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If not, then what if as well- Unexpected Trigonometric Insights

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If not, then what if as well? *Unexpected Trigonometric Insights*

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Abstract

In performing an exercise of "What if not", one can end up with a paucity of structure. Adding alternative structure can be a rich source of discovery, as we present here. The framework of this presentation is the original voyage of discovery, from a trivial geometric problem to the derivation of some unexpected trigonometric formulae based on regular polygons. The original "voyage" has been changed only sufficiently to make the text readable.

Keywords: Trigonometric identities; Problem posing; Euclidean Geometry

Introduction

Consider the following construction problem: *ABC* is a triangle inscribed in its circumscribing circle. Using a compass, take the measure of one of the triangle's sides, which is also a chord of the circle, and mark off identical chords around the circle, starting at either vertex on the selected chord. Can you characterize the triangles for which the construction will return exactly to the starting vertex? Can you compute how many revolutions are required?

In figure 1, starting from *C*, using chord length *BC*, the suggested construction goes through $CD = BC$, $DE = BC$ and $EF = BC$. Since *F* is beyond *B* it is clear that this

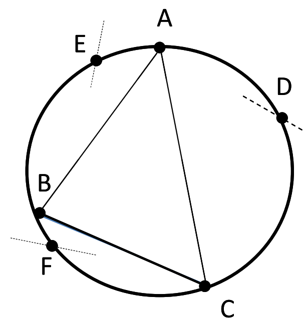

Figure 1

construction will not return to *C*.

A construction that returns to the starting point in one revolution will follow the vertices of a regular polygon since the chords are all equal and are circumscribed in a circle. If the construction returns to the starting vertex after several revolutions, an obvious next question is whether the points on the circumference that are visited will constitute points on a regular polygon. We can, in fact, pose a general question: Is there an explicit property of the angles or sides (or both) that will *characterize* which triangles support constructions, of the above type, that return to their starting point in a discrete number of revolutions?

The answer to both questions is yes and the characterizing property is surprisingly simple.

The question, however, did not arise spontaneously. It came at the end of a surprising voyage of discovery whose source was an innocuous what-if-not exercise (Brown & Walter, 1990). This what-if-not exercise took an interesting route I have labeled "If not, then what-if-as-well".

What follows, documents my personal voyage and of course answers the question raised above. Some curious by-products of the answer are also presented.

The Starting Point

The starting point for this voyage of discovery was a desire to compose a small research activity for students in the 10th or 11th grade who are studying Euclidean geometry. In doing so, a common textbook problem serves as a convenient starting point.

The original textbook problem is illustrated in Figure 2:

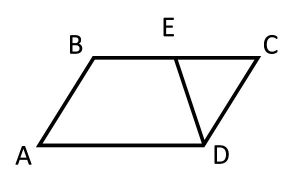

Figure 2

ABCD is a parallelogram. *E* is a point on *BC*.

1. Prove $\dfrac{S_{ECD}}{S_{BEDA}} = \dfrac{BC - BE}{BC + BE}$.

2. Then ask yourself: what if not?

Proof:

1. Suppose the distance between parallel sides *BC* and *AD* is *h*. Then

$$\frac{S_{ECD}}{S_{BEDA}} = \frac{EC \cdot h/2}{BC \cdot h - EC \cdot h/2} = \frac{EC}{2 \cdot BC - EC} = \frac{BC - BE}{BC + (BC - EC)} = \frac{BC - BE}{BC + BE}$$

2. For the what-if-not exercise, I started by writing down problem attributes one might consider changing. There are many alternatives, of which the following are some examples:

   1. *E* is on *BC*. Suppose *E* is not on *BC*:

      a) *E* is on the extension of *BC* to the right or to the left.

      b) *E* is on *AB*

      c) *E* is interior to the parallelogram.

   2. The problem is about areas. Suppose the problem is not about areas:

      a) Consider some relationship between perimeters.

   3. *E* is connected to *D*. Suppose *E* is not connected to *D* but to *F*:

      a) *F* is on the extension of *CD*.

      b) *F* is on *AD* and the triangle *ECD* becomes the trapeze *ECDF*.

      c) *F* is on an extension of *AD* to the left or to the right.

   4. *ABCD* is a parallelogram. Suppose it is not:

      a) *ABCD* is an arbitrary quadrilateral.

      b) *ABCD* is an arbitrary polygon with more than 4 sides.

An initial run through the above what-if-not scenarios raised some moderately interesting situations but nothing really "meaty". Having made no real headway, I decided I would at least extract a minor victory by writing up the formulae for the general case of 4b, as illustrated in Figure 3.

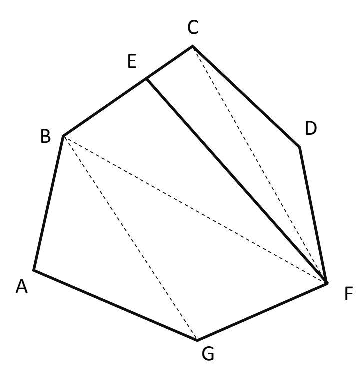

Figure 3

The correspondence to the original problem is that *E* is a point on *BC* of *polygon ABCDFG*, which is connected to vertex *F* by a line, and we are interested in the area proportion $\frac{S_{ECDF}}{S_{EFGAB}}$.

Counting included triangles reveals that $\frac{S_{ECDF}}{S_{EFGAB}} = \frac{S_{CDF} + S_{ECF}}{S_{EFB} + S_{BFG} + S_{BGA}}$. Can we simplify or find an interesting rule to compute that? The answer seemed to be a disappointing no. There is just too little structure, as there was with other alternative what-if-nots I tried earlier.

However*, if not, what if* ***as well***? Suppose we *add* some structure to the problem and see if something interesting comes up. Suppose the polygon to be a *regular* polygon. Can we find an expression for the area proportion that will be a function of, say, only the length *d* of a side and the distance *EC*? I decided to examine an octagon (it seemed easiest to draw!), as a representative of a generic regular polygon with *k=8* sides and side length *d*, illustrated in Figure 4 below. Note that although the illustration is for an octagon, the calculations are written in a general form that applies to a regular polygon with any number of sides.

The Main Derivation

In the general case of the problem, as illustrated in Figure 4, we wish to find a simple expression for the ratio of the areas of the two polygons, here ECDFG and EGHIAB: $\frac{S_{ECDFG}}{S_{EGHIAB}} = \frac{S_{CDF} + S_{CFG} + S_{CGE}}{S_{EGB} + S_{BGH} + S_{BHI} + S_{BIAI}}$

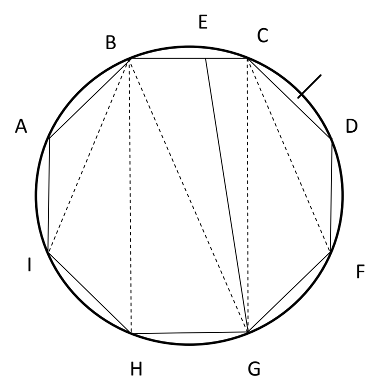

Figure 4

Consider triangle ΔCDF and let us compute its area.

$$\angle D = 180° \cdot \frac{(n-2)}{n} = \pi - \frac{2\pi}{n} \qquad (1)$$

This is the value for all angles between adjacent sides of a regular polygon.

Since ΔCDF is an isosceles triangle and the sum of the angles of a triangle sum to π:

$$\angle DCF = \angle DFC = \frac{1}{2} \cdot [\pi - \angle D] =$$

$$= \frac{1}{2} \cdot \left[\pi - \left(\pi - \frac{2\pi}{n}\right)\right] = \frac{\pi}{n} \qquad (2)$$

Using the sine law we can deduce the length of CF:

$$\frac{d}{\sin\frac{\pi}{n}} = \frac{CF}{\sin\left(\pi - \frac{2\pi}{n}\right)} = 2R \qquad (3)$$

where R is the radius of the circumscribing triangle. Therefore $CF = \dfrac{d\sin\frac{2\pi}{n}}{\sin\frac{\pi}{n}}$ and the area of ΔCDF is obtained from the sine rule for area: $S_{\Delta CDF} = \dfrac{a\cdot c\cdot \sin B}{2} = \dfrac{d^2 \cdot \sin\frac{2\pi}{n}}{2}$

Continuing with computing the area to the right of EG, let us now compute the area of triangle ΔCFG:

$\angle CFG = \angle DFG - \angle DFC = \left(\pi - \dfrac{2\pi}{n}\right) - \dfrac{\pi}{n} = \pi - \dfrac{3\pi}{n}$. Application of the sine rule to triangle ΔCFG yields $\dfrac{CG}{\sin\angle CFG} = 2R$. We now use the convenient and elegant fact that *all* triangles that can be constructed by connecting three vertices of a regular polygon have the *same value of R* for their circumscribed circle because *they are all the same circle!*

So $\dfrac{CG}{\sin\angle CFG} = 2R = \dfrac{d}{\sin\frac{\pi}{n}}$ from which it follows that

$CG = \dfrac{d\cdot\sin\left(\pi - \frac{3\pi}{n}\right)}{\sin\frac{\pi}{n}}$. To determine the remaining angles of ΔCFG we apply the sine rule once again:

$\dfrac{d}{\sin\frac{\pi}{n}} = 2R = \dfrac{FG}{\sin\angle FCG} = \dfrac{d}{\sin\angle FCG}$. Since $n \geq 2$, $\angle FCG$ is an acute angle and therefore $\angle FCG = \dfrac{\pi}{n}$.

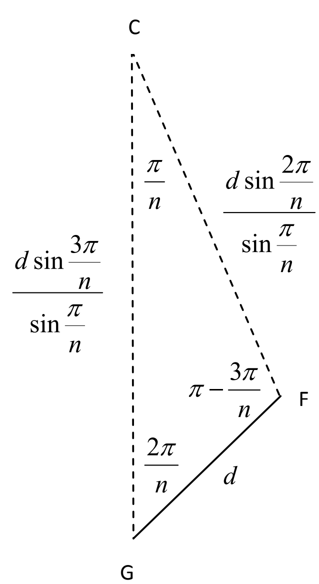

Figure 5

Finally $\angle FGC = \pi - \left(\pi - \frac{3\pi}{n}\right) - \frac{\pi}{n} = \frac{2\pi}{n}$.

The parameters of ΔCFG are summarized in Figure 5:

By the sine rule for area, and using the fact that $\sin\alpha = \sin(\pi - \alpha)$, the area of ΔCFG is:

$$S_{\Delta CFG} = \frac{FG \cdot CG \cdot \sin\frac{2\pi}{n}}{2} = \frac{d^2 \sin\frac{2\pi}{n} \sin\frac{3\pi}{n}}{2\sin\frac{\pi}{n}}$$

The expression looked so tidy that I asked myself if it might be an instance of a *generic* formula for the area of *any such triangle* in the regular polygon.

The generic expression would be of the form:

$$S = \frac{d^2 \sin\frac{k\pi}{n} \sin\frac{(k+1)\pi}{n}}{2\sin\frac{\pi}{n}} \qquad (4)$$

where k is the number of sides between the given side and the given vertex. In the case of the previous triangle (ΔCFG), *k=2*.

The first test of its validity would be to see if the area of triangle ΔCDF is of the same form. To test, I set $k = 1$ in equation (4) and found:

$$S_{\Delta CDF} = \frac{d^2 \sin\frac{1\cdot\pi}{n} \sin\frac{(1+1)\cdot\pi}{n}}{2\sin\frac{\pi}{n}} = \frac{d^2 \sin\frac{\pi}{n} \sin\frac{2\pi}{n}}{2\sin\frac{\pi}{n}} = \frac{d^2 \sin\frac{2\pi}{n}}{2}$$, so the generic expression applies to ΔCDF as well.

I now had reason to believe in the following claim:

Lemma: In a regular polygon with n sides, the angles of the triangle constructed from a given side and vertex of the polygon are $\frac{\pi}{n}, \frac{k\pi}{n}, \frac{(n-(k+1))\pi}{n}$ where *k* is the number of sides between the given side and the given vertex.

Proof: The proof is by mathematical induction for *k=1, 2, ..., n-2*. These cover all the *n-2* triangles in a regular polygon of *n* sides.

For *k=1*, we have $\frac{\pi}{n}, \frac{\pi}{n}, \frac{(n-(1+1))\pi}{n} = \pi - \frac{2\pi}{n}$ which is correct as shown in (1) and (2) above.

Suppose the claim is true for *k = j, j < n-2*, i.e. the sizes of the angles of triangle ΔABC

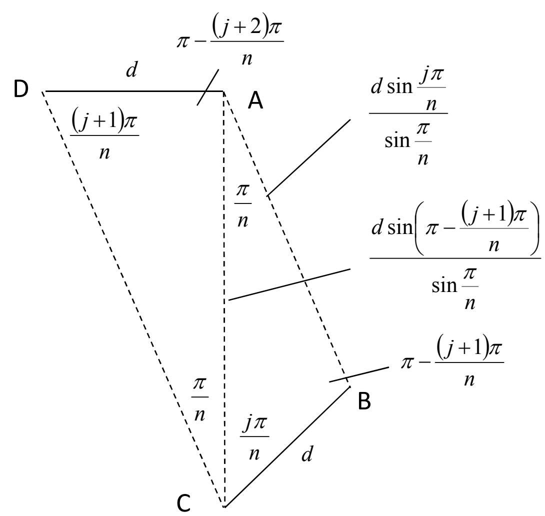

Figure 6

are: $\frac{\pi}{n}, \frac{j\pi}{n}, \frac{(n-(j+1))\pi}{n} = \pi - \frac{2\pi}{n}$ as illustrated in triangle ΔABC in Figure 6.

The sides of triangle ΔABC are determined using the sine law, as was shown above.

We show that the claim holds for *k = j+1*, i.e. that the triangle ΔACD has angle sizes $\frac{\pi}{n}, \frac{(j+1)\pi}{n}, \frac{(n-(j+2))\pi}{n}$.

By the sine law applied to triangles ΔABC and ΔACD,

$2R = \frac{d}{\sin \frac{\pi}{n}} = \frac{d}{\sin \angle ACD} \Rightarrow \angle ACD = \frac{\pi}{n}$. Also,

$$\frac{AC}{\sin\angle ADC}=\frac{d\sin\left(\pi-\frac{(j+1)\pi}{n}\right)}{\sin\frac{\pi}{n}\cdot\sin\angle ADC}=\frac{d}{\sin\frac{\pi}{n}}\Rightarrow\sin\angle ADC=\sin\left(\pi-\frac{(j+1)\pi}{n}\right).$$

Since the angle $\angle ABC$ is not equal to the angle $\angle ADC$ (if it were, the polygon would not be regular), angle $\angle ADC=\frac{(j+1)\pi}{n}$.

It follows that angle $\angle DAC=\pi-\frac{(j+1)\pi}{n}-\frac{\pi}{n}=\pi-\frac{(j+2)\pi}{n}$.

By the principle of mathematical induction on the set *k=1,.., n-2*, the claim holds for all the triangles of the defined type in the regular polygon of size n. Since n is arbitrary, the lemma holds for all finite size n. *Q.E.D.*

Corollary: Using a version of the sine rule for area, the area $S_{\Delta k}$ of a generic polygonal triangle can be written:

$$S_{\Delta k}=\frac{a^2\cdot\sin B\cdot\sin C}{2\cdot\sin A}=\frac{d^2\sin\frac{k\pi}{n}\sin\frac{(k+1)\pi}{n}}{2\sin\frac{\pi}{n}}.$$

Going back to the question of the ratio between the areas to the right and left of a diagonal line connecting a side to a vertex of a regular polygon, as illustrated in Figure 7, we can now write down the parametric solution.

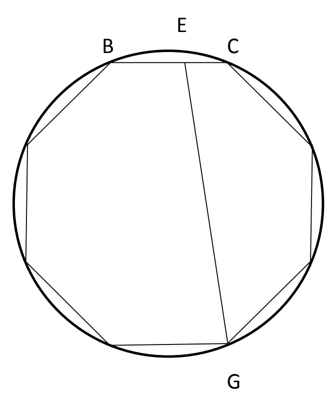

Figure 7

Solution:

$$\boxed{\frac{S_{E\ldots G(clockwise)}}{S_{E\ldots G(anticlockwiase)}}=\frac{\sum_{j=1}^{k-1}\frac{d^2\sin\frac{j\pi}{n}\sin\frac{(j+1)\pi}{n}}{2\sin\frac{\pi}{n}}+\frac{EC}{d}\sin\frac{k\pi}{n}\sin\frac{(k+1)\pi}{n}}{\sum_{j=k+1}^{n-2}\frac{d^2\sin\frac{j\pi}{n}\sin\frac{(j+1)\pi}{n}}{2\sin\frac{\pi}{n}+}+\left(1-\frac{EC}{d}\right)\sin\frac{k\pi}{n}\sin\frac{(k+1)\pi}{n}}}$$

where *k* is the number of sides counting anticlockwise from *G* to the side containing *E*.

For the octagon illustrated above, *k=3* and we obtain:

$$\frac{S_{ECDFG}}{S_{EGHIAB}}=\frac{\sum_{j=1}^{2}\frac{d^2\sin\frac{j\pi}{n}\sin\frac{(j+1)\pi}{n}}{2\sin\frac{\pi}{n}}+\frac{EC}{d}\sin\frac{3\pi}{n}\sin\frac{4\pi}{n}}{\sum_{j=k+1}^{6}\frac{d^2\sin\frac{j\pi}{n}\sin\frac{(j+1)\pi}{n}}{2\sin\frac{\pi}{n}+}+\left(1-\frac{EC}{d}\right)\sin\frac{3\pi}{n}\sin\frac{4\pi}{n}}$$

So we have solved the original “what if not” problem for the ratio of the two areas in a regular polygon formed by a dividing line from a point on one side to an opposing vertex.

However, the reader may still recall that the opening question to this paper was about how to characterize triangles that can support a construction where a series of chords of identical length to one of the sides, when counted off along the circumference of the circumscribing circle, returns to the starting vertex.

It turns out that using the lemma above we can now answer that question. A sufficient condition for a triangle to support the construction is given in the following theorem:

Theorem: ABC is a triangle inscribed in a circle. Using a compass, take the measure of the smallest chord and mark off identical chords around the circle, starting at the vertex at either end of the chord. It is a sufficient condition for the construction to return exactly to the starting vertex that the angles of the triangle satisfy the relation $\angle A : \angle B : \angle C = 1 : p : q$ where p,q are natural numbers with no common divisor.

Proof: To prove sufficiency, let $1 + p + q = n$. Since the sum of the angles in the triangle is π, the angles are in proportion $\frac{\pi}{n}, \frac{p\pi}{n}, \frac{(n-(p+1))\pi}{n}$. From the lemma, we know that this is a triangle that can be embedded in a regular polygon of n sides such that the smallest side (subtending the smallest angle) is a side of the polygon. We can construct the remainder of the polygon from the inscribed triangle by marking off the sequence of identical-sized chords around the circumference of the circumscribing circle. Since the chords of the construction are the sides of the polygon, the construction will return to the starting vertex. *Q.E.D.*

The condition is not a necessary condition because it is easy to change the two longer sides of the triangle so they still intersect on the circle but do not fulfill the stated proportions. The construction using the smallest chord will still return to the starting vertex.

But what if we demand that the construction hold for *all* sides of the triangle, where we allow any discrete number of revolutions to return to the starting vertex.

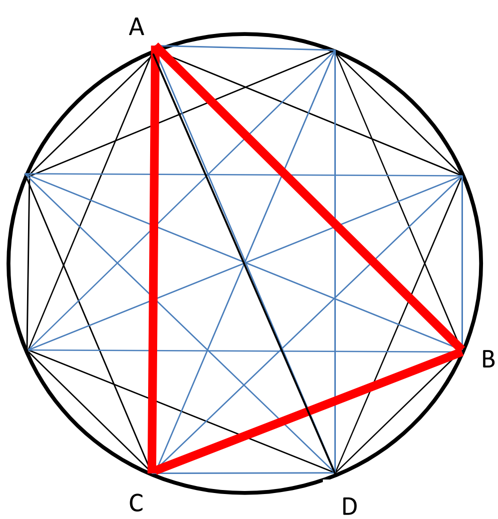

Figure 8

This situation can be observed in Figure 8, which shows an octagon in a circumscribed circle with all pairs of vertices connected by straight lines and a trangle constructed from three of the lines. The ratios of the angles in *ΔABC* is not *1:p:q.* In fact, as we will see, it is

2:3:3. However, side CB can be counted off around the circumference starting from vertex C and will return to its starting vertex C in one revolution.

Careful contemplation of the above construction (where all pairs of vertices are connected), helped reveal the full characterization as an extension to the previous theorem.

Theorem: ABC is a triangle inscribed in a circle. Using a compass, take the measure of any of the chords and mark off identical chords around the circle, starting at the vertex at either end of the selected chord. It is a necessary and sufficient condition for the construction to return exactly to the starting vertex, that the angles of the triangle satisfy the relation: $\angle A : \angle B : \angle C = p : q : r$ where p, q, and r are natural numbers with no common divisor. If p is associated with the angle subtending the selected chord, then the number of circumferences required to return to the starting vertex is the smallest l for which $2 \cdot l \cdot (p+q+r) = 0 \pmod p$.

Proof: The proof of sufficiency relies on a construction around the given triangle, of the type illustrated in Figure 8.

First, circumscribe the triangle with a circle. For each angle of the triangle, divide the arc it subtends into k sub-arcs, where k equals p, q or r, according to the relative size of the angle. (This is not a compass and edge construction but a virtual construction, based on the fact that the angle is divisible by a natural number.) Connect the start and end points of each sub-arc with a chord. The result is a regular polygon of n sides where $p + q + r = n$. This follows from the fact that chords that subtend the same angle are equal in length and from the claim stated in the lemma. Each triangle created from the subtended chord of the sub-arc and the lines enclosing the subtended angle, is a triangle of the type described in the lemma. Hence all the chords subtend an angle of size $\frac{\pi}{n}$.

Figure 8 illustrates the result for a polygon with $n = 8$. Observe that angle A subtends two sub-arcs and angles B and C subtend 3 sub-arcs. as specified by the construction. (In Figure 8 we have also connected the other vertices in order to appreciate the symmetry).

The proof that the construction returns to the starting vertex follows with the aid of some algebraic manipulation. Without loss of generality, assume the selected side of the given triangle is a chord that subtends the angle $\frac{p\pi}{n}$. If we mark off similar size chords from a starting vertex on the source chord, then the construction will return to the starting vertex if the number of revolutions *k*, when multiplied by the angle is a multiple of 2π. Let the multiple of 2π be *l*. Then we can write:

$$k \cdot \frac{p\pi}{n} = k \cdot \frac{p\pi}{p+q+r} = l \cdot 2\pi \quad \text{or} \quad k = \frac{2 \cdot l \cdot (p+q+r)}{p}$$

Taking $l = p$ proves sufficiency since it leads to a natural number $k = 2\cdot(p+q+r)$. However, the lowest number of revolutions is achieved at the smallest *l* for which $2\cdot l \cdot(p+q+r)$ is divisible by p, that is, when $2\cdot l \cdot(p+q+r) = 0 \pmod p$.

The proof of necessity is obtained with the aid of similar algebraic manipulation. Since the sides of the triangle fulfill the conditions of the construction and all return to their starting vertex after a discrete number of revolutions, there exist constants l_A, l_B, l_C and k_A, k_B, k_C, such that:

$$\begin{aligned}
&k_A \cdot 2\pi = l_A \cdot \angle A \\
&k_B \cdot 2\pi = l_B \cdot \angle B \\
&k_C \cdot 2\pi = l_C \cdot \angle C \\
&\angle A : \angle B : \angle C = \frac{k_A}{l_A} : \frac{k_B}{l_B} : \frac{k_C}{l_C}
\end{aligned}$$

Since proportions do not change when multiplied by a constant, we can write:

$$\angle A : \angle B : \angle C = \frac{(l_A \cdot l_B \cdot l_C)\cdot k_A}{l_A} : \frac{(l_A \cdot l_B \cdot l_C)\cdot k_B}{l_B} : \frac{(l_A \cdot l_B \cdot l_C)\cdot k_C}{l_C}$$
$$\angle A : \angle B : \angle C = (l_B \cdot l_C \cdot k_A) : (l_A \cdot l_C \cdot k_B) : (l_A \cdot l_B \cdot k_C)$$

After dividing by the highest common factor of the three numbers in the last equation, we arrive at the required condition $\angle A : \angle B : \angle C = p : q : r$. *Q.E.D.*

The construction in Figure 8 reveals a further intriguing fact: all possible triangles whose angles are a multiple of $\frac{\pi}{8}$ can be found among the constructed angles. Similarly, a regular polygon of 360 sides with all pairs of vertices connected contains all possible triangles whose angles are drawn to an accuracy of 1º!

I leave it to the reader to contemplate the construction and convince himself of the truth of these statements.

Finite trigonometric identities

At this point, pleased with the progress of the what-if-as-well exploration, it seemed appropriate to stop. But the temptation was too great: Suppose we examine the expression for the full area of the polygon, rather than a ratio of two areas?

The area of a regular polygon is usually derived by locating the center of the circumscribed circle and dividing the polygon into n congruent triangles as illustrated in Figure 9a. Applying the sine rule to each triangle finds a relationship between the radius *R* and the side *d* and enables computing the area of a single triangle and therefore the sum of the areas of all n triangles.

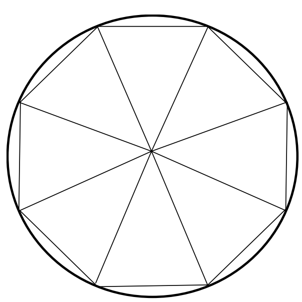

Figure 9a

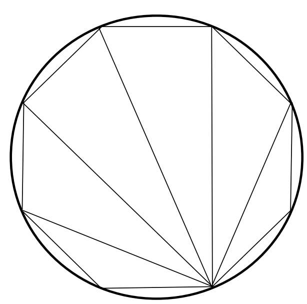

Figure 9b

The area of a single triangle in Figure 9a is $S = \frac{R^2 \cdot \sin\frac{2\pi}{n}}{2}$. Using the fact that the radius *R* of the circumscribed circle is the same *R* as in equation (3), it follows that:

$$R = \frac{d}{2\sin\frac{\pi}{n}}$$

Summing all the triangles in Figure 9a, the area of a regular polygon with n sides of length d is: $S = n \cdot \dfrac{R^2 \cdot \sin\frac{2\pi}{n}}{2} = n \cdot \dfrac{\frac{d^2}{4\sin^2\frac{\pi}{n}} \cdot \sin\frac{2\pi}{n}}{2} = \dfrac{n \cdot d^2 \cdot \sin\frac{2\pi}{n}}{8\sin^2\frac{\pi}{n}}$

Using the trigonometric identity for the sine of a double angle we obtain

$$S = \frac{n \cdot d^2 \cdot \cos\frac{\pi}{n}}{4 \cdot \sin\frac{\pi}{n}}.$$

But we have already expressed the area of the polygon as the sum of the triangles constructed from connecting vertices of the polygon, as illustrated in Figure 9b or in Figure 4 above. Equating the two results leads to the equation:

$$S = \sum_{k=1}^{n-2} \frac{d^2 \sin\frac{k\pi}{n} \sin\frac{(k+1)\pi}{n}}{2\sin\frac{\pi}{n}} = \frac{nd^2 \cos\frac{\pi}{n}}{4\sin\frac{\pi}{n}} \qquad (5)$$

Since the sum is over k, we extract terms not dependent on k from the sum, cancel equal terms from both sides of the equation, and arrive at:

$$\boxed{\sum_{k=1}^{n-2} \sin\frac{k\pi}{n} \sin\frac{(k+1)\pi}{n} = \frac{n}{2}\cos\frac{\pi}{n}}$$

Substituting for n we obtain a family of trigonometric identities. The following are the identities for the first few values of n:

$n = 3$ (equilateral triangle): $\sin\frac{\pi}{3}\sin\frac{2\pi}{3} = \frac{3}{2}\cos\frac{\pi}{3}$

$n = 4$ (square): $\sin\frac{\pi}{4}\sin\frac{\pi}{2} + \sin\frac{\pi}{2}\sin\frac{3\pi}{4} = 2\cos\frac{\pi}{4}$

$n = 5$ (pentagon): $\sin\frac{\pi}{5}\sin\frac{2\pi}{5} + \sin\frac{2\pi}{5}\sin\frac{3\pi}{5} + \sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{2}\cos\frac{\pi}{5}$

The last identity is not an obvious one. Of course, the really intriguing property of this family of identities is that they have an elegant constructive proof!

Infinite trigonometric identities

Having seen two ways of dividing a regular polygon into triangles, it became irresistible to try and find a third way.

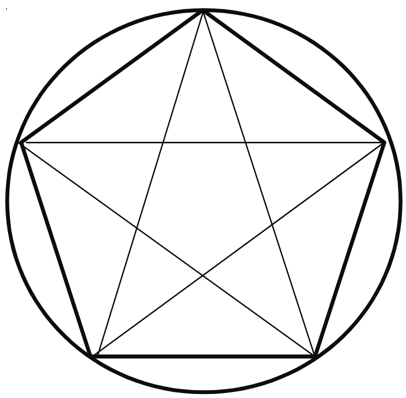

Figure 10

Consider the perimeter triangles created by connecting all pairs of vertices spaced one vertex apart, as illustrated in Figure 10 for an even and odd number of sides.

Notice that the inner polygon circumscribed by all the perimeter triangles is again a regular polygon *of the same order* as the original (outer) polygon, with reduced side length.

Let us derive the relevant quantities:

The areas of all the perimeter triangles are equal. Each is an isosceles triangle with equal side d and angles $\frac{\pi}{n}, \frac{\pi}{n}, \frac{(n-2)\pi}{n}$. The area of overlap between two perimeter triangles is also an isosceles triangle (the "overlap" triangle) with base d and base angles $\frac{\pi}{n}$. These quantities follow directly from the constructions in Figures 5 and 6.

Using the sine rule on the overlap triangle we find $\frac{d}{\sin\left(\pi - \frac{2\pi}{n}\right)} = \frac{l}{\sin\frac{\pi}{n}}$ where l is the short side of the overlap triangle, which is also the side of the inner regular polygon. So the inner polygon has a side length of $l = \dfrac{d \sin \frac{\pi}{n}}{\sin \frac{2\pi}{n}}$. Using equation (5), the area of the inner polygon is then

$$S_{inner} = \sum_{k=1}^{n-2} \frac{l^2 \sin \frac{k\pi}{n} \sin \frac{(k+1)\pi}{n}}{2 \sin \frac{\pi}{n}} = \frac{\sin^2 \frac{\pi}{n}}{\sin^2 \frac{2\pi}{n}} \cdot \sum_{k=1}^{n-2} \frac{d^2 \sin \frac{k\pi}{n} \sin \frac{(k+1)\pi}{n}}{2 \sin \frac{\pi}{n}}.$$

So $S_{inner} = \dfrac{\sin^2 \frac{\pi}{n}}{\sin^2 \frac{2\pi}{n}} S_{outer}$. From this we can compute the area of the difference between the two polygons: $S_{diff} = \left(1 - \dfrac{\sin^2 \frac{\pi}{n}}{\sin^2 \frac{2\pi}{n}} \right) S_{outer}$.

If we continue to construct perimeter triangles and sum the differences between outer and inner polygons in an infinite progression, we will attain the area of the original outer polygon. Therefore

$\displaystyle\sum_{k=1}^{\infty} S_{diff} = \sum_{k=1}^{\infty} \left(1 - \frac{\sin^2 \frac{\pi}{n}}{\sin^2 \frac{2\pi}{n}} \right) \cdot S_{outer} = S_{outer}$. Cancelling the common factor and rearranging terms, we obtain: $\displaystyle\sum_{k=1}^{\infty} \left(\frac{\sin^2 \frac{2\pi}{n} - \sin^2 \frac{\pi}{n}}{\sin^2 \frac{2\pi}{n}} \right) = 1$

At this point I called it a day!

Conclusion

The voyage has brought us a long way from the original problem. The process taught me several lessons.

Firstly, to get an interesting result requires a certain amount of structure. Too little structure yields dilute results and too much structure is likely to be stifling. An ideal amount goes a long way. Using what-if-as-well seems to be an effective means for enhancing the what-if-not technique.

Secondly, the domain of regular polygons appears to be a goldmine of interesting structures. I have scratched the surface. I leave it to the reader to discover more. I should add that not all the proofs fell out immediately. Some of them were originally haphazard, inaccurate, and not entirely understood. I returned to them later to add rigor.

Thirdly, I experienced very vividly what can be gained by setting a challenge to your students using simple problems. Moreover, in "What if not" exercises, we should encourage the students to change assumptions and add novel structure. They may well become as interested as I was in the process of discovery itself.

References

Brown , S. I. & Walter, M. I. (1990). *The Art of Problem Posing* (2nd ed.). Hillsdale, NJ: Lawrence Erlbaum Associates.