On the idea of Learning Trajectories: Promises and Pitfalls

Susan B. Empson

On the Idea of Learning Trajectories: Promises and Pitfalls[1]

Susan B. Empson[2]
The University of Texas at Austin

Learning mathematics is a complex and multidimensional if not an inherently indeterminate process. A necessary goal of research on learning is to simplify this complexity without sacrificing the ability of research to inform teaching. This goal has been addressed in part by researchers focusing on how to represent research on learning for teachers and on how to support teachers to use and generate models of students' learning (e.g., Franke, Carpenter, Levi, & Fennema, et al., 2001; Hammer & Schifter, 2001; Simon & Tzur, 2004; Steffe, 2004). Recently, the idea of *learning trajectories* has gained attention as a way to focus research on learning in service of instruction and assessment. It is influencing curriculum standards, assessment design, and funding priorities. In this paper – which grew out of my response to Michael Battista's keynote address on learning trajectories at the last annual meeting of the North American chapter of Psychology in Mathematics Education (Battista, 2010) – I examine the idea of learning trajectories and speculate on its usefulness in mathematics education.

---

[1] Thanks to Jennifer Knudsen of SRI International for invaluable feedback.
[2] empson@mail.utexas.edu

The National Research Council (2007) described learning progressions as "successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic" (p. 214). The recently released *Common Core Standards in Mathematics* (CCSM) (2010), noted that the "development of these Standards began with research-based learning progressions detailing what is known today about how students' mathematical knowledge, skill, and understanding develop over time" (p. 4). The idea of learning trajectories has a great deal of intuitive appeal and may offer a way to bring coherence to how we think about learning and the curriculum. As research on learning trajectories proliferates and is brought to bear on some of the most vexing problems in teaching and learning mathematics, however, it is worth considering what it foregrounds and what it may obscure.

In this paper, I briefly describe the origins of *learning trajectories* in mathematics education and then consider three points for us to keep in mind as we study learning and apply our findings to serve the purposes of understanding and addressing the problems of practice.

1) The idea that learning progresses is not especially new. What do we know about learning mathematics and how does it fit with the idea of a trajectory?

2) Learning trajectories focus on specific domains of conceptual development and may be limited in characterizing other valued aspects of the mathematics curriculum.

3) Learning in school is function of teaching. Too tight a focus on learning trajectories may lead us to oversimplify or ignore critical drivers of learning associated with teaching.

My goal in making these points is not to state the obvious but to foreground the question of what the idea of learning trajectories affords us education researchers and practitioners, and what it might obscure.

Origins of Learning Trajectories

The term *learning trajectory* appears to have been first used in mathematics education in Marty Simon's oft-cited 1995 paper, "Reconstructing Mathematics Pedagogy from a Constructivist Perspective." As I reread this paper, the most important things I noticed – besides the fact that that the actual words "learning trajectory" did not appear until 21 pages into the article – were that a) a learning trajectory did not exist for Simon in the absence of an agent and a purpose and b) it was introduced in the context of a theory of teaching. According to Simon, a *hypothetical learning trajectory* is a teaching construct – something a teacher conjectures as a way to make sense of where students are and where the teacher might take them. It is hypothetical because an "actual learning trajectory is not knowable in advance" (p. 135). Teachers are agents who hypothesize learning trajectories for the purposes of planning tasks that connect students' current thinking activity with possible future thinking activity. A teacher might ask, "What does this student understand? What could this student learn next and

how could they learn it?" and create a hypothetical learning trajectory as a way to prospectively grapple with these questions.

The idea of *learning progressions* appears to have emerged first in the context of science education and is now virtually synonymous with learning trajectory. In a special issue of the *Canadian Journal of Science, Mathematics, and Technology Education* devoted to the topic of "long-term studies" of learning in science education, Shapiro (2004) traced the notion of learning progression in part to Rosalind Driver in her 1989 article, "Students' Conceptions and the Learning of Science." In it, Driver drew attention to the increasing number of studies of the development of children's thinking in specific science domains that documented patterns in what she called *conceptual progressions* and sequences of conceptual progressions, which she termed *conceptual trajectories* (Shapiro, 2004, p. 3). In contrast to Simon, the focus in that special issue of CJSMT was on describing children's learning as it had actually occurred under a given set of conditions, rather than on a thought experiment about how it could occur. Neither of these senses of *learning trajectory* – as a teacher-conjectured possible progression or a researcher-documented progression of actual learners– predominates in current conceptions of the notion.

Since 2004, there has been a groundswell of research that explicitly identifies itself as concerned with learning trajectories or progressions, as reflected in conferences and special journal issues (Clements & Sarama, 2004; Duncan & Hmelo-Silver, 2009), reports (Catley, Lehrer, & Reiser, 2005; Cocoran, Mosher, &

Rogat, 2009; Daro, Mosher, & Cocoran, 2011), and books (Clements & Sarama, 2009). A report by the Center for Continuous Improvement in Instruction (Daro, et al., 2011) treats learning trajectories as interchangeable with learning progressions, reflecting the general trend.

Because the metaphor of trajectory implies a sequenced path, researchers who focus explicitly on learning trajectories have taken pains to draw attention to their multidimensional character. For example, Clements and Sarama (2004) defined learning trajectories as complex constructions that include "the simultaneous consideration of mathematics goals, models of children's thinking, teachers' and researchers' models of children's thinking, sequences of instructional tasks, and the interaction of these at a detailed level of analysis of processes" (p. 87). Confrey and colleagues (2009) defined them as "researcher-conjectured, empirically-supported description[s] of the ordered network of experiences a student encounters through instruction … in order to move from informal ideas … towards increasingly complex concepts over time" (p. 2).

Three Points to Keep in Mind

Learning Trajectories are Not Really New – So What does the Metaphor Buy Us?

The idea that students' learning progresses in some way as a result of instruction is at the very heart of the enterprise of mathematics education. Researchers have been studying students' mathematics thinking and what it could mean for that thinking to progress in identifiable ways since long before the term *learning trajectories* was introduced. Chains of inquiry focused on

children's mathematics learning – we could call these research trajectories – have stretched over decades. For example, Glenadine Gibbs's (1956) study of students' thinking about subtraction word problems helped to pave the way for later researchers such as Carpenter and Moser (1984) to create frameworks portraying the development of children's thinking about addition and subtraction, and for Carpenter, Fennema, and Peterson to study how teachers used this information about children's thinking to teach for understanding (Carpenter, et al., 1989; Carpenter, et al., 1999). Les Steffe and John Olive's recent (2010) book on *Children's Fractional Knowledge* detailing the evolution of children's conceptual schemes for operating on fractions synthesized two decades' worth of prior research, as did Karen Fuson's findings on the development children's multidigit operations (1992). None this work mentioned learning trajectories as such, but each focused on elucidating the development of children's understanding and identifying major conceptual advances.

Why then talk about learning trajectories now? The metaphor emphasizes the orderly development of children's thinking and draws our attention to learning targets and possible milestones along the way.

To what extent is this kind of assumption about learning warranted? That is, in what sense does children's mathematics learning follow predictable trajectories? Some domains appear to readily lend themselves to analysis in terms of a pathway, such as the development of young children's counting skills (Gelman & Gallistel, 1986). The progression of children's strategies for addition

and subtraction story problems from direct modeling, to counting, to the use of derived and recalled facts also has been well established (Carpenter et al., 1999; Carpenter, 1985; Fuson, 1992). Yet even given such a robust progression in a basic content domain, how and when – and sometimes whether – children come to understand and use these strategies depends on a variety of factors differing from classroom to classroom and from child to child. Trying to represent research on learning in terms of trajectories quickly gets complicated, even for as fundamental a concept as rational number (e.g., Figure 2 "Learning Trajectories Map for Rational Number Reasoning," in Confrey, Maloney, Nguyen, et al., 2009) or measurement (Figure 1).
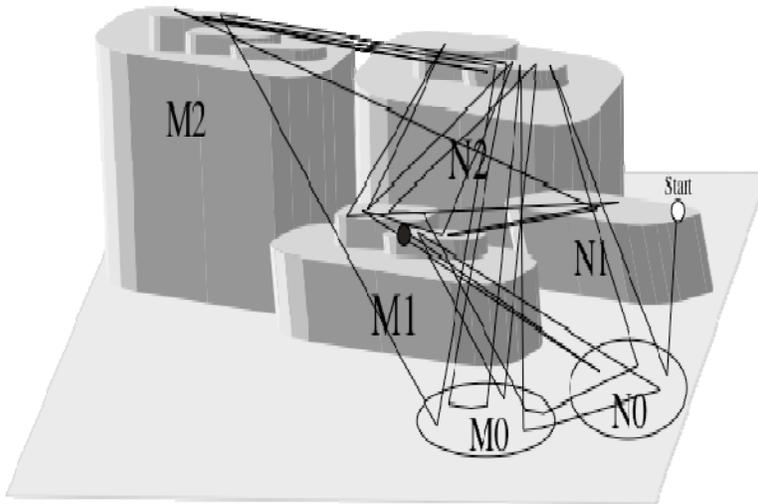
Figure 1. Battista's (2010) representation of one student's actual learning path in measurement

Other research suggests that the development of much of children's thinking is more piece-meal and context-dependent than representations of learning trajectories might lead us believe (DiSessa, 2000; Greeno & MMAP, 1998). For

example, in a cross-sectional, cross-cultural study, Liu and Tang (2004) found differences in progressions of students' conceptions of energy in Canada and China over several years of schooling, which they attributed to differences in curriculum and instruction in each country. The topic of rational numbers in mathematics has an ample research base that illustrates, in some cases meticulously, how children's thinking about fractions could progress (Behr, Harel, Post, & Lesh, 1992; Davydov & Tsvetkovich, 1991; Empson & Levi, 2011; Hackenberg, 2010; Steffe & Olive, 2010; Streefland, 1991; Tzur, 1999). Taken collectively this research does not appear to converge on a single trajectory of learning.

Why might this be? In practice, learning cannot be separated from tasks and the instructional context; the "selection of learning tasks and the hypotheses about the process of student learning are interdependent" (Simon & Tzur, 2004, p. 93). What children learn is sensitive to the context in which they learn it – a context that is constituted by many factors, including most immediately the types of instructional tasks and how teachers organize students' engagement with these tasks.

For example, in classrooms where part-whole tasks (Fig. 2a) dominate instruction on fractions, children learn to think about fractions in terms of counting parts rather than as magnitudes (Thompson & Saldanha, 2003). Students are likely to think about 5/8 as "5 out of 8" and of 8/5 as an impossible fraction. In classrooms where teachers have students solve and discuss equal

sharing tasks (Fig. 2b), children learn to think about fractions in terms of relationships between quantities and later in terms of a multiplicative relationship between numerator and denominator (Empson, Junk, Dominguez, & Turner, 2005; Empson & Levi, 2011). They are more likely to think of 5/8, for example, as 5 groups each of size 1/8, instead of "5 out of 8." In classrooms where teachers engage students in reasoning about multiplicative comparisons of measures (Fig. 2c), students learn to think about fractions as a ratio of measures (Brousseau, Brousseau, & Warfield, 2004; Davydov &Tsvetkovich, 1991; Steffe & Olive, 2010). Children learn to interpret 5/8 as a multiplicative comparison between 5 and 8. Both of these latter types of tasks – equal sharing and measuring – coupled with norms for engaging in tasks that put a premium on intellectual effort and agency (Hiebert & Grouws, 2007) – appear to constitute productive approaches to learning fractions.

How much pizza is left on the plate?
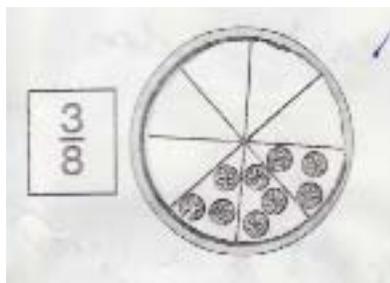
(a)

8 children want to share 10 candy bars so that each one gets the same amount. How many candy bars can each child have?
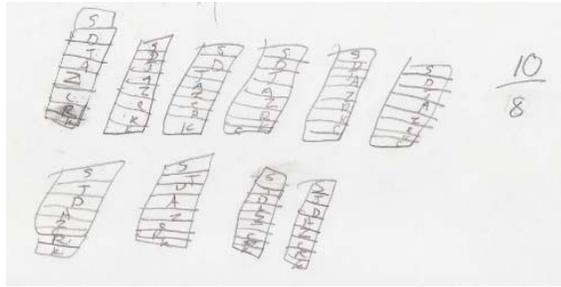
(b)

How long is B compared to A? A compared to B?

(c)

Figure 2. Examples of types of tasks to teach fractions: a) part-whole, b) equal sharing (with sample solution), and c) measurement

Within the context of documenting regularities and patterns in the development of children's thinking, however, it's important to recognize individual children's ways of reasoning and the significant contributions this reasoning could make to a group's learning. To return to my research on equal sharing, for example, we found that students frequently produced strategies for solving problems that were, from the perspective of a trajectory, "out of sequence" *and* presented rich learning opportunities for other students (e.g., Turner et al., in press). There was a progression in what students learned but "deviations" were consistent and numerous, and, I am suggesting, fruitful – not anomalies to be ignored but significant occurrences that teachers could use to advance everyone's learning.

Consider first a simple progression of strategies for equal sharing (Empson &

Levi, 2011). To figure out how much one person got if 8 people were sharing 6 burritos equally, a child using a basic strategy might draw all 6 burritos, decide to split each burrito into 8 pieces, and give each person 1 piece from each burrito for a total of 6 pieces. A more sophisticated strategy would involve imagining that each burrito could be split into 8 pieces and mentally combining those pieces to conclude that one person's share consisted of 6 groups of 1/8 burrito or 6/8 burrito. Ultimately, children come to the understanding that the problem can be represented by 6÷8, which is the same as 6/8.

Within this simplified progression, there are several other ways to solve the problem that do not fall into a sequence and do not appear as an inevitable consequence of development. These other strategies were a function of specific quantities in a problem as well as what tools children were using and children's prior knowledge. For example, a fifth grader solved the problem by reducing it to an equivalent ratio involving 1 1/2 burritos and 2 children, which she easily solved by finding half of 1 and half of 1/2 and combining the amounts (Fig. 3). Another fifth grader used a similar strategy, but used cubes to represent each quantity (8 total cubes for sharers, 6 total cubes for burritos), specifically highlighting the ratio character of the strategy. These strategies were appropriated by several children who saw them as more efficient and they provided an opportunity for the teachers to address concepts of fraction and ratio equivalence.
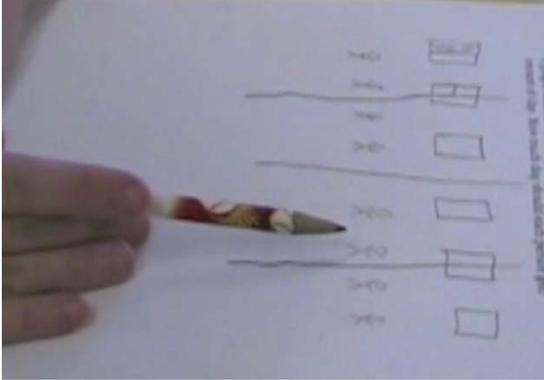
Figure 3. A solution for 6÷8 involving the equivalent ratios 3 for 4 and 1 1/2 for 2

As students' understanding develops and diversifies, they become more likely to see and make connections between their ways of thinking and different ways of thinking expressed by their fellow students. Making these connections enriches learners' understanding and cultivates their ability to recognize and pursue new avenues of reasoning independently of the teacher's direction and to monitor their thinking. The balance in instruction between supporting students' agentic initiative and aiming to instill specific conceptions can be difficult to manage. Indeed, some researchers have cautioned that representing learning as progressive sequences of content understanding could lead teachers to direct students through the sequences at the expense of allowing students to "express, test, and revise their own ways of thinking" (Lesh & Yoon, 2004, p. 206; Sikorski & Hammer, 2010). At the same time, other research suggests that, at the right level of abstraction, representations of the progressive development of students' understanding can enhance teachers' ability to respond to students' thinking in ways that open up or are generative of new possibilities (e.g., Franke et al. 2001).

In either case, it's important to recognize that research on learning in specific mathematics domains has a long history that, while concerned with progress, may not fit easily into the idea of a single trajectory.

Learning Trajectories Involve Specific Domains of Conceptual Development – So Their Reach May be Limited

Researchers have made the study of mathematics learning more tractable by focusing in particular on conceptual development in specific content domains, represented by sets of well-defined, interrelated tasks. Steffe and Olive's (2010) research on the development of fraction concepts and Clements and Sarama's (2009) research on children's understanding of measurement are examples of such an approach. This work, like a great deal of the research in mathematics education including my own, is informed by a Piagetian-like view of learning, if not in its emphasis on levels, then certainly in its emphasis on a conceptual trajectory, in which less sophisticated concepts give way to more sophisticated concepts. Because this work is based on children's thinking about specific types of tasks, its power lies in its capacity to inform teachers' use and interpretation of these tasks to foster students' conceptual development in a coherent unit of study (e.g., Fennema et al, 1996; Simon & Tzur, 2004).

However powerful, these kinds of portrayals of learning necessarily represent only one dimension or a small set of what we value as a field about mathematics and wish for students to learn. Learning is a multidimensional process, comprised of a variety of intertwined cognitive and social processes. In

particular, since the publication of *Everyone Counts* (National Research Council, 1989) and the *Curriculum and Evaluation Standards* (National Council of Teachers of Mathematics, 1988), mathematics educators have increasingly focused on teaching students to engage in practices such as posing and solving problems (Hiebert et al., 1996), constructing models (Lesh & Doerr, 2003), and making convincing arguments (Lehrer & Schauble, 2007) – that is, to do mathematics. Doing mathematics involves a complex and integrated set of content understanding *and* disciplinary practices (Bass, 2011; Kilpatrick, Swafford, & Findell, 2001) as well as the ability to monitor the interplay between these things (Schoenfeld, 1992).

The ability to engage in mathematical practices such as the ones above is as critical as content knowledge to a well-developed capacity to think mathematically, but it is less amenable to analysis in terms of sequences of development. For example, students engaged in mathematical modeling or problem solving may draw on multiple content domains and work collaboratively on tasks that have many possible resolutions such that the solutions they produce appear to follow no predictable trajectory over time. Examples of such tasks include creating simulations of disease spread (Stroup, Ares, & Hurford, 2005), optimizing the occupancy of a hotel during tourist season (Aliprantis & Carmona, 2003), and designing a template to generate a quilt pattern (Lesh & Doerr, 2003). These kinds of tasks and thinking practices pose considerable challenge for researchers seeking to codify and systematically

represent learning in terms of a trajectory, because of the variety of understanding and practices that students bring to bear in the their solutions.

Learning trajectories may be limited in what they can and cannot specify in terms of learning mathematics over time; and in particular, they may not be applicable to certain critical aspects of the mathematics curriculum. Catley, Lehrer, and Reiser (2005) recognized this potential limitation when they argued that "scientific concepts are never developed without participation in specialized forms of practice" and "concepts are contingent on these practices" (p. 4) – such as the ones listed in the Common Core Standards in Mathematics (2010). Among others, these practices include making sense of problems and persevering in solving them; using appropriate tools strategically; attending to precision; and looking for and making use of structure (CCSM, 2010, pp. 6-8). Most, if not all, current characterizations of learning trajectories do not address the practices that engender the development of concepts – although it's worth thinking about alternative ways to characterize curriculum standards and learning trajectories that draw teachers' attention to specific aspects of students' mathematical practices as well as the content that might be the aim of that practice.

What is a reasonable unit of students' mathematical activity for teachers to notice? If a unit is too small or requires a great deal of inference (e.g., a mental operation), then teachers in their moment-to-moment decision-making may not be able to detect it and respond to it; likewise if a unit is too broad or stretches over too long a period of time (e.g., "critical thinking"), teachers may not

recognize it when they are seeing it. The most productive kinds of units of mathematical activity would allow teachers to see and respond to clearly defined instances of student's thinking during instruction and to gather information about students' progress relative to instructional goals. For example, in research in elementary mathematics, *strategies* and *types of reasoning* are productive units because we know that teachers can learn to differentiate students' strategies and use what they learn about students' thinking to successfully guide instruction (e.g., Fennema, et al., 1996). Catley and colleagues (2005) proposed "learning performances" as a way to represent the "cognitive processes and associated practices linked to particular standards" (p. 5). Formative assessments that include a variety of points of access and possible solutions and that require students to engage in various mathematics practices could also yield rich information about students' understanding of and engagement in mathematics (cf., Aliprantis & Carmona, 2003; Lesh & Doerr, 2003). The important thing is to take into account the interplay of practices and content in students' learning over time.

Teaching is Integral to Learning and Learning Trajectories

Learning school mathematics depends on teaching. To support learning, teachers need be able to "understand, plan, and react instructionally, on a moment-to-moment basis, to students' developing reasoning" and coordinate these interactions with learning goals (Battista, 2010). Similarly, Daro and colleagues (2011) concluded that:

Teachers are going to have to find ways to attend more closely and regularly to each of their students during instruction to determine where they are in their progress toward meeting the standards, and the kinds of problems they might be having along the way. Then teachers must use that information to decide what to do to help each student continue to progress, to provide students with feedback, and help them overcome their particular problems to get back on a path to success. (Daro et al., 2011, p. 15)

We know very little about how teachers do these things, in contrast to what we know about children's learning, whether it falls under the rubric of learning trajectory research or not. As teachers interact with students and decide how to proceed, there are many types of decisions to be made – how to gather information about children's thinking, how to respond to it appropriately in the moment, how to design tasks that extend it, and even what to pay attention to. With the right tools, teachers have access to the most up-to-date information about each student, what they understand and are able to do, their disposition, their history, and so on, and can make decisions based on their own informed understanding of these things and their relationships. Good tools, such as formative assessment frameworks in particular, enhance this knowledge and support teachers to engage in the active, contingent process of creating instructional trajectories informed by knowledge of actual children's learning.

Further, learning mathematics in school takes work and depends fundamentally on interpersonal relationships of trust and respect, which cannot be designed into a tool or a list of learning goals. Teaching is a relational act and the relationship between the teacher and the student is at the center of students' learning in school (Gergen, 2009; Grossman & McDonald, 2008). These

relationships can have a profound effect on what students learn and how they come to see themselves.

In the face of what can seem like a tidal wave of top-down mandates, I suggest that we mathematics educators keep sight of the fact that teaching is driven essentially by interpersonal relationships and happens from the bottom up, beginning with the teacher and the student relating to each other and the content. We need to be sure that teachers are equipped with knowledge of the domain and its learning milestones without forgetting that both teachers and students are active agents in learning.

Closing Thoughts

"Clearly … the trajectories followed by those who learn will be extremely diverse and may not be predictable" (Lave & Wenger, 1991)

In choosing to focus on learning trajectories, we embrace a metaphor that, for all its appeal, implies that learning unfolds following a predictable, sequenced path. Everyone knows it is not that simple; researchers and educators alike acknowledge the complexity of learning. As Simon (1995) emphasized, learning trajectories are essentially provisional. We can think of them as the provisional creation of teachers who are deliberating about how to support students' learning and we can think of them as the provisional creation of researchers attempting to understand students' learning and to represent it in a way that is useful for teachers, curriculum designers, and test makers.

I firmly believe that a critical part of our mission as researchers is to produce something that is of use to the field and serves as a resource for teachers and curriculum designers to optimize student learning. No doubt this includes creating, testing, and refining empirically based representations of students' learning for teachers to use in professional decision-making and, further, investigating ways to support teachers' decision-making without stripping teachers of the agency needed to hypothesize learning trajectories for individual children as they teach. This focus would add a layer of complexity to our research on learning and invite us to think seriously about how to support teachers to incorporate knowledge of children's learning into their purposeful decision-making about instruction. Further, I suggest we consider, in the end, "Whose responsibility is it to construct learning trajectories?" (Steffe, 2004, p. 130). If we researchers can figure out how to supply teachers with knowledge frameworks and formative assessment tools to facilitate their work, teachers will be able to exercise this responsibility with increasing skill, professionalism, and effectiveness.

Because of the growing popularity of *learning trajectories* in education circles, it is worth thinking hard about the role of learning trajectory representations in teaching, and in particular, whether a learning trajectory can exist meaningfully apart from the relationship between a teacher and a student at a specific time and place. Simon's (1995) perspective on teaching and learning suggests not. As the field moves forward with research on learning trajectories and strive for

coherence in learning across the grades, I would like to remain mindful of both the affordances and constraints this particular type of representation offers for teachers and students alike.

References

Aliprantis, C. & Carmona, G. (2003). Introduction to an economic problem: A models and modeling perspective. In R. Lesh & H. Doerr (Eds.) *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching,* 255-264. Mahway, NJ: Erlbaum.

Bass, H. (2011). A vignette of doing mathematics: A meta-cognitive tour of the production of some elementary mathematics. *The Montana Mathematics Enthusiast,8*(1-2), 3-34.

Battista, M. (2010). Representations of learning for teaching: Learning progressions, learning trajectories, and levels of sophistication. In P. *Brosnan, D. B. Erchick, & L. Flevares* (Eds.), *Proceedings of the Thirty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* Vol. pp. 60-71. Columbus, OH: The Ohio State University.

Behr, M., Harel, G., Post, T. & Lesh, R. (1992). Rational number, ratio, and proportion. In D. Grouws (Ed.). *Handbook of research on mathematics teaching and learning* (pp. 296 -333). New York: Macmillan.

Brousseau, G., Brousseau, N., & Warfield, V. (2004). Rationals and decimals as required in the school curriculum Part 1: Rationals as measurements. *Journal of Mathematical Behavior, 23,* 1-20.

Carpenter, T. P. (1985). Learning to add and subtract: An exercise in problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 17-40). Hillsdale, NJ: Erlbaum.

Carpenter, T. P., Fennema, E., Franke, M., Levi, L. & Empson, S. B. (1999). *Children's Mathematics: Cognitively Guided Instruction.* Portsmouth, NH: Heinemann.

Carpenter, T. P., Fennema, E., Peterson, P., Chiang, C. & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*(4), 499 – 531.

Carpenter, T. P. & Moser, J. (1983). The acquisition of addition and subtraction concepts. In R. Lesh & M. Landau (Eds.), *The acquisition of mathematical concepts and processes* (pp. 7 - 44). New York: Academic.

Catley, K. M., Lehrer, R., & Reiser, B. J. (2005). Tracing a prospective learning progression for developing understanding of evolution. Paper commissioned by the National Academies Committee on Test Design for K-12 Science Achievement.

Clements, D. & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6*(2), 81-89.

Clements, D. &Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach.* New York: Routledge.

Common Core State Standards Initiative (2010). *Mathematics.* Washington, DC: Council of Chief State School Officers & National Governers Association

Center for Best Practices. Downloaded 8/15/10 from http://www.corestandards.org/the-standards/mathematics.

Confrey, J., Maloney, A., Nguyen, K., Mojica, G., & Myers, M. (2009). Equipartitioning/splitting as a foundation of rational number reasoning. In M. Tzekaki, M. Kaldrimidou & C. Sakonidis (Eds.) *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education,* Vol. 1. Thessaloniki, Greece: PME.

Daro, P., Mosher, F., & Cocoran, T. (2011). *Learning trajectories in mathematics: A foundation for standards, curriculum, assessment, and instruction.* Philadelphia, PA: Consortium for Policy Research in Education.

Davydov, V. V. & Tsvetkovich, Z. H. (1991). The object sources of the concept of fractions. In L. Steffe (Ed.) *Soviet studies in mathematics, vol. 6: Psychological abilities of primary school children in learning mathematics* (J. Teller, Trans.). Reston, VA: National Council of Teachers of Mathematics. (Original work published in 1969)

DiSessa, A. (2000). *Changing minds.* Cambridge, MA: MIT.

Driver, R. (1989). Students' conception and learning of science. *International Journal of Science Education, 11*(5), 481-490.

Duncan, R. & Hmelo-Silver, C. (2009). Learning progressions: Aligning curriculum, instruction, and assessment. *Journal of Research in Science Teaching, 46*(6), 606-609.

Empson, S. B., Junk, D., Dominguez, H., & Turner, E. E. (2005). Fractions as the coordination of multiplicatively related quantities: A cross-sectional study of children's thinking. *Educational Studies in Mathematics, 63*, 1-28

Empson, S. B. & Levi, L. (2011). *Extending children's mathematics: Fractions and decimals.* Portsmouth, NH: Heinemann.

Fennema, E., Carpenter, T. P., Franke, M., Levi, L., Jacobs, V. & Empson, S. B. (1996). Mathematics instruction and teachers' beliefs: A longitudinal study of using children's thinking. *Journal for Research in Mathematics Education, 27*(4), 403-434.

Franke, M. L., Carpenter, T. P., Levi, L. & Fennema, E. (2001). Capturing teachers' generative change: A follow-up study of professional development in mathematics. *American Educational Research Journal, 38*, 653-689.

Fuson, K. (1992). Research on whole number addition and subtraction. In D. Grouws (Ed.), *Handbook of research on mathematics education.* New York: Macmillan.

Gelman, R. & Gallistel, C. R. (1986). *The child's understanding of number.* Boston: Harvard University Press.

Gergen, K. J. (2009). *Relational being: Beyond self and community.* New York: Oxford.

Gibb, E. G. (1956). Children's thinking in the process of subtraction. *The Journal of Experimental Education, 25*(1), 71-80.

Greeno, J. & Middle-School Math through Applications. (1998). The situativity of knowing, learning, and research. *American Psychologist, 53*(1), 5-26.

Grossman, P. & McDonald, M. (2008). Back to the future: Directions for research in teaching and teacher education. *American Educational Research Journal, 45*(1), 184-205.

Hackenberg, A. (2010). Students' reasoning with reversible multiplicative relationships. *Cognition and Instruction, 28*(4), 383-432.

Hammer, D. & Schifter, D. (2001). Practices of inquiry in teaching and research. *Cognition and Instruction, 19*, 441-478.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H., Olivier, A. & Wearne, D. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational Researcher, 25*(4), 12-21.

Hiebert, J. & Grouws, D. (2007). The effects of classroom mathematics teaching on students' learning. In F. Lester (Ed.), *Second handbook of research on teaching and learning mathematics, Vol. I* (pp. 371-404). Reston, VA: National Council of Teachers of Mathematics.

Kilpatrick, J., Swafford, J. & Findell, B. (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.

Lave, J. & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation.* New York: Cambridge.

Leach, J. (1998). Rosalind Driver (1941 – 1997): A tribute to her contribution to science education. In M. Meheut & G. Rebmann (Eds.) *Proceedings of the 4th European Science Education Summerschool,* Marly-le-Roi, France.

Lehrer, R. & Schauble, L. (2007). A developmental approach for supporting the epistemology of modeling. *Modeling and applications in mathematics education, vol. 10, part 3, sec. 3.1,* 153-160.

Lesh, R. & Doerr, H. (Eds.) (2003). *Beyond constructivism: Models and modeling perspectives on mathematics teaching, learning, and problem solving.* Mahwah, NJ: Lawrence Erlbaum Associates.

Liu, X. & Tang, L. (2004). The progression of students' conceptions of energy: A cross-grade, cross-cultural study. *Canadian Journal Science, Mathematics, and Technology Education, 4*(1), 43-57.

National Council of Teachers of Mathematics. (1988). *Curriculum and evaluation standards for school mathematics.* Reston, VA: author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: author.

National Research Council. (1989). *Everyone counts: A report to the nation on the future of mathematics education.* Washington, DC: National Academy Press.

National Research Council. (2001). *Knowing what students know: The science and design of educational assessment.* Pelligrino, J., Chudowsky, N., and Glaser, R., (Eds.). Washington, DC: National Academy Press.

Resnick, L. & Ford, W. (1981). *The psychology of mathematics for instruction.* Hillsdale, NJ: Erlbaum.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. Grouws (Ed.), *The handbook of research in mathematics teaching and learning* (pp. 334-370). New York: Macmillan Publishing Company.

Shapiro, B. (2004). Developing understanding: Research on science learning and teaching over time. *Canadian Journal Science, Mathematics, and Technology Education, 4*(1), 1-6.

Sikorski, T-R. & Hammer, D. (2010). A critique of how learning progressions research conceptualizes sophistication and progress. In Gomez, K., Lyons, L., & Radinsky, J. (Eds.) *Learning in the Disciplines: Proceedings of the 9th International Conference of the Learning Sciences (ICLS 2010) - Volume 2, Full Papers,* 1032-1039. International Society of the Learning Sciences: Chicago IL.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.

Simon, M. & Tzur, R. (2004). Explicating the role of tasks in conceptual learning: An elaboration of a hypothetical learning trajectory. *Mathematical Thinking and Learning, 6*(2), 91-104.

Smith, C. L., Wiser, M., Anderson, C. W., & Krajcik, J. (2006). Implications of research on children's learning for standards and assessment: A proposed

learning progression for matter and the atomic-molecular theory. *Measurement*, *14*(1&2), 1-98.

Steffe, L. & Olive, J. (2010). *Children's fractional knowledge.* New York: Springer.

Streefland, L. (1991). *Fractions in realistic mathematics education.* Boston: Kluwer.

Thompson, P. & Saldanha, L. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, W. G. Mateo, & D. Schifter (Eds.) *A research companion to principles and standards for school mathematics* (pp. 95-113). Reston, VA: National Council of Teachers of Mathematics.

Turner, E. E., Dominguez, H., Maldonado, L., & Empson, S. B. (in press). English learners' participation in mathematical discussion: Shifting positionings, dynamic identities. In R. Guttiérez, *Equity, identity, and power.* Special issue of *Journal for Research in Mathematics Education*.