The Mathematics Enthusiast

Volume 8
Number 3 *Number 3*

Article 5

7-2011

Conceptualizations and Issues related to Learning Progressions, Learning Trajectories, and Levels of Sophistication

Michael T. Battista

Conceptualizations and Issues Related to Learning Progressions, Learning Trajectories, and Levels of Sophistication[1]

Michael T. Battista[2]
Ohio State University

Abstract: *In this paper the nature of learning progressions and related concepts are discussed. The notions of learning progressions and learning trajectories are conceptualized and their usage is illustrated with the help of examples. In particular the nuances of instructional interventions utilizing these concepts are also discussed with implications for the teaching and learning of mathematics.*

Keywords: Learning progressions; Learning trajectories; teaching; Instructional interventions

Learning progressions (LP) are playing an increasingly important role in mathematics and science education (NRC, 2001, 2007; Smith, Wiser, Anderson, & Krajcik, 2006). They are strongly suggested for use in assessment, standards, and teaching. In this article, I discuss the nature of learning progressions and related concepts in mathematics education, and I illustrate issues in their construction and use. I emphasize the different ways that LP and related constructs *represent* learning for teaching. Finally, I illustrate the need that teachers have for LP.

[1] The research described in this article was supported by the National Science Foundation under Grant numbers 0099047, 0352898, 554470, 838137. The opinions, findings, conclusions, and recommendations, however, are those of the author and do not necessarily reflect the views of the National Science Foundation.
[2] battista.23@osu.edu

The Mathematics Enthusiast, **ISSN 1551-3440, Vol. 8, no.3**, pp.507-570

Definitions and Constructs

According to the National Research Council, “Learning progressions are descriptions of the successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic” (2007, p. 214). A similar description of learning progressions is given by Smith et al. who define a learning progression “as a sequence of successively more complex ways of thinking about an idea that might reasonably follow one another in a student’s learning” (2006, pp. 5-6). Unlike Piaget's stages, but similar to van Hiele's levels[3], it is assumed that progress through learning progressions is "not developmentally inevitable" but depends on instruction (Smith et al., 2006).

Common Characteristics of the LP Construct

In the research literature, descriptions of the LP construct possess both differences and similarities. The characteristics that seem most common to different views of learning progressions are as follows:

- LP "are based on research syntheses and conceptual analyses” (Smith et al., 2006, p. 1); "Learning progressions should make systematic use of current research on children’s learning " (NRC, 2007, p. 219).
- LP "are anchored on one end by what is known about the concepts and reasoning of students. … At the other end, learning progressions are

[3] Because many of my examples refer to the van Hiele levels, I have included a very brief synopsis of the levels in Appendix 1.

anchored by societal expectations. … [LP also] propose the *intermediate* understandings between these anchor points that … contribute to building a more mature understanding" (NRC, 2007, p. 220).

- LP focus on core ideas, conceptual knowledge, and connected procedural knowledge, not just skills. LP organize "conceptual knowledge around core ideas" (NRC, 2007, p. 220). LP "Suggest how well-grounded conceptual understanding can develop" (NRC, 2007, p. 219).
- LP "recognize that all students will follow not one general sequence, but multiple (often interacting) sequences" (NRC, 2007, p. 220).

Differences in LP Construct

There are several differences in how the learning progressions construct is used in the literature.

- LP differ in the time spans they describe. Some progressions describe the development of students' thinking over a span of years; others describe the progression of thinking through a particular topic or instructional unit.
- LP differ in the grain size of their descriptions. Some are appropriate for describing minute-to-minute changes in students' development of thought, while others better describe more global progressions through school curricula.

- LP differ in the audience for which they are written. Some LP are written for researchers, some for standards writers, some for assessment developers (formative and summative), and some for teachers.
- LP differ in the research foundation on which they are built. Some LP are syntheses of extant research; some synthesize extant research then perform additional research that elaborates the syntheses (the additional research may be cross-sectional or longitudinal).
- LP differ in how they describe student learning. Some focus on numerically "measuring" student progress, while others focus on describing the nature or categories of students' cognitive structures and reasoning.

Learning Trajectories

Another important construct that is similar to, different from, and importantly related to, learning progressions is that of a "learning trajectory[4]." I define a learning trajectory as a detailed description of the sequence of thoughts, ways of reasoning, and strategies that a student employs while involved in learning a topic, including specification of how the student deals with all instructional tasks and social interactions during this sequence. There are two types of learning trajectories, hypothetical and actual. Simon (1995) proposed that a "*hypothetical learning trajectory* is made up of three components: the

[4] Although some people use the terms "learning progression" and "learning trajectory" similarly, I think it is extremely useful to carefully distinguish learning progressions and learning trajectories.

learning goal…, the learning activities, and the hypothetical learning process—a prediction of how the students' thinking and understanding will evolve in the context of the learning activities" (p. 136). In contrast, descriptions of *actual learning trajectories* can be specified only during and after a student has progressed through such a learning path. Simon states that an "actual learning trajectory is not knowable in advance" (p. 135). Steffe described an actual learning trajectory as "a model of [children's] initial concepts and operations, an account of the observable changes in those concepts and operations as a result of the children's interactive mathematical activity in the situations of learning, and an account of the mathematical interactions that were involved in the changes. Such a learning trajectory of children is constructed during and after the experience in intensively interacting with children" (2004, p. 131).

Clements and Sarama's (2004) "conceptualize learning trajectories as descriptions of children's thinking and learning in a specific mathematical domain and a related, *conjectured* route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain" (2004, p. 83). In their *hypothetical learning trajectories*, they specify instructional tasks that promote (and assess) progression through their levels of thinking.

One critical difference between my definition of learning progressions and my definition of learning trajectories is that *trajectories include descriptions of instruction, progressions do not.* One of the most difficult issues facing researchers who are constructing learning trajectories for curriculum development is determining how instructional variation affects trajectories. That is, how specific is the trajectory to the instructional sequence in which it is embedded? If the sequence has been tested for one curriculum, how well does it apply to other curricula? Also, how do actual trajectories for individual students vary from the hypothetical trajectory for a curriculum? That is, a learning trajectory for a curriculum is in some sense an "average" of actual trajectories for a sample of individual students—and, as an average, it is a prediction for a target population, and thus it is necessarily hypothetical. And the "standard deviation" of the distribution of actual trajectories may be as relevant as the mean.

Pedagogical Uses of LP

Beyond the scientific value of LP/LT descriptions of students' mathematics learning, these descriptions are powerful tools for teaching. LP/LT can be used for formative and summative assessment, and to guide instructional decisions made in curriculum development and moment-to-moment teaching. Indeed, Simon states, "I choose to use 'hypothetical learning trajectory' … to emphasize aspects of teacher thinking that are grounded in a constructivist perspective and that are common to *both advanced planning and spontaneous decision making*" (1995, p. 135). Such a hypothesized trajectory (or LP) helps teachers make instructional decisions based on their "best guess of how learning might proceed" (Simon, 1995, p. 135). Thus, from the constructivist perspective, LP and LT should ideally help teachers not only plan instruction, but understand students' learning on a moment-to-moment basis and appropriately and continuously adjust instruction to meet students' evolving learning needs.

Another difference between learning progressions and learning trajectories derives from their intended use and consequent development. If one is designing and testing a curriculum, one is more likely to develop a learning trajectory based on the fixed sequence of learning tasks in that curriculum. If, in contrast, one is focusing on a formative assessment system that applies to many curricula, one is more likely to develop a learning progression based on many assessment tasks, not those in a fixed sequence. A general learning progression describes students' various ways of reasoning about a topic, irrespective of curriculum; it focuses on understanding and reacting to students' current cognitive structures. A curriculum-based learning trajectory describes students' ways of reasoning within a fixed curriculum; it focuses on understanding and reacting to students' cognitive structures, relative to the curriculum sequence. The advantage of learning progressions is that they are widely applicable and focus tightly on general student cognition. The advantage of learning trajectories is their specificity in tracing students' movement through a fixed curriculum.

LP as Cognitive Terrain

It is useful to think of learning progressions as describing the terrain on a mental mountain slope that students must ascend to learn and become fluent with particular mathematical topic. From a curriculum-development, instructional planning perspective, we try to determine the most efficacious ascent path (the one for which most students are most likely to succeed), as depicted by the fixed path in Figure 1a (the hypothetical prototypical learning trajectory). But to meet individual students' learning needs, often we must zoom in on individual deviations from the path to more precisely determine the next steps that students can make successfully. Critical to aiding a student's moment-to-moment climb is flexibly and reactively choosing tasks that provide them with successful hand- and foot-holds in this cognitive terrain (Figure 1b).

Figure 1a

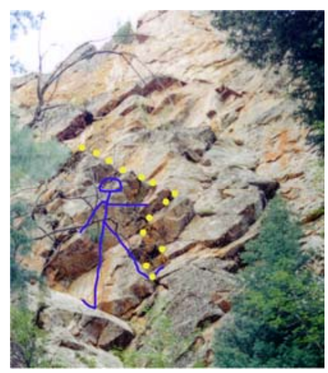


Figure 1b

Theoretical Frameworks for Learning Progressions

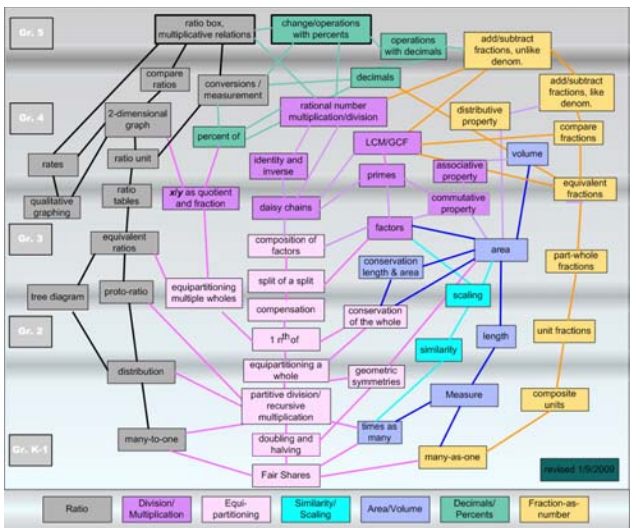
Another way to understand differences between learning progressions is to examine their postulated learning mechanisms. For instance, the original van Hiele theory relates progression through the levels of geometric thinking to phases of instruction. In contrast, Battista uses constructivist constructs such as levels of abstraction to describe students' progression through the van Hiele levels (see also the theories of abstraction of Simon, et al. (2004) and Mitchelmore & White (2000), as well as Pegg & Davey's analysis of geometric learning, 1998).

We might also contrast a constructivist approach to teaching to the approach taken in Gagne's "programmed learning" hierarchies[5], which seem much more fixed, logical, prescribed, and less interactive.

> Beginning with the final task, the question, is asked, What kind of capability would an individual have to possess if he were able to perform this task successfully, were we to give him only instructions? … Having done this, it was natural to think next of repeating the procedure with this newly defined entity (task). What would the individual have to know in order to be capable of doing *this* task without undertaking any learning, but given only some instructions? … Continuing to follow this procedure, we found that what we were defining was a *hierarchy* of subordinate knowledges [sic], growing increasingly "simple" … Our hypothesis was that (a) no individual could perform the final task without having these subordinate capabilities … and (b) that any superordinate task in the hierarchy could be performed by an individual provided suitable instructions were given, and provided the relevant subordinate knowledges could be recalled by him (Gagne, 1962, p. 356).

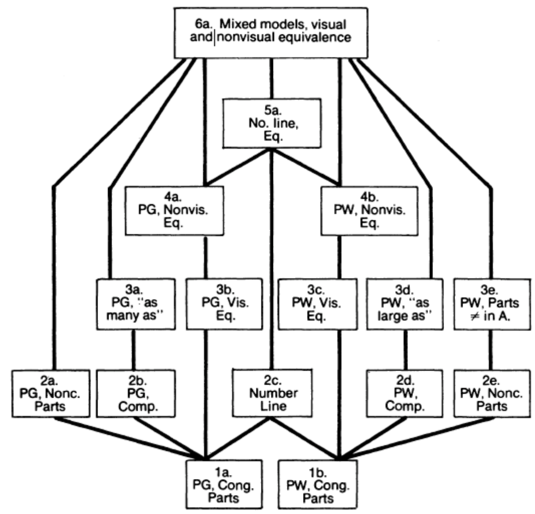
[5] A hierarchy was empirically validated by examining student success rates on various items in the hierarchy (similar to examining item difficulties in current quantitative approaches). So it was not intended that hierarchies be developed strictly logically.

It is interesting that, on the surface at least, representations of learning progressions from different theoretical frameworks can look similar. For instance, compare the overall appearance of the learning progression of Confrey et al. (from a more constructivist perspective) to the Gagne-like hierarchy described by Novillis (see Figure 2). It would be revealing to analyze how these progressions differ at a micro- versus macro-level.



Confrey et al., 2009, p. 1-4

Learning Trajectories Map for Rational Number Reasoning.



Novillis, 1976, p. 132

A Hierarchy of selected subconcepts of the fraction concept

Figure 2. Trajectory versus Hierarchy

The Nature of Levels

A critical component of learning progressions is the notion of "levels." Because the concept of level is not straightforward, and because how one defines level determines how one views (and measures) level attainment, I examine this concept in more detail, using the van Hiele levels as an example. Indeed, the issues discussed below for the van Hiele levels are critical because any attempt to develop, assess, and use levels in learning progressions must address these issues in some way.

Levels, Stages, and Hierarchies

Clements and Battista (1992) described the difference between researchers' use of the terms *stage* and *level* as follows. A *stage* is a substantive period of time in which a particular type of cognition occurs across a variety of domains (as with Piagetian stages of cognitive development). In contrast, a *level* is a period of time in which a distinct type of cognition occurs for a specific domain (but the size of the domain may be an issue). Battista defines a third construct—a *level of sophistication* in student reasoning as a qualitatively distinct type of cognition that occurs within a hierarchy of cognition levels for a specific domain.

Example: The van Hiele Levels

In discussing the van Hiele levels, Clements and Battista (1992) suggested several characteristics that might apply to levels.

- "Learning is a discontinuous process. That is, there are 'jumps' in the learning curve which reveal the presence of discrete, qualitatively different levels of thinking.
- The levels are sequential and hierarchical. For students to function adequately at one of the advanced levels in the van Hiele hierarchy, they must have mastered large portions of the lower levels. … Progress from one level to the next is more dependent upon instruction than on age or biological maturation. … Students cannot bypass levels and achieve understanding (memorization is not an important feature of any level). The latter requires working through certain "phases" of instruction" (1992, pp. 426-7).

Types of Levels-Hierarchies

When considering *hierarchies* of levels in learning progressions, it is helpful to distinguish two types. A "weak" levels-hierarchy refers to a set of levels that are ranked in order of sophistication, one above another, with no class inclusion relationship between the levels necessary. A "strong" levels-hierarchy refers to a set of levels ranked in order of sophistication, one above another, with class inclusion relationships between the levels required. That is, in a "strong" levels-hierarchy, students who are reasoning at level *n* are assumed to have progressed through reasoning at levels 1, 2, … (*n*-1). The van Hiele levels were originally hypothesized to form a strong levels-hierarchy (which is generally supported by the research—but there are issues), while Battista's levels of sophistication in reasoning about length to be discussed below form a weak levels-hierarchy. (I will return to this idea when I discuss quantitative methods for examining learning progressions.)

Being "At" a Level

What, precisely, does it mean to be "at" a level? Battista (2007) argued that students are *at* a van Hiele level when their overall cognitive structures and processing causes them to be disposed to and capable of thinking about a topic in a particular way. So students are "at" van Hiele Level 1 when their overall cognitive organization and processing disposes them to think about geometric shapes in terms of visual wholes; they are at Level 2 when their overall cognitive organization disposes and enables them to think about shapes in terms of their properties. Also in this view, when students move from familiar content to unfamiliar content, their level of thinking might decrease temporarily; but because students are disposed to operate at the higher level, they look to use that level on the new material, and quickly become capable of using that level (Battista, 2007). So, for instance, in moving from studying quadrilaterals to studying triangles, students who are at Level 2 for quadrilaterals might initially process triangles as visual wholes, but right from the start they look for, and fairly quickly discover and use, triangle properties.

A Different Approach: Vectors and Overlapping Waves

Some studies indicate that people exhibit behaviors indicative of different van Hiele levels on different subtopics of geometry, or even on different kinds of tasks (Clements & Battista, 2001). So an alternate view of the development of geometric reasoning is that students develop several van Hiele levels simultaneously. To represent this view, Gutiérrez et al. (1991) used a vector with four components to indicate the degrees of acquisition of each of van Hiele levels 1 through 4. For example, a student's degree of acquisition vector might be: 96.67% for Level 1, 82.50% for Level 2, 50.00% for Level 3, and 3.75% for Level 4. Using this vector approach, Gutiérrez et al. described six profiles of level-configurations in students' reasoning about 3d geometry. To illustrate, Profile 2 was characterized by complete acquisition of Levels 1 and 2, high acquisition of Level 3, and low acquisition of Level 4. However, even though level acquisition was described in terms of the vector model, the profiles could easily be re-interpreted in terms of levels only. For instance, Profile 2 could be thought of as Level 2 or transition to Level 3.

Similar to the vector approach to the van Hiele levels, several researchers have posited that different types of reasoning characteristic of the van Hiele levels develop simultaneously at different rates, and that at different periods of development, different types of reasoning are dominant, depending on the relative competence students exhibit with each type of reasoning (Clements & Battista, 2001; Lehrer et al., 1998; see Figure 3). The "waves" depicted in Figure 3 are the competence growth curves for the different types of reasoning.

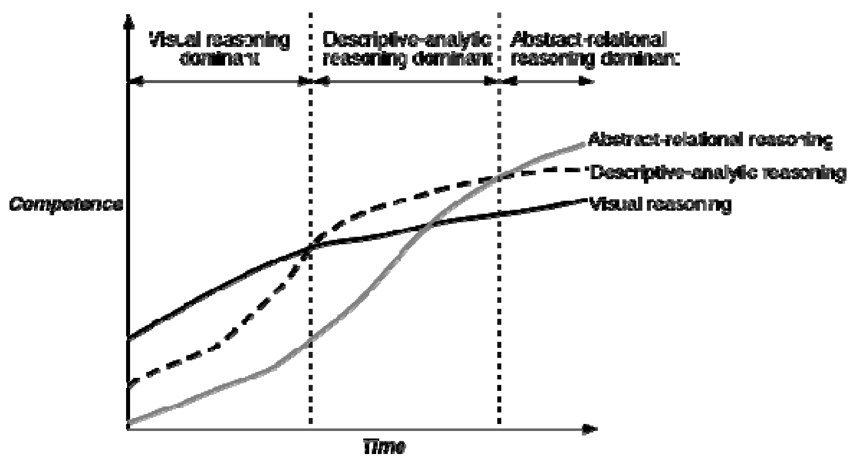


Figure 3. Waves of acquisition of van Hiele levels

Lehrer et al. (1998) argued that … geometric development should be characterized "by which 'waves' or forms of reasoning are most dominant at any single period of time" (p. 163). Clements and Battista (2001) also proposed the view that the van Hiele levels (seen as types of reasoning) develop simultaneously but at different rates. Visual-holistic knowledge, descriptive verbal knowledge, and, to a lesser extent initially, abstract symbolic knowledge grow simultaneously, as do interconnections between levels. However, although these different types of reasoning grow in tandem, one level tends to become ascendant or privileged in a child's orientation toward geometric problems. Which level is privileged is influenced by age, experience, intentions, tasks, and skill in use of the various types of reasoning.

Although the vector and wave models for the van Hiele theory have merit, embedded within both is a difficult issue—distinguishing *type of reasoning* from *level of reasoning.* That is, sometimes the term *visual-holistic* is used to refer to that type of reasoning that is strictly visual in nature, and sometimes it is used to refer to a period of development of geometric thinking when an individual's thinking is dominated and characterized by visual-holistic thinking. For instance, Gutiérrez et al. (1991) used vectors to indicate students' "capacity to use each one of the van Hiele *levels*" (p. 238). This statement makes sense only if van Hiele levels are taken as types of reasoning, not periods of development characterized by qualitatively different kinds of thought. Similarly, Clements and Battista (2001), along with Lehrer et al. (1998), talked about "waves of acquisition" of *levels* of reasoning defined by van Hiele. Thus, broadly speaking, researchers have intermingled and not yet completely sorted out (a) van Hiele levels as types of reasoning, and (b) van Hiele levels as periods of development of geometric reasoning. The waves theory described above is similar, but not identical, to Siegler's overlapping waves theory (2005). Indeed, the vertical axis in Siegler's theory is "relative frequency" of use, not competence, as shown in the van Hiele interpretation above. Frequency of use many be connected to competence, but also to other factors such as personal preference, social pressure, and so on.

In summary, given the variability in strategy use and reasoning that seems to accompany learning, even if we develop an adequate definition for what it means for a student to be "at" a level, the periods of time when students meet the strict requirement for being at levels may be short, with students spending most of the time "in transition."

Level Determination

Empirical determination of levels of reasoning is a major issue in the van Hiele theory, and LP/LT levels in general, because it operationalizes researchers' conceptions of the qualitatively different types of reasoning that occur in the LP/LT. For instance, consider some of the different ways that researchers have determined van Hiele levels. Some studies (Carroll 1998; Usiskin, 1982) used paper and pencil tests, judging that a level was achieved if a given number of items designed to assess that level were *answered* correctly. In other studies (Fuys et al., 1988; Clements & Battista, 1992; Battista, in prep) students' reasoning (as recorded in interviews or open response written tasks) was coded by matching students' reasoning to characteristics of the van Hiele levels. Beyond the answers versus reasoning dichotomy, there have been additional differences in level determination. For instance, in the Usiskin van Hiele test, three of the tasks used to assess property-based (Level 2) reasoning about quadrilaterals involved diagonals, but the Battista and Clements and Battista studies focused on visually salient "defining" properties of shapes. Thus, the properties assessed by Usiskin's test were more likely to be unfamiliar to students than those assessed by Battista and Clements and Battista.

A totally different approach to assessing van Hiele levels was devised by another group of researchers (Battista, 2007). In a collaborative effort to find ways to assess elementary students' acquisition of the van Hiele levels in interview situations, Battista, Clements, and Lehrer developed a triad sorting task, that, with variations, both Clements and Battista (2001), Lehrer et al. (1998), and Battista (in progress) used in separate research efforts. In this task, students were presented with three polygons, such as those shown in Figure 4, and were asked, "Which two are most alike? Why?" Choosing B and C and saying that they "look the same, except that B is bent in" was taken as a Level 1 response. Choosing A and B and saying either that they both have two pairs of congruent sides or that they both have four sides was taken as a Level 2 response. The purpose of this task was to determine the type of reasoning used on a task that students had not seen before (so it was unlikely to elicit instructionally programmed responses).

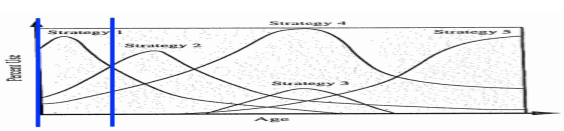


Figure 4. Triad polygon sorting task.

One difficulty with this analysis is that giving the number of sides of a polygon is a "low-level" use of properties. That is, there are different types of geometric properties. The simplest property involves describing the number of components in a shape. For instance, a quadrilateral has four sides; a triangle has three angles. A second, more sophisticated type of property describes spatial relationships that are particularly salient in identifying shapes (e.g., opposite sides of a rectangle are congruent and all angles are right angles). In some sense, these properties are the "psychological defining characteristics" of shapes for Level 2 students. The third type of property describes other interesting but less salient relationships (e.g., the diagonals of a rectangle are congruent and bisect each other[6]). These properties are likely to be derived once students understand the meaning of shape classification—so they are more likely to occur in Level 3.

The distinction in properties described above suggests that students' use of number of sides of a polygon may not be a very good indicator of Level 2 thinking, which should focus on relational properties. Thus some jumps in levels on triad tasks observed by Lehrer et al. (1998) may have been caused by coding students' use of number of sides as Level 2. Because a critical factor used in distinguishing van Hiele levels is how students deal with geometric properties, clarifying the meaning of *properties*, as it relates to the van Hiele levels, is important.

Another factor that should be considered with the triad task is that saying Shape B is more like Shape C is not necessarily a less sophisticated response than focusing on number of sides. That is, Shape B is actually more like Shape C if we

[6] Of course, it is true that some "interesting" properties *logically* can be used to define shapes.

consider how much movement it takes to transform B into C, compared to B into A. In fact, one could imagine a metric that quantifies the amount of movement required. Thus, the "morphing" response described by Lehrer et al. (1998), and also observed by Clements and Battista (2001), may be an intuitive version of a notion whose mathematization is far beyond the reach of elementary students.

Another issue with the triad-task approach is pointed out by differences in the ways the researchers used the triads. Lehrer et al. (1998) construed *each* triad task as an indicator of type of reasoning. So students' use of different types/levels of reasoning on different triads was taken as evidence of differences in levels of response. In contrast, Clements and Battista (2001) used a *set* of 9 triad items as an indicator of level of students. To be classified at a given level, a student had to give at least 5 responses at that level. If a student gave 5 responses at one level and at least 3 at a higher level, the student was considered to be in transition to the next higher level. Of course, because it aggregates responses, this approach obscures intertask differences and variability in reasoning. It focuses on determining the *predominant* level of reasoning that a student used on the triad tasks.

Another difference between the researchers' approaches is also important. In analyzing students' reasoning on the triad tasks, Lehrer et al. (1998) classified student responses solely on the basis of the type of reasoning that students employed. In contrast, in determining students' van Hiele levels, Clements and Battista (2001) attempted to also account for the "quality" of students' reasoning—each reason for choosing a pair in a triad was assessed to see if it correctly discriminated the pair that was chosen from the third item in the triad. In this scheme, the van Hiele levels for students were determined based on a complicated algorithm that accounted for both type of reasoning and discrimination score[7,8].

Cognition Based Assessment (CBA): Levels, Progressions, Trajectories, and Profiles

I now describe my work on the Cognition Based Assessment project to illustrate the relationship between learning progressions and learning trajectories as *representations of learning for teaching*[9]. The description of CBA also illustrates that to be useful for teachers, learning progressions must be embedded within an interconnected system of *LP-based* formative assessments, interpretations of students' reasoning, and instruction.

The CBA View of Learning and Instruction

According to the "psychological constructivist" view of how students learn mathematics *with understanding,* the way students construct, interpret, think about, and make sense of mathematical ideas is determined by the elements and

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[7] Additional discussions of van Hiele levels measurement issues can be seen in articles by Wilson (1990), and Usiskin and Senk (1990).

[8] It is worth noting that quantitative methods for determining levels face the same issues described here for qualitative methods. For instance, using the Saltus method can still leave us with many students who cannot be clearly placed in a level (e.g., Draney & Wilson, 2007).

[9] CBA development was partially supported by the National Science Foundation under Grant Nos. ESI 0099047and 0352898. Opinions, findings, conclusions, or recommendations, however, are those of the author and do not necessarily reflect the views of the National Science Foundation.

organization of the relevant mental structures that the students are currently using to process their mathematical worlds (e.g., Battista, 2004). To construct new knowledge and make sense of novel situations, students build on and revise their current mental structures through the processes of action, reflection, and abstraction. A major component of psychological constructivist research on mathematics learning and teaching is its attention to students' construction of meaning for specific mathematical topics. For numerous mathematical topics, researchers have found that students' development of conceptualizations and reasoning can be characterized in terms of "levels of sophistication" (e.g. Battista & Clements, 1996; Battista et al., 1998; Cobb & Wheatley, 1988; Steffe, 1992; van Hiele, 1986). These levels lie at the heart of the CBA conceptual framework for understanding and building upon students' learning progress. Selecting/creating instructional tasks, adapting instruction to students' needs, and assessing students' learning progress require detailed, cognition-based knowledge of how students construct meanings for the specific mathematical topics targeted by instruction.

CBA Assessment and Instruction

To implement mathematics instruction that genuinely and effectively supports students' construction of mathematical meaning and competence, teachers must not only understand cognition-based research on students' learning of particular topics, they must be able to use that knowledge to determine, monitor, and guide the development of their own students' reasoning. Cognition-Based Assessment supports these activities by including the following five critical components.

1. Descriptions of core mathematical ideas and reasoning processes that form the foundation for students' sense making and understanding of elementary school mathematics.
2. For each core idea, research-based descriptions of levels of sophistication in the development of students' understanding of and reasoning about the idea (these are CBA LP).
3. For each core idea, coherent sets of assessment tasks that enable teachers to investigate their students' mathematical thinking and precisely locate students' positions in the cognitive terrain for learning that idea.
4. For each assessment task, a description of what each level of reasoning might look like for the task.
5. For each core idea, descriptions of instructional activities specifically targeted for students at various levels to help them move to the next higher level.

These five components are critical for an assessment "system" that focuses on understanding and guiding the development of students' mathematical reasoning.

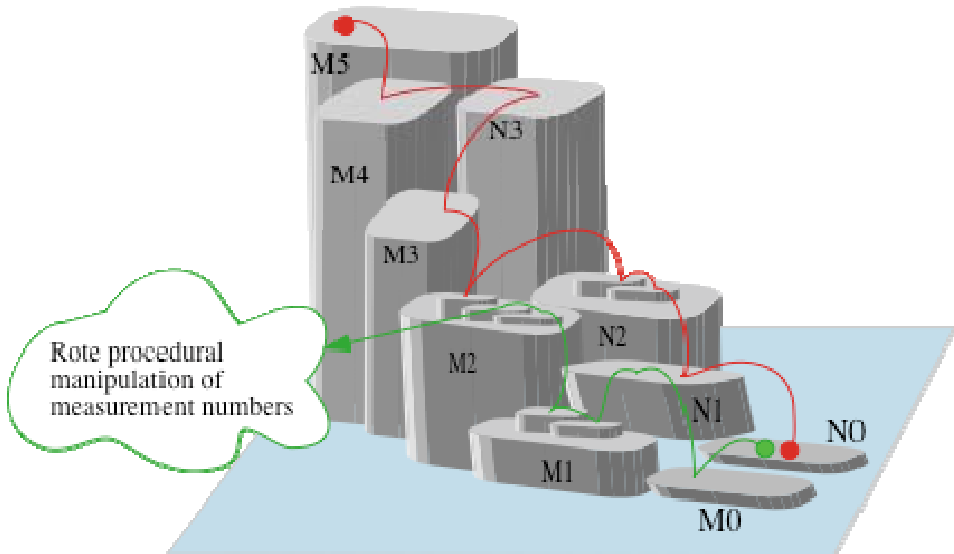
Learning Progressions and Trajectories for Length

The CBA levels of sophistication, or learning progressions, for a topic (a) start with the informal, pre-instructional reasoning typically possessed by students; (b) end with the formal mathematical concepts targeted by instruction; and (c) indicate cognitive plateaus reached by students in moving from (a) to (b). As an example, Figure 5 outlines the CBA levels of sophistication for the concept of length.

Non-Measurement Reasoning	Measurement Reasoning
N0: Student Compares Objects' Lengths in Vague Visual Ways N1: Student Correctly Compares Whole Objects' Lengths Directly or Indirectly N2: Student Compares Objects' Lengths by Systematically Manipulating or Matching Their Parts *N2.1. Rearranging Parts to Directly Compare Whole Shapes* *N2.2. One-to-One Matching of Parts* N3: Student Compares Objects' Lengths Using Geometric Properties	M0: Student Uses Numbers in Ways Unconnected to Iteration of Unit-Lengths M1: Student Iterates Units Incorrectly *M1.1: Iterates Non-Length Units (e.g., Squares, Cubes, Dots) and Gets Incorrect Count of Unit-Lengths* *M1.2: Iterates Unit-Lengths but Gets Incorrect Count* M2. Student Correctly Iterates ALL Unit-Lengths One-by-One *M2.1: Iterates Non-Length Units (e.g., Squares, Cubes) and Gets Correct Count of Unit-Lengths for Straight Paths* *M2.2: Iterates Non-Length Units (e.g., Squares, Cubes) To Correctly Count Unit-Lengths for Non-Straight Paths* *M2.3: Explicitly Iterates Unit-Lengths and Gets Correct Counts for Straight and Non-Straight Paths* M3: Student Correctly Operates on Composites of Visible Unit-Lengths M4: Student Correctly and Meaningfully Determines Length Using only Numbers—No Visible Units or Iteration M5: Student Understands and Uses Procedures/Formulas for Perimeter Formulas for Non-Rectangular Shapes

Figure 5. CBA Levels for Students' Reasoning about Length (Battista, accepted)

The set of CBA levels of sophistication for the topic of length are graphically depicted in Figure 6. Also shown, are an ideal hypothetical learning trajectory (in red) and a typical actual learning trajectory for students (in green). The CBA levels represent the "cognitive terrain" that students must ascend during an actual learning trajectory.



Ideal Summary LT: N0, N1, N2.1, N2.2, M2.3, M3, N3, M4, M5

Typical Summary LT: N0, M0, N1, M1.1, M1.2, M2.1, M2.2, Rote Procedures

Figure 6. Levels of Sophistication Plateaus and Learning Trajectories for CBA Length

A CBA levels-model for a topic describes not only cognitive plateaus, but what students can and cannot do, students' conceptualizations and reasoning, cognitive obstacles that obstruct learning progress, and mental processes needed both for functioning at a level and for progressing to higher levels. The levels are derived from analysis of both the mathematics to be learned and empirical research on students' developing conceptualizations of the topic. The jumps in the ascending plateau structure of a CBA levels-model represent cognitive restructurings evidenced by observable increases in sophistication in students' reasoning about a topic. Furthermore, an ideal CBA levels-of-sophistication model for a topic provides indications of jumps in sophistication that are small enough to fall within students "zones of construction." That is, a student should be able to accomplish the jump from conceptualizing and reasoning at Level N to conceptualizing and reasoning at Level N+1 by making a significant abstraction, in a particular context, while working to solve an appropriate problem or set of problems[10]. For instance (See Figure 7), in Situation A the student has to make a cognitive jump that is too great. In Situation B, the student can progress from Level 1 to Level 2 by making cognitive jumps to successive sublevels.

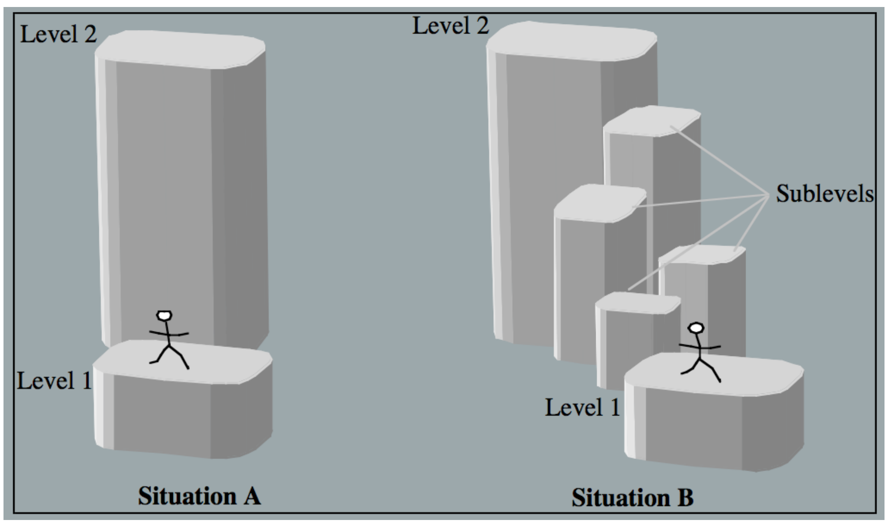


Figure 7

However, because the levels are compilations of empirical observations of the thinking of many students, and because students' learning backgrounds and mental processing differ, a particular student might not pass through every level for a topic; he or she might skip some levels or pass through them so quickly that the passage is difficult to detect. Even with this variability, however, the levels still describe the plateaus that students achieve in their development of reasoning about a topic. They indicate major landmarks that research has shown students often pass through in "constructive itineraries" or learning trajectories

[10] The jump in reasoning may apply to restricted contexts, not to all contexts connected with the mathematical topic. That is, the jump may be tightly situated rather than global.

for these topics. Thus, such levels provide an excellent conceptual framework for understanding the paths students travel to achieve meaningful learning of a topic.

Digging Deeper into the LP/LT Representations

As hypothetical or average learning trajectories, the trajectories depicted in Figure 6 are still simplifications of actual learning trajectories traversed by individual students. To illustrate, I describe one portion of the actual learning trajectory of a fifth grader, RC, who was having particular difficulty with the concept of length (the trajectories of most other students were much simpler). Figure 8 shows RC's learning trajectory for 34 consecutive length items (start with the green point, end with the red point). This actual learning trajectory is extremely complex because it contains so much back-and-forth movement between levels. Note that RC's performance is consistent with the variability in strategy choice described by Siegler (2007, var).

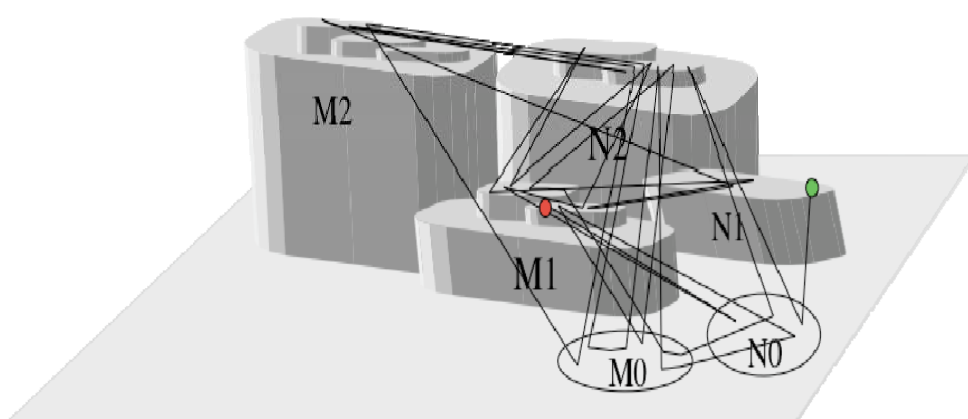


Figure 8 RC's learning trajectory for 34 consecutive length items

Figure 9 provides a better representation of this complicated portion of RC's learning trajectory. This figure starts with RC's levels on initial assessment items, moves to his responses during an instructional intervention, and ends with his reasoning on reassessment items.

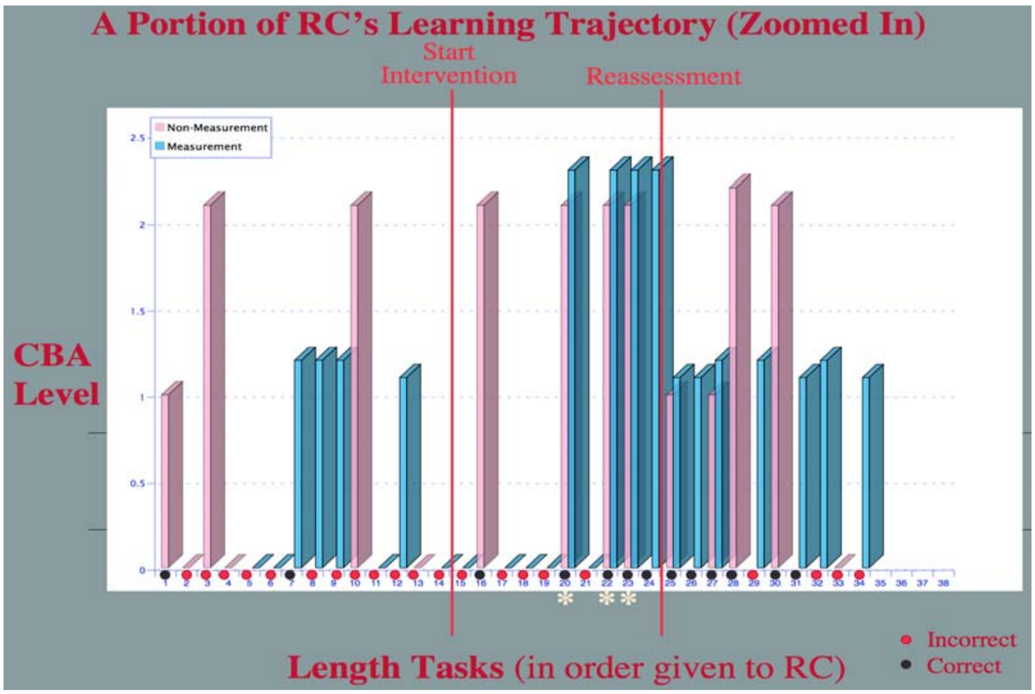


Figure 9. Another representation of RC's learning progress

But even Figure 9 does not represent RC's learning trajectory with enough detail to be maximally useful for instruction. We need a *narrative description* of (a) what tasks he was attempting, and (b) his level of reasoning on each task. Below, this information is provided for the critical period of instructional intervention in which RC made progress (see the three starred items in Figure 9).

During the instructional intervention, RC was given items of the following type.

Item 23 (see Figure 10). Suppose I pull the wires so they are straight. Which wire would be longer, or would they be the same? How do you know? Predict an answer, then check with inch rods (the black/gray sections on the student sheet were each 1 inch in length). [Items 20-22 were similar.]

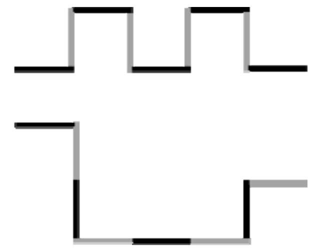


Figure 10

On Item 23, RC counted unit lengths as shown in Figure 11 and concluded that the top wire was longer. He checked his answer by placing inch rods on both wires then straightening each set of rods to compare the lengths directly.

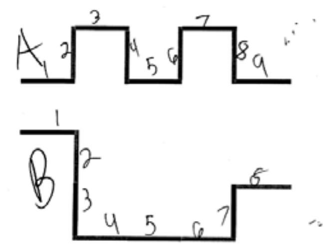


Figure 11

Importantly, on Item 23 and several other problems, RC used *both* M2.3 and N2.1 reasoning. On the last problem of this type (Item 24), RC did not check his answer by straightening—he seemed sure of his prediction, having empirically abstracted that comparing counts of unit lengths predicted the results of comparing straightened wires.

In the reassessment period, RC's thinking regressed when he attempted problems that were different from the ones he successfully used M2.3 reasoning on. For instance, on Item 27, at first, RC counted unequal segments, then dots (Measurement Reasoning), then imagined straightening paths (Non-Measurement reasoning), forming contradictory conclusions.

Item 27. Which path is shorter, or are they the same? How do you know?

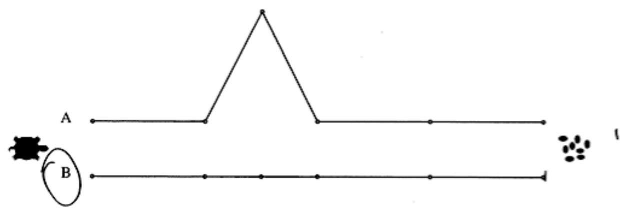


Figure 12

RC: [Counts gaps between dots on the bottom path, then on the top path] 1, 2, 3, 4, 5. 1, 2, 3, 4, 5. Hmm. Which one do I think is shorter… [Counts dots on the top path, then on the bottom path] 1, 2, 3, 4, 5, 6. 1, 2, 3, 4, 5, 6. This one's [pointing to the bottom path] shorter. …

I: Okay. Now you got 6 both times? But you still think this one [pointing to path B] is shorter?

RC: Yep.

I: Why is it shorter?

RC: Because if you pull this one out [pinching the endpoints of A with his fingers]…it'll be like right there [moving his fingers horizontally outwards to just past the endpoints of A]. You can't pull this [pinching the endpoints of B] out anymore.

So in the face of a seeming conflict between measurement and non-measurement reasoning, RC correctly relies on his non-measurement reasoning.

I: Okay. Could using these rods help you think about this problem [placing inch rods on the paper]?

RC: Yes. [Counts the segments on path A.] 1, 2, 3, 4, 5. … So here and right up here [draws marks at the two ends of the straight line of 5 rods he places above path A].
[RC counts gaps between dots on path B, then counts 5 inch rods that he places at the top of the sheet, between the two marks that he previously made when rearranging path A] 1, 2, 3, 4, 5. 1, 2, 3, 4, 5.

I: Okay, now how did you get 5? Is that for the bottom path? For B?

RC: Yeah.

I: How did you get B? Show me.

RC: Because [counting gaps between dots on path B, then on path A] 1, 2, 3, 4, 5. 1, 2, 3, 4, 5.

I: … Do 5 of these [rods] fit? Like if you put 1 here [placing a rod on the leftmost segment of the bottom path], and one here [places a second rod over the second and third gaps between dots from the left end of the bottom path].

RC: Well, I can make it like a string.

I: Do you want to use this [hands RC a line of cylindrical inch rods strung on a wire]?

RC: [Places the first 4 rods over path B and makes a mark at the right end with his pen.]

I: So what are you thinking?

RC: This one [pointing at line A] is longer. This one [pointing at path B] is shorter.

I: Okay. And how did you figure that out?

RC: I lined these up [pointing at the string of rods]. And there was some more right there [pointing to the 'hill' on the top path]. …

In the above episode, the interviewer attempted to get RC to use inch rods and measurement M2 reasoning. However, RC used the inch rods mainly to correctly implement the non-measurement N2.1 strategy.

In the episode below, the interviewer was even more directive in encouraging RC to develop correct *measurement* reasoning.

Item 32

I: [See Figure 13a] If these are wires and I pull them so they are straight, which will be longer, or will they be the same? Is there any way that counting can help you solve this problem?

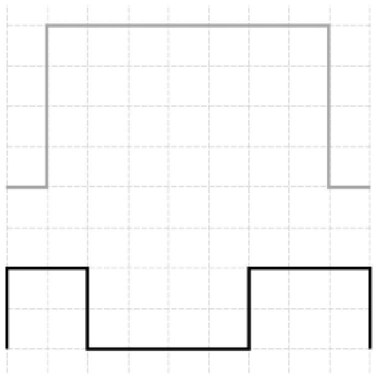


Figure 13a

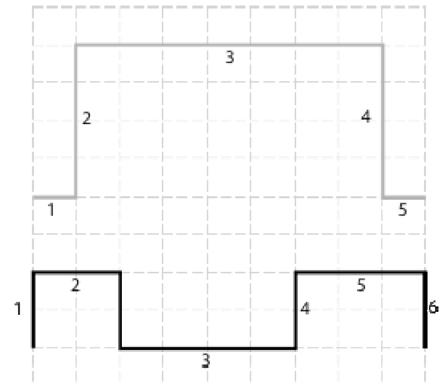


Figure 13b

RC: Um, yes.
I: So what would you do?
RC: Count these [counts the unequal straight portions of the bottom wire, but skips the second vertical segment from the left—see Figure 13b] 1, 2, 3, 4, 5, 6. [Counts the unequal straight portions on the top wire] 1, 2, 3, 4, 5.
RC: But this one [pointing to the top wire] would actually be longer. Because if you pull it out it'll come right there [pulling his hands out from the endpoints of the top wire to several inches past the endpoints]. And if you pull it out, it'll come right there [pulling his hands out from the endpoints of the bottom wire to a few inches past the margins].

So RC uses incorrect measurement and non-measurement reasoning on this task.

Item 34

I: Okay, could counting rods like this [tracing the unit segment at the left end of the top wire] help at all?

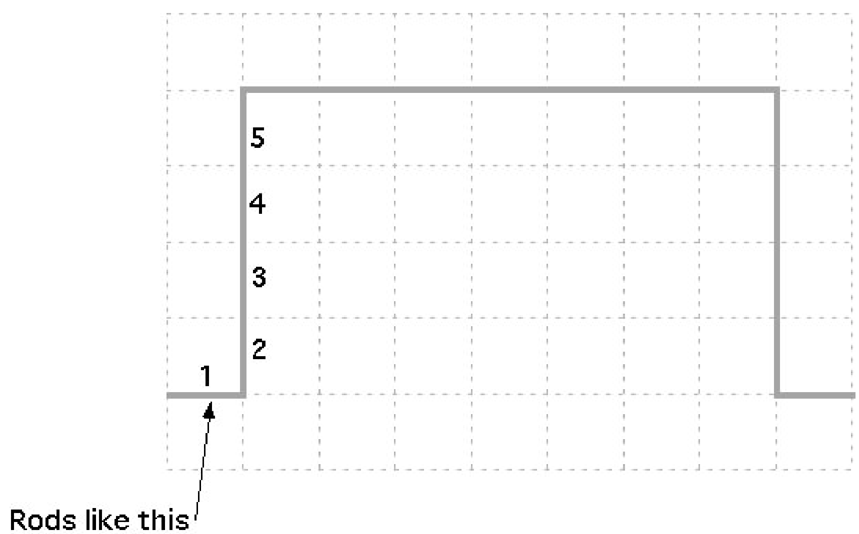


Figure 14

RC: I already counted that. [Pointing at each straight portion of the top wire again] 1, 2, 3, 4, 5.
I: Oh, but I was wondering if, could you count like [counting a few unit segments on the top wire, moving from left to right] 1, 2, 3, 4, 5 like that [see Figure 14]? Would that help?…
RC: I think so. [Counting squares along the top wire; see Figure 15] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15. [Counting squares along the bottom wire] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15.

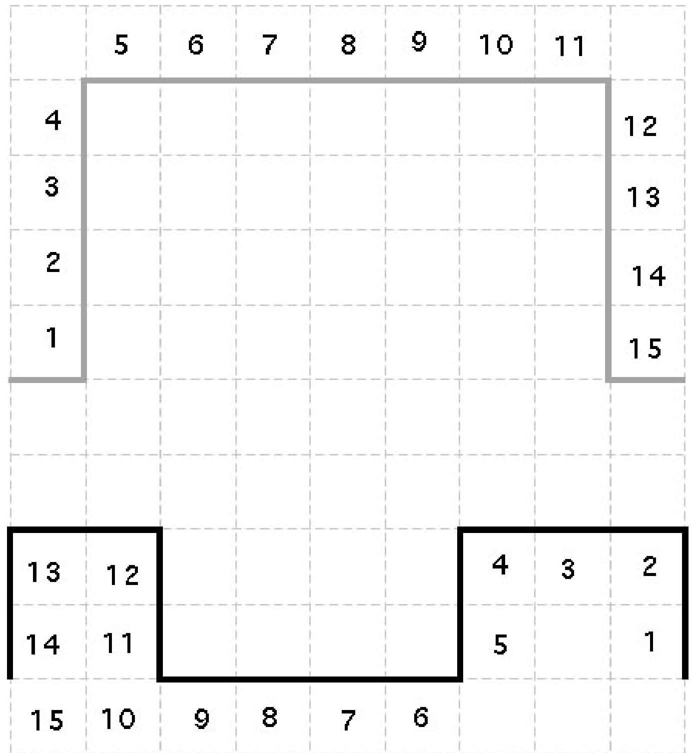


Figure 15

I: So what do you think?
RC: Probably the same length.

Given this narrative data on RC's reasoning, how should we represent his current knowledge structure with respect to length in a way that is most helpful for instruction? Rather than using an actual learning trajectory, the CBA approach is to construct a "profile" of RC's reasoning, using CBA LP levels of sophistication as the conceptual framework. To see what this profile looks like, note that in the context of problems like Item 23, in which the "wires" could be straightened using actual inch rods, RC had seen *empirically* that counting unit lengths could predict which was longer. So, for the last of these problems, he adopted the scheme of comparing wires by counting unit lengths in them. At first, he checked his answers by physically straightening a set of inch rods for each wire; but he curtailed this physical check on the last problem. We can conclude that in this context, RC had abstracted a particular reasoning scheme. However, for problems in different contexts, where dots or squares were salient, RC did not apply his new scheme (but he also did not apply his original M0 scheme). Furthermore, throughout the sessions, RC kept returning to the non-measurement scheme of straightening the paths (N2.1). So, the profile of RC's reasoning in terms of the CBA LP for length is: (a) he still relies heavily on non-measurement N2.1 reasoning; (b) he has started to see that measurement reasoning M1.2 (counting rods) can help him determine which path is longer; but (c) he does not yet understand the critical properties of unit length iteration (no gaps, overlaps; uniform lengths—M2.3).

So, future instruction must help RC (a) connect his iteration of inch rods (M2) to straightening paths (N2.1), (b) develop understanding of the properties of unit length iteration (M2.3), and (c) generalize a correct unit iteration scheme to new contexts (M2.3). For instance, in problems like Item 32, we would encourage RC to use inch rods (matching square size) to check his answers by *counting and straightening*. In response to this type of instruction/intervention, many students constructed generally applicable schemes, overcoming the fixation on the visually salient squares. Additionally, we also need to give RC tasks that highlight the importance of unit length iteration properties. For instance, we need to give RC problems in which he can determine by straightening that counting unequal segments gives incorrect comparisons.

It is the condensed, synthesized narrative profile of RC's reasoning, described in terms of CBA tasks and levels, that enables us to appropriately characterize and diagnose RC's reasoning in a way that is most useful for designing instruction that best matches his learning needs. Knowing the average CBA level for these tasks, or having a numerically valued vector or table of CBA level numbers, is insufficient for proper diagnosis and remediation.

Qualitative versus Quantitative Approaches to Developing LP

Both qualitative and quantitative methods have been used to develop learning progressions in mathematics (and science). *Both approaches are equally careful and scientific.* Generally, both approaches involve (a) synthesizing, integrating, and extending previous research to develop conceptual models of the development of student reasoning about a topic (hypothesized learning progressions); (b) developing and iteratively testing assessment tasks; (c) conducting several rounds of student interviews in support of steps (a) and (b); and (d) iteratively refining LP levels. In qualitative approaches, the cycle of iteration, testing, and revising eventually "stabilizes" into final levels, as determined by current level descriptions being used to reliably code all data. In contrast, quantitative methods compare the data to statistical model predictions (which often are derived using mathematical iteration), and, if needed, make adjustments to assessment item sets and levels.

Rash Rush to Rasch? Issues with Quantitative Methods

There have been numerous recommendations (sometimes demands) to use quantitative techniques to develop learning progressions (e.g., NRC, 2001), with a hint that using non-quantitative techniques is less "scientific." For example, Stacey & Steinle state that there have been "repeated suggestions made by colleagues over the years, which implied that we had been remiss in not using this Rasch analysis with our data" (2006, p. 89). However, using Rasch and other IRT approaches raises often-ignored serious issues that I now highlight.

First, Rasch/IRT models are "measurement" models. For instance, Masters and Mislevy state that "The probabilistic partial credit model … enables *measures* of achievement to be constructed" [italics added] (1991, p. 16). Or, Wilson, who describes the Saltus model as an example of "psychometric[11] models suitable for the analysis of data from assessments of cognitive development" (Wilson, 1989, p. 276). However, the whole enterprise of "measuring" in psychological research has been criticized, with less than compelling rebuttals (Michell, 2008).

Second, many of the assumptions of numerical models do not seem to fit our understanding of the process of learning and reasoning in mathematics. For instance, the Saltus model "assumes that each member of group *h* applies the strategies typical of that level consistently across all items" (Wilson, 1989, p. 278). Or, "The saltus model assumes that all persons in class *c* answer all items in a

[11] Of course psychometrics is "the measurement of mental capacity, thought processes, aspects of personality, etc., esp. by mathematical or statistical analysis of quantitative data" (OED online).

manner consistent with membership in that class…. In a Piagetian context, this means that a child in, say, the concrete operational stage is always in that stage, and answers all items accordingly. The child does not show formal operational development for some items and concrete operational development for others" (Draney & Wilson, 2007, p. 121). But, as has been discussed earlier, the levels in learning progressions are not necessarily stages, and often do not form a strong levels-hierarchy, making quantitative models problematic:

> "From this research, one can only conclude that there are situations in which students appear to reason systematically…When these situations arise, evidence about student understanding can be summarized by [numerical] learning progression level diagnoses, and educators can draw valid inferences about students' current states of understanding. Unfortunately, inconsistent responding across problem contexts poses challenges to locating students at a single learning progression level and makes it unclear how to interpret students' diagnostic scores. For example, how should one interpret a score of 2.6? A student with this score could be reasoning with a mixture of ideas from levels 2 and 3, but the student could also be reasoning with a mixture of ideas from levels 1, 2, 3, and 4. Such challenges prompt additional studies to support the valid interpretation of learning progression diagnoses" (Steedle & Shavelson, 2009, p. 704).

Thus, use of Rasch-like models to examine cognitive development, such as Wilson's Saltus model or latent class analysis, assumes that students are "at a level" (Briggs…; Draney & Wilson), which returns us to the problem discussed earlier about a student being at a level. Research on learning suggests that quite often, the state of student learning is not neatly characterized as "being at a specified level," which causes problems for interpretation of model results: "The results from this study suggest that students cannot always be located at a single level of the learning progression studied here. Consequently, learning progression level diagnoses resulting from item response patterns cannot always be interpreted validly" (Steedle & Shavelson, 2009, p. 713). See also my previous discussion of overlapping waves.

Third, Rasch/IRT models are based on item difficulty, which does not capture critical aspects of the nature of student reasoning, as Stacey and Steinle argue:

> Being correct on an item for the wrong reason characterises DCT2 [their decimal knowledge assessment]. It is one of the reasons why the DCT2 data do not fit the Rasch model, because these items break with the normal assumption that correctness on an item indicates an advance in knowledge (or ability) that will not be 'lost' as the student further advances. … A student's total score on this test might increase or decrease depending on the particular misconception and the mix of items in the test. This does not fit the property of Rasch scaling stated in Swaminathan (1999), that 'the number right score contains all the information regarding an examinee's proficiency level, that is, two

examinees who have the same number correct score have the same proficiency level' (p. 49). Neither the total score … nor Rasch measurement estimates provides a felicitous summary of student performance on the decimal comparison items of the DCT2 test" (2006, pp. 87-88).

Indeed, Stacey and Steinle further state that, "Conceptual learning may not always be able to be measured on a scale, which is an essential feature of the Rasch approach. Instead, students move between categories of interpretations, which do not necessarily provide more correct answers even when they are based on an improved understanding of fundamental principles" (2006, p. 77). Even more, how to place rote performance on items becomes extremely problematic in such models. For instance, in Noelting's hierarchy for proportional reasoning, the highest level is the formal operational stage in which the "child learns to deal formally with fractions, ratios, and percentages" (Draney & Wilson, 123). But using a formal procedure rotely is not a valid indication of formal operational reasoning. Stacey and Steinle concluded that there is nothing to gain in using the Rasch approach to the case of decimals that they studied and many other contexts. "Learning as revealed by answers to test items is not always of the type that is best regarded as 'measurable', but instead learning may be better mapped across a landscape of conceptions and misconceptions" (2006, p. 89).

Methods for Determining Levels in Learning Progressions

The most accurate way to determine levels in learning progressions (once the framework has been developed) is administering individual interviews, which are then coded by experts, using the LP levels framework. The difficulty with this approach is that it is time consuming. However, many teachers can learn to make such determinations, both with individual interviews and during class discussions. Another way to gather such data is using open-ended questions. Again, students' written responses must be coded, and many students do not write enough for proper coding. However, if teachers help students learn how to accurately describe their reasoning in writing, written responses can be a valuable means for gathering strategy information.

An alternate, less time-consuming, way to gather data is through multiple choice items that have distracters that are generated from interviews and that correspond to specific levels (Briggs et al., 2006, have labeled format "Ordered Multiple-Choice"). CBA has also experimented with teacher coding sheets—students describe their reasoning but the teacher or a classroom volunteer chooses the options in a multiple-choice-like coding sheet. However, beyond convenience, there are several issues that one must consider when using these alternate formats (e.g., Alonzo & Steedle, 2008; Briggs & Alonzo, 2009; Steedle &

Shavelson, 2009)[12]? For instance, students may not recognize which multiple-choice description matches the strategy they used to solve the problem.

When assessments are used summatively, however, taking a numerical approach can be both practical and useful. However, if one stores the data as numerical levels codes, in order to use the data for individual diagnoses, teachers must consult the theoretical model on learning that underlies the levels framework.

In Summary

When using quantitative methods to develop levels in learning progressions, the validity and usefulness of interpretations of results depends on (a) the adequacy of the underlying conceptual model of learning, (b) the fit between the statistical/mathematical model (including its assumptions) and the conceptual model of learning, and (c) the fit between the data and the statistical/mathematical model's predictions. Unfortunately, use of quantitative methods often ignores factor (b). For example, adopting the Saltus model might cause one to neglect explicit consideration of the critical issue of what it means to be at a level. Also, although many users of quantitative approaches argue that implementing such approaches enables them to test their models, too often, these tests are restricted to factor (c). Researchers in mathematics education need to resist external pressures to apply quantitative techniques without deeply

[12] Also at issue is whether Rasch techniques are the appropriate model when Ordered Multiple-Choice format tasks are employed (Briggs & Alonzo, 2009).

questioning their validity, because such adoptions result in the techniques being applied in ways that we would call in other contexts instrumental or rote procedural. Instead, researchers must investigate much more carefully the conceptual foundations of these techniques (a daunting task, given the statistical/mathematical complexity underlying the procedures)[13].

Learning Progressions and Curriculum/Assessment Standards

In the current era of "high standards," testing, and accountability, it seems reasonable to base both the content and grade-level locations of standards on research-based learning progressions. Indeed, the CCSSM state, "the development of these Standards began with research-based learning progressions detailing what is known today about how students' mathematical knowledge, skill, and understanding develop over time" (CCSSM, 2010, p. 4). However, there are aspects of the CCSSM, in particular for geometry, that seem to contradict this claim. As an example, consider the consistency of the CCSSM with the van Hiele levels. Although modern researchers have expressed several misgivings about the nature of the levels, recent reviews agree that "research generally supports that the van Hiele levels are useful in describing students' geometric concept development" (Clements, 2003, p. 153; Battista, 2007).

A major landmark in the van Hiele levels is when students develop

[13] One way to investigate the conceptual foundations of the approaches is to apply both to the same sets of data.

property-based reasoning about geometric shapes. For instance, at van Hiele Level 2, a student conceptualizes a rectangle, not as a visual gestalt, but, say, as a figure that has the properties:[14] "4 right angles," and "opposite sides parallel and equal." The CCSSM rightly recognize the critical importance of Level 2 reasoning. However, they specify that the development of this reasoning occurs at grades 4 and 5, which ignores van Hiele-based research that strongly suggests that, for most students, this reasoning is very difficult to achieve before ninth grade (Battista, manuscript in preparation). Indeed, the percent of students at or above Level 2, before and after high school geometry, has been reported as 31% before and 72% after by Usiskin (1982), and 51% before and 76% after by Frykholm (1994). Even after high quality instruction specifically targeting increasing students' van Hiele levels, research shows that the highest percent of students in grades 5-7 that achieved Level 2 reasoning or above was about 58%. So existing research casts serious doubt on the achievability of the CCSSM geometry standards *for most student*s.

It should be noted, however, that this research often uses different kinds of level indicators. For instance, in the Usiskin assessment of van Hiele levels (which was also used by Frykholm), property assessment tasks involved diagonals of quadrilaterals, which may have been studied less as opposed to basic defining, and more familiar, properties of classes of quadrilaterals.

[14] At Level 2, students do not understand minimal definitions. Instead, definitions tend to be lists of all the visually salient properties that students know (stated in terms of formal geometric concepts).

Furthermore, in Battista's study of fifth grade students working in his *Shape Makers* curriculum, if Level 2 was assessed by the triad tasks described above (which should be considered "transfer" tasks), 58% achieved Level 2 or higher on the posttest. But if Level 2 was assessed by students' knowledge of properties of shapes that had been explicitly explored in the curriculum, 83% were judged as achieving Level 2 or higher. However, Battista's research also suggests that, *in general*, junior high students' level of reasoning on these same familiar quadrilaterals is quite low (only 22% achieving Level 2).

This example illustrates several issues:

1. Standards too often are not sufficiently based on research. For example, given the research cited above, expecting ALL fourth or fifth graders to achieve Level 2 reasoning seems unreasonable.

2. Integrating various research studies into coherent learning progressions can be difficult because of variability in methods and assessments. For instance, assessments of van Hiele Level 2 have variably focused on knowledge of properties of familiar shapes, use of properties in transfer tasks, and knowledge of derived/secondary, as opposed to defining, properties (Battista, in preparation).

3. Although it is sometimes possible for students to make great progress in LP when using LP-based curricula and being taught by excellent teachers, this situation is not the norm. Basing standards on what happens in the best situations seems unwise.

4. For learning progressions to be useful in standards setting, the goals of the standards must closely match the knowledge acquisition described in the progressions. For instance, exactly which properties are targeted by CCSSM—familiar defining properties, or unfamiliar derived properties?

5. Should standards set benchmarks that all or most (say 80%) students can achieve, or should they target benchmarks that only, say, 50% (or 30%) of students might reasonably be expected to achieve? This is a critically important issue that may inadvertently place equity concerns in opposition to concerns about ensuring that sufficient numbers of students enter advanced mathematics and science careers in the US.

Teachers' Use of and Need for Learning Progressions

Professional recommendations and research advocate that mathematics teachers possess extensive knowledge of students' mathematical thinking (An, Kulm, Wu, 2004; Carpenter & Fennema, 1991; Clarke & Clarke, 2004; Fennema & Franke, 1992; Saxe et al., 2001; Schifter, 1998; Tirosh, 2000). Teachers must "have an understanding of the general stages that students pass through in acquiring the concepts and procedures in the domain, the processes that are used to solve different problems at each stage, and the nature of the knowledge that underlies these processes" (Carpenter & Fennema, 1991, p. 11). Research shows that such knowledge can improve students' learning (Fennema & Franke, 1992; Fennema et al., 1996). Indeed, "There is a good deal of evidence that learning is enhanced when teachers pay attention to the knowledge and beliefs that learners bring to a learning task, use this knowledge as a starting point for new instruction, and monitor students' changing conceptions as instruction proceeds" (Bransford et al., 1999, p. 11). Thus, there is a great need to study teachers' learning, understanding, and use of learning progressions in mathematics.

Related to the study of teachers' use of learning progressions, there is much research investigating the nature of the knowledge teachers have and need to teach mathematics, with the scope of this work described by the "egg" domain-map of Hill, Ball, and Schilling (2008, p. 377) (see Figure 16). Battista's Cognition Based Assessment, Phase 2 (CBA2) research project is focusing on one component in this domain, "Knowledge of Content and Students" (KCS), which Hill et al. define as, "Content knowledge intertwined with knowledge of how students think about, know, or learn that content" (p. 378).

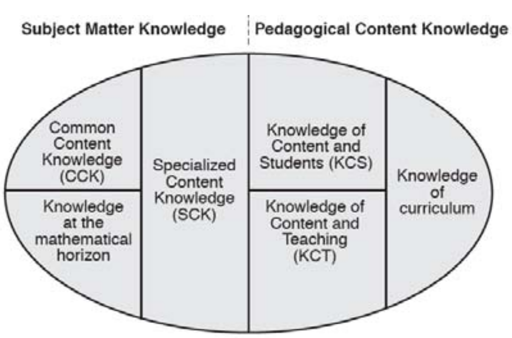


Figure 16

The Hill/Ball/Schilling framework puts mathematical knowledge at the forefront in describing mathematics-related teacher knowledge. Consistent with this content-primary perspective, Park and Oliver state, "it is transformation of subject matter knowledge for the purpose of teaching that is at the heart of the definition of PCK" (2008, p. 264).

In contrast, the CBA2 approach to studying KCS focuses on teachers' "cognitive/psychological knowledge" of students' mathematical thinking, and a major component of this research is connected to teachers' understanding and use of learning progressions. Although cognitive/psychological knowledge and mathematical knowledge are distinct, they are intertwined with each other and with knowledge of teaching and curricula. See Figure 17.

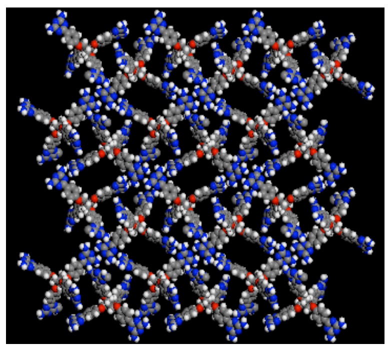


Figure 17. Intertwined Teacher Knowledge [Mathematics gray, Psychological white, Teaching blue, Curricula red]

In the CBA2 project, we are conducting case studies that qualitatively describe (a) the nature of teachers' conceptualizations of students' mathematical thinking, (b) the processes by which teachers come to understand research-based knowledge on the development of students' mathematical thinking (as represented in CBA LP), and (c) how teachers use this knowledge (including CBA assessments and instructional guidance) in assessment and teaching.

One Teacher's Use of CBA

Before describing several issues in teachers' understanding and use of learning progressions, it is worthwhile to note the power that many teachers obtain with CBA's linked LP, assessments, and instructional guidance. So I quickly summarize a case study of one teacher in the CBA project who used several extremely detailed CBA learning progressions in his teaching and assessment. As Teacher 19 learned and used CBA ideas and materials, he made major progress in:

- understanding students' learning progressions
- understanding assessment tasks
- deciding what's most important in the curriculum
- diagnosing and remediating students' learning difficulties
- deciding on the effectiveness of instruction—are there problems in the teaching, or are students not quite ready to learn a particular concept
- improving informal assessments by helping to him ask better questions and more quickly understanding what students say
- understanding and building on students' reasoning and procedures as they occurred in frequent class discussion
- helping parents understand their children's mathematics program and progress through it.

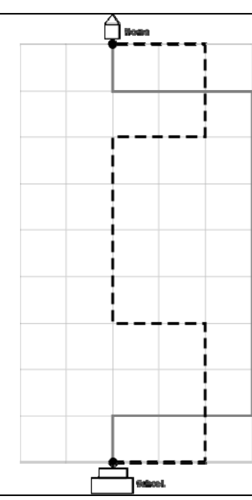
In much of T19's discussion of CBA, he described how important it was for him to be able to say to himself, "Well, they're here and this is where I need to take them," a major affordance of CBA LP. This is practical, decision-making information needed for everyday mathematics teaching. Finally, T19 was impressed by the great progress in learning his students made (especially those who were struggling), which he attributed to his learning about and use of CBA materials. As an especially important example for him, he described how one of his struggling students started the school year at a kindergarten level in mathematics and by mid-year was functioning at a third grade level.

Teachers' Understanding of Students' Reasoning about Length: The Need for LP

To illustrate why research-based LP are so important for teachers, I describe one example of teachers' understanding and misunderstanding of students' reasoning about length measurement, a topic that almost all elementary students have difficulty with. Examination of this example illustrates the kind of content that is needed in LP written for teachers.

Teachers were shown the work of Student X and asked to analyze it (see Figure 18).

Student Problem
Which sidewalk from home to school is longer, the dotted one, the gray one, or are they the same?



Teacher Task
Consider Student X who used the strategy below on the Student Problem (above).

Student X: [Counts squares along the gray path 1-14, then along the dotted path 1-15.] The gray path is shorter because it has less squares.

(a) Is Student X's reasoning correct or incorrect? If it is incorrect, what is wrong with it?

(b) What would you do instructionally [to help Student X]?

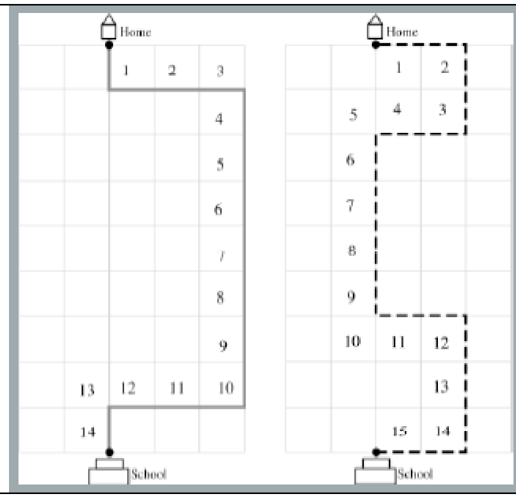


Figure 18. Student problem and teacher task

To illustrate the difficulties that teachers had with analyzing Student X's reasoning, I describe two examples of how teachers conceptualized (a) X's reasoning, and (b) subsequent instruction for X.

Teacher1: [X's reasoning] is incorrect because … she is counting the boxes instead of the side length for the unit. Like on this first box [in the gray path; see Figure 19] <u>she is just counting it as one unit</u> even though there are two sides there that should be measured.

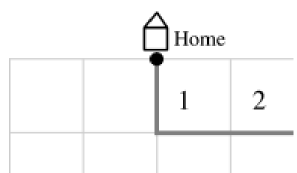


Figure 19.

Teacher3: [X's reasoning] is incorrect. She is not recognizing that she is counting two segments as one [pointing to the first turn in the gray path] <u>because she is looking at area</u>. So she is looking at the area of the squares, not counting the sides or segments.

Although both teachers understood that X's reasoning is incorrect, several features of the teachers' conceptualizations of X's reasoning are problematic. First, there was no evidence in any of X's work that she was mistaking area for length. Instead, X implemented the procedure of "placing" squares along a path, without properly relating this procedure to unit-length iteration. X did not understand the concept of unit-length iteration or the procedure for implementing it. Conceptualizing X's error as looking at area mis-conceptualizes X's reasoning *psychologically*. One of the key features of LP is that they provide psychologically sound, and pedagogically useful, interpretations of students' reasoning.

The second important feature is the statement by both teachers that X is counting 2 length units instead of 1. Thus, both teachers misinterpret X's conceptualization and error. *X is iterating squares, not different-sized linear units.* Both teachers focus on the mathematical consequences of X's errant strategy, rather than its psychological root.

To further examine this misinterpretation and its consequences, we look at how the teachers' conceptualizations of X's reasoning affects their view of the instruction X needs.

Int: What would you do instructionally to move X to this next type of reasoning [correct iteration]?
T1: Well I think she needs to understand what the unit is, and that the units have to be … consistent as she is measuring. So she would need to see that this unit <u>that she labeled as one</u> [draws Figure 20A] is more than this unit [draws Figure 20B].

Figure 20.

So like you could show her that this unit and this unit are not the same cause if you straighten it out this would be two units, and this would just be the one unit.
T3: We used inch rods cut out of straws … and physically put those along [the paths] … And that helped them to recognize that they weren't counting the sides when they were using squares. They were missing something.

T1 has a valid *long-term* instructional goal—X must learn to iterate a constant unit-length. However, because T1 misinterprets X's conceptualization, she chooses an inappropriate *short-term/immediate* instructional goal. Telling X that she counted 2 units instead of 1 would confuse X. Students who are conceptualizing length measurement as iterating squares along a path must first see that a totally different kind of unit—linear—must be iterated. This is surprisingly difficult for many elementary school students. Understanding the properties of unit-length iteration—equal-length units, no gaps/overlaps—comes after understanding the nature of the iterated unit. LP provide not only long-term instructional goals but the kind short-term/immediate instructional goals that are critical for guiding and supporting students' moment-to-moment learning.

In summary, T1 understands X's reasoning *mathematically* but not *psychologically*. *It seems that focusing on the mathematical consequences of counting squares, while critical to determining the validity of X's reasoning, caused T1 to incorrectly conceptualize the nature of X's reasoning.* Consequently, although T1's instructional goal was worthwhile, her plan does not adequately build on X's current reasoning. Interestingly, although T3's conceptualization of X's reasoning was also problematic, probably because she had previously been interactively guided in the appropriate use of length activities by CBA staff, she was still able to appropriately choose which CBA instructional activity was appropriate for X. Nevertheless, to appropriately build on X's current reasoning, teachers must fully and psychologically understand the nature of her reasoning. It is insufficient to merely know that X is getting incorrect counts of units, or even where the incorrect counts occur. Teachers must understand that X is using the wrong kind of measurement unit. And it is the CBA LP on length that provides the appropriate framework for this understanding.

To fully understand and respond to misconceptions like those of X and other students, teachers need research-based learning progressions that describe the range of conceptualizations that students possess about length and length measurement. Knowledge of such progressions not only helps teachers understand students' thinking psychologically, it expands a teacher's focus beyond mathematical, to pedagogically critical psychological, interpretations of students' mathematical thinking. And for LP to be maximally useful for teachers in instruction, LP must be linked to (a) appropriate assessment tasks that reveal students' reasoning, and (b) instructional tasks specifically designed to address students' learning needs at various locations in the LP.

Balance: How Much Detail Is Needed in LP for Teachers?

In discussing the use of learning progressions for formative assessment by teachers, Popham states, "It's important to stress that there must always be a balance between (1) the level of analytic sophistication that goes into a learning progression and (2) the likelihood of the learning progression being used by teachers and students" (2008, p. 29). So a central issue in describing learning progressions written for teachers is how much detail teachers can handle in the progression descriptions.

Although space does not permit me to provide a full analysis of this issue in the CBA2 project, it is true that almost all of the teachers who participated in the CBA2 project for at least a year did learn to use the great amount of detail in CBA LP. However, a comment made by many teachers who participated in the CBA2 project is that most/many teachers would have difficulty learning the great amount of detail in the CBA materials. Consequently, some of these teachers suggested giving teachers simplified versions of the CBA materials. The following episode illustrates that this approach, if it oversimplifies a LP, can lead to difficulties.

Misinterpretation of "Simplified" Level Descriptions

One idea that we experimented with in the CBA2 project is providing "simplified" descriptions of CBA levels to teachers. As an example, in the regular CBA materials, Level M1 for length was described and numerous examples of student work were provided. In contrast, some teachers were given the very abbreviated description of Level M1 below. Notice that in this abbreviated description, the terms "gaps" and "overlaps" were not elaborated or illustrated. It was assumed that teachers would understand the terms, given the context.

> *Abbreviated Version*
> ***"CBA Length Level M1. Incorrect Unit Iteration***
> *Students do not fully understand the process of unit-length iteration; their iterations contain gaps, overlaps, or different length units, and are incorrect."*

How several teachers misinterpreted the terms "gaps" and "overlaps" is revealing. T17[15] made the following comment in deciding which CBA level Student X evidenced on the home-to-school problem.

T17: Well what do you mean gap? An opening that is not counted. … [X] didn't count it [pointing at the "2" on the dotted path, See Figure 21a]. … So it has to be a gap.

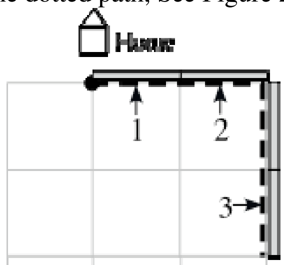


Figure 21a

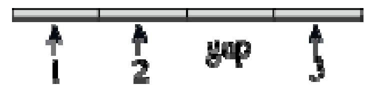


Figure 21b

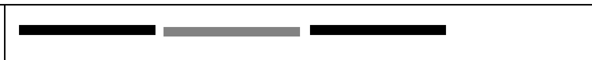
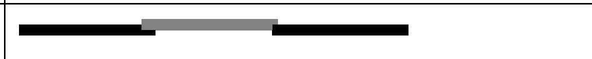
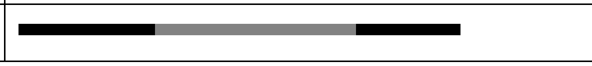
In this case, T17 interpreted the term "gap" as a mismatch in the correspondence between the number sequence "1, 2, 3" and the sequence of 4 unit-lengths that should have been iterated along the portion of the dotted path shown above. If X were counting unit lengths, she should have counted "1, 2, 3, 4" for this portion of the path. But she omitted the count for the third segment; so there was a "gap" in her counting sequence (see Figure 21b). T17's interpretation of *gap* was very different from the meaning of *gap* that the CBA author intended (see CBA document excerpt below). And, like T3 and T1 above, T17's interpretation of gap seemed to contribute to her mis-interpretation of X's conceptual difficulty.

"M1.2: Iterates Unit-Lengths but Gets Incorrect Count

Students iterate unit-lengths rather than shapes. So when iterating unit-lengths, they draw line segments, not squares, rectangles, or rods. However, because they do not understand the properties of

[15] T1 and T3 had read the full CBA document on length; T17 had not read any CBA length material other than the abbreviated desriptions like that shown above.

unit-length iteration, their iterations contain gaps, overlaps, or different length units (see below)" (Battista, in press).

gaps	
overlaps	
different length units	

The issue of determining how teachers can use learning progressions in their teaching and formative assessment, and how learning progressions should be described to facilitate this use, is central to supporting mathematics teaching that develops deep conceptual knowledge and problem-solving proficiency in students.

References

Alonzo, A. C., & Steedle, J. T. (2008). *Developing and assessing a force and motion learning progression.* Published online 2 September 2008 in Wiley InterScience (www.interscience.wiley.com).

An, S., Kulm, G., & Wu, Z. (2004). The pedagogical content knowledge of middle school, mathematics teachers in China and the U.S. *Journal of Mathematics Teacher Education, 7*, 145-172.

Battista, M. T. (2001). How do children learn mathematics? Research and reform in mathematics education. In T. Loveless (Ed.), *The Great Curriculum Debate: How Should We Teach Reading and Math?* Washington D.C.: Brookings Press.

Battista, M.T. (2004). Applying cognition-based assessment to elementary school students' development of understanding of area and volume measurement. *Mathematical Thinking and Learning, 6*(2), 185-204.

Battista, M. T. (2007). The development of geometric and spatial thinking. In Lester, F. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 843-908). NCTM. Reston, VA: National Council of Teachers of Mathematics.

Battista, M. T. (2009). Highlights of research on learning school geometry. In T. Craine & R. Rubenstein (eds.), *2009 Yearbook, Understanding Geometry for a Changing World*. NCTM.

Battista, M. T. (manuscript in preparation). *At What Grade Should CCSSM Expect Students to Achieve the Properties Level in the van Hiele Learning Progression?*

Battista, M. T. & Clements, D. H. (1996). Students' understanding of three-dimensional rectangular arrays of cubes. *Journal for Research in Mathematics Education*, *27*(3), 258-292.

Clements, D. H., Battista, M. T. (2001). Logo and Geometry. *Journal for Research in Mathematics Education Monograph Series*, Number 10.

Battista, M. T., Clements, D. H., Arnoff, J., Battista, K., & Borrow, C. V. A. (1998 November). Students' spatial structuring and enumeration of 2D arrays of squares. *Journal for Research in Mathematics Education 29*(5), 503-532.

Battista, M. T. (accepted). *Cognition Based Assessment for elementary school mathematics: Length and length measurement.* Heinemann.

Bransford, J. D., Brown, A. L., & Cocking, R. R. (1999). *How people learn: Brain, mind, experience, and school*. Washington, D.C.: National Research Council.

Briggs, D. C., & Alonzo, A. C. (2009). *The psychometric modeling of ordered multiple-choice item responses for diagnostic assessment with a learning progression.* Paper presented at the Learning Progressions in Science (LeaPS) Conference, Iowa City, IA.

Briggs, D.C., Alonzo, A. C., Schwab, C., & Wilson, M. (2006). Diagnostic assessment with ordered multiple-choice items. *Educational Assessment*, *11*(1), 33-63.

Carpenter, T. P., & Fennema, E. (1991). Research and cognitively guided instruction. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 1-16). Albany: State University of New York Press.

Carroll, W. M. Geometric knowledge of middle school students in a reform-based mathematics curriculum. *School Science and Mathematics*, Volume 98(4), April 1998, 188-197.

Clarke, B., & Clarke, D. (2004). A framework for growth points as a powerful professional-development tool. *Annual meeting of the American Educational Research Association*.

Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420-464). New York, NY: NCTM/Macmillan Publishing Co.

Clements, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning*, *6*(2), 81-89.

Cobb, P., & Wheatley, G. (1988). Children's initial understanding of ten. *Focus on learning problems in mathematics, 10*(3), 1-28.

Confrey, J. et al. (2009). Equipartitioning/splitting as a foundation of rational number reasoning using learning trajectories. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, C. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, 1-8. Thessaloniki, Greece: PME.

Draney, K., & Wilson, M. (2007). Application of the Saltus model to stagelike data: Some applications and current developments. In M. von Davier & C. H. Carstensen (Eds.), *Statistics for social and behavioral sciences: Multivariate and mixture distribution Rasch models* (pp. 119-130). Springer.

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching* (pp. 127-164). Reston, VA: National Council of Teachers of Mathematics/Macmillan.

Fennema, E., Carpenter, T. P., Franke, M. L., Levi , L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education, 27(4)*, 403-434.

Frykholm, J. A. (1994). *External variables as predictors of van Hiele levels in algebra and geometry students*. ERIC.

Fuys, D., Geddes, D., & Tischler, R. *The van Hiele Model of Thinking in Geometry among Adolescents. Journal for Research in Mathematics Education* Monograph No. 3. Reston, Va.: National Council of Teachers of Mathematics, 1988.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39(4)*, 372-400.

Lehrer, R., Jenkins, M., & Osana, H. (1998). Longitudinal study of children's reasoning about space and geometry. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 137-167). Mahwah, NJ: Lawrence Erlbaum Associates.

Masters, G. N, & Mislevy, R. J. (1991). *New views of student learning: Implications for educational measurement.* Final report, Office of Naval Research.

Michell, J. (2008). Is psychometrics pathological science? *Measurement*, *6*, 7–24.

Mitchelmore, M. C., & White, P. (2000). Development of angle concepts by progressive abstraction and generalisation. *Educational Studies in Mathematics, 41*(3), 209-238.

National Research Council. (2001). *Knowing what students know: The science and design of educational assessment.* Committee on the Foundations of Assessment. Pelligrino, J., Chudowsky, N., and Glaser, R., editors. Board on Testing and Assessment, Center for Education. Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.

National Research Council. (2007). *Taking Science to School: Learning and Teaching Science in Grades K-8.* Committee on Science Learning, Kindergarten Through

Eighth Grade. R. A. Duschl, H. A. Schweingruber, and A. W. Shouse, Editors. Board on Science Education, Center for Education. Division of Behavioral and Social Sciences and Education. Washington, DC: The National Academies Press.

Novillis, C. F. (1976). An analysis of the fraction concept into a hierarchy of selected subconcepts and the testing of the hierarchical dependencies. *Journal for Research in Mathematics Education, 7*, 131-144.

Park, S. & Oliver, J. S. (2008). Revisiting the conceptualisation of pedagogical content knowledge (PCK): PCK as a conceptual tool to understand teachers as professionals. *Research in Science Education, 38*, 261–284.

Pegg, J., & Davey, G. (1998). Interpreting student understanding in geometry: A synthesis of two models. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 109-133). Mahwah, NJ: Lawrence Erlbaum Associates.

Saxe, G. B., Gearhart, M., & Nasir, N. S. (2001). Enhancing students' understanding of mathematics: A study of three contrasting approaches to professional support. *Journal of Mathematics Teacher Education, 4*, 55-79.

Schifter, D. (1998). Learning mathematics for teaching: From the teachers' seminar to the classroom. *Journal for Mathematics Teacher Education, 1*(1), 55-87.

Siegler, R. S. (2005). Children's learning. *American Psychologist, November*, 769-778.

Siegler, R. S. (2007). Cognitive variability. *Developmental Science, 10*(1), 104-109.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*(2), 114-145.

Simon, M. A., Tzur, R., Heinz, K., & Kinzel, M. (2004). Explicating a mechanism for conceptual learning: Elaborating the construct of reflective abstraction. *Journal for Research in Mathematics Education, 35*(5), 305-329.

Smith, C. L., Wiser, M., Anderson, C. W., & Krajcik, J. (2006). Implications of research on children's learning for standards and assessment: A proposed learning progression for matter and the atomic-molecular theory. *Measurement*, *14*(1&2), 1-98.

Stacey, K., & Steinle, V. (2006). A case of the inapplicability of the Rasch model: Mapping conceptual learning. *Mathematics Education Research Journal*, *18*(2), 77-92.

Steedle, J. T., & Shavelson, R. J. (2009). Supporting valid interpretations of learning progression level diagnoses. *Journal of Research in Science Teaching*, *46*(6), 699-715.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and individual differences, 4*(3), 259-309.

Steffe, L. P. (2004). On the construction of learning trajectories of children: The case of commensurate fractions. *Mathematical Thinking and Learning*, *6*(2), 129-162.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31*(1), 5-25.

Usiskin, Z. (1982). *van Hiele levels and achievement in secondary school geometry: Final report of the cognitive development and achievement in secondary school geometry project*. Chicago: University of Chicago.

Usiskin; Z., & Senk, S. (1990). Evaluating a test of van Hiele levels: A response to Crowley and Wilson. *Journal for Research in Mathematics Education, 21*, 242-245.

van Hiele, P. M. (1986). *Structure and insight*. Orlando: Academic Press, Inc.

Wilson, M. (1989). Saltus: A psychometric model of discontinuity in cognitive development. *Psychological Bulletin*, *105*(2), 276-289.

Wilson, M. (1990). Measuring a van Hiele geometry sequence: A reanalysis. *Journal for Research in Mathematics Education, 21*, 230-237.

Appendix 1: The van Hiele Levels

Below I describe the van Hiele levels in a way that is consistent with Clements' and Battista's (1992) analysis and synthesis of research on the levels. My recent elaborations and extensions of the levels are described in Battista (2007, 2009).

Level 0 Pre–recognition

At the pre-recognition level[16], children perceive geometric shapes, but perhaps because of a deficiency in perceptual activity, may attend to only a subset of a shape's visual characteristics. They are unable to identify many common shapes. They may distinguish between figures that are curvilinear and those that are rectilinear but not among figures in the same class. That is, they may differentiate between a square and a circle, but not between a square and a triangle.

Level 1 Visual

Students identify and operate on geometric shapes according to their appearance. They recognize figures as visual gestalts. In identifying figures, they often use visual prototypes, saying that a given figure is a rectangle, for instance, because "it looks like a door." They do not, however, attend to geometric properties or traits that are characteristic of the class of figures represented. That is, although figures are determined by their properties, students at this level are not conscious of the properties. For example, they might distinguish one figure from another without being able to name a single property of either figure, or they might judge that two figures are congruent because they look the same; "There is no why, one just sees it" (van Hiele, 1986, p. 83). By the statement "This figure is a rhombus," the student means "This figure has the shape I have learned to call 'rhombus'" (van Hiele, 1986, p. 109).

Level 2 Descriptive/analytic

Students recognize and can characterize shapes by their *properties.* For instance, a student might think of a rhombus as a figure that has four equal sides; so the term "rhombus" refers to a collection of “properties that he has learned to call 'rhombus'" (van Hiele, 1986, p. 109). Students see figures as wholes, but now as collections of properties rather than as visual gestalts; the image begins to fall into the background. The objects about which students reason are classes of figures, thought about in terms of the sets of properties that the students associate with those figures. Students experientially discover that some combinations of properties signal a class of figures and some do not. Students at this level do not see relationships between classes of figures (e.g., a student might contend that figure is not a rectangle because it is a square).

[16] Not described by van Hiele, but argued for by Clements and Battista (1992).

Level 3 Abstract/relational

Students can form abstract definitions, distinguish between necessary and sufficient sets of conditions for a concept, and understand and sometimes even provide logical arguments in the geometric domain. They can classify figures hierarchically (by ordering their properties) and give informal arguments to justify their classifications (e.g., a square is identified as a rhombus because it can be thought of as a "rhombus with some extra properties"). Thus, for instance, the "properties are ordered, and the person will know that the figure is a rhombus if it satisfies the definition of quadrangle with four equal sides" [van Hiele, 1986], p. 109).

As students discover properties of various shapes, they feel a need to organize the properties. One property can signal other properties, so definitions can be seen not merely as descriptions but as a way of logically organizing properties. It becomes clear why, for example, a square is a rectangle. The students still, however, do not grasp that logical deduction is the method for establishing geometric truths.

Level 4 Formal deduction

Students establish theorems within an axiomatic system. They recognize the difference among undefined terms, definitions, axioms, and theorems. They are capable of constructing original proofs. That is, they can produce a sequence of statements that logically justifies a conclusion as a consequence of the "givens."

Level 5 Rigor/metamathematical

Students reason formally about mathematical systems. They can analyze and compare axiom sets.