The Mathematics Enthusiast

Volume 8
Number 3 *Number 3*

Article 7

7-2011

Can Dual processing Theories of Thinking inform conceptual learning in Mathematics?

Ron Tzur

Can Dual Processing Theories of Thinking Inform Conceptual Learning in Mathematics?

Ron Tzur[1]
University of Colorado Denver

Abstract: *Concurring with Uri Leron's (2010) cross-disciplinary approach to two distinct modes of mathematical thinking, intuitive and analytic, I discuss his elaboration and adaptation to mathematics education of the cognitive psychology dual-processing theory (DPT) in terms of (a) the problem significance and (b) features of the theory he adapts. Then, I discuss DPT in light of a constructivist stance on the inseparability between thinking and learning. In particular, I propose a brain-based account of conceptual learning—the Reflection on Activity-Effect Relationship (Ref*AER) framework—as a plausible alternative to DPT. I discuss advantages of the Ref*AER framework over DPT for mathematics education.*

Key Words: constructivism, reflection, anticipation, activity-effect, dual processing, heuristic-and-bias, intuitive, analytic, brain.

This theoretical paper extends an article (Tzur, 2010b) in which I discussed Uri Leron's (2010) plenary address during the last annual meeting of PME-NA. Being invited to discuss his paper re-acquainted me with the inspiring empirical and theoretical work that he and his colleagues were conducting in the last two decades (Leron & Hazzan, 2006, 2009). It also provided me with an important window into literature outside mathematics education (e.g., cognitive

[1] ron.tzur@ucdenver.edu

The Mathematics Enthusiast, **ISSN 1551-3440, Vol. 8, no.3**, pp.597-636

psychology), which I consider as both thought provoking and relevant to our field. Last but not least, after reading his paper(s) I realized how naturally his approach linked with recent efforts in which I have been participating—to relate mathematics education research with cognitive neuroscience (brain studies). I concur with Leron's belief that bridging between intuition and analytical thinking can contribute to optimizing student mathematical understandings and am delighted to provide my reflections on this endeavor.

In itself, the main thesis that human thinking and judgment (or rationality) consist of two qualitatively distinct modes is not new to mathematics education. Skemp's (1979) seminal work has already articulated and linked both modes, which he termed *intuitive* and *reflective* intelligences. To the best of my knowledge, Skemp's constructivist theory evolved independently of the commencement of the 'heuristic and bias' approach (Kahneman, Slovic, & Tversky, 1982; Kahneman & Tversky, 1973; Tversky & Kahneman, 1973, 1983). Moreover, I believe that, in mathematics education, this distinction can be traced back to Dewey's (1933) notion of *reflective* thought (contrasted with *unconscious* mental processes), and to Vygotsky's (1986) notion of ZPD and his related distinction between *spontaneous* and *scientific* concepts.

However, two novelties in Leron's contribution seemed very useful for mathematics education. First, his review of cognitive psychology literature pointed out to empirical studies in which a dual view of thinking processes has been robustly elaborated on (Evans, 2006; Kahneman & Frederick, 2002;

Stanovich, 2008) and 'mapped' onto corresponding, differentiated brain regions (Lieberman, 2003, 2008). Thus, a similarly important and timely direction, of linking mathematics education with brain studies (Medina, 2008), is supported by relevant findings from cognitive psychology (see Section 2). Second, he reported on studies (Leron & Hazzan, 2006, 2009) informed by DPT that demonstrated its applicability to our field, including articulation of instructional goals and design criteria. Next, I further discuss both contributions.

1. SIGNIFICANT QUESTIONS! USEFUL THEORY?

1.1 Significance of DPT

Like many teachers of mathematics and mathematics educators, Leron and his colleagues noticed a phenomenon that seemed to equally puzzle researchers in other fields. Quite often, researchers observing people's solutions to various problems framed them as recurring faulty judgments (reasoning processes and conclusions). Examples of such solutions abound in the aforementioned papers; I will present three of my own below. Interestingly, studies of such examples in the '80s and '90s fueled a debate about human rationality that quite tightly conjoined epistemology and psychology (Goldman, 1994; Kim, 1994; Nisbett & Ross, 1994; Quine, 1994). For example, alluding to computational complexity, Cherniak (1994) considered 'ideal' (normative) rationality as intractable. Instead, using the example of mathematicians working on unfeasibly long proofs he proposed 'minimal' rationality, owing much of its functionality to 'quick-and-dirty' heuristics that evade practical (mental) paralysis.

As I see it, addressing this puzzling phenomenon and significant problem by mathematics educators is more pressing and weighty than by cognitive psychologists and/or economists. As challenging as it might be to solidly explain why/how the human mind produces erroneous judgments, in those other fields it may suffice. The works of Leron (Leron, 2010; Leron & Hazzan, 2006, 2009) and others (Katz & Katz, 2010; Viholainen, 2008) indicate, however, that in our field such an explanation is but a start. In this sense, Leron made two key contributions: (a) *clarifying a goal* for student and teacher learning—closing the rather prevalent gap between intuitive and analytic reasoning, and (b) *explicating mathematics educators' duty* to figure out ways of thinking about, designing, and implementing teaching that can foster student development of and disposition toward analytic reasoning. To these ends, Leron identified four vital questions for mathematics educators:

i) What differentiates among those who solve problems correctly and incorrectly, that is, why do the latter fail to use analytic reasoning whereas the former do so?
ii) Using the above as a basis—how can we explain observations about the '*cueing impact*' of changes in a problem format or context have on correctly solving a problem, and what does this entail for instructional design?

iii) When using puzzling problem situations in our teaching (e.g., earth circumference), what strategies can be used to effectively capitalize on students' "Aha" moments that follow those puzzlements?

iv) How may we design instruction to promote (a) students' (and teachers') *awareness* of the potential use of improper intuitive reasoning and (b) disposition toward constant activation of analytic reasoning to override the faulty intuitions (i.e., resist and critique the intuitive)?

1.2 Dual Processing Theory (DPT): Is It Useful for Mathematics Education?

To articulate what purposes DPT can serve in mathematics education, I first briefly present its key features by alluding to one of Leron's examples and three of mine (to keep it short, language does not precisely replicate the original problems).

A. Adults with college education were asked: Two items cost $1.10; the difference in price is $1. How much does each item cost? (Over 50% submit to impulse and respond: $1 & $0.10)

B. In the elevator, the 7th floor button is already lit. A person who also wanted that floor gets on the elevator and, though seeing the lit button, pressed it again.

C. Grade 3 students were asked to reason which side will a next (fair) coin flip show, 'Head or Tail', after it showed 4 'Heads' in a row. Roughly 50% said 'Head', because it's always been the case; the rest said 'Tail', because it could not always be 'Head'. Virtually no one reasoned 50-50, and that previous flips were irrelevant.

D. As a Sudoku enthusiast, I made two careless errors while solving a 'black-belt' puzzle (see Figure 1). In the puzzle on the left (1a), I considered and *almost* wrote '4' in the bottom-middle square while transposing the digits to a different cell and ignoring the vertical 'conflict'. Two minutes later, while solving the puzzle on the right (1b), I actually committed a similar error (considering only vertical '9' and writing the small '9' digits where the top one conflicts with a horizontal, given '9').

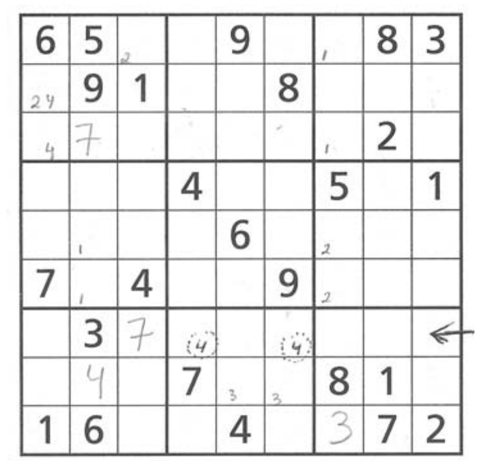

Figure 1a. Processing error not committed; almost placing '4' in mid-lower left cell (transpose row, ignore vertical)

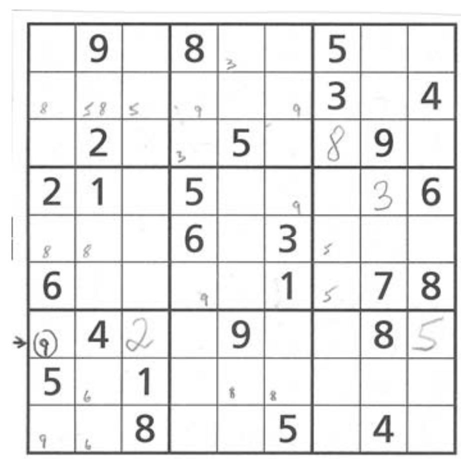

Figure 1b. Same error repeated & committed; '9' in left-lower cell (checked for vertical only)

The key insight about human thinking, which led to different variants of DPT, is that responses to vastly diverse problems, faulty or correct, may all share a common root. As implied by its name, the basic tenet of DPT is that two different modes of brain processing are at work (Evans, 2003, 2006; Stanovich, 2008; Stanovich & West, 2000). The first mode, '*intuitive* reasoning' (or 'heuristic'), is considered evolutionary more ancient and shared with animals. It is characterized by automatic (reflexive, sub-conscious), rapid, and parallel in nature processing, with only its final product available to consciousness. The second mode, '*analytic* reasoning', is evolutionary recent and considered unique to humans. It is intentional (reflective, conscious), relatively slow, and sequential in nature. The principal roles attributed by DPT to the second mode are monitoring, critiquing, and correcting judgments produced by the first mode. Said differently, the second mode of processing suppresses/inhibits default responses; it serves as a failure-prevention-and-correction mental device. As Leron (2010) pointed out, some cognitive psychologists refer to the intuitive mode as System-1 (S1) and to the analytic mode as System-2 (S2). They further emphasize that, quite often, both systems work in tandem, which basically means that S1 produced a proper judgment that S2 did not need to correct.

A second tenet of DPT is that, in essence, faulty responses given by problem solvers reflect failure of their analytic processes to prevent-and-correct output from their intuitive processes. A key, corresponding assumption that seems to be taken-as-shared by most proponents of DPT and to underlie the notion of '*rational* judgment/actions', is that at any given problem situation a person intends to accomplish a correct solution that serves her or his own purposes (e.g., economic benefit, academic success, etc.). In the four examples above, a person would like to properly solve the problems but, as DPT explains, the fast-reacting insuppressible S1 tends to "hijack" the subject's attention and thus yields a non-normative answer (Leron, 2010). Thus, in Example *A*, S1 'falls prey' to the cost of one item ($1) being equal to the difference. In Example *B*, S1 brings forth and directs execution of the planned action (get on elevator, identify-and-press 7th floor button) before S2 could re-evaluate necessity in the circumstances. In Example *D* (Figure 1b), S1 directed my actions to place the digits with only partial checking before S2 detected that partiality. This occurred *soon after* I actually thought of placing the '4' where it is shown in Figure 1a, but then consciously (S2 override) avoided this error. Example *C* (predicting results of a coin flip) was selected to highlight a few hurdles with DPT, particularly the impact of problem solvers' cognitive abilities on their solutions (Stanovich & West, 2000). Clearly, what to an observer would appear as *non-normative* responses (e.g., it's most likely to be 'Head') was the proper response within the children's cognitive system—a case of S1 and S2 working in tandem for the reasoner, though erroneously for an observer.

Before turning to hindrances I find in DPT, a few more comments seem noteworthy. Evans (2006) highlighted a key distinction to keep in mind—between dual *processes* and dual *systems*. This is important for mathematics education particularly because, as he asserted, dual system theories are too broad. Thus, he asserted the need to elaborate specific dual-reasoning accounts at an intermediate level that explains solutions to *particular tasks*. To me, his goal (particular task) seems primary whereas the means (dual accounts, or singular, or triple) seems secondary.

This leads to my second comment—the need to pay particular attention to solution processes—and kinds of problem situations—in which analytic/reflective processes successfully monitor and correct S1's 'run' *before* reaching and submitting to the latter's judgment. For example, when I first read Example *A* in Leron's paper, I immediately identified the task as 'inviting' the faulty conclusion. I also immediately noticed my conscious, pro-active 'flagging' of this tendency and, consequently, selected an analytic process instead. This mental adjustment happened before I even calculated the faulty difference (90 cents), precisely the desired state of affairs indicated in question #*iv* above. My case indicates the need for precisely analyzing the way intuitive and analytic processes *interact*. Initial forms of DPT assumed sequential operation, where outcomes of intuitive processes (or S1) serve as input for analytic processes only when/if S2 identified S1's output as a faulty response. Recently, the possibility for parallel processing of both modes was postulated, including the idea that they often compete for the immediate or final judgment in a given problem situation (Evans, 2006). To further theorize such interaction, Evans suggested 3 principles: (a) *singularity*—epistemic mental models are generated and judged one-at-a-time, (b) *relevance*—intuitive (heuristic) processes contextualize problems to maximize relevance to the person's current goals, and (c) *satisficing*—analytic processes tend to accept intuitive judgments unless there is a good reason to reject and override them. While essential, it seems that these principles fall short of accounting for how I solved Example *A*.

My last comment refers to factors that were found to make a difference in ways groups of people, or even an individual, solve particular problems. Stanovich and colleagues (2008; Stanovich & West, 2000) provided a good review of those. Here, I refer to a critical factor for mathematics education that was highlighted in Leron's (2010) address, namely, the impact of problem format ('packaging') on suppression of intuitive judgments. A substantial portion of Leron's work, which I see as a major contribution to our field, focused on the design of bridging tasks that are more likely to trigger what he considered solvers' available analytic processes. These tasks, in turn, enabled student solutions of the mathematically congruent tasks that were difficult to unpack without such bridging. This indirect allusion to assimilatory conceptions of those for whom bridging is required points to a hindrance.

From a constructivist perspective, a major theoretical and practical hindrance I find in DPT is the unproblematic application of an observer's frame of reference—considered as 'normative'—to the evaluation of people's responses—considered as 'rational' (or not, or partial). In essence, if the 'same' task is solved differently by people of different cognitive abilities (the observing researchers included), and if many who failed on a structurally identical task can solve a bridging task (and later also the failed one), then what a problem solver brings to the task must be explicitly distinguished from the observer's cognitive toolbox. Simply put, the presence of two cognitive frames of reference is glossed over by DPT's equating of normative with rational (for more about this, see Nisbett & Ross, 1994).

Theoretically, and crucial for mathematics education, what this lack of distinction fails to acknowledge is both the different interpretation(s) of a task and different mental activities available to the observed person for solving it. That is, it fails to acknowledge the core construct of *assimilation* (Piaget, 1980, 1985; von Glasersfeld, 1995). Recent research in cognitive psychology did point out to possible differences between observer and observed interpretations (Stanovich & West, 2000), but the key theoretical implication of those findings—simultaneously addressing two frames of reference—did not seem to follow. In my view, distinguishing the observer (Roth & Bautista, 2011; Steffe, 1995; von Glasersfeld, 1991) and using assimilation as a starting point is necessary in our field in order to move beyond cognitive psychology's focus on thinking and reasoning into accounts of learning as a conceptual advance that can be observed, and fostered, in other people's minds. And, as Skemp (1979) so eloquently asserted, for a mathematics education theory of teaching to be useful—at its core one must articulate learning as a process of cognitive change in what the learner already knows.

Practically, overlooking learners' available conceptions when analyzing their solutions, correct or faulty, precludes the powerful design of bridging tasks demonstrated in Leron's (2010) paper. Indirectly, both the specific features of those tasks (e.g., the need to cue for a nested sub-set, or steps to 'see' the invariant length of string-around-earth when different shapes increase) and the rationale and criteria he provided for introducing those features (e.g., make the problem *accessible* to the *solver's intuition*), draw on conjectured inferences about how a person may interpret and solve the alternative tasks. That is, such tasks require inferences into students' existing (assimilatory) conceptions. This leads to the discussion of DPT's core hindrance.

2. A CONSTRUCTIVIST LENS ON DPT: 'BRAINY' MATHEMATICS EDUCATION

2.1 Taking Issue with DPT

As a constructivist, I adhere to the core premise common to Piaget's (1970, 1971, 1985), Dewey's (Dewey, 1902; 1949), and Vygotsky's (1978, 1986) grand theories, that knowing (thinking, reasoning) cannot be understood apart from the 'historical process' in which one's knowing evolved. This premise entails my twofold thesis about hurdles in adopting and adapting DPT to mathematics education. First, a sole focus on normative and faulty modes of thinking/reasoning in mathematics or other domains (aka cognitive psychology), falls short of the theoretical accounts needed to intentionally foster optimal student (and teacher) understandings. Second, although DPT can inform our work, mathematics education already has frameworks that interweave articulated accounts of knowing, coming to know (learning), and teaching (Dreyfus, 2002; Dubinsky & Lewin, 1986; Hershkowitz, Schwarz, & Dreyfus, 2001; Pirie & Kieren, 1992, 1994; Sfard, 1991, 2000; Steffe, 1990, 2010; Tall & Vinner, 1981; Thompson, 2002, 2010; Thompson, Carlson, & Silverman, 2007). As I shall discuss below, one framework that my colleagues and I have been developing—reflection on activity-effect relationship (*Ref*AER*)—seems to (a) singularly resolve issues of faulty/normative *reasoning* and of *conceptual learning* (with or without teaching) and (b) explain different modes of thinking without alluding to 2 systems (or distinct processes). Moreover, the *Ref*AER* framework is supported by and gives support to cognitive neuroscience models of the brain. Due to space limitations, the brief exposition below makes wide use of references to comprehensive versions. I begin by listing seven critical questions for mathematics education that Leron's work and accounts of DPT raised, and a framework such as *Ref*AER* needs to address:

1. Why does the mental system of some people make an error (e.g., selects $1 and 10 cents in the price example *A*) whereas other people focus also on the difference? Unless one considers solvers' assimilatory conceptions, this question (and 2-4 below) cannot be resolved by DPT assumptions that S2 has no direct access to the perceived information or that S2 selects accessible instead of relevant information.
2. When a person's response is non-normative, is it a case of (a) having the required conceptions but failing to trigger them (e.g., Sudoku and

elevator examples), (b) having a rudimentary form of those conceptions that require explicit prompting (e.g., sub-set in Leron's (2010) bridging task; renegotiating the difference aspect in the price problem and/or making the numbers more 'difficult'), or (c) not having a conception for monitoring S1 (e.g., my next coin-flip example and the original medical base-rate example in Leron's paper)? And how can we distinguish among these three cases?

3. How does S2, which failed to monitor S1 in a specific task, become capable of doing so? Is the process of learning different for each of the three cases above?
4. How do new monitoring capacities learned by S2 'migrate' to S1 (become automatic)?
5. What is the source of learners' surprise (e.g., string-around-earth example), how may it be linked to learning, and how might teaching capitalize on this?
6. What role do specific examples play in learning (by S2 and/or S1)?
7. Can we explain why particular bridging tasks promote some learning in some students but not others, and provide explicit ideas for changing them in the latter case?

2.2 A Brain-Based Model of Knowing and Learning

In recent years, a few cross-disciplinary meetings among cognitive neuroscientists and mathematics educators took place. One of those (Vanderbilt, 2006) focused on the design of tasks that (a) reveal difficult milestones in mathematics and (b) can be examined at the brain level (e.g., fMRI). Using the *Ref*AER* framework of knowing and learning (Simon & Tzur, 2004; Simon, Tzur, Heinz, & Kinzel, 2004; Tzur, 2007; Tzur & Simon, 2004; Tzur, Xin, Si, Woodward, & Jin, 2009), I presented fractional tasks to the group. This presentation, and the fertile dialogue with brain researchers that ensued, led to an elaborated, brain-based *Ref*AER* account (Tzur, accepted for publication) that seems highly consistent with DPT studies of the brain (Lieberman, 2003, 2008).

Briefly, *Ref*AER* depicts knowing (having a conception) as *anticipating* and *justifying* an *invariant relationship* between a single (goal-directed) activity-sequence the mental system executes at any given moment (Evans' *Singularity* principle; see also Medina, 2008), potentially or actually, and the effect it must bring forth. Learning is explained as transformation in such anticipation via two basic types of reflection. Reflection *Type-I* consists of ongoing, automatic comparison the mental system executes continually *between the goal* it sets for the activity-sequence and *subsequent effects* produced and noticed. As Piaget (1985) asserted, the internal global goal (anticipated effect) serves as a regulator of the execution for both interim effects and the final one (Evans' *Relevance* principle) (see also Stich, 1994). The effects either match the anticipation or not (Evans' *Satisficing* principle). By default, the mental system runs an activity-sequence to its completion as determined by the goal (e.g., the elevator example). Yet, the execution may stop earlier if (a) the goal detects unanticipated sub-effects (e.g., Sudoku example in Figure 1a) or (b) a different goal became the regulator, including possibly a sub-goal within the activity-sequence overriding the global goal. Reflection *Type-II* consists of comparison *across (mental) records of experiences*, each containing a linked, re-presented bit of a 'run' of the activity and its effect (*AER*), sorted as match or no-match. Critically, Type-II reflection does not happen automatically—the brain may or may not execute it. The recurring, invariant *AER* across those experiences are linked with the situation(s) in which they were found anticipatory of the proper goal and registered as a new conception.

Accordingly, *Ref*AER* postulates that the construction of a new conception proceeds through two stages. The first, *participatory*, necessitates reflection Type-I and is marked by an anticipation that a problem solver can access *only when and if somehow prompted* for the novel, provisional *AER* (Tzur & Lambert, in press, linked this stage with the Zone of Proximal Development—ZPD). The second, *anticipatory* stage necessitates reflection Type-II and is marked by independent, spontaneous bringing forth, running, and possibly justifying the novel anticipation. It should be noted that although developed independently, *Ref*AER* is consistent with Skemp's (1979) theory; the reflection types and stage distinctions extend his work.

To link the *Ref*AER* framework with brain studies, I separated and 'distributed' von Glasersfeld's (1995) tripartite notion of scheme—situation, activity, and result—across three major neuronal systems in which they are postulated to be processed. The assumption regarding both knowing and learning is that the fundamental unit of analysis in the brain is not a single synaptic connection or a neuron (Hebb, 1949, cited in Baars & Gage, 2007; Crick & Koch, 2003; Fuster, 1997, 2003). To stress neuronal 'firing' in the brain and the life-long growth, change, and decay of neuronal networks (Medina, 2008), I use the term Synapse Inhibition/Excitation Constellation (*SIEC*)—any-size aggregate of synapses of connected neurons that, once 'firing' and updating, forms a stable pattern of activity (Baars, 2007b). The roles and functions of *SIECs* are described in terms of the three neuronal networks where they may be activated (Baars, 2007a): a 'Recognition System' (RecSys), which includes the sensory input/buffer and various long-term memories; a 'Strategic System' (StrSys), which includes the Central Executive; and an 'Engagement-Emotive System' (EngSys). Within these networks, solving a problem, as well as learning through problem solving, is postulated as follows (indices in the diagram correspond to those in the text below):

1. Solving a problem begins with assimilating it via one's sensory modalities into the *Situation* part of an extant scheme in the RecSys. This *SIEC* is firing and updating until reaching its activity pattern

(recognizing state), and activates firing and updating of a Goal *SIEC* in the StrSys.

2. A Goal *SIEC* is set in the StrSys as a desired inhibition-excitation state that *regulates* the *execution* and *termination* of an activity sequence. The goal *SIEC* also triggers:
   a. Corresponding *SIECs* in the EngSys that set the desirability of the experience and the sense of control the learner has over the activity (McGaugh, 2002; Medina, 2008; Tzur, 1996; Zull, 2002). These were found linked to activity in the anterior cingulate cortex (Bush, Luu, & Posner, 2000; Lieberman, 2003, 2008).
   b. A temporary auxiliary *SIEC* checks if an activity has already been partly executed and can thus be resumed. If its output is 'Yes', it re-triggers the *AER's* execution in the StrSys from the stopping point (go to #4); if 'No,' it triggers the Goal *SIEC* to trigger #3 below.

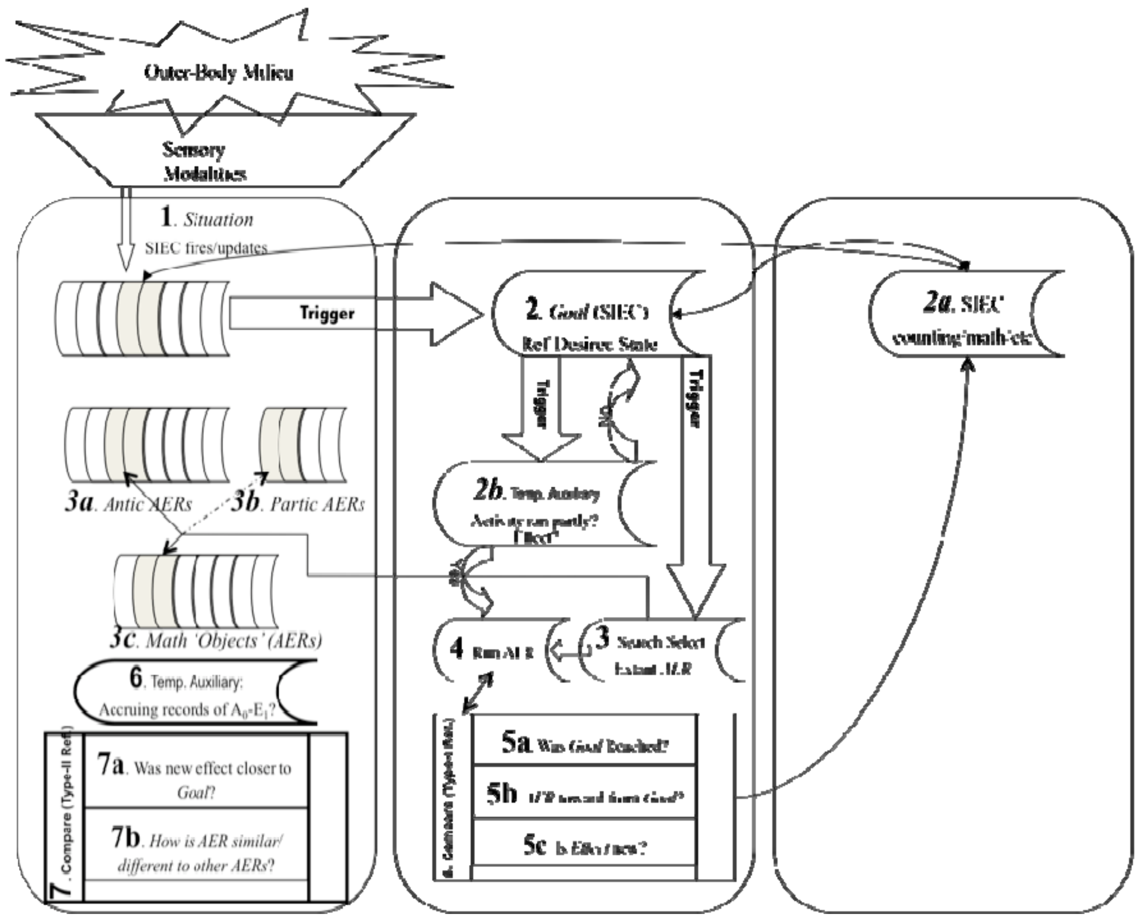

Figure 2. Brain problem solving and learning processes

3. A *SIEC* responsible for searching-and-selecting an available *AER* is triggered by the Goal *SIEC*. The search operates on three different long-term memory 'storages' of *SIECs* (3a, 3b, 3c below). Using a metaphor of 'road-map', Skemp (1979) explained that, within every universe of discourse (e.g., math, economy), the 'path' from a present state to a goal state may consist of multiple activity-sequences, among which one that is eventually executed is selected (see also multiple-

trace theory in Nadel, Samsonovich, Ryan, & Moscovitch, 2000). Searched and selected *AERs* include:

a. Anticipatory *AERs* – a mental operation carried out and its anticipated effect;
b. Participatory *AERs* that the learner is currently forming and can thus be called up only if prompted, as indicated by the dotted arrow;
c. Mental (e.g., mathematical) 'objects,' which are essentially anticipatory *AERs* established and encapsulated previously (e.g., 'number' is the anticipated effect of a counting operation).

4. Once an operation and an 'object' *AERs* were selected, the brain executes them while monitoring progress to the goal via a meta-cognitive *SIEC* in the StrSys responsible for Type-I reflections. Skemp's (1979, see ch. 11) model articulates this component in great details, including how it can be carried out automatically (intuitive) and/or reflectively (analytic). This goal-based monitoring component seems compatible with Norman and Shallice's (2000) model of schema activation, Corbetta and Sulman's (2002) notion of 'circuit breaker', and Kalbfleisch, Van Meter, and Zeffiro's (2006) identification of brain internal evaluation of response correctness. Mathematical operations are mainly activated in the Intraparietal Sulcus (IPS, see Nieder, 2005).

5. The execution of the selected *AER* is constantly monitored by Type-I reflection to determine 3 features:
    a. Was the learner's goal, as set in *SIEC* 2a, met?
    b. Is the *AER* execution moving toward or away from the goal (see McGovern, 2007 for relevant emotions)?
    c. Is the final effect of the executed portion of the *AER* different from the anticipated, set goal? Goldberg and Bougakov (2007) suggested that this is a function of prefrontal cortex (PFC).

   Each feature (5a, 5b, 5c) can stop the currently executed *AER* (e.g., seeing the lit elevator button halts the process leading to pressing it again). If the output of 5c is 'No', that 'run' of the *AER* is registered as another record of experience of the existing scheme (see Zull, 2002). Symbolically, such no-novelty can be written: Situation$_0$-Goal$_0$-AER_0 (Tzur & Simon, 2004). If the output is 'Yes', symbolized as Situation$_0$-Goal$_0$-AER_1, a new conceptualization may commence (see next). This perturbing state of the mental system (von Glasersfeld, 1995), seems related to anticorrelations of brain networks (Fox, et al., 2005).

6. Type-II reflective comparisons *may* then operate on the output records of Type-I reflection. Whenever the output of Type-I question 5c is 'Yes,' the brain *updates a new SIEC* for that recently run *AER* and stores it in a temporary auxiliary in the *RecSys* (symbolized A_0-E_1, or AER_1).

Each repetition of the solution process for which the output of 5c is 'Yes' adds another such record to the temporary auxiliary.

7. The accruing records of temporary AER_1 (novel) compounds are continually monitored by Type-II reflective comparison *SIEC* in terms of two features:
   a. Is the effect of the new *AER* (E_1) closer to or further away from the Goal?
   b. How is the new AER_1 similar to or different from the extant anticipatory and/or participatory *AERs* in the *RecSys*? This aspect of Type-II reflection seems supported by Moscovitch et al.'s (2007) articulation of the constant interchanges between MTL and PFC.

The output of recurring Type-II reflective comparisons is a new *SIEC* (AER_1). The anticipatory-participatory stage distinction implies that a new *SIEC* can initially be accessed by the Search-an-Select *SIEC* (#3) only if the learner is prompted for the activity (A_0), which generates the noticed effect (E_1) and thus 'opens' the neuronal path to using AER_1 in response to the triggering situation ($Situation_0$). Over time, Type-II comparisons of the repeated use of AER_1 for $Situation_0$ produces a new neuronal pathway from the $Situation_0$ *SIEC* to the newly formed AER_1, that is, to the construction of a new, directly retrievable, anticipatory *SIEC* (scheme symbolized as $Situation_1$-$Goal_1$-AER_1). This construction of an anticipatory *AER* seems to explain how repeatedly correct analytic judgments may become intuitive (automatic).

3. DISCUSSION: BRAIN-BASED *REF*AER* VS. DPT

I contend that *Ref*AER*, with its brain-based elaboration, simultaneously resolves not only the reasoning puzzlement addressed by DPT, but also central problems of mathematics learning and teaching. Concerning what an observer considers *normative* solutions, *Ref*AER explains and predicts* their production as the outcome of either an anticipatory conception, which can run automatically and/or reflectively, or a compatible participatory conception that was made accessible by a prompt—self/internal (e.g., Soduku-1a) or external (e.g., Leron's bridging task, apple falling on Newton's head). Accordingly, *faulty* solutions may be the outcome of (a) partial, inefficient, and/or flawed execution of a suitable anticipatory conception (e.g., Soduku-1, elevator), (b) prompt-dependent inability to access a suitable participatory conception (e.g., solving the $1.10 incorrectly when difference=$1 and correctly with other amounts), and, quite often, (c) lack of a suitable conception for correctly solving the given problem (e.g., 3rd graders facing the next coin-flip problem, Leron's students who could not solve the bridging task).

I further contend that, for mathematics education purposes, and possibly also cognitive psychology, *Ref*AER* resolves DPT problems better. Instead of postulating two systems (or processes), it explains how the brain gives rise to a multi-part *single thought process* by which a problem solver may reach a normative or a faulty *answer*. Furthermore, it stresses that a '*solution*' must encompass not only the answer, but also the crucial (inferred) *solver's* reasoning processes used for producing it. A good demonstration of such analysis, and the vitality of intuitive solutions (e.g., for finding limits of sequences), were provided by Hersh (2011). *Ref*AER* accomplishes such inferences via analyzing the solver's: (i) goal and sub-goals (see Stanovich & West, 2000, for differing researcher/subject goals), (ii) entire or partial activity-sequence selected and executed (see Kahneman & Frederick, 2002, for the notion of Attribute Substitution), (iii) suitability of objects operated on (see Leron's, 2010, specific explication of objects, such as length *gap* in the string-around-earth task and the *nested sub-set* in his RMP task), (iv) sub- and final effects noticed, and (v) successful/failed reflections (both types).

Most importantly, *Ref*AER* analyses are rooted in an explicit distinction between two frames of reference operating in the evaluation of solvers' judgments—the observer's advanced, well-justified frame and the observed's evolving and sensible frame in terms of his or her extant conceptions (Roth & Bautista, 2011; Steffe, 1995). Thus, consistent with Stich's (1994) assertion that cognitive systems serve one's goals and not absolute truths, *Ref*AER* evades the pitfalls of equating normative with rational. Instead, it clarifies that upon a solver's assimilation of a problem situation and setting her/his goal(s), one path among multiple extant activity-sequences (spontaneously known or prompted) is selected, executed and being monitored by the goal. By default, the brain runs the sequence to its completion, which is signaled via Type-I comparison (goal *SIEC*), and can thus be portrayed by an observer as intuitive/automatic.

However, at any given moment during the activity-sequence execution or after its completion, the system's regulator (goal *SIEC*) may notice effects that require interruption and/or correction to the run and/or even to the goal itself (portrayed as analytic/reflective). In paraphrasing Gigerenzer's (2005) "I think, therefore I err", we shall say: "I learn to think, therefore I may adjust (initially) erroneous anticipations."

Consequently, *Ref*AER* seems to provide a basis for resolving two problems that, while not addressed by DPT, are vital for mathematics education, namely, explaining (a) how *learning* to reason—both intuitively and analytically—may occur and (b) how can *teaching* capitalize on it and foster (optimize) students' mathematical progress. The former has been articulated above in a way that seems to address each of the 7 questions presented in Section 2. The latter (implications for teaching) exceeds the scope of this paper; it was articulated elsewhere (Tzur, 2008a, 2008b, 2010a) as a 7-step cycle that proceeds from analysis of students' extant conceptions. To briefly convey the potential of this *Ref*AER*-based 7-step cycle, I return to Leron's example of a bridging (RMP) task.

In designing that task, Leron made explicit the two-phase activity-sequence of considering base-rate (1/1000) *and* diagnostic information (5% false positive) as necessarily linked sub-goals. What's more, the 'objects' on which his alternative sequence would operate were replaced, from multiplicatively related quantities (fractions, percents) to frequencies of whole numbers considered additively up to the final multiplicative calculation. In terms of *Ref*AER*, these alterations explain why some of the students who incorrectly solved the DMP problem could correctly solve the RMP problem. The alteration was more likely to orient solvers to (a) explicitly coordinated sub-goals (*specifying* each of the nested sub-sets) of the task's global goal and (b) selection of and operation on *accessible quantities*—anticipatory *AER* ('objects')—in place of quantities that are notoriously prompt-dependent (or lacking) in youngsters and adults and thus, not surprisingly, '*neglected*'. Accordingly, these insightfully designed task alterations explain the educative power of a bridging task. It seemed to bring forth an anticipatory *AER* that, I conjecture, could have served Leron's students as an internal prompt for correctly selecting-and-executing the entire activity-sequence for operating similarly on the more difficult-to-grasp multiplicative quantities and relationships.

Leron's design of bridging task not only fits well within the *Ref*AER*-based, 7-step teaching cycle, but also with a teaching practice we recently found in China (Gu, Huang, & Marton, 2006; Jin & Tzur, 2011). Our study was based on Xianyan Jin's dissertation, which provided a penetrating inspection of how bridging ('*xianjie*') tasks are consistently fitted within a 4-component lesson structure in Chinese mathematics teaching. She further 'mapped' the 7-step cycle onto the Chinese lesson structure, while highlighting the role that bridging tasks, like those designed by Leron et al. (Leron, 2010; Leron & Hazzan, 2009), can play in the cycle's critical first step—activating students' extant (assimilatory) conceptions. Alluding to Leron's (2010) closing slogan, I believe that, without positing thinking dualities, mathematics teaching informed by the brain-based *Ref*AER* framework, and designed to bridge between available (assimilatory) and intended mathematical ideas, can nurture the power of natural (intuitive) thinking, address the challenge of stretching it, and inform the beauty of overcoming it (via anticipatory analytic processes).

4. REFERENCES

Baars, B. J. (2007a). A framework. In B. J. Baars & N. M. Gage (Eds.), *Cognition, brain, and consciousness: Introduction to cognitive neuroscience* (pp. 31-57). Amsterdam, The Netherlands: Elsevier.

Baars, B. J. (2007b). Neurons and their connections. In B. J. Baars & N. M. Gage (Eds.), *Cognition, brain, and consciousness: Introduction to cognitive neuroscience* (pp. 59-85). Amsterdam, The Netherlands: Elsevier.

Baars, B. J., & Gage, N. M. (Eds.). (2007). *Cognition, brain, and consciousness: Introduction to cognitive neuroscience*. Amsterdam, The Netherlands: Elsevier.

Bush, G., Luu, P., & Posner, M. I. (2000). Cognitive and emotional influence in anterior cingulate cortex. *Trends in Cognitive Science, 4*(6), 215-222.

Cherniak, C. (1994). Computational complexity and the universal acceptance of logic. In H. Kornblith (Ed.), *Naturalizing epistemology* (2nd ed., pp. 239-260). Cambridge, MA: MIT Press.

Corbetta, M., & Shulman, G. L. (2002). Control of goal-directed and stimulus-driven attention in the brain. *National Review in Neuroscience, 3*(3), 201-215.

Crick, F., & Koch, C. (2003). A framework for consicouness. *National Neuroscience, 6*(2), 119-126.

Dewey, J. (1902). *The child and the curriculum*. Chicago: The University of Chicago.

Dewey, J. (1933). *How we think: A restatement of the relation of reflective thinking to the educative process*. Lexington, MA: D.C. Heath.

Dewey, J., & Bentley, A. F. (1949). *Knowing and the known*. Boston: Beacon.

Dreyfus, T. (2002). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25-41). New York: Springer.

Dubinsky, E., & Lewin, P. (1986). Reflective abstraction and mathematics education: The genetic decomposition of induction and compactness. *Journal of Mathematical Behavior, 5*, 55-92.

Evans, J. (2003). In two minds: Dual process accounts of reasoning. *TRENDS in Cognitive Sciences, 7*(10), 454-459.

Evans, J. (2006). The heuristic-analytic theory of reasoning: Extension and evaluation. *Psychonomic Bulletin & Review, 13*(3), 378-395.

Fox, M. D., Snyder, A. Z., Vincent, J. L., Corbetta, M., Van Essen, D. C., & Raichle, M. E. (2005). The human brain is intrinsically organized into dynamic, anticorrelated functional networks. *Proceedings of the National Academy of Science (USA), 102*(27), 9673-9678.

Fuster, J. M. (1997). Network memory. *Trends in Neuroscience, 20*, 451-459.

Fuster, J. M. (2003). *Cortex and mind: Unifying cognition*. New York: Oxford University Press.

Gigerenzer, G. (2005). I think, therefore I err. *Social Research, 72*(1), 1-24.

Goldberg, E., & Bougakov, D. (2007). Goals, executive control, and action. In B. J. Baars & N. M. Gage (Eds.), *Cognition, brain, and consciousness: Introduction to cognitive neuroscience* (pp. 343-367). Amsterdam, The Netherlands: Elsevier.

Goldman, A. I. (1994). What is justified belief? In H. Kornblith (Ed.), *Naturalizing epistemology* (2nd ed., pp. 105-130). Cambridge, MA: MIT Press.

Gu, L., Huang, R., & Marton, F. (2006). Teaching with variation: A Chinese way of promoting effective mathematics learning. In L. Fan, N.-Y. Wong, J. Cai

& L. Shiqi (Eds.), *How Chinese learn mathematics: Perspectives from insiders* (pp. 309-347). Singapore: World Scientific Publishing.

Hersh, R. (2011). Mathematical intuition (Poincaré, Polya, Dewey). *The Montana Mathematics Enthusiast, 8*(1&2), 35-50.

Hershkowitz, R., Schwarz, B. B., & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education, 32*(2), 195-222.

Jin, X., & Tzur, R. (2011). *Progressive incorporation of new into known: A perspective on and practice of mathematics learning and teaching in China*. Paper presented at the Association of Mathematics Teacher Educators.

Kahneman, D., & Frederick, S. (2002). Representativeness revisited: Attribute substitution in intuitive judgment. In T. Gilovich, D. Griffin & D. Kahneman (Eds.), *Heurisics and biases: The psychology of intuitive judgment* (pp. 49-81). Cambridge: Cambridge University Press.

Kahneman, D., Slovic, P., & Tversky, A. (1982). *Judgment under uncertainty: Heuristics and biases*. Cambridge: Cambridge University Press.

Kahneman, D., & Tversky, A. (1973). Subjective probability: A judgment representativeness. *Cognitive Psychology, 3*, 430-454.

Kalbfleisch, M. L., Van Meter, J., & Zeffiro, T. A. (2006). The influences of task difficulty and response correctness on neural systems supporting fluid reasoning. *Cognitive Neurodynamics*.

Katz, K. U., & Katz, M. G. (2010). When is .999... less than 1? *The Montana Mathematics Enthusiast, 7*(1), 3-30.

Kim, J. (1994). What is "Naturalized Epistemology?". In H. Kornblith (Ed.), *Naturalizing epistemology* (2nd ed., pp. 33-55). Cambridge, MA: MIT Press.

Leron, U. (2010). The power of natural thinking: Applications of cognitive psychology to mathematics education. In P. Brosnan, D. B. Erchick & L. Flevares (Eds.), *Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. VI, pp. 9-20). Columbus, OH: The Ohio State University.

Leron, U., & Hazzan, O. (2006). The rationality debate: Application of cognitive psychology to mathematics education. *Educational Studies in Mathematics, 62*, 105-126.

Leron, U., & Hazzan, O. (2009). Intuitive vs analytical thinking: four perspectives. *Educational Studies in Mathematics, 71*, 263-278.

Lieberman, M. D. (2003). Reflective and reflexive judgement processes: A social cognitive neuroscience approach. In J. P. Forgas, K. R. Williams & W. von

Hippel (Eds.), *Social judgements: Explicit and implicit processes* (pp. 44-67). New york: Cambridge Universiy Press.

Lieberman, M. D. (2008). What zombies can't do: A social cognitive neuroscience approach to the irreducibility of reflective consciousness. In J. Evan & K. Frankish (Eds.), *In two minds: Dual processes and beyond* (pp. 293-316). Oxford: Oxford University Press.

McGaugh, J. L. (2002). Memory consolidation and the amygdala: A systems perspective. *Trends in Neurosciences, 25*(9), 456-461.

McGovern, K. (2007). Emotion. In B. J. Baars & N. M. Gage (Eds.), *Cognition, brain, and consciousness: Introduction to cognitive neuroscience* (pp. 389-443). Amsterdam, The Netherlands: Elsevier.

Medina, J. (2008). *Brin rules: 12 principles for surviving and thriving at work, home, and school*. Seattle, WA: Pear.

Moscovitch, M., Chein, J. M., Talmi, D., & Cohn, M. (2007). Learning and memory. In B. J. Baars & N. M. Gage (Eds.), *Cognition, brain, and consciousness: Introduction to cognitive neuroscience* (pp. 255-291). Amsterdam, The Netherlands: Elsevier.

Nadel, L., Samsonovich, A., Ryan, L., & Moscovitch, M. (2000). Multiple trace theory of human memory: Computational, Neuroimaging, and neuopsychological results. *Hippocampus, 10*, 352-368.

Nieder, A. (2005). Counting on neurons: The neurobiology of numerical competence. *Nature Reviews, 6*, 177-190.

Nisbett, R. E., & Ross, L. (1994). Judgmental heuristics and knowledge structures. In H. Kornblith (Ed.), *Naturalizing epistemology* (2nd ed., pp. 261-290). Cambridge, MA: MIT Press.

Norman, D. A., & Shallice, T. (2000). Attention to action: Willed and automatic control of behavior. In M. S. Gazzaniga (Ed.), *Cognitive Neuroscience: A Reader* (pp. 376-390). Malden, MA: Blackwell.

Piaget, J. (1970). *Genetic epistemology* (E. Duckworth, Trans.). New York: Columbia University.

Piaget, J. (1971). *Biology and knowledge* (B. Walsh, Trans.). Chicago: The University of Chicago.

Piaget, J. (1980). Opening the debate: The psychogenesis of knowledge and its epistemological significance. In M. Piattelli-Palmarini (Ed.), *Language and learning: The debate between Jean Piaget and Noam Chomsky* (pp. 23-34). Cambridge, MA: Harvard University.

Piaget, J. (1985). *The equilibration of cognitive structures: The central problem of intellectual development* (T. Brown & K. J. Thampy, Trans.). Chicago: The University of Chicago.

Pirie, S. E. B., & Kieren, T. E. (1992). Creating constructivist environments and constructing creative mathematics. *Educational Studies in Mathematics, 23*(5), 505-528.

Pirie, S. E. B., & Kieren, T. E. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics, 26*(2-3), 165-190.

Quine, W. V. O. (1994). Epistemology naturalized. In H. Kornblith (Ed.), *Naturalizing epistemology* (2nd ed., pp. 15-31). Cambridge, MA: MIT Press.

Roth, W.-M., & Bautista, A. (2011). Transcriptions, mathematical cognition, and epistemology. *The Montana Mathematics Enthusiast, 8*(1&2), 51-76.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 26*, 114-145.

Sfard, A. (2000). Symbolizing mathematical reality into being--or how mathematical discourse and mathematical objects create each other. In P. Cobb, E. Yackel & K. McClain (Eds.), *Symbolizing and communicating in* *mathematics classrooms: Perspectives on discourse, tools, and instructional design* (pp. 37-98). Mahwah, NJ: Lawrence Erlbaum.

Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning, 6*(2), 91-104.

Simon, M. A., Tzur, R., Heinz, K., & Kinzel, M. (2004). Explicating a mechanism for conceptual learning: Elaborating the construct of reflective abstraction. *Journal for Research in Mathematics Education, 35*(3), 305-329.

Skemp, R. R. (1979). *Intelligence, Learning, and Action*. New York: Wiley.

Stanovich, K. E. (2008). Distinguishing the reflective, algorithmic, and autonomous minds: Is it time for a tri-process theory? In J. Evan & K. Frankish (Eds.), *In two minds: Dual processes and beyond*. Oxford: Oxford University.

Stanovich, K. E., & West, R. F. (2000). Individual differences in reasoning: Implications for the rationality debate? *Behavioral and Brain Sciences, 23*, 645-665.

Steffe, L. P. (1990). Mathematical curriculum design: A constructivist's perspective. In L. P. Steffe & T. Wood (Eds.), *Transforming children's mathematics education* (pp. 389-398). Hillsdale, NJ: Lawrence Erlbaum.

Steffe, L. P. (1995). Alternative epistemologies: An educator's perspective. In L. P. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 489-523). Hillsdale, NJ: Lawrence Erlbaum.

Steffe, L. P. (2010). A New Hypothesis Concerning Children's Fractional Knowledge. In L. P. Steffe & J. Olive (Eds.), *Children's fractional knowledge* (pp. 1-12). New York: Springer.

Stich, S. P. (1994). Could man be an irrationl animal? Some notes on the epistemology of rationality. In H. Kornblith (Ed.), *Naturalizing epistemology* (2nd ed., pp. 337-357). Cambridge, MA: MIT Press.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.

Thompson, P. W. (2002). Didactic objects and didactic models in radical constructivism. In K. Gravemeijer, R. Lehrer, B. Van Oers & L. Verschaffel (Eds.), *Symbolizing, modeling, and tool use in mathematics education* (pp. 197-220). Dordrecht, The Netherlands: Kluwer.

Thompson, P. W. (2010). *Towards a foundtion of mathematics education*. Paper presented at the AERA.

Thompson, P. W., Carlson, M., & Silverman, J. (2007). The design of tasks in support of teachers' development of coherent mathematical meanings. *Journal of Mathematics Teacher Education, 10*(4), 415-432.

Tversky, A., & Kahneman, D. (1973). Availability: A heuristic for judging frequency and probability. *Cognitive Psychology, 5*, 207-232.

Tversky, A., & Kahneman, D. (1983). Extensional versus intuitive reasoning: The conjunction fallacy in probability judgment. *Psychological Review, 90*, 293-315.

Tzur, R. (1996). *Interaction and children's fraction learning*. Ann Arbor, MI: UMI Dissertation Services (Bell & Howell).

Tzur, R. (2007). Fine grain assessment of students' mathematical understanding: Participatory and anticipatory stages in learning a new mathematical conception. *Educational Studies in Mathematics, 66*(3), 273-291.

Tzur, R. (2008a). Profound awareness of the learning paradox (PALP): A journey towards epistemologically regulated pedagogy in mathematics teaching and teacher education. In B. Jaworski & T. Wood (Eds.), *The international handbook of mathematics teacher education: The mathematics teacher educator as a developing professional* (Vol. 4, pp. 137-156). Rotterdam, the Netherlands: Sense.

Tzur, R. (2008b). A researcher perplexity: Why do mathematical tasks undergo metamorphosis in teacher hands? . In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sepulveda (Eds.), *Proceedings of the 32nd Annual Conference of the International Group of Psychology of Mathematics Education* (Vol. 1, pp. 139-146). Morelia, Michoacán, Mexico: Michoacán University of Saint Nicholas of Hidalgo.

Tzur, R. (2010a). How and why might teachers learn through teaching mathematics: Contributios to closing an unspoken gap. In R. Leikin & R. Zaskis (Eds.), *Learning through teaching mathematics: Development of teachers' knowledge and expertise in practice* (pp. 49-67). New York: Springer.

Tzur, R. (2010b). How may conceptual learning in mathematics benefit from dual processing theories of thinking? In P. Brosnan, D. B. Erchick & L. Flevares (Eds.), *Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. VI, pp. 21-32). Columbus, OH: The Ohio State University.

Tzur, R. (accepted for publication). Brain-mind theory-driven task design. In E. Kelly, S. Jensen & J. Baek (Eds.), *NeuroMathEducation*. Cambridge, MA: Cambridge University Press.

Tzur, R., & Lambert, M. A. (in press). Intermediate participatory stages as ZPD correlate in constructing counting-on: A plausible conceptual source for children's transitory 'regress' to counting-all. *Journal for Research in Mathematics Education*.

Tzur, R., & Simon, M. A. (2004). Distinguishing two stages of mathematics conceptual learning. *International Journal of Science and Mathematics Education, 2*, 287-304.

Tzur, R., Xin, Y. P., Si, L., Woodward, J., & Jin, X. (2009). Promoting transition from participatory to anticipatory stage: Chad's case of multiplicative mixed-unit coordination (MMUC). In M. Tzekaki, M. Kaldrimidou & H. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 249-256). Thessaloniki, Greece: PME.

Viholainen, A. (2008). Incoherence of a concept image and erroneous conclusions in the case of differentiability. *The Montana Mathematics Enthusiast, 5*(2&3), 231-248.

von Glasersfeld, E. (1991). Distinguishing the observer: An attempt at interpreting Maturana. *Methodologia, 8*, 57-68.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Washington, D.C.: Falmer.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Vygotsky, L. S. (1986). *Thought and language*. Cambridge, Massachusetts: MIT.

Zull, J. E. (2002). *The art of changing the brain: Enriching the practice of teaching by exploring the iology of learning*. Sterling, VA: Stylus.