The Mathematics Enthusiast

Volume 8
Number 3 *Number 3*

Article 2

7-2011

Research on Practical Rationality: Studying the justification of actions in mathematics teaching

Patricio Herbst

Daniel Chazan

Research on Practical Rationality: Studying the Justification of Actions in Mathematics Teaching

Patricio Herbst[1]
University of Michigan
&
Daniel Chazan
University of Maryland

Abstract: *Building on our earlier work conceptualizing teaching as the management of instructional exchanges, we lay out a theory of the practical rationality of mathematics teaching – that is, a theory of the grounds upon which instructional actions specific to mathematics can be justified or rebuffed. We do that from a perspective informed by what experienced practitioners consider viable but also in ways that suggest operational avenues for the study of instructional improvement, in particular for improvements that enable students to do more authentic mathematical work. We show how different kinds of experiments can be used to engage in theory building and provide examples of initial work in building this theory.*

Keywords: Mathematics instruction; Practical Rationality; Theory of teaching; Teacher education

Introduction

In this paper we address the work of the mathematics teacher in instruction and the rationality behind this work. We first sketch out how the teacher's work could conceivably contribute to the creation of opportunities for students to do authentic mathematical work. In that sense we expect that the paper will add to

[1] pgherbst@umich.edu

our collective sense of what is conceivable and perhaps desirable to happen in classrooms. Most of the paper, however, is concentrated on elaborating on the grounds for possibility and justification of teachers' actions. In particular, what is the rationality that might (or might not) support teachers' management of authentic mathematical work by students?

In accounting for the rationality beneath teachers' actions and in regard to the possibility of enabling authentic mathematical work by students, we take some distance from two relatively commonplace ways of responding to a vision sketch. In one of these approaches, a vision of conceivable mathematical work in classrooms might be followed by an acknowledgment and analysis of the forces and structures that make the vision not viable. Such an approach would summon us to be like social critics of the current educational system, and to endorse a new educational system that would bring all our hopes to fruition. In the other approach, the vision sketch is followed by a busy shaping of persuasive rhetoric, design of efforts, and organization of resources, all of them aimed at making the vision happen against all odds. Such an approach would summon us to be like social engineers, relentlessly working to realize the vision, as if the only thing that separated the conceivable from the viable was the existence of the will to make the vision happen.

Without meaning to disrespect proponents of either of those approaches, we take a third approach, which combines the orientation to improvement of the second with the analytic disposition of the first but poses questions that call neither for critique nor for engineering but rather for theory and research. We elaborate on the notion that the actions of teachers in classrooms are not mere expressions of their free will and personal resources; rather their actions also attest to adaptations to conditions and constraints in which they work. And yet that realization does not necessarily condemn us to accept the status quo; rather, it can suggest ways of working toward improvement in viable, incremental, and sustainable ways.

How can we think about the distinction, and the gap, between what is conceivable and what is viable in mathematics teaching? How can we find out how much of the vision can be realized within existing conditions and constraints? We argue below that what is required is first to understand and then to co-opt what we have been calling the practical rationality of mathematics teaching (Herbst & Chazan, 2003; Herbst, Nachlieli, & Chazan, 2011). We first recount how the story of practical rationality began and the big picture it serves.

How We Started Our Efforts to Explain Teaching

We started to work together back in 2000, following our common interest in understanding the teaching of mathematics at the secondary level and our shared sense of the importance of learning the wisdom of the practice (Shulman, 2004). But while the focus of our interest was convergent, our theoretical perspectives and our methods required some work. Chazan had been doing what Ball (2000) calls first-person research: He had been using his own practice teaching Algebra I to investigate the dilemmas and dynamics that a teacher needs to manage (see Chazan, 2000; Chazan & Ball, 1999; also Lampert, 1985). Herbst had been using the more structuralist notions of didactical contract (Brousseau, 1997) and didactical transposition (Chevallard, 1985) to provide detached observer descriptions and explanations of the work of teaching and its effects on the classroom representation of knowledge (see Herbst, 1998; 1999; 2002a; 2002b). Our conversations at the time had found a good anchor concept in Bourdieu's (1998) notion of *disposition*: an element of practical reason that could be conceived as having two sides, like a coin. Dispositions could be seen by an observer as ordinances to which the individual is subject given the position in which they are, but dispositions could also be experienced as tendencies emanating from the individual and compelling them to act in particular ways (see Herbst & Chazan, 2003; cf. how Lampert, 1985, speaks of *commitments*). Early on the conversation was mostly theoretical, as we searched for ways to complement our perspectives; but then our conversation took a methodological turn.

At about the same time that we started talking about dispositions, the educational research community was dealing with a renewed interest in the use of experimental methods in education, which culminated with reports like Shavelson and Towne (2002) and the establishment of the What Works Clearinghouse by the US Department of Education (http://ies.ed.gov/ncee/wwc/). The notion was in the air that educational research should aspire to the gold standard of using experimental design, randomly assigning participants to conditions; and our conversations started to include considerations of hypothesis testing in research on mathematics teaching. As we considered what experimental research in mathematics instruction could look like, it was odd to us that the image that first came to mind was that of research on whether the implementation of an instructional intervention might affect students' performance: Does curriculum X produce better gains than curriculum A on the scale N? To be clear, nothing is odd about thinking of curriculum or pedagogy implementation in terms of experimental research. What seemed odd to us was that those types of questions would appear as the prototypical examples of how our field might take on the challenge of experimental research.

Experimental research that gauged the achievement gains that could be caused by a particular treatment were clearly worthwhile questions, important for policy and practice, but they were also applied questions, not necessarily illuminating the fundamental phenomena of mathematics instruction. We wondered whether embracing an experimental paradigm would necessarily mean that research on mathematics instruction would be limited to asking questions of an applied nature, questions that took for granted that we knew the nature of mathematics instruction well and just had to design and test ways of improving it. Given our experience as classroom researchers, we knew that, at the time, mathematics education research (for a long time focused on learning and the learner, and later on the individual teacher) still had some ways to go as far as understanding the nature of the activity of mathematics teaching. We thought there was a great need for basic (as opposed to applied) research on mathematics teaching, not just basic research on students or teachers. And so we wondered whether basic research on mathematics instruction had some use for an experimental paradigm.

Instructional Situations and their Norms:
A Focus for Basic Research on Mathematics Teaching

The fundamental idea, proposed by David K. Cohen among others (see Cohen, Raudenbush, and Ball, 2003; also Chevallard, 1985; Hawkins, 1974; Henderson, 1963), that instruction consists of the interactions among teacher, students, and content in environments was compelling to us and essential for defining an emerging field. We pondered what basic research on the nature of mathematics instruction could look like if it embraced an experimental paradigm: What kind of interventions could reveal aspects of the nature of *mathematics* instruction? And what aspects of mathematics instruction could we expect to find out about? These questions seemed important, on the one hand, in order to respond to the challenge of using an experimental paradigm. Those questions seemed important, on the other hand, in order to establish a foundation for basic (rather than applied) research on instructional practice in mathematics--research that asked questions distinct from the study of instruction writ large (which might assume that the subject does not matter or that it matters the same regardless of the particular discipline from which it comes) as well as questions distinct from the study of people (teachers or students) which might perpetuate the reduction of mathematics education research to psychology.

One key idea presented itself as an aspect of mathematics instruction that we wanted to find out more about: If the subject matters in instruction, that is, if mathematics instruction in geometry is a practice distinct from instruction say in Calculus, American History, or Organic Chemistry, we would expect to see regularities of some sort across different cases of instruction in a specific domain. This was anchored by our mutual interest in justification and proof and our question of why, while those practices were current in geometry, they continued to be absent in algebra, in spite of calls for it in reports over the decades: How could it be that the same teacher with the same class, but perhaps at one year's remove, would talk and act so differently in regard to the source of mathematical truth simply due to a shift from geometry to algebra instruction? Additionally, if the regularities observed concerned mathematics instruction as an activity, we would expect to observe regularities that went beyond the knowledge being transacted to include similar ways in which teacher and students managed those knowledge transactions. The word "norm" used in the sociological sense as the normal or unmarked behavior that is tacitly expected in a setting, suggested itself as the name of the object of study. We hypothesized that instruction in specific courses of mathematical study (algebra, geometry, etc.) could be described as abiding by consistent sets of norms, much as other human practices like eating in a formal dinner or getting a table in a restaurant abide by consistent sets of norms (Garfinkel & Sacks, 1970). And we thought that experimental research could be used to confirm that those norms exist.

Instructional Situations, their Norms, and the Notion of Breaching Experiment

While the observance of norms could be found at various layers of classroom activity (as we indicate below, in particular at the layer of the didactical contract and the layer of the mathematical task), we concentrated on studying norms at the layer that we've called the *instructional situation* (Herbst, 2006). Conceptually, an instructional situation is a type of encounter where an exchange can happen between (1) specific mathematical work done by students and their teacher in moment-to-moment interaction and (2) a claim on students' knowing of a specific item of knowledge at stake. Intuitively one could think of an instructional situation as including a mathematical task and the element of the curriculum that the completion of the task enables the teacher to lay claim on. We model instructional situations by spelling out norms that describe the knowledge and the work being exchanged, who is expected to do what, and when those different actions are supposed to happen (see Herbst & Miyakawa, 2008; Herbst, Chen, Weiss, & González, 2009).

Herbst's own research studying the work of the teacher managing the instructional situation of 'doing proofs' in high school geometry provided an example of a norm: students are expected to justify a statement in a proof with a reason before they move on to make the next statement. In proposing it as a norm, we did not mean to endorse the norm as appropriate, but to describe what classroom participants—teacher and students—would consider appropriate. We were not willing to posit that those norms would necessarily be explicit for teachers or students: We expected that people might act as if they followed norms but not necessarily bring them up if and when they were asked to describe the activities they do. And we realized also that, unlike physical laws those norms of human activity could not be thought of as inevitable; they could in fact be broken—one could conceive of and actually find a teacher who had let a student make a new statement without having justified the previous one. While one would expect that a large number of observations of a similar instructional situation would reveal compliance with norms more often than non compliance, the notion that mathematics instruction is regulated by norms could not be validated solely through the observation of regularities in action. We needed empirical ways of attesting that even if a norm had actually been breached, people familiar with the practice would have expected it to be fulfilled.

The notion that basic research on mathematics instruction could consist of finding out about the norms of instruction in subject specific situations, along with the particular notion of a norm as a tacit, shared expectation for action, led us to an idea for how to pick up the challenge of doing experimental research. We were inspired by the ethnomethodological notion of breaching experiments (Garfinkel & Sacks, 1970; Mehan & Wood, 1975), which the first author was already adapting for use in classroom research (Herbst, 2003, 2006). We thought this notion could be adapted to deliberately bring to the surface practitioners' sense of the norms of instruction. If we could represent to practitioners (for example, through a videotaped episode of instruction, but also possibly through an animation or through a virtual reality experience) action that purported to be of the same kind of what they would ordinarily do, but where a hypothesized norm of that action had been breached, we might be able to hear from practitioners whether they had expected the norm to hold. In that sense, a representation of teaching that included the breach of a norm could be expected to reproduce deliberately the phenomenon of interest, namely, that practitioners expected that norm to hold. The extent to which those procedures could be called *experiments* refers to Francis Bacon's notion of experiment in scientific inquiry: "there remains simple experience; which, if taken as it comes, is called accident," "if sought for, experiment" (cited in Durant, 1926, p. 146). That is, our earlier conception of doing experimental research only abided by the notion of experiment as the deliberate reproduction of a phenomenon. But one could also see at least as a possibility that the modern conception of experiment, which emphasizes reproduction of the phenomenon under controlled conditions by way of random assignment of participants to conditions, could be used to confirm that a norm holds: Imagine having two representations of teaching that differed only in that in one of them (the control condition) a hypothesized norm held while in the other (the treatment condition) the hypothesized norm has been breached. Imagine a sample of practitioners who have a comparable degree of socialization in the practice where the norm is supposed to hold. Imagine randomly assigning those participants to one or another representation. Imagine having a way of gauging their satisfaction with the instruction experienced and comparing both groups in regard to that assessment. That gave us a skeleton of what basic experimental research on instruction could look like and some impetus for initial work on a project that we would later call Thought Experiments in Mathematics Teaching (ThEMaT).

Thought Experiments in Mathematics Teaching

The notions of instructional situation, norm, and breaching experiments led us first to gather video records from a geometry lesson on proofs where the teacher allowed a student at the board to omit the justification of a statement and to move on with the proof. We started by gathering focus groups of geometry teachers that looked at that video record and then examining the discourse of those focus groups for comments that might provide evidence that teachers in the focus groups had seen the actions of the videotaped teacher as breaching a norm (Herbst & Chazan, 2003; Nachlieli & Herbst, 2009; Weiss, Herbst, & Chen, 2009). At the same time that this work was being done we started exploring the use of animations to represent classroom scenarios and we wrote a grant proposal for Thought Experiments in Mathematics Teaching to the National Science Foundation, asking for support to create animations that helped us study what by then we had started calling the *practical rationality of mathematics teaching*.

Thought Experiments in Mathematics Teaching (ThEMaT) was funded in 2004 and, among other things, it enabled us to create seventeen families of animated classroom stories (the stories can be seen in LessonSketch, www.lessonsketch.org). The animations use simple cartoon characters and voice over to represent scenarios of classroom instruction. The use of animations allowed us to control the content of those scenarios, allowing us to design scenarios that breach a norm but comply with others. Animations also allowed us to produce breaches that had not been observed in actual classrooms (thus showing one important advantage over video records). And this media also allowed us to create stories that branched, thus depicting alternative scenarios that proceeded from a common trunk (thus our reference to families of stories, since many of them have several alternative stories; see Chazan & Herbst, 2012; Herbst, Chazan, Chen, Chieu, & Weiss, 2011; Herbst, Nachlieli, & Chazan, 2011). The generous support of the National Science Foundation has been crucial for us to maintain a research program that, in our view, has contributed to the field not only an important technique for data collection but also some useful theoretical and methodological ideas.

The goals of the research program are quite ambitious: To develop and test a theory of the rationality of instructional practices in mathematics. This theory of the rationality of instruction explains what instructional actions are justifiable by drawing on two elements (1) the norms that the practice of teaching a particular mathematics course imposes on whoever plays the role of teacher, and (2) the obligations that the profession of mathematics teaching requires of anybody taking the position of mathematics teacher. Combined with the personal assets (including knowledge, skills, and beliefs) that an individual teacher brings with them to that position and that role, those norms and obligations can help explain teacher action and decision-making. The project is now on its second funding cycle in which we are designing and using an online interface (LessonSketch, www.lessonsketch.org) to deliver online multimedia experiences that include animations and other cartoon-based representations of teaching. The project designs multimedia experiences and questionnaires that confront individuals or groups of teachers with representations of teaching; the project will investigate how responses to those questionnaires correlate with measures of mathematical knowledge for teaching (MKT; Ball, Thames, & Phelps, 2008). Over the years, project ThEMaT has allowed us not only to probe and ground our ideas about norms and develop instruments but also to deepen the theory and make progress, though we have not yet used an experimental paradigm in quite the sense described above. Our interventions thus far are experiments in the sense that they reproduce predicted phenomena (evidence of the breach of a norm), but they have not yet reached the gold standard of controlled conditions by random assignment. These conditions may be fulfilled through our current efforts with LessonSketch: An authoring tool in the LessonSketch environment allows us to create online multimedia experiences that may be randomly assigned to participants (see Inglis & Mejía Ramos, 2009, for an example of a similar use of the internet in experimental research in mathematics education). While the foregoing describes the story of our work, we use the following sections to expand on the ideas and some of the methods.

Explicating Practical Rationality

A Classroom Scenario

Consider what we would call a thought experiment in mathematics teaching. The action happens in a high school geometry course in late November. The class has spent some time learning to use triangle congruence to prove statements and has begun the study of quadrilaterals. The teacher, Mr. Jones, has drawn a figure on the board (see Figure 1) and wants the class to prove a statement about the relationship between the sides of the rectangle *ABCD*. There is some hesitation. Somebody asks whether they could prove that $\overline{AB}$ is longer than $\overline{BC}$ while another student asks what they have to go on; the teacher lets those comments pass. A student asks whether triangles *ADE* and *BCE* are congruent. Mr. Jones writes this question on the board and draws two arrows from it. One arrow points toward a question he writes, "how would it help to know that those triangles are congruent?" The other arrow points toward another question he also writes, "what would you need to assume to be able to say that those triangles are congruent?" You can hear somebody say that it's obvious that they are congruent while another says that they could then say the triangles are isosceles. Another student says, "you'd need to know that *AEB* is a right angle;" Mr. Jones writes this on the board and asks the class what they have to say about that (see Figure 2). Some students claim to not really know what the teacher means with that question but others raise their hands. One of these students says that she thinks it would be useful if the angle were right because then the angles at the top would be congruent with the small angles at E. Some kids perk up and one kid says, “and you could then say that AB is twice BC.” The teacher asks them to take a few minutes and see if they can prove that the ratio between the sides is 2 assuming as little as possible. You see a kid write, “Prove: The ratio is 2 ” while others have written “Given:” and are pensive.

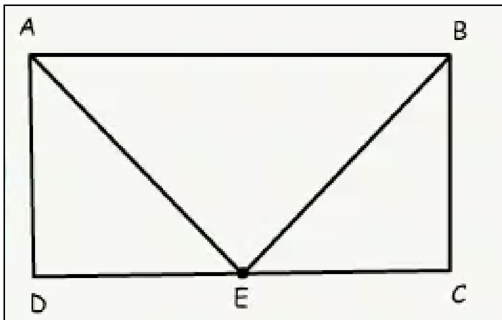


Figure 1. Mr. Jones diagram

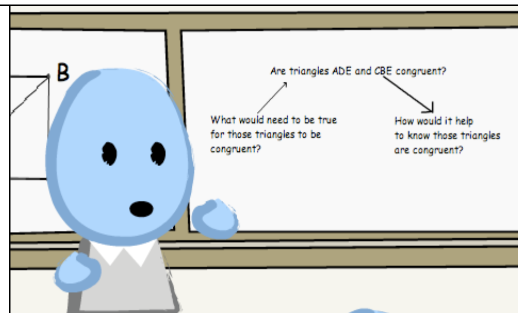


Figure 2. Discussing given and prove

For a few years now, in the context of the project Thought Experiments in Mathematics Teaching, we have been creating cartoon-based representations of teaching that illustrate conceivable scenarios of instruction. One of them is the story “A Proof about Rectangles,” a version of which we’ve just described. Now we want to use that episode to raise a few questions about mathematics instruction in school classrooms and to elaborate on the ideas that this kind of material has helped us explore.

Some of these questions concern the substance of this conceivable episode: What opportunities for students' mathematical work are made possible by how the teacher has been managing the instruction? Other questions are about theory: What kind of considerations about classroom instruction could help us describe and explain how teacher and students ordinarily transact mathematical ideas, in such a way that we could also account for possible avenues for improvement and foresee their consequences? Finally, other questions are about research methodology: What kind of data can help us ground those theoretical considerations? How to obtain it? These questions, though large, serve to explicate the program of research that we call the *practical rationality of mathematics teaching* (Herbst & Chazan, 2003; Herbst, Nachlieli, & Chazan, 2011).

Desirable and Customary Mathematical Work

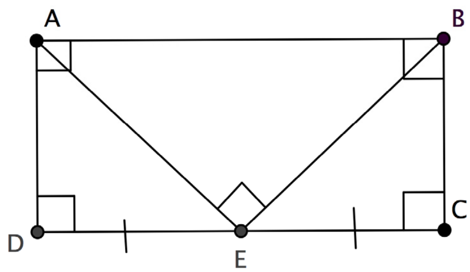
What mathematical work are students doing in the episode described above? We could describe it as listing plausible statements about a figure and considering whether these plausible statements could be connected through logical necessity. The source of some of those statements seems to be perceptual—for example, the observation that angle *AEB* is right. Other statements seem to result from deduction—notably, the observation that if the angle AEB was right then one could conclude that side $\overline{AB}$ would be twice as long as side $\overline{BC}$. But regardless of the origin of each of those statements, the teacher is helping students connect all statements through abduction and deduction: Asking what assumptions would enable one to infer the plausible statement made and asking what inferences could be made if one took that plausible statement for granted. The assertion about the relative length of the sides of the rectangle eventually derives from the plausible truth of those earlier statements. The teacher is thus helping students reduce a question of truth (what could be true about an object) to a question of deducibility from possible statements about an object. They are using proof as a method to find things out.

Such use of proof as method in knowledge inquiry is essential to the discipline of mathematics (Lakatos, 1976). It is also behind the drive to mathematically model other fields of experience: The expectation that in those fields it will also be possible to reduce the problem of truth to a quest for deducibility, which can then warrant new, still unknown, possible truths is important in pure and applied science. Hanna & Jahnke (1996) have argued that, by using an empirical theory to predict empirical phenomena, scientists engage in modeling the world and deductively producing inferences based on assumptions, predictions that are eventually subject to confirmation by experimentation.

Being able to master such a form of inquiry can make a child resourceful in ways that can add to methodological resources they get from the study of other disciplines. Mathematical work of the kind depicted in the scenario is not only authentic mathematical work (Weiss, Herbst, & Chen, 2009) but also embodies skills and processes that might empower students to contribute to knowledge production writ large. In that sense we would argue that Mr. Jones's questions to students about what could be deduced from a given statement, or what statement could entail what they think is true, are helpful ways of educating his students in the use of mathematical reasoning for making predictions about the world, in this case about the world of diagrams. A scenario where students could work on connecting plausible statements deductively is therefore conceivable and it could be represented using animations or comic strips with cartoon characters.

However, it is likely the case that few students encounter such opportunities to engage with proof in school mathematics in the way outlined by the foregoing scenario. The work they do during their school years rarely includes chances to acquire the skill or the appreciation of the methodological, model-making function of proof or even experiences doing work that could have had that exchange value.

It is more likely that the problem above would be presented to high school geometry students as shown in Figure 3. In particular, while students are ordinarily expected to prove propositions in high school geometry, it is ordinarily the teacher (or the book) who will state the givens and the conclusion of the propositions they prove. While efforts to change these norms have been made (e.g., the work with the Geometric Supposers reported in Schwartz, Yerushalmy, & Wilson, 1993), it rarely falls on the students to determine the givens for a plausible conclusion, to deduce the conclusion from a set of givens, or to find both the givens and the conclusion for a theorem that relates to some plausible naïve conjecture.



• Given:

$ABCD$ rectangle,

E midpoint of $\overline{DC}$,

$\angle AEB$ right angle

Prove: $\frac{AB}{BC} = 2$

Figure 3. A more likely proof problem.

The Scenario as an Example of Norms and Instructional Situations

The expectation that, if students are to be held accountable for producing a proof, the teacher will have to provide for them the givens and the “prove” statement is an example of what we call a *norm* of the instructional situation “doing proofs.” It is a norm in the sense that an observer can describe teachers and students acting *as if* they expected that this would be the case. In consequence, if students and teacher were involved in an interaction about a problem for which the teacher did not provide the given and the “prove”, then it is likely that neither teachers nor students would describe those activities as doing proofs—they might describe them as something else (e.g., having a discussion). The norm is that anytime the students are expected to produce a proof, teachers are expected to provide the givens and the conclusion to prove. Of course by “norm” we don’t mean ‘the correct thing to do’; it is certainly not “correct” from our perspective informed by our understanding of mathematical practice, though it may be experienced as correct or appropriate by teachers and students. We use “norm” and “normative” in two complementary senses: First, the sense in which ‘normative’ means ‘frequent’ or ‘usual;’ this could be corroborated empirically by observing, over a large number of high school geometry classrooms, the recurrence of this feature in proof activity. Second, the sense in which the participants in the situation act as if they expected such behaviors to be appropriate or correct.

Such norms are not just arbitrary belief systems, idiosyncratic and completely changeable; they are norms of interaction between teacher, students, and specific content and are thus ascribed not to individuals but to the specific instructional situation where that interaction happens. They have a particular purpose; they regulate the division of labor over time between student and teacher vis-à-vis a specific kind of instructional exchange. In this case, this norm regulates the exchange between the work students do when proving a proposition and the claim (that the teacher needs to substantiate in high school Geometry) that students know how to do proofs. In that sense, this norm is different in scope than the more general norms of the didactical contract, which are present across different instructional exchanges (e.g., the expectation that when the teacher asks students a question, she already knows the answer). In trying to understand the practical rationality that underlies that norm and the possibilities to depart from it we are therefore asking not a question about instruction in general (e.g., Do teachers see it as possible, desirable, or appropriate to have students work on tasks where they determine the givens or the goal?) but rather a question about what counts as doing a proof in high school geometry: Do teachers see it as possible, desirable, or appropriate to hold students accountable for doing a proof and to do so in the context of tasks where students are in charge of providing the givens or the conclusion of the proof problem? To us it seems that such tasks would enable students to experience and learn about the methodological role of proof: Its instrumentality in finding new knowledge. But, such tasks are not common in classrooms.

"Doing proofs" in high school geometry illustrates what we mean by an instructional situation. These are frames for the encounter among teacher, students, and specific content: In these encounters an instructional exchange takes place—the exchange between the work that students do, for example, on a particular task, and the knowledge claim that such work enables the teacher to make by virtue of having done that work. Instructional situations can be modeled as systems of norms such as the one described above. Instructional situations are content-specific in two regards: They accommodate or make room for specific tasks, and they permit the exchange of work on those tasks for specific items of knowledge. The instructional situation "doing proofs" does not customarily accommodate students' work in which students produce the givens or the 'prove' for a proof problem; rather, if they are ever involved in such work, their involvement does not count as knowledge of proof. Based on our understanding of the methodological role of proof in mathematics (Lakatos, 1976) we argue that such work (figuring out the givens or the conclusion) does not always precede but it is often part of the work of proving in mathematics.

Is it Feasible to Change Instructional Situations?

A motivation for our work has been to understand better whether the kind of mathematical work described above—the use of proof as a tool to know with—could feasibly be deployed in classrooms. One way of addressing that question focuses on the design of resources that can support that work. And some of our instructional experiments (e.g., Herbst, 2003, 2006) have included developing resources, including special lessons and units co-developed with teachers. In those, problems were designed to create contexts where proving could help students come up with an answer to the problem. Our focus on the feasibility of that work led us not only to investigate whether proof could play a role as a tool to know with (see Herbst, 2005) but also to investigate what kinds of disruptions of the work of teaching those tasks would cause (Herbst, 2003) and what sorts of negotiations a teacher needed to make to restore a sense of normalcy (Herbst, 2006).

Another way of addressing the feasibility question goes beyond investigating what is possible when teachers use different tasks to engage students in proving and taps into the source of arguments that teachers could draw upon to justify or rebuff such tasks. Behind that version of the feasibility question is the fundamental hypothesis that classrooms are complex systems where actions are not merely a projection of the will or capacity of the actors or the richness of their resources. Rather, actions of individual actors contribute to the deployment of a joint activity system whose performance also feeds back, and thus gives shape, to the actions that the participants can take in that system. And at least tacitly and as a group, teachers of a given course know the demands of that system to the point that we should be well advised to canvass that knowledge if we intend to understand whether a particular improvement will be feasible or not. The question then is not simply how to design materials that enable desirable mathematical work or how to create in teachers the desire to promote that work. We also need to ask about the structure and function of the activity system where that work might be deployed and how this system might accommodate or resist attempts to deploy that work. In particular this requires thinking of mathematics instruction in school classrooms as a system of relationships that are deployed under various conditions and constraints. A conceptualization of this system could enable us to think in a more sophisticated and potentially accurate way about what teacher and students do and thus be able to foresee if given improvement efforts have a prospect of success.

An analogy with how mathematics educators have evolved in their thinking about students' errors can illuminate this conceptualization of instruction as a system. There used to be a time when student errors were seen as indications of misfit, mishaps, or forgetfulness. Things changed when research on students' mathematical work started to be treated within a cognitive paradigm. For example, an international study led by Lauren Resnick, Pearla Nesher, and François Leonard (see Resnick, Nesher, Leonard, Magone, Omanson, & Peled, 1989) on students' sorting of decimal fractions showed that students' errors had a conceptual basis: Their errors could be explained by the existence of conceptual, tacit controls such as the "fraction rule" or the "natural number rule." These were mathematical quasi-truths, or epistemological obstacles (Brousseau, 1997), true within a limited domain but false when that domain was extended. Students that made errors did so not out of the lack of knowledge but out of the possession of some knowledge. As a field, our stance toward students' errors thus changed from an early judgment stance to a later inquiry stance: Rather than judging students as irrational when they make errors, we now strive to understand what rationality leads them to make those errors.

We propose that we should think of the actions of teachers (and students) in the classroom by analogy with how we have come to think about error in children's mathematical thinking. The analogy we propose is that we could think of "error" in instruction—really teaching that deviates from what might be deemed desirable—not as an indication of misfit, ill will, or lack of knowledge, on the part of the practitioner. Rather, we should think of this "error" as an indication of the possible presence of some knowledge, knowledge of what to do, which is subject to a practical rationality that justifies it. This is a rationality that we should try to understand better before judging teachers or attempting to legislate their practice. It is this rationality, rather than simple stubbornness, that explains why many reforms are not able to make their way into classrooms. Teachers and students act in classrooms in ways that attest to the existence of specialized knowledge of what to do; knowledge that outsiders to those classrooms are less likely to have even if they know the knowledge domain being taught and learned. For example, as it relates to the scenario narrated above, teachers and students of geometry would likely see it as strange for Mr. Jones to ask the students for the givens of the problem. We focus here on the rationality associated with the role of the teacher and how this might warrant or refute actions like that one.

Practical Rationality and the Role of the Teacher

The "teacher" of a specific course of mathematical studies, such as high school geometry, is an institutional role, not just a name to describe an aspect of an individual's identity (Buchmann, 1986). There is a person who plays the role, for sure; that person comes to play the role with personal assets that are likely to matter in what he or she chooses to do. These assets are likely to include mathematical knowledge for teaching and skill at doing some tasks of teaching (Ball, Thames, and Phelps, 2008). It is widely believed that those assets make a difference; that teachers who have those assets may be able to figure out and do things that others may not be able to do. But while teachers' causes and motives to do things may have personal grounds, it is unlikely that their actions could be justified on personal grounds. One could imagine that Mr. Jones in the scenario above might have been bored with the prospect of giving his students another routine proof exercise or wanted to have a fun day teaching geometry. But we could not really expect him to use any of that as the warrant for doing what he did—his job is not to find activities that amuse him, but rather to teach geometry to his students. Even if the actual basis for his actions had been his own amusement, how could he justify having done that when talking with his peers? Those grounds for justification are what we call practical rationality.

The notion of practical rationality points to a container of dispositions that could have currency in a collective, for example, within the set of colleagues who teach geometry in similar settings. These are dispositions to abide by the *norms* of the specific instructional situation a teacher is engaged in (i.e., the norms of the situation of doing proofs in high school geometry) as well as dispositions to honor the *obligations* to the profession of mathematics teaching.

By dispositions we mean what Bourdieu (1998) describes as the categories of perception and appreciation that compel agents in a practice to act in specific ways. We interpret categories of perception to include the taken as shared ways in which practitioners perceive people, events, things, and ideas in the shared world of the classroom, as instantiated, for example in the language tokens they use to talk about the world of the classroom. We interpret categories of appreciation to include the principles and qualities on which practitioners rely to establish an attitude toward people, events, things, or ideas. Dispositions tend to be tacit but they can be articulated to others when justifying to one's peers (or to other stakeholders) why one might or might not do something like what Mr. Jones did with that proof problem. The high school geometry course and the work of doing proofs, in particular, have been particularly fertile grounds for us to develop theory about instruction and the practical rationality of mathematics teaching.

Didactical Contract and the Role of the Teacher

To conceptualize the work of the teacher as the playing of a role, we start from the notion of the didactical contract (Brousseau, 1997): The hypothesis that student and teacher have some basic roles and responsibilities vis-à-vis a body of knowledge at stake. What does it mean that there is knowledge *at stake*? The relationship between teacher and students exists because of the assumption that there is knowledge that can be communicated from one to the other; this knowledge is at stake because such communication may or may not happen. The didactical contract is a tacit assignment of rights and responsibilities between teacher and student vis-à-vis the communication of that knowledge. These responsibilities include the expectation for the teacher to give students work to do that is supposed to create opportunities to learn elements of that body of knowledge, and the expectation for the student to engage in the work assigned, producing work that can be assessed as evidence of having (or not yet having) acquired that knowledge.

We use the word *norm* to designate each of those statements that an observer makes in an effort to articulate what regulates a practice: Actors act *as if* they held such statement as a norm, though they may be quite unaware of it. Each class has a didactical contract that can be modeled by listing its norms. From the perspective of the teacher, the didactical contract authorizes a basic exchange economy of knowledge that he or she has to manage: An exchange between work designed for, assigned to, and completed by students and elements of knowledge, prescribed by the curriculum, at stake in that work, and hopefully embodied in students' productions. The role of the teacher includes managing those exchanges between work and knowledge. This management includes, first, enabling and supporting mathematical work; and second, interpreting the results of this work, exchanging it for the knowledge at stake.

The hypothesis of a didactical contract only says that a contract exists that fulfills those goals; the hypothesis means to describe any mathematics teaching inside an educational institution. But it is also obvious that the teacher and student roles and responsibilities are under-described by that hypothesis: There are many ways in which the didactical contract could be enacted that would have at least those characteristics; contracts could be quite different from each other not the least because the mathematics at stake could be very different from course to course and thus require very different forms of work to be learned. Even for the same course of studies, say high school geometry, different contracts could further stipulate the roles and responsibilities of teacher and student differently. In particular, it is conceivable that some contracts might include the expectation that every new task would require negotiation about how the general norms of the contract apply (e.g., What is it required of the teacher to get students to work on a particular task? What does it mean for students to work on that task?). It is also conceivable, and we argue more likely, that contracts rely on a manifold of instructional situations that forego the need for some of those negotiations much of the time. These instructional situations include mostly tacit but specific norms that specify how the didactical contract applies for a range of tasks and the specific items of knowledge to be exchanged for the students' work on those tasks.

While some research has endeavored to conceptualize, enact, and study the characteristics of alternative contracts (e.g., Chazan, 2000; Lampert, 1990, 2001; Yackel & Cobb, 1996), the first author has been interested in using a variety of approaches to study the usual high school geometry contract and the practical rationality behind the teachers' work managing the exchanges enabled by that contract. The reason for that is founded on the considerations about improvement made earlier. Sustainable improvement in instruction will not only need to provide new and better resources but also to be able to deal constructively with the inertia and possible reactions from established practice. Knowledge of how instruction usually works and what rationality underpins its usual operations is key for the design of reforms that are viable and sustainable. Furthermore, knowledge of how usual instruction works can encourage piecemeal, incremental changes that don't throw the proverbial baby with the bathwater.

Instructional Situations and the Role of the Teacher

The situation of "doing proofs" has been a useful starting point in that research agenda. Historical analysis (Herbst, 2002b; González & Herbst, 2006) has showed how the general skill "how to do proofs" became an object of study in and of itself, leaving behind the important relationships between proofs and specific concepts, theorems, and theories. The work that students do has also evolved to the current state in which *what* a student can prove from available givens matters much less than *whether* and *how well* they carry out a proof. In exchange for a claim on that knowledge (to show that they know "how to do proofs") students are to show that they can connect a "given" with a "prove" by making a sequence of statements justified with prior knowledge (regardless of the strength or the importance of the proposition proved): In other words students are learning the logical form of proof at the expense of its methodological function. In describing such exchange as an instructional situation, we posit that this exchange is facilitated by a specialized set of norms that elaborate how the didactical contract applies.

From observing work in geometry classrooms we have noted that implicit expectations of who is to do what and when vary depending on the specifics of the object of study. In relation to diagrams, for example, the extent to which students can draw objects into a diagram or draw observations from a diagram varies according to whether the work is framed as a construction, an exploration, or a proof (Herbst, 2004). While the didactical contract for a course may have some general norms that differentiate it from a contract for a different course, there is also differentiation between the more specific norms within a given course of studies, depending again on what is at stake. Much of those rules are cued in classroom interaction through the use of selected words such as *prove*, *construct*, or *conjecture*. These words frame classroom interaction by summoning special, mutual expectations, or *norms*, of who can do what and when. As noted above, we use the expression *instructional situation* to refer to each of those frames. Instructional situations are specialized, local versions of the didactical contract that frame particular exchanges of work for knowledge, obviating the need to negotiate how the contract applies for a specific chunk of work.

"Doing proofs" is an example of an instructional situation in high school geometry; "solving equations" is an example of an instructional situation in algebra I (Chazan & Lueke, 2009). We contend that these frames for classroom interaction, these instructional situations, are defaults for classroom interaction, tacit knowledge held by the classroom as an organization (Cook & Brown, 1999) that specifies what to do; knowledge perpetuated through socialization (and with the aid of textbooks and colleagues) that, in particular, provides cues for the teacher on what to do and what to expect the student to do. Instructional situations are sociotechnical units of analysis; they organize joint action with specific content.

Our perspective centers on the situation rather than the individual and has the power to explain why the same individual might happen to do quite different things in different situations by no fault of their own. To implement this focus on the situations thus far we have created models of those situations. A model is not a portrait of what is desirable but rather a simplified operational description of a reality, in this case a human activity. Our models consist of arrays of norms that describe each situation in terms of who has to do what and when (Herbst & Miyakawa, 2008). Those models facilitate research on the content of practical rationality.

Practical rationality is a container whose content includes the categories of perception and appreciation that are viable within the profession of mathematics teaching to warrant (or refute) courses of action in teaching. The notions of instructional situation, norm, and breach of a norm are the points of departure to study this rationality empirically. Based on the ethnomethodological notion of a *breaching experiment* (Mehan & Wood, 1975) we propose, as a methodological hypothesis, that if participants in an instructional situation are immersed in an instance of a situation where one of its norms has been breached, they will engage in repair strategies that not only confirm the existence of the norm but also elaborate on the role that the norm plays in the situation or on what might justify departing from the norm.

Our data collection technique relies on representations of breached instances of instructional situations—representations made in videos, slideshows, or comic strips, sometimes using real teachers and students (e.g., Nachlieli & Herbst, 2009) or using cartoon characters (Herbst, Nachlieli, & Chazan, 2011). We confront usual participants in an instructional situation with a breached representation. For example, the classroom scenario narrated above is quite close in content to an animated classroom story, "A Proof about Rectangles," that we produced in order to study the rationality behind the tacit norm that the teacher is in charge of spelling out the givens and the prove. To find out about that rationality we attend to participants' reactions to the representation: Do they perceive the breach of the norm? Do they accept the situation in spite of the breach? What do they identify as being at risk because of the breach? What opportunities, if any, do they see being created or lost because of the breach?

Our aim is not to understand the participants themselves; our aim is to use the participants' experience with the situation to understand the *situation* better. In particular we want to discover the elements of the practical rationality of mathematics teaching that teachers consider viable justifications of breaches of situations that would arguably be desirable, say because they might create a more authentic kind of mathematical work (see Weiss, Herbst, & Chen, 2009). In the case of the story narrated above we would pose the following concrete question: On what account could a teacher justify (or rebuff) an action like the one Mr. Jones took? Clearly, researchers might be able to justify Mr. Jones' action and we have tried to articulate that from a mathematical perspective. But in spite of the fact that some of us have had experience teaching we don't know teaching now in the way practitioners do. By virtue of the role that they play and the position from which they take on that role, teachers have to respond to specific obligations that shape their decisions.

Experimentation and Teachers' Responses to a Breach of a Norm

In the previous section we noted that our technique to study the practical rationality with which practitioners might justify abiding by or departing from a norm in an instructional situation consists in creating a representation of practice that instantiates the situation and where the norm in question has been breached, then listening to how teachers respond to that representation. When teachers respond to a breach in an instructional situation, they might reject the situation or might repair the situation. By *reject the situation* we mean that they would come across as saying "this class is not doing a proof;" key in such a categorization is (1) the recognition that someone might argue that the target situation (doing proofs) describes the scenario being enacted and (2) their denial of the validity of such a description. By *repair the situation* we mean a softer version of rejection: participants come across as describing the events using a different situation or as conforming to a contract different than the normative. For example, some teachers have said that Mr. Jones is leading students in an *exploration* rather than a proof.

> "the only thing I could see him doing is that he was trying to get them the idea of making conjectures, okay? What, what can we assume about this picture" (ITH062806, 4, 81, Tina)[2]
>
> "Maybe it's just like a -- kind of a like a blank canvas for just discussing without all of the restrictions tied on at this point just y'know lighter form of conversation y'know." (ThEMaT082206, 10, 109, Lucille)

Key in categorizing those expressions as *repairs of the situation* are that (1) participants are describing the events in terms of the larger grain size of the teacher's instructional goal and that (2) participants are using some conventional labels for recurrent classroom activity to describe what happened in ways that fail to recognize the situation as one of "doing proofs" (e.g., conversation, making conjectures).

A third alternative, also present in our data, can be described as participants' *acceptance* of the situation, namely recognized it as a case of "doing proofs." For the sake of coding data, whenever participants don't reject or repair the situation we take that as an acceptance, even if this is tacit. In some of these cases their acceptance of the situation came with comments that indicated that something about the particular task in which "doing proofs" was embodied had not been done as it should have been done. For example, some of our participants said

[2] References to session data follow the convention (sessionid, interval, turn, speaker). All names are pseudonyms.

> "So the fact that he's y'know not marking anything and asking them to kinda trust that drawing is kind of odd" (ThEMaT082206, 20,227, Edwin)

"we tell them not to assume anything that we draw." (ThEMaT082206, 5,112,Tina)

Among those comments accepting the situation as doing proofs, some comments indicated a positive appraisal of what the teacher had done. For example:

> "In the books we always go given-prove, right? So we don't really give them the option to even explore some of the nature of the figures." (ThEMaT082206, 10,116, Jillian)

We describe those responses as accepting the situation (the participant identifies or at least does not deny that the goal of the activity is to "do a proof") but *repairing the task* (while the participant does not cast the situation as different than doing proofs, the participant recognizes some actions as deviating from the norm in that situation). A complete enumeration of contingencies includes, at least conceptually, the possibility that participants may accept the situation and accept the task: However, empirically one might observe those cases to be unmarked (e.g., the participant talks about something other than the breach). Incidentally, note that in this discussion we are proceeding rather globally and omitting considerations of the possible complexities of the unit of analysis for the sake of proposing how the experimental data could be aggregated: While the present considerations might be used to examine data gathered from individual practitioners providing a one-time response to a representation (for example, responding to a multimedia questionnaire), data gathered from groups of practitioners in more extended conversations (such as those reported by Chazan & Herbst, 2012, or Herbst, Nachlieli, & Chazan, 2011) require more sophisticated considerations of the unit of analysis.

From those broad considerations about the way we might code data from practitioners' responses to a representation of an instructional situation we can anticipate a way of using this data to gauge the extent to which a hypothesized norm pertains to the situation under consideration—and in that way use experimentation to build basic knowledge about the practice of mathematics teaching. Consider first the case of practitioners responding to a representation of an instructional situation in which a hypothesized norm of that situation has been breached (e.g., the teacher asks students to provide the givens for a proof exercise). Consider further that the encounter between practitioners and representations is framed for them as a case of the situation (e.g., the instrument declares something to the effect of "we are going to see how a class works on a proof") but no mention is made of the possibility that a norm might be breached nor is attention explicitly directed to the actions by which the breach is manifest. After the encounter, participants are asked to comment on how appropriately the teacher handled the situation (e.g., "what do you think of the way the teacher managed the class's engagement in proving"). The data is then coded in ways that permit the aggregation shown in the contingency table below (and drawing on the definitions of reject, repair, and accept given above).

	Accept Situation	Reject or Repair Situation
Reject or Repair Task	2	1
Accept Task	4	3

The hypothesis that the norm breached is a norm for the situation being represented would justify the expectation that data would aggregate in cells 2 and 3. Cell 2 represents responses of the kind 'in this situation you'd rather do this other work instead' (e.g., if you want students to do a proof, you give them the givens and the prove). Cell 3 represents responses of the kind 'the kind of work you are doing there fits better in this other kind of situation' (e.g., a question like that would be better off in a conversation than in a proof). Data that could be classified in any of those cells would provide evidence that adds credibility to the hypothesis that the norm applies. (Note that this evidence could but would not solely include repairs that specifically mention the norm breached—norms could stay tacit in spite of being breached and the evidence provided by participants might just reveal their sense that something has gone awry.) In contrast, cells 1 and 4 provide evidence that contradicts or at least provides no evidence in favor of the normative nature of the hypothesized norm.

Intuitively, under the hypotheses that the norm applies to the situation, that the representation breaches the norm, and that the participants are experienced enactors of the situation, one would expect the aggregate of Cells 2 and 3 (repairs of situation or of task) to be higher than the aggregate of Cells 1 and 4. One could define a measure of the extent to which the representation elicits repairs (2 + 3) or percentage of teachers who repaired over those who provided comments.. More generally, given a representation (related to a norm N of a situation S) and a sample of practitioners, the representation could be classified a priori as breaching or non breaching N, and each practitioner could be classified as experienced or not experienced in S. The percentages of repairs could be used in particular, to test (this time using the modern sense of experiment) the extent to which experienced practitioners in a situation hold norm N.

Imagine a sample of experienced practitioners randomly assigned to one of the following two conditions. In the experimental condition the practitioners consider a breached representation, while in the control condition the practitioners consider a compliant representation. The responses from practitioners would then be summarized in corresponding repair ratios $r_{1,e}$ and $r_{0,e}$ as defined above and the difference between these proportions could be tested for significance. Similarly, one could pose the question of whether this norm is significantly more salient for teachers experienced in the situation of interest than for teachers who do not have such experience. This question could lead one to compare the ratios $r_{1,1}$ and $r_{1,0}$, that is, the repair ratios for experienced and non experienced practitioners confronting a breached representation. Finally, one could consider randomly assigning practitioners who are either experienced or inexperienced in the situation to either a breached or a compliant representation, and analyzing the table of contingencies below. The Chi Square test could be used to examine whether acknowledgment of Norm N is specific to teachers experienced in Situation S.

	Experienced in S	Inexperienced in S
Breached Representation (of N_S)	$r_{1,1}$	$r_{1,0}$
Compliant Representation (of S)	$r_{0,1}$	$r_{0,0}$

Of course the preceding argument is only a sketch of what the research ahead requires. In addition to the problem of determining the unit of analysis noted above, there remains the problem of finding operational ways of determining repairs, rejections, and acceptances of task and situation. While we have made some important progress identifying norms of situations to be researched and creating representations that breach those norms, the work of developing measures of the repairs that practitioners produce in response to those representations is still incipient. Our current work in this area investigates the use of elements of systemic functional linguistics, particularly the notions of modality and appraisal (Halliday & Matthiessen, 2004; Martin & White, 2007), to anchor the notion of repair in linguistic performance. Furthermore, as far as the implementation of the technique, these considerations oversimplify the certainty with which one can say that a representation of a situation breaches a norm or complies with all norms—it isn't only that the provisional nature of models challenges the extent to which one can ever say that a representation will be compliant, but the multidimensional and interactive nature of human activity makes it hard to represent breaches of a norm without other remarkable entailments needed for continuity's sake. Along those lines, and because of the extent to which an instance of a situation may instantiate more than the actions specific to a norm, a third challenge consists of being able to reproduce the phenomenon (participants' recognition of the norm) independently of the representation used: Would representations R and R' of different instances of the same situation S, each of which breaches the same norm N, produce similar responses from practitioners experienced in S? Considering those methodological challenges, it is fitting to say that so far we have only been able to show how our theoretical agenda and basic research goals could use an experimental paradigm and within that to indicate more specific methodological goals.

The sketch above does indicate a path for using an experimental approach in basic research on mathematics teaching—specifically, research that identifies and confirms the existence of specific norms for specific instructional situations. But as noted above, practical rationality includes more than the norms of instructional situations; it includes the categories of perception and appreciation with which practitioners can relate to actual and possible actions in teaching. In particular, practical rationality includes the grounds on which a breach of a norm might be recognized as a breach and yet appraised favorably. Notwithstanding the possible use of the experimental design sketched above to test hypotheses, it is probably just as important for theory and practice to deepen the descriptive research that can lead to more refined hypotheses, especially hypotheses that can account for the difference between justifiable and unjustifiable breaches of norms.

Practical Rationality and the Justifications for Breaches of Norms

The data that we collect from practitioners in response to breached representations usually contains more than repairs of those breaches. Practitioners not only recognize the presence of a norm when they repair its breach, quite often they do so using discourse that commits a stance toward such a breach. Those stances are not always negative; when these stances are positive, practitioners may engage in a rather visible practical argument to justify an action in spite of the norm against it. As part of the agenda to flesh out the content of practical rationality we are interested in inventorying and accounting for the dispositions used by practitioners to warrant actions that breach norms (as well as those actions that comply with norms).

Sometimes, teachers' responses to breaches of a norm may indict the teacher for breaching a norm and justify it with an argument that explicates why the norm exists. In the case presented above, the evidence we found suggests that the norm of providing the given and the "prove" may be justified on the grounds that it keeps students from making knowledge claims by relying on the looks of a diagram. Indeed the line between, on the one hand, assuming something as given so as to start drawing necessary consequences from it and, on the other hand, assuming something else as true while one is drawing those consequences, may be blurry enough to justify keeping students from having to manage it. One could represent this argument for a norm by adapting Toulmin's (1969; see also Inglis, Mejía-Ramos, & Simpson, 2007) argument layout, as shown in Figure 4 (where instead of *data* and *claim* we use *circumstances* and *action* respectively).

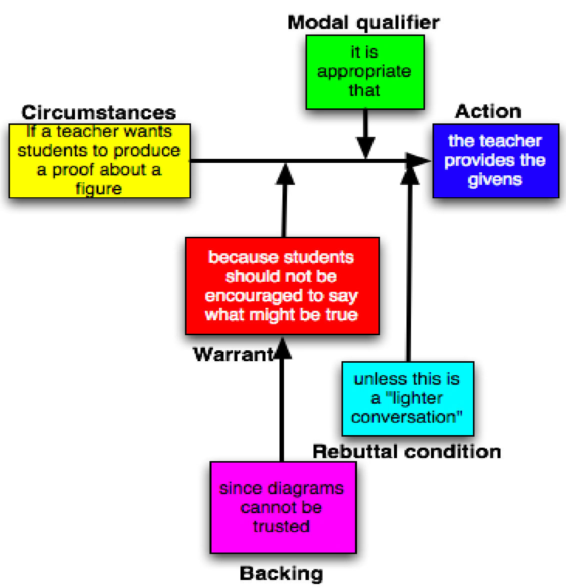


Figure 4. A practical argument using Toulmin's layout.

The data also shows that teachers' responses sometimes acknowledge the breach, but rather than indicting the teacher for the breach they might justify it while relaying whatever reasons they might have for that justification. In this sense, the breaching experiments give access to other elements of the practical rationality of mathematics teaching. In the data shown above, one of the comments appeared to justify the breach by elaborating on the grounds for exception noted above.

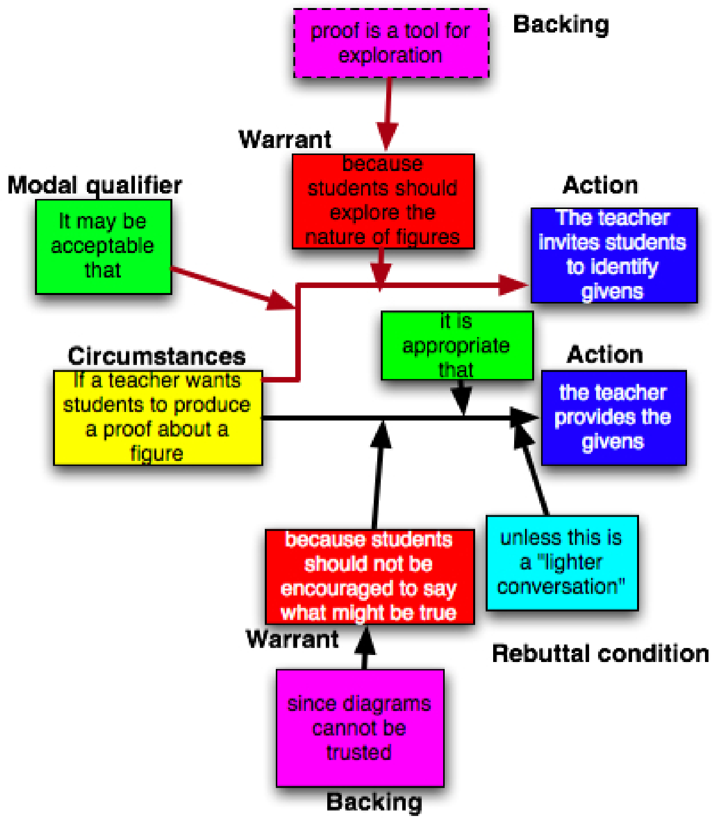


Figure 5. A practical argument for and against an action using Toulmin's layout.

The Norms and Obligations that Span Practical Rationality

From our work in the past five years, looking at the responses from teachers to animations that represent breaches of situations in geometry and algebra, we have built an initial model of this practical rationality. In this model, conceivable moves by a teacher are justified or rebuffed on the basis of principles or warrants that attest to the presence of two sets of regulatory elements. One of those sets of regulatory elements describes the roles the teacher is called to play in the contract, the instructional situations, or in mathematical tasks. As noted above, we call all of those *norms*: Some are norms of the contract (they regulate work across the many objects of knowledge in a course of studies), while others are norms of the instructional situation (they regulate work that is specific to an object of knowledge). A third kind of norms, norms of the task (regulating how the teacher supports the milieu for the students' mathematical task) is also part of the model but is not discussed here (see Herbst, 2003; also Brousseau, 1997). The other set of regulations, which we explicate below, includes the professional obligations that tie an individual to the position of mathematics teacher, beyond the specific demands of a particular contract, situation, or task.

In general, the first set of regulations for actions in teaching come from the structure of the different 'games' the teacher and the student play with specific content. The various norms that justify teachers' actions respond to the requirements of the role the teacher is called to play in the contract for a course of studies, the situation that frames the different kinds of work that exchange for a particular object of knowledge, and a specific mathematical task. But these norms by themselves don't explain why practitioners see some breaches of norms as acceptable (see, for example, Nachlieli & Herbst, 2009; Herbst, Nachlieli, & Chazan, 2011). The data that we have gathered shows not only that the norm exists and what problems it would help solve, but also on what grounds it could be breached. As we analyze the data from study groups that considered the many animations we created in ThEMaT, a more systematic way of accounting for those warrants has become useful to us.

Both the presence of norms and the breaches of norms can be accounted for by appeal to various professional *obligations* that we posit apply to the mathematics teacher (to some extent these obligations may also apply to the elementary teacher who teaches mathematics part of their time, but they likely need to be adapted). We propose that four professional obligations can organize the justifications (or refutations) that participants might give to actions that depart from a situational (or contractual) norm. We call these four obligations *disciplinary*, *individual*, *interpersonal*, and *institutional* (Herbst & Balacheff, 2009; see also Ball, 1993).

The *disciplinary obligation* says that the mathematics teacher is obligated to steward a valid representation of the discipline of mathematics. This may include the obligation to steward representations of mathematical knowledge, mathematical practices, and mathematical applications.

The *individual obligation* says that a teacher is obligated to attend to the well being of the individual student. This may include being obligated to attend to individual students' identities and to their behavioral, cognitive, emotional, or social needs.

The *interpersonal obligation* says that the teacher is obligated to share and steward their medium of interaction with other human beings in the classroom.

This may include attending to the needs and resources of shared discursive, physical, and social spaces within shared time.

And the *institutional (schooling) obligation* says that the teacher is obligated to observe various aspects of the schooling regime. These include attending to school policies, calendars, schedules, examinations, curriculum, extra curricular activities, and so on.

These obligations are not specific to a contract for a course of studies; they describe equally the teacher of AP Calculus and the teacher of informal geometry. They coalesce to justify contracts and their instructional situations; and they may combine with norms of contract, situation, or task in order to justify extraordinary actions. In general, combined with the norms of contracts, situations, and tasks these obligations span the practical rationality of mathematics teaching. The dispositions that compose practical rationality could be accounted for as combinations of norms and obligations. One can then say that the justifications for actions in teaching, either those actions that are usual or those that are unusual but viable, can be found by combining norms of the contract and situations that the teacher is enacting with obligations the teacher has to the profession of mathematics teaching.

Within that rationality one can see specific contracts (high school geometry, algebra I) and their instructional situations (doing proofs, solving equations) as sociohistorical constructions that have persisted over time by complying in some way with those obligations. To the extent that the obligations could contradict each other, it is quite an accomplishment for teaching to have been able to develop stable contracts and situations over time (Herbst, 2002a).

Conclusion: Practical Rationality and Instructional Improvement

The theory of practical rationality is a way of accounting for existing, stable practices. To the extent that our interest in improving practice stresses the need for improvements to be responsible, incremental, and sustainable, it is appropriate for us to try to understand what justifies the norms of stable contracts and situations, even if we might want to modify or do away with some of them: Understanding stable systems of practices as well as understanding how those systems react to perturbations is fundamental for the design of new practices. Indeed, since improved practices will need to subject themselves to similar grounds for justification, practices that are close to those that are normal in existing instructional situations (as gauged by how many norms of a situation a practice breaches) may be easier to justify than others.

The theory also provides the means for the researcher to anticipate how instruction may respond to new practices: A novel task such as "what is something interesting that could be proved about the object in Figure 1" conjures up by resemblance one or more instructional situations (e.g., "doing proofs" and "exploration") as possible frames for the work to be done. Models of those situations provide the researcher with a baseline of norms that could be breached as the work proceeds. Researchers can then use the obligations to anticipate what kinds of reactions teachers may have to the enactment those breaches. This anticipation can be useful in examining the potential derailments in the implementation of new practices in classrooms. That anticipation may also be useful in the examination of teachers' responses to assessments or development, or their reactions to instructional interventions.

Thus the theory provides not only the basis for the design of probes for the rationality of teaching (Herbst & Miyakawa, 2008) but also a framework for an analysis of the reactions from participants. Combined with finer tools from discourse analysis (e.g., Halliday & Matthiessen, 2004) teachers' responses to representations of breaching (but arguably valuable) instances of an instructional situation can help us understand not only what justifies teaching as it exists today but also whether and how proposed new practices could be justified in ways that practitioners find compelling.

Along these lines, the theory also provides a framework for teacher development. This framework puts a premium on the teachers' noticing of actions in teaching, their consideration of alternative actions, and the consideration of justifications for those different actions. The various tools we have created, which include not only the animations and the cartoon characters but also software to create scenarios with them, software to annotate the scenarios individually or in forums, and software to author online sessions[3] that

[3] A dedicated software tool enables teacher educators to create an agenda for users to interact with representations of practice (e.g., videos, images), prompts and questions, and tools for the user to interact with the media (e.g., annotating, marking moments, etc.) and with each other.

use the materials, can be useful in implementing this development program.[4] It is important to note that at the core of these developments there is a theory of teaching and its rationality that accounts for the teaching that is customarily seen in classrooms: At its base the theory attempts to be descriptive and explanatory rather than axiological or prescriptive. This is particularly visible in our identification of the obligations: We posit the institutional obligation in all its strength not necessarily out of advocacy for it but out of our recognition that practitioners are obligated to it regardless of anybody's feelings about it.

The theory does identify mechanisms for exploring empirically teaching that might be conceivable and desirable: The notions of situation, norm, breach, repair, and obligation can help examine a priori attempts to improve teaching and examine a posteriori the data from implementation. In that sense, the theory can support the piecemeal exploration of instructional improvement. The theory is a basic theory of mathematics instruction, a basic account of the activity of teaching mathematics in the school classroom—not an applied theory that reduces that phenomenon to the psychology of individual teachers. The psychology of mathematics teachers may still be useful to inform what enables and motivates individual teachers to do things, but the logic of action in mathematics teaching addressed by practical rationality may help us understand why some of those actions can be responsible, viable, and sustainable.

---

[4] These tools and content, including examples of these learning experiences are available at www.lessonsketch.org

An important limitation of the theory in its current formulation is that it does not quite incorporate an explicit account of learning[5] either by students or by teachers. Indeed the theory described above represents instruction as composed of stable patches of specific practices (contracts, situations, and tasks) and one might conclude that the theory describes only how knowledge is used by students and attested by teachers. Building on situated and socio-cultural accounts of learning and practice (e.g., Engestrom, 1992; Wenger, 1999) we contend that learning (by students and by the teacher) is accomplished in and through their practice in contracts, situations, and tasks. Additionally, the notion that contracts and situations can be breached by tasks that fall outside the norms of a situation or a contract is key in describing how the teacher might promote adaptive learning deliberately; and it has been foundational for Brousseau's (1997) theory of didactical situations. An explicit account of how this theory of instructional practice interfaces or complements accounts of student and teacher learning is needed and it remains a goal as we move ahead.

[5] We appreciate Ron Tzur's comment to this effect in the occasion of the first author's plenary lecture at the 2010 PME-NA Conference.

References

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. Elementary School Journal, 93 (4), 373-397.

Ball, D.L. (2000). Working on the inside: Using one's own practice as a site for studying teaching and learning. In A.E. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp.365-402). Mahwah, NJ: Lawrence Erlbaum.

Ball, D. L., Thames, M., and Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389-407.

Bourdieu, P. (1998). *Practical reason*. Palo Alto, CA: Stanford University Press.

Brousseau, G. (1997). *Theory of didactical situations in mathematics: Didactique des Mathematiques 1970-1990*. Dordrecht, The Netherlands: Kluwer.

Buchmann, M. (1986). Role over person: Morality and authenticity in teaching. *Teachers' College Record*, *87*(4), 529-543.

Chazan, D. (2000). *Beyond formulas in mathematics and teaching: Dynamics of the high school algebra classroom*. New York, NY: Teachers College.

Chazan, D. and Ball, D. (1999). Beyond being told not to tell. *For the Learning of Mathematics, 19*(2), 2-10.

Chazan, D. and Herbst, P. (2012). Animations of classroom interaction: Expanding the boundaries of video records of practice. *Teachers' College*

Record, 114(3), p. - http://www.tcrecord.org ID Number: 16319, Date Accessed: 3/5/2011 6:26:14 AM

Chazan, D. and Lueke, H. M. (2009). Exploring tensions between disciplinary knowledge and school mathematics: Implications for reasoning and proof in school mathematics. In D. Stylianou, et al. (Eds.), *Teaching and Learning Mathematics Proof Across the Grades: A K-16 perspective*. New York, NY: Routledge.

Chevallard, Y. (1985). *La transposition didactique. Du savoir savant au savoir enseignè*. Grenoble: La Pensèe Sauvage.

Cohen, D. K., Raudenbush, S., and Ball, D. L. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis. 25, 119-142.*

Cook, S. D., and Brown, J. S. (1999). Bridging epistemologies: The generative dance between organizational knowledge and organizational knowing. *Organization Science*, *10*(4), 381–400.

Engestrom, Y. (1992). *Interactive expertise: Studies in distributed working intelligence* (Research Bulletin No.83). Department of Education, Helsinki University.

Garfinkel, H., & Sacks, H. (1970). On formal structures of practical action. In J. McKinney & E. Tiryakian (Eds.), *Theoretical sociology: Perspectives and development* (pp. 337–366). New York, NY: Appleton-Century-Crofts.

González, G. and Herbst, P. (2006). Competing arguments for the geometry course: Why were American high school students supposed to study geometry in the twentieth century? *International Journal for the History of Mathematics Education, 1*(1), 7-33.

Halliday, M. A. K and Matthiessen, M. I. M. (2004). *An Introduction to Functional Grammar* (Third Edition). London: Hodder Arnold.

Hanna, G., and Jahnke, H.N. (1996). Proof and proving. In A., Bishop, K. Clements, C. Keitel, J. Kilpatrick & C. Laborde (Eds.), *International Handbook of Mathematics Education* (pp.877-908). Dordrecht: Kluwer Academic Publishers.

Henderson, K.B. (1963). Research on teaching secondary school mathematics. In N. Gage (Eds.), *Handbook of Research on Teaching* (pp.1007-1030). Chicago, IL: Rand McNally.

Herbst, P. (1998). *What works as proof in the mathematics classroom*. Unpublished doctoral dissertation. University of Georgia, Athens.

Herbst, P. (1999). Pour lire Brousseau*, For the Learning of Mathematics*, *19*(1), 3-10.

Herbst, P. (2002a). Engaging students in proving: A double bind on the teacher. *Journal for Research in Mathematics Education*, *33*(3), 176-203.

Herbst, P. (2002b). Establishing a custom of proving in American school geometry: Evolution of the two-column proof in the early twentieth century. *Educational Studies in Mathematics*, *49*(3), 283-312.

Herbst, P. (2003). Using novel tasks in teaching mathematics: Three tensions affecting the work of the teacher. *American Educational Research Journal, 40*(1), 197-238.

Herbst, P. (2004). Interactions with diagrams and the making of reasoned conjectures in geometry. *ZDM Mathematics Education, 36*(5), 129-139.

Herbst, P. (2006). Teaching geometry with problems: Negotiating instructional situations and mathematical tasks. *Journal for Research in Mathematics Education, 37*, 313-347.

Herbst, P. and Balacheff, N. (2009). Proving and Knowing in Public: What Counts as Proof in a Classroom In M. Blanton, D. Stylianou, and E. Knuth (Eds.), *Teaching and learning of proof across the grades: A K-16 perspective* (pp. 40-63). New York: Routledge.

Herbst, P. and Brach, C. (2006). Proving and 'doing proofs' in high school geometry classes: What is 'it' that is going on for students? *Cognition and Instruction, 24,* 73-122.

Herbst, P. and Chazan, D. (2003). Exploring the practical rationality of mathematics teaching through conversations about videotaped episodes. *For the Learning of Mathematics*, *23*(1), 2-14.

Herbst, P., Chen, C., Weiss, M., and González, G., with Nachlieli, T., Hamlin, M., and Brach, C. (2009). "Doing proofs" in geometry classrooms. In D. Stylianou, et al. (Eds.), *Teaching and learning proof across the grades* (pp. 250-268). New York, NY: Routledge.

Herbst, P. and Miyakawa, T. (2008). When, how, and why prove theorems: A methodology to study the perspective of geometry teachers. *ZDM - Mathematics Education*, *40*(3), 469-486

Herbst, P., Nachlieli, T., and Chazan, D. (2011). Studying the practical rationality of mathematics teaching: What goes into "installing" a theorem in geometry? *Cognition and Instruction*, *29*(2), 218-255.

Inglis, M., and Mejía-Ramos, J.P. (2009). The effect of authority on the persuasiveness of mathematical arguments. *Cognition and Instruction*, *27*(1), 25-50.

Inglis, M., Mejía-Ramos, J.P., and Simpson, A. (2007). Modelling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics*, *66*, 3-21.

Jahnke, H. (2007). Proofs and hypotheses. *ZDM Mathematics Education*, *39*, 79–86

Lakatos, I. (1976). *Proofs and Refutations: The logic of mathematical discovery*. New York: Cambridge University Press.

Lampert, M. (1985). How do teachers manage to teach? *Harvard Educational Review*, *55*(2), 178-194.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, *27*(1), 29-63.

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven, CT: Yale.

Mehan, H., and Wood, H. (1975). *The reality of ethnomethodology*. Malabar, FL: Krieger.

Nachlieli, T. and Herbst, P. with González, G. (2009). Seeing a colleague encourage a student to make an assumption while proving: What teachers put to play in casting an episode of geometry instruction. *Journal for Research in Mathematics Education, 40*(4), 427-459.

Resnick, L. B., Nesher, P., Leonard, F., Magone, M., Omanson, S., and Peled, I. (1989). Conceptual bases of arithmetic errors. *Journal for Research in Mathematics Education 20*(1), 8-27.

Shulman, L. (2004). *The wisdom of practice: Essays on teaching, learning, and learning to teach.* San Francisco: Jossey Bass.

Schwartz, J., Yerushalmy, M. and Wilson, B. (Eds.). (1993). *The Geometric Supposer: What Is It a Case of?* Hillsdale, NJ: Lawrence Erlbaum Associates.

Stein, M. K., Grover, B. W., and Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning. *American Educational Research Journal, 33*(2), 455-488.

Toulmin, S. (1969). *The uses of argument*. Cambridge: Cambridge University Press.

Weiss, M., Herbst, P., and Chen, C. (2009). Teachers' Perspectives on Mathematical Proof and the Two-Column Form. *Educational Studies in Mathematics*, *70*(3), 275-293.

Wenger, E. (1999). *Communities of practice: Learning, meanings, and identity*. Cambridge, UK: Cambridge University Press.

Yackel, E., and Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*, 458-477.