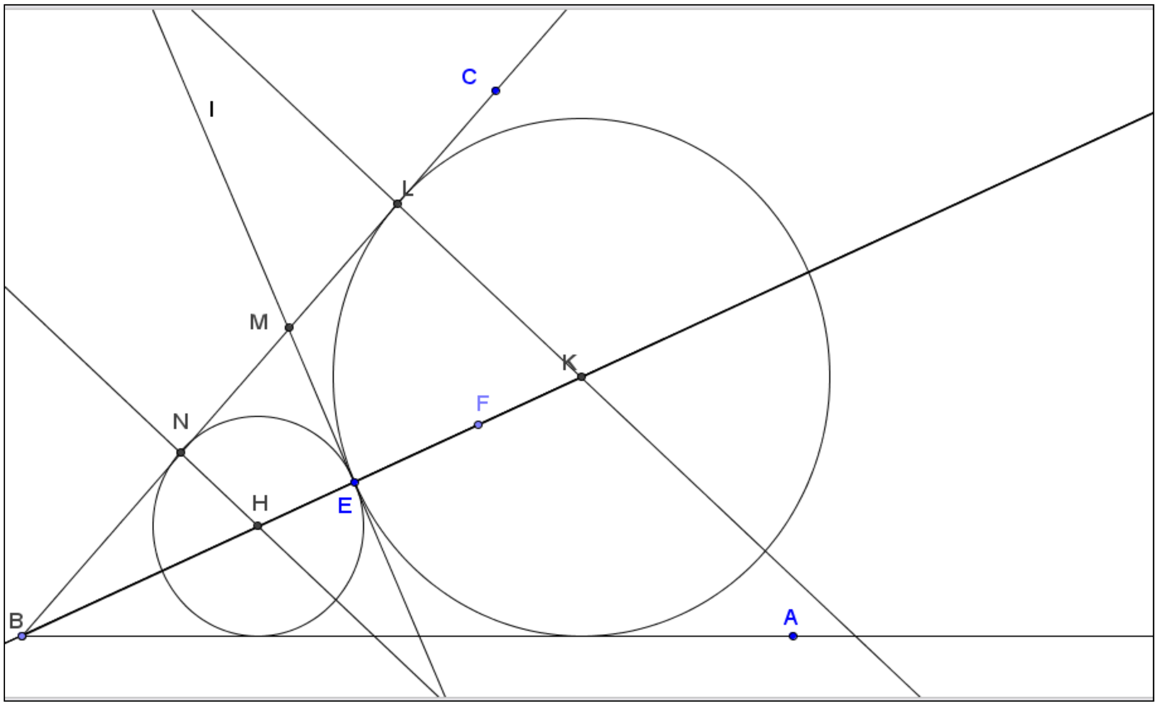

Figure 3a. Special case: point E lies on the angle bisector.

First, we draw a line perpendicular to BF passing through point E. This new line intersects the side BC at point M. We put points L and N on side BC such that LM=ME=MN. Two lines perpendicular to the side BC and passing through points L and N intersect the angular bisector at points K and H respectively. These are the centers of the required circles. Similarly, if E lies on one of the angle's sides, say, AB, we find the center of the circle as an intersection of the angular bisector BF and the line perpendicular to the side AB and passing through E.

C. Parabola-based approach:

I. We first draw the angular bisector of ABC.

II. Our second step is to draw a parabola with focus at given point E and the directrix being one of the angle's sides, say AB. Recall that *parabolais the set of points which are equidistant from given point* (called *focus*) *and a given line* (called *directrix*). Thus we draw it in the following way (Figure 4):

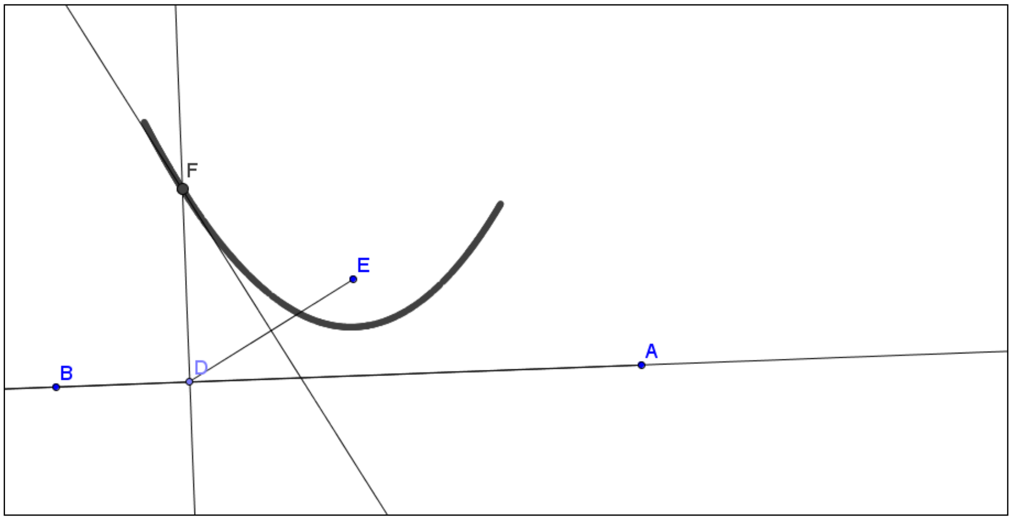

Figure 4: Drawing a parabola with focus at E and directrix AB. Here EF=FD

1. Take an arbitrary point D on side AB.
2. Draw a perpendicular to the side AB through point D.
3. Draw a perpendicular bisector to the segment ED.
4. These two lines intersect at a point F which lies on the parabola.
5. As D moves along the line AB, the intersection points form the parabola.

The parabola is a locus of centers of all circles which pass through point E and are tangent to the side AB. This parabola intersects with the angular bisector at two points, call them H and G (Figure 5). We claim that these two points are the centers of the circles we need to construct. Note that the second step, the drawing of a parabola with given focus and directrix, can alternatively be performed with a help of special mechanisms (linkages) known to ancient Greeks and widely used in the Middle Ages (see e.g. Henderson and Taimina, 2005, p.300). Modern geometry software such as *GeoGebra* has this tool as a built in option.

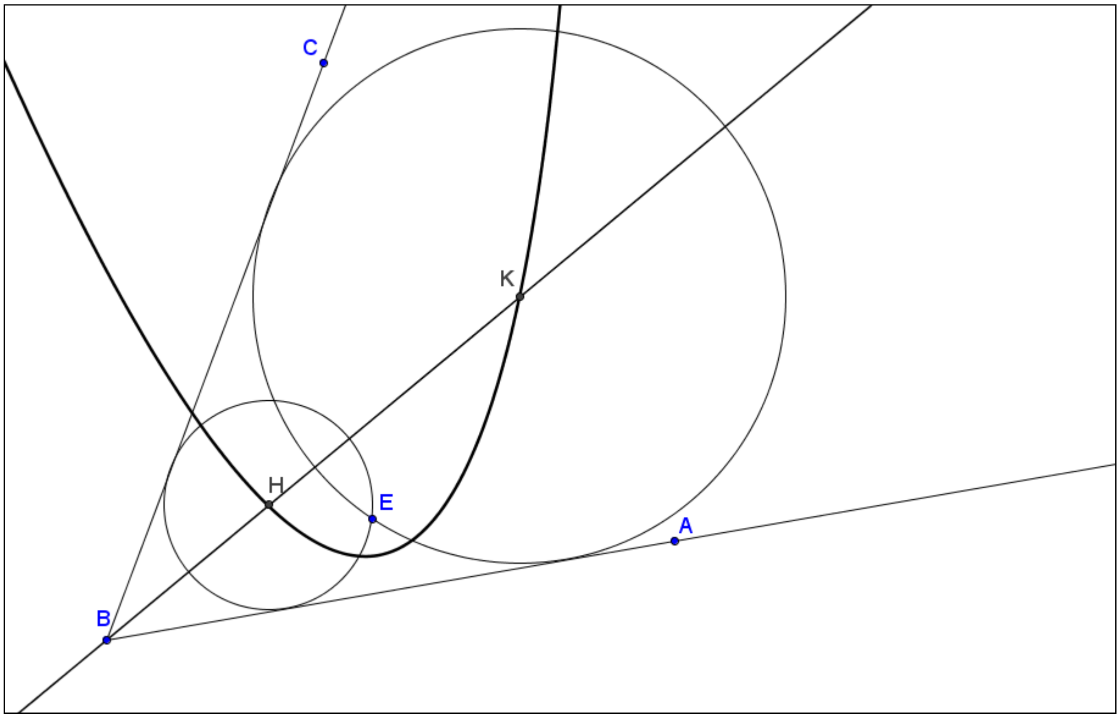

Figure 5: Approach involving geometrical definition of parabola.

The idea of the parabola-based approach could be converted into an algebraic method by a learner who knows how to describe geometrical objects such as lines and circles analytically, to reformulate the question in terms of related algebraic equations and solve those equations. We outline this approach in the following subsection.

D. Algebraic approach:

Let the angle measurement be α , where $0<\alpha<\pi$. Consider a coordinate system in which the angle is formed by the ray AB with equation $y=0, x\geq 0$ and ray BC with equation $y = x\tan(\alpha)$ in the first quadrant or second quadrant (Figure 5a). Let a given point E lie inside the angles and have coordinates (x_0, y_0) . We are looking for the coordinates (x, y) of the center of a circle which passes through E and is inscribed in the angle. As we previously observed, the center lies on the angular bisector, and thus we have one relation $y = kx$, where $k = \tan(\alpha/2)$. The ray representing the angular bisector lies in the first quadrant. Another relation comes from the observation that the distance between the center and point E must be equal to the ordinate of the center. Squaring both values, we obtain $(x-x_0)^2+(y-y_0)^2=y^2$. We note that since both values, the distance and the ordinate, are nonnegative, squaring does not affect the roots of the equation.

Now, the system of two equations leads to one equation with respect to the abscissa of the unknown center, $(x-x_0)^2+(kx-y_0)^2=k^2x^2$. After a simplification it becomes a quadratic equation $x^2-2x(x_0+ky_0)+x_0^2+y_0^2=0$, and thus we find two possible solutions $x=x_0+ky_0\pm\sqrt{2kx_0y_0+y_0^2(k^2-1)}$, which correspond to the abscissas x_1 and x_2 of the centers H and K of the two circles. Consequently, the ordinates y_1 and y_2 of the centers are $y=kx=k(x_0+ky_0\pm\sqrt{2kx_0y_0+y_0^2(k^2-1)})$. By construction we have y_1 =EH and y_2 =EK. An analysis of these formulas reveals the cases when there is only one solution possible: when point E lies on the side of the angle, that is either $y_0=0$ or $y_0=x_0\tan(\alpha)$. In the first case, the center has coordinates (x_0,kx_0), and in the second we get $(x_0(1+k^2)/(1-k^2),kx_0(1+k^2)/(1-k^2))$.

Also, note that the formula simplifies when point E lies on the angular bisector, i.e. $y_0=kx_o$. Then we obtain $x=x_0(1+k^2\pm k\sqrt{1+k^2})$, $y=kx_0(1+k^2\pm k\sqrt{1+k^2})$.

This approach is essentially an algebraic realization of the second geometrical approach, C, based on the intersection of a ray with a parabola.

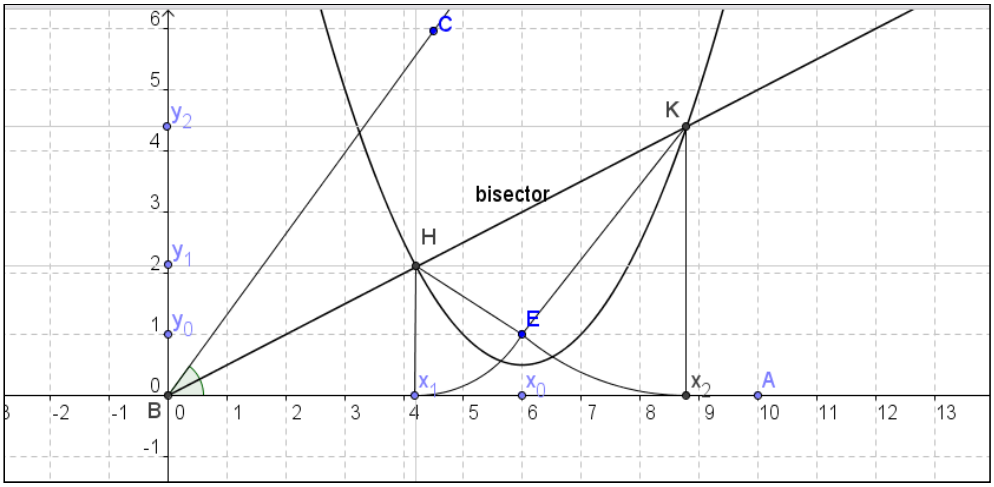

Figure 5a: Algebraic approach: Graphs in the coordinate plane.

The parabola, which consists of centers of all circles passing through E (x_0, y_0) and tangent to the ray $y = 0, x \geq 0$ has equation $y = (x - x_0)^2 / (2y_0) + y_0 / 2$ because its focus lies at E and the x-axis is its directrix. Together with the equation of the ray $y = kx$, this yields exactly the same quadratic equation as we have analyzed above in approach D.

4. Discussion of the key ideas of each of the four approaches.

Gifted students often grasp the formal structure of the problem and produce their solutions from exploration of certain key ideas associated with this perceived structure (Krutetskii, 1976). Polya (1973) distinguishes between the stages of designing a plan in problem solving and implementing the plan. The design is based on the conceptual grasp of the problem situation, whereas its implementation requires more of instrumental knowledge. Since identification of concepts and ideas relevant to a given problem is essential for the solvers' success, training of able students must include a deep analysis of each solution accompanied by the explicit identification of its main ideas. Observe that approaches B, C, and D, if presented to a student as such, will indeed guide him/her to the right answer. Yet, without an appropriate reflection by the learner, without identification and understanding of the reason for each step of the construction, the solutions remain useless for learning to solve problems in general. In this section I listsome ideas and concepts associated with more algorithmic step-by-step solutions presented in the previous section.

The approach A based on the experiment with an inflating balloon is not quite a solution of the problem but it plays an important role in the exploration, visualization and internalization of the situation. It shows that a solution exists and can be found as a result of a continuous process. Embedding this problem in 3D, we allow for a physical realization of the question. Similarly, using modern dynamic geometry (or engineering) software one can easily perform the task approximately just by a trial and error method in the interactive 2D environment. The size and position of the circle can be continuously adjusted in order to obey the requirements of the problem. Most of students (and engineers!) would employ this approach sufficient for a particular configuration. Thus it may take some effort to convince them to find a solution for a general configuration based on mathematical concepts and ideas. Some of them are as follows.

Each of the other three mathematically more advanced approaches B, C, and D uses the fact that the *center of the circle inscribed in an angle lies on the angular bisector*. This observation is essentially based on one's *embodied knowledge* because it refers to the axial symmetry of the geometrical figure and may be demonstrated to a child by folding the picture along the angular bisector. In addition, every approach has its key mathematical ideas, which I outline below.

The fact that *similarity results from dilatation (or uniform scaling)* is the key idea of the first geometrical solution (approach B). Figure 6 shows two circles inscribed in an angle. An inner ray started at the vertex of the angle intersects each of the circles in two points, I, J and K, L respectively. Triangles IJD and KLF, formed by the points of intersection with the ray and the centers D and F of the circles, are similar. Again, one can appeal to the embodied cognition, the natural sense of geometrical perspective, to view the second circle as a magnified copy of the first. This view implies that the sides of the triangles are parallel, which forms the basis for the construction employed by approach B.

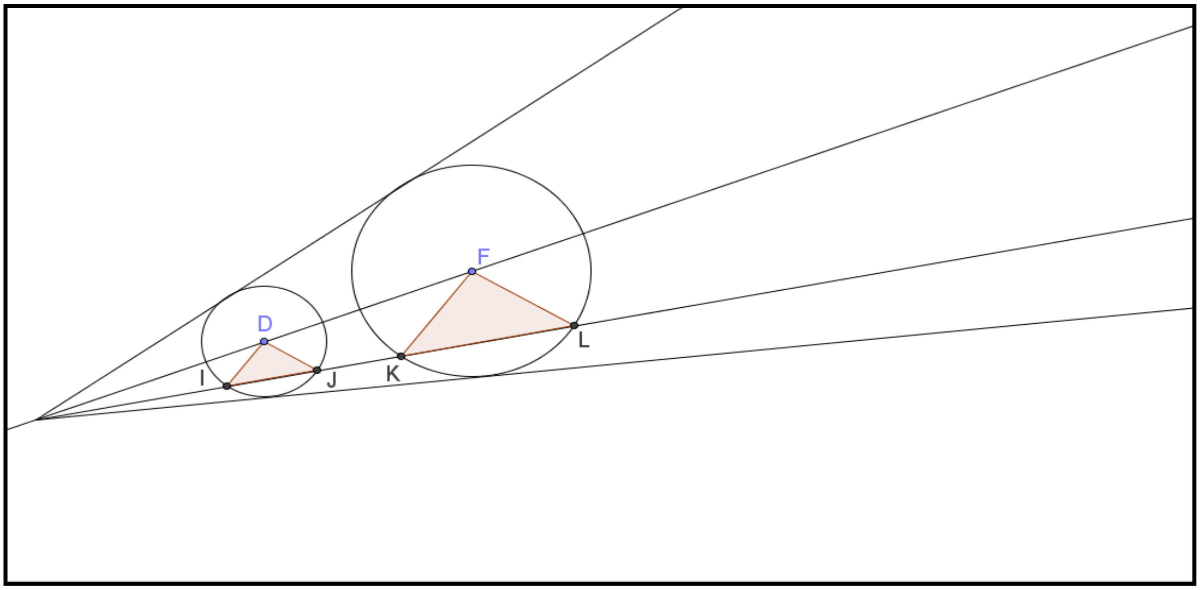

Figure 6: Two similar triangles IJD and KLF viewed as a result of dilatation.

The following key ideas form a foundation for the solution with a parabola (approach C): the set of all circles inscribed in an angle form a family; their centers lie on the ray which is the angle bisector. Similarly, the set of circles passing through E and tangent to one side of an angle form another family; their centers lie on a parabola with focus at E and the directrix being the side of the angle. The center of the required circle is at the same distance from the angle's sides as it is from the given point E, thus the *elements common to both families give the required circles*.

The algebraic solution (approach D) is based on the following key ideas: In an appropriate system of coordinates, an equation of the angular bisector involves a homogeneous linear function with slope expressed via the value of given angle. The distance between two points given by their coordinates is calculated by the Pythagorean Theorem. This leads to the equation of a circle, which is a set of points equidistant from one given point, its center. *In order to find intersection points of two curves, one needs to solve a system of equations describing the curves.*

Note that in this paper I only listed elementary solutions accessible for students in grade school. One may also identify some approaches from university mathematics curriculum, e.g. methods of complex analysis, relevant to the problem. But even if solved by elementary methods, we see that the problem offers a range of mathematical ideas to be explored. These ideas become connected as learners discover them one by one in a course of continuous engagement with the problem. Furthermore, this long-term commitment to the same problem helps to develop students' "capacity for work on one interesting problem for a long period of time", which was found to be one of the characteristics of "creative-productive giftedness in mathematics" (Velikova et all , 2004). If we want our students to make sense of mathematics "we cannot expect any brief program on problem solving to do the job. Instead we must seek the kind of long term engagement in mathematical thinking" (Resnik, 1988, p.58), and this thinking can be organized around an interconnecting problem, its possible solutions and their interplay.

I conclude this section with an illustration of the effect of such an interplay or interconnectivity of ideas employed in different solutions. The following geometrical fact emerges from a comparison of approaches B and C.

Theorem. *Consider an arbitrary circle and parabola drawn in such a way that the same line is tangent to the circle and is the directrix of the parabola, and both the circle and the parabola lie on the same side from the line (see Figure 7). Pick arbitrary point A on this line. Let O denote the center of the circle and F the focus of the parabola. Assuming that line passing through point A and O intersects the parabola in two points, call points of the intersection D and E. Assuming that the line passing through point A and F intersects the circle, call points of the intersection B and C. Then segments FD and CO are parallel and so are segments FE and BO.*

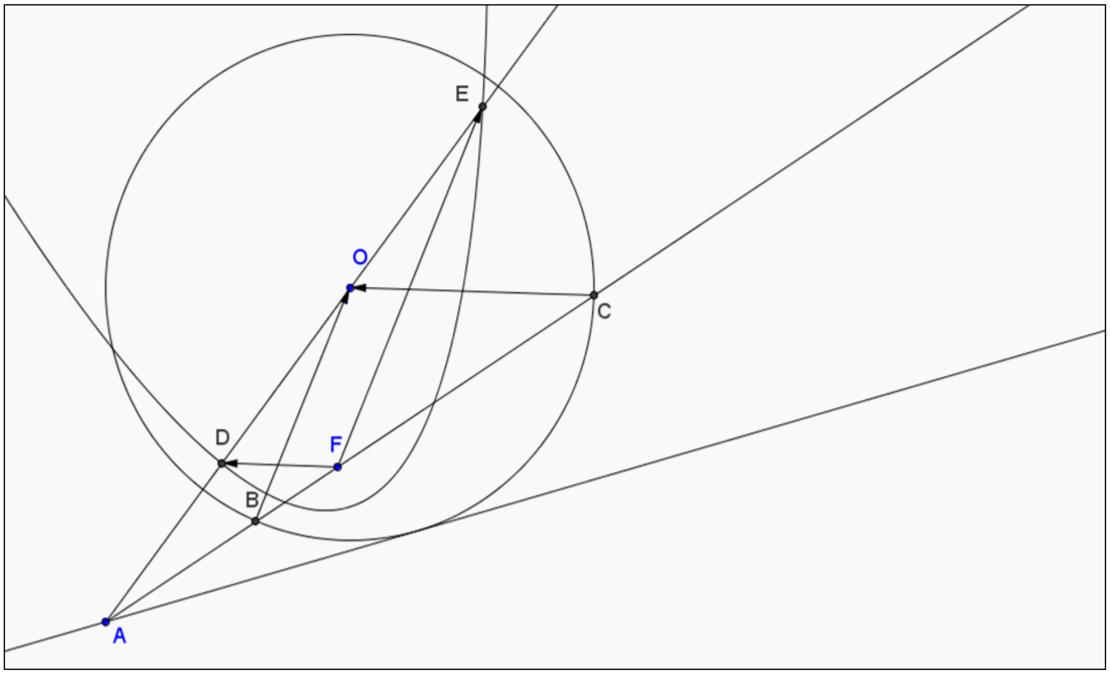

Figure 7: New theorem emerged from approaches B and C to the initial problem.

Proving this statement would be a challenging task for a majority of secondary school students. It would constitute a good question in a mathematical contest and thus can be used for identifying and fostering mathematical giftedness. Note however, that the statement becomes obvious if one identifies points D and E in Figures 7 and 6 with points H and K in Figures 3 and 5, or in other words, if one connects the ideas learned in two approaches to our initial problem. We leave it for the reader to reproduce the proof in full details. While doing this, the reader is advised to focus on his/her own experience and observe how familiarity with an interconnecting problem may lead to understanding of new mathematical facts in the process of rewiring various mathematical ideas.

5. Teaching issues related to interconnecting problems

Mathematics' teachers can play a pivotal role in helping students make connections. Teachers' commitment to this role is reflected in how they select curriculum materials, express personal interest in solving problems, explore and learn new connections in mathematics, negotiate meaning, and search for adequate pedagogical approaches (Koshy, 2001, p.123). The success of the interconnecting problems approach implementation depends on mathematics teachers' readiness to implement it in general, and as a method of nurturing mathematical talent, in particular.

Today's teachers have access to many problems and mathematical activities through books, Internet, journals, conferences, and other channels. Thus, it is unreasonable to say that the teachers are in need of more problems. But precisely because the number of available problems is large, teachers necessitate a systematic approach which would help them select problems appropriate for creating a coherent and connected representation of mathematical ideas for their students. By making this choice teachers would need to deal with such issues as ensuring that problems make mathematical sense, are clear and non-ambiguous. But the real challenge the teachers face is not just to pick a good problem and discuss it with the students, but also let the students experience usefulness of previously learned methods as well as develop an understanding of needs and possibilities of more advances approaches. Interconnecting problems also allow teachers to form a strategic vision and use it in their choice of tasks and actions in a classroom.

However, to be able to successfully implement the interconnecting problem approach, and especially if teaching a gifted group, teachers would benefit from (Barbeau et al., 2010):

- Having personal experiences of problem-solving (in particular, having experience with multiple-solution connected tasks and ability to identify the place of each solution within mathematical curriculum) and investigations to draw upon. This would also help teachers to distinguish the markers of giftedness from just getting good marks in standard assessments or memorizing and following procedures diligently.
- The ability to accept that some of the pupils they encounter will indeed be quicker and more intelligent than they are, but also that they have a role in nurturing whatever talent they find; put more emphasis on modeling the process of problem solving by their own example of thinking out loud rather that just providing student with information and techniques;
- Becoming familiar with the resources so that they can orchestrate a program that will benefit their pupils, and having peers outside the school available for advice, assistance and mentoring. All of these presuppose a level of self-confidence that many teachers lack;

- Having administrative support for working with the same group of students for a longer period of time. It is possible that a proper assessment of giftedness requires contact over a long time, as the teacher needs to understand how a given student thinks. Instead of having a new teacher each year at school, perhaps pupils need fewer teachers, each for several years. This allows a dynamic to be created between the teacher and the class and allows the teacher to get to know the student in a way not possible over a single year.

In relation to this new approach, it would be helpful to find out what teachers' views are on good mathematical problems, what they value, how they select questions for their students; what their beliefs about useful learning recourses are and how close are teachers' descriptions of good problems to the idea I am developing in this paper. In short, the following two questions are essential for the successful use of the approach: (1) Would practicing teachers identify interconnecting problems as good problems? (2) Would teachers be able to see good problems as interconnecting ones? A discussion of teachers' perspective on interconnecting problems goes beyond the scope of this paper. Further investigation of teachers' readiness to implement the approach and their related understandings, knowledge, perspectives and experiences will provide some empirical evidence of benefits of proposed approach and guide its effective implementation in practice.

Conclusion

Being an instructor of mathematics, I often find myself leading a classroom discussion around problems illuminating the essence of a mathematical method. Some of the problems I bring into play appear to be universally useful in a variety of courses. Students attending my classes enjoy recognizing them and comparing how different ideas and techniques can be applied to address the same mathematical question. My observations suggested identification of problems useful for systematical use in various university level courses. Similar practices are discussed in literature. For example, Mingus (2002) refers to "calculation of *n*-th roots of unity" as a problem which "encourages students to see connections between geometry, vectors, group theory, algebra and long division". By means of investigation of this problem in different courses "students were able to review concepts from previous courses and improve their understanding of the old and new concepts" (Mingus, 2002, p.32). Further discussion reveals that "proving identities involving the Fibonacci numbers provide a solid connection between linear algebra, discrete mathematics, number theory and abstract algebra". In my view, these are examples of interconnecting problems. The practice of using such problems effectively responds to the proposal that students' achievements at university level courses are greatly influenced by the degree of interconnectedness of their basic mathematical knowledge, in particular, by connectedness between mathematical terminology, images, and the properties of the objects represented by these terms (Kondratieva & Radu, 2009). My own experiences resonated with like-minded instructors' practices led me to the formulation of the approach described in this paper, which I propose to apply to the whole mathematics curriculum with particular consideration of the needs of gifted students.

Modern curriculum is moving from a formal approach towards more exploration-based and inquiry-based study of mathematics. While making connections and multiple representations of mathematical ideas are recognized as primary goals in teaching and learning mathematics, it is not always clear how teachers can implement this agenda. House & Coxford (1995) argued that presenting mathematics as a "woven fabric rather than a patchwork of discrete topics" is one of the most important outcomes of mathematics education. However, there is also a need for practical teaching strategies "for engaging students in exploring the connectedness of mathematics" (House & Coxford, 1995, p. vii).

The interconnecting problem approach is one of such strategies. I hope that this article shows the potential of interconnecting problems and provides some practical ideas for teachers who pursue this direction in mathematics education.

I suggest that the use of the interconnecting problem approach at different stages of students' cognitive growth can foster the intellectual ability of the best students, identify mathematically-able students and engage them in analysis of connections between various ideas and methods. In addition, the application of different methods to the same mathematical problem throughout the years of schooling can:

- save classroom time devoted for exploration in high school by having necessary investigations and hands on experiences in earlier grades;
- foster earlier transitions to the study of algebraic methods by means of reference to pictorial or other previously employed representations of the problem;
- motivate students through freedom of exploration and experimental observations;
- improve students' logical skills by letting them reason in familiar terms;
- improve retention of basic facts by using them in the context of the problem and connect to other basic facts used in the same problem earlier;
- develop students' visualization skills and rely on their hand-on experience with geometrical objects when a more advances mathematical method is employed.
- help with producing multi-step solutions by building connections between various topics.

One may point at the obstacles the use of interconnecting problems may face because by the time students are in high school they may forget what they have done in previous years. Therefore, I emphasize the importance of very careful planning through the years of school curriculum for using of this approach. Elementary and secondary level teachers may need to collaborate in order to identify useful interconnecting problems and outline the direction of emphasis through elementary grades required for the secondary level studies appealing to the same problem. Teachers need to ensure that the experience with interconnecting problems obtained in earlier years of education is memorable. For that, each investigation needs to be concluded with a concise summary of the key ideas and perhaps illustrated by special schematic images which students will associate with the problem in the future. The purpose of such images is to allow the students quickly evoke previous experiences associated with the problem and thus prepare them for learning new skill related to the old ones. As an example one may consider the notion of "procept" viewed as an amalgam of processes, an object emerged from them and the symbol which both represents and evokes it (Gray & Tall, 1994). Another example is the Shatalov's "support signals" also helpful for "to reward successes—however small—and thus build up the child's natural enthusiasm for learning and confidence to be creative (Johnson, 1992, p. 59).

To summarize, I am not claiming that the interconnecting problem approach is easy to implement but it is worth trying because students equipped with a comprehensive view of one interconnecting mathematical problem will likely exhibit more confidence, mathematical insight, and elegancy in problem solving than those who have studied an equivalent number of disconnected and arbitrarily contextualized mathematical facts. Teachers who care about coherent picture of mathematics they teach may observe more signs of giftedness in their classrooms.

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