The Mathematics Enthusiast

Volume 9
Number 1 *Numbers 1 & 2*

Article 9

1-2012

Number Theory and the Queen of Mathematics

Megan Wagner

Recommended Citation

Wagner, Megan (2012) "Number Theory and the Queen of Mathematics," *The Mathematics Enthusiast*: Vol. 9 : No. 1 , Article 9.
Available at: https://scholarworks.umt.edu/tme/vol9/iss1/9

Number Theory and the Queen of Mathematics

Megan Wagner

The University of Montana

Introduction

Geometry is an integral component of secondary mathematics curriculum. From my experience in a mathematics teacher preparation program, I have seen a real push to connect geometry to other areas of mathematics. Secondary geometry can often be presented without clear or any connections to other areas of mathematics. One main purpose of this paper is to explore geometry and its rightful connection to other areas of mathematics, specifically number theory. Such strong emphasis is placed on drawing connections to number theory because of its intrinsic value in enhancing understanding of mathematical concepts. Learning number theory has positive ramifications for students, making the transition from arithmetic to an introduction to algebra. It helps "students develop better understandings of the abstract conceptual structure of whole numbers and integers," and it has important algebraic characteristics, which relate to variables and mathematical reasoning (Campbell & Zazkis, 2006, p.28). Another purpose of this paper is to explore not only number theory as it relates to geometry, but also the basic history of number theory. Number theory has a beauty, accessibility, history, formal and cognitive nature, and intrinsic merits all to its own (Campbell & Zazkis, 2006, p. 13). For the sake of all the intrigue of number theory, I have a desire to learn more about it to fuel my own pursuit of a better understanding of effectively teaching mathematics, but also to use it to encourage and engage students in their personal pursuit of mathematical understanding.

Because of the purposes of this paper, this research was compiled and organized from a mathematics education perspective. Although, in order to meet the purposes of this paper, the history of number theory and its history of interaction of geometry (This is more of the history of the acknowledgment of number theory and its interaction of geometry.) will play an important role. Despite this, however, this research was not compiled strictly from a historical perspective, but rather from a mathematics education perspective, with a historical perspective that is inherently a part of it.

Ancient Geometry—An Early look at the Union of Geometry and Number Theory

Geometry consistently played an important role in the mathematics of early civilizations. The discovery and study of approximately 500 clay tablets from the area once known as Mesopotamia indicate that the Babylonians were no exception. Babylonians' interest in

***The Mathematics Enthusiast*, ISSN 1551-3440, Vol. 9, nos.1&2**, pp.193- 206

geometry is evident. One tablet, Plimpton 322[1]—a tablet of what appears to be pythagorean triples—indicates that the Babylonians knew of the relationship between the sides of a right triangle. Plimpton 322, dated to the mid-18th century B.C., has been the subject of research and study for several decades (Robson, 2001, p. 170). Plimpton 322 appears to be incomplete; there is an evident break along its left edge. Also, this tablet does not include any scratch work that would shed light on the methods of the Babylonians' pythagorean triple generation. The question on the minds of researchers is *how did the Babylonians generate this list of pythagorean triples*?

The significance of answering this question relates to the study of the origins of Babylonians generation of pythagorean triples. To study these origins, it is imperative to unveil our eyes of our modern and cultural views embedded into our mathematical understanding and try to see Babylonian mathematics through their cultural veil. In answering this question, it will be important to acknowledge Babylonian culture. There are significant distinctions between the Babylonians' culturally view of mathematics and our modern view. Also, in answering this question, an important union of two mathematical branches, geometry and number theory, will reveal itself. Modern mathematical views and Babylonian mathematical views are distinct without question. There needs to be the use of a common medium, which will facilitate understanding of how Babylonians generated pythagorean triples. This medium is the theory of numbers, after all numbers belong to a universal language, which has been used by researchers to formulate theories, trying to answer the question of how Babylonians generated the list of pythagorean triples included on Plimpton 322. Three theories on Plimpton 322 have received the most attention. The formulation and deciphering of these theories illustrate the importance of taking cultural factors into consideration and using number theory.

The trigonometric table theory is not so much a theory on the generation of the pythagorean triples, but it is rather a theory/interpretation of Plimpton 322's contents. This interpretation originates from the fact that the first column appears to be tan^2 of the angle opposite of the short side of the triangle. Calculating this angle, θ, shows that θ decreases about 1^o row to row, showing some type of order to the arrangement of the rows. Also, the calculations show that θ is between 30 degrees and 45 degrees (Buck, 1980, p. 344). This theory, however, violates the importance of cultural considerations. Through her study of Babylonian tablets, Robson has deduced that Babylonians used the circumference of a circle, not the radius, to define circles and find their areas. Instead of using $A = \pi r^2$, they appear to have used $A = \frac{c^2}{4\pi}$, where π is approximately equal to 3 (Robson, 2002, p.111). According to Robson, there is no evidence of the Babylonians' rotating of radii, and without this evidence, there is no "conceptual framework" for trigonometry (2002, p. 112). The researchers that authored this theory did not look through the veil of Babylonian culture, so this theory is deemed invalid.

The second theory to be discussed in this section is the generating pair method. Otto Neugebauer and Abraham J. Sachs introduced this theory in their 1945 book, *Mathematical*

Cuneiform Texts. The generating pair theory is essentially taken from Book X of *The Elements*, which presents an understanding of rational and irrational lines using the concepts of commensurable and incommensurable[2] lengths and squares (Roskam, 2009, p. 277). The generating pair theory advocates that the Babylonians used a formula, comparable to Euclid's formula, for generating pythagorean triples, as seen in Book X. With this formula, triples are produced with an m and n, satisfying the following conditions:

- $m > n$, $\gcd(m,n)=1$
- m,n are not both odd
- $a = mn$, $b = m^2-n^2$, $c = m^2+n^2$.

This theory is dismissed on a few grounds. No evidence has been found to suggest that Babylonians knew the concepts of odd and even numbers and coprime numbers (Robson, 2001, p. 177). This violates the importance of looking at Babylonian mathematics through the veil of their cultural view of mathematics; since there is no evidence of these concepts, they cannot be assumed to be a part of their mathematics. Another reason for this dismal is the fact that Plimpton 322 has a clear, purposeful order. Other Babylonian tablets indicate that order was important to Babylonians. This theory and the number theory behind this theory do not support the formulation of a tablet with structure with such order. According to Robson, the scribe would have hundreds of pairs from the standard reciprocal table from which to choose, so creating Plimpton 322, as it is, would be exceedingly difficult (2001, p. 177-178; 2002, p. 110-111). The third of the theories, discussed in this paper, is revered as the theory with the most merit because of its unveiled view and the use of the medium of the theory of numbers, respecting the cultural context of the Babylonians.

The reciprocal theory was introduced in 1949 by E.M. Bruins. The reciprocal theory advocates that the tablet was constructed with the use of reciprocals and cut-and-paste geometry. Another Babylonian clay tablet, YBC 6967, offers evidence that these techniques were indeed used regularly by the Babylonians. According to this theory, Babylonians used reciprocals to concretely compete squares to generate integral pythagorean triples (Robson, 2001, p. 183-185). This theory not only views the mathematics of Plimpton 322 within the cultural context of Babylonians, but it also uses number theory effectively to interpret the method of Plimpton 322's generation of pythagorean triples; therefore, the reciprocal theory has been given the most merit among all the theories of Babylonian pythagorean triple generation, and the consensus of the mathematical community is that this theory is most plausible.

The important thing to take away from the study of Plimpton 322 is that it illustrates that mathematics is not culture-free; however, most importantly, it illustrates a powerful application of uniting geometry and number theory. Plimpton 322, itself,—without deciphering the method of pythagorean triple generation— is an artifact of ancient "modern" number theory. It, after all, makes use of a particular case of Fermat's Last Theorem, $a^n + b^n = c^n$ when $n = 2$. The Babylonians' work with pythagorean triples and this case of Fermat's Last Theorem can be viewed as a prelude to modern number theory. The discovery of Plimpton 322 has been called one of the "most surprising discoveries in twentieth century archaeology" because it shows that the Babylonians were working with this type of problem for centuries before Diophantus, Euclid, and Fermat (Edwards, 1977, p. 4). It appears that number theory and geometry have always had some connection, but this connection has not always been known or acknowledged. In fact, in the 17^{th} century, geometers vehemently insisted that geometry be untainted by arithmetic (Mahoney, 1994, p. 3). Pierre de Fermat entered the mathematics scene in 17^{th} century Europe. His work indicates that he had a similar fascination with the particular case of his last theorem of when $n = 2$ to that of the Babylonians. Fermat is credited as being the father of modern number theory, the queen of mathematics. His time spent working with mathematics is marked by his efforts to end the segregation of number and geometry.

Mathematical Climate in 17^{th} Century Europe

To better understand Fermat and his contributions to mathematics, it is important to become familiar with the mathematical climate in which he worked. Mathematics was a fragmented system. The lack of a unifying term for the work of mathematics contributed to a fragmented mathematical community. The term mathematician was not used in reference to those that work with mathematics (although for the purposes of this paper, the term *mathematician* will be used to refer to those working in mathematics). "*Mathematicus* retained in the sixteenth and seventeenth centuries the meaning it had for the Middle Ages; it meant; 'astrologer' or 'astronomer'" (Mahoney, 1994, p. 14). Geometers called themselves *geomatre*. Germans called themselves *Rechenmeister*. Also, mathematics was divided into six distinct branches with distinct philosophies on mathematics, further fragmenting the mathematical community.

Most individuals working in mathematics identified themselves as one or more of the following: classical geometers, cossist algebraists, applied mathematicians, mystics, artists and artisans, and analysts. The philosophies on the purpose and nature of mathematics and the mathematical styles of each of these groups mostly contrasted (Mahoney, 1994, p. 2). Classical geometers viewed Greek tradition as the superior model for behavior, and thus exclusively used techniques and developments that had Greek roots. They were more interested in presentation style than in new and unique results. Limiting themselves to a purely Greek style, they refused to adopt new and useful non-Greek methods (Mahoney, 1994, p. 3-4). Cossist algebraists valued efficiency and novelty in problem-solving, oftentimes at the expense of presentation style. When they wrote their solutions for the public, they would do so in a complicated manner, which only other cossists could understand, and for the purpose of announcing their triumphs to their peers. "A cossist's ability to solve problems his competitors could not solve gave him an advantage he was loath to surrender through publication" (Mahoney, 1994, p. 5-6). Applied mathematicians valued Greek methods, but they did not exclusively seek these methods. They were most interested in developing operational mathematics for uses such as business and navigation. They placed more emphasis on computation than proof, which sharply contrasted with most of the mathematicians in the other branches (Mahoney, 1994, p. 9-10). Mystics were interested in reviving ancient number theory. They searched for "the secrets of integers." They saw mathematics as a means to reveal the secrets of the universe (Mahoney, 1994, p. 11). The final branch of 17th century European mathematics was the group of which Fermat was a member, the analysts. They borrowed philosophies from the other five branches. They valued methods with Greek heritage, borrowed from the geometers. From the cossists, they borrowed the advocacy of algebra as a powerful tool for solving problems. They viewed Greek models as a foundation on which other non-Greek mathematics could build (applied mathematicians). Finally, they desired to unite mathematics with a system of symbolic reasoning (mystics) (Mahoney, 1994, p.12).

Some other important characteristics of this time to consider are the university setting in relation to mathematics and the communication of mathematical inquiry and results. Most universities in early 17th century Europe had curricula that promoted very little mathematics. The focus of these universities was mainly law, medicine, and theology. Toward the middle of the century, mathematics was more readily integrated into university curricula. Even with this shift of mathematics' presence in the universities, most mathematical influence was outside higher education; most mathematical discovery and advancement was outside universities. Most mathematicians of that time may have been university graduates, but they received their mathematical training from their peers or they were self-taught. René Descartes, John Wallis, Sir Isaac Newton, Christiaan Huygens, and Fermat are among this self-taught/peer-taught community (Mahoney, 1994, p. 13). There were no mathematical journals in the 17th century, so mathematical discoveries or inquiries were shared through the correspondence of letters. Marin Mersenne was unofficially the overseer of letters shared among mathematicians. Mathematicians wrote to Mersenne about their discoveries, and then Mersenne relayed this information to other mathematicians through written letters (Dudley, 2008, p. 60). This correspondence adds an intriguing element to this period of math. Fermat and his works were shaped by this mathematical climate and these characteristics of the 17th century.

Background on Fermat

Pierre de Fermat (1601-1665) was not a mathematician by profession; he was trained in law. Mathematics was a pursuit of leisure for Fermat as is was for most mathematicians. Working as an amateur mathematician allowed Fermat freedom. He was a "free agent." If another mathematician rejected his methods or work, it affected nothing more than his self-esteem. Once again, mathematics was a fragmented system, so there was a lot of rejection of others' works. Since mathematicians' professional careers were not a stake, he could feud as he wanted to with others because even if his opposition gained the favor of all other mathematicians, it did not have a real effect on his career (Mahoney, 1994, p. 21). Of course, he may have taken offense at his opposition's negative opinion of his work (and he did let his opposition frustrate him, which will be addressed later), but his professional life was safe from the impacts of disagreement. In fact, Mahoney claims that Fermat gravitated toward mathematics because it was a safe haven from the disputes and controversies that he saw in his career of law. Controversies in mathematics seemed less intense, which Mahoney thinks is interesting, since many mathematical disputes were, in fact, very intense (1994, p. 23).

Fermat wrote to several mathematicians. He started correspondence with Mersenne in 1636. It was not until about 1662 that this correspondence ended; Pierre de Carcavi took over Mersenne's role of mathematical mediator after Mersenne's death in 1648 (Weil, 1984, p. 41-42). Even though neither of these men were "creative mathematicians," themselves, they enthusiastically relayed information to and from the most prominent mathematicians of the day (Goldman, 1998, p.13). Correspondence played an especially important role in Fermat's leisurely study of mathematics. The only known personal contact that Fermat had with another mathematician was a three-day visit with Mersenne in 1646. Fermat corresponded with men including the following: Bernard Frénicle de Bessy, (a fellow "number lover"), Descartes, Étienne Pascal, Blaise Pascal, Gilles Personne de Roberval, and Wallis (Weil, 1984, p. 41, 53, 81). Letters took the place of personal contact. Fermat's correspondence with Frenicle was very valuable. Frenicle, interested in number theory, challenged Fermat's discoveries, seeking reasoning behind his number theory discoveries. This questioning of his discoveries led Fermat to reveal a few of his "carefully guarded secrets." This correspondence of Fermat and Frenicle quite possibly yielded some of the most important information on Fermat's number theory (Mahoney, 1994, p.293). Letters from correspondence, in general, have played a significant role in unveiling Fermat's work because of his shying away from the publication of his work.

Fermat chose not to publish much of his work. In fact, there are no formal publications of any of his work in number theory (Goldman, 1998, p. 12). There are several reasons that he may have chosen to not publish much of his work. Publishing was a stressful endeavor. If the mathematician handed over his work to be printed, there was a substantial risk that, if the printer was not familiar with the mathematician's notation and style, errors would be made. "Only too often, once the book had come out, did it become the butt of acrimonious controversies to which there was no end" (Weil, 1984, p. 44). Perhaps out of this fear, Fermat declined to publish his work. Also, Fermat struggled with writing up his proofs. He especially was plagued with awkwardness in writing up his proofs concerning number theory. This is mainly because he had no models of number theory publications to emulate (Weil, 1984, p. 44). It was the fear of Fermat's admirers that Fermat's work would be simply lost and forgotten, if his work was not published. After his death, some of his writing on geometry, algebra, differential and integral calculus were published posthumously. Also, many of the letters that Fermat wrote to fellow mathematicians have been published. Samuel de Fermat is responsible for publishing much of his father's work. In fact, it appears that only one of Fermat's proofs in number theory has been published. This proof was published by Samuel posthumously as Observation 45 on Diophantus.

This proof is of the proposition that "the area of a right triangle cannot be a square" (Edwards, 1977, 9.10-11). This proof will be further discussed in a later section of this paper. It is because of Samuel that Fermat's Last Theorem was published for the world to see (Weil, 1984, p.44; Edwards, 1977, p.1-2).

Fermat's Interest in Number Theory

Fermat's interest in number theory was fostered by the works of Diophantus. At this time, the only sources on number theory were Diophantus' *Arithmetic* and Books VII-IX of *The Elements* (Kleiner, 2005, p. 4). Ironically, Fermat meant to revive old, classical mathematical traditions, but he actually ended up laying the foundation for a "new, modern tradition," modern number theory (Mahoney, 1994, p. 283). He did return to one ancient tradition that had been discarded by his peers. This ancient tradition was the belief that arithmetic was "the doctrine of whole numbers and their properties" (Mahoney, 1994, p. 283-284). Plato advocated this ancient tradition. In *The Republic*, Plato states, "Good mathematicians, as of course you know, scornfully reject any attempt to cut up the unit itself into parts..." (Mahoney, 1994, p. 284). *Arithemtic* contains about 200 problems, which require the use of one or more indeterminate equations to solve. Diophantus[3] sought rational solutions to these equations (Kleiner, 2005, 4). Although Fermat was captivated by Diophantus' work in *Arithmetic*, he rejected a lot of his work because it allowed rational solutions. Motivated by his intention to renew arithmetic's commitment to integers, he felt that the only solutions sought ought to be integral solutions (Mahoney, 1994, p. 284). Nevertheless, Fermat was truly inspired by *Arithmetic*, as illustrated in *Observations on Diophantus*, which is the publication of the abounding notes that Fermat wrote in the marginalia of *Arithmetic* (Mahoney, 1994, p. 286). His particular fascination with indeterminate equations is evident in much of his work.

One proposition of Diophantus' *Arithemtic* which piqued Fermat's interest and undoubtedly, significantly impacted much of Fermat's work with number theory. This proposition, "to write a square as the sum of two squares," is one of mathematics' oldest problems (Edwards, 1977, p. 3); after all, it dates back to the ancient mathematics of the Babylonians, as discussed earlier in this paper. Edwards claims that this proposition is a great inspirer of Fermat's Last Theorem (1977, p. 3); clearly, this proposition is the particular case of Fermat's Last Theorem that has been referenced throughout this paper. A geometric expression of Diophantus' proposition is "find right triangles in which the lengths of the sides are commensurable, that is, in ratio of whole numbers" (Edwards, 1977, p. 4). Fermat worked with the geometric expression of Diophantus' proposition throughout his career. It not only inspired Fermat's Last Theorem, but it also led Fermat to discover intriguing details relating to pythagorean triples and to pen some theorems on right-angled triangles. Most importantly, its inspiration led Fermat to blend his work with geometry and number theory and to pave the way for the idea that geometry and number theory could live harmoniously in the realm of mathematics.

Fermat's Use of Number Theory in Geometry

It is only natural that Fermat worked with pythagorean triples, given his fascination with indeterminate equations. His work with pythagorean triples illustrates his interest in joining number theory and geometry. He posed and solved several problems involving right-angled triangles. As briefly mentioned earlier, pythagorean triples relate to Fermat's Last Theorem. This theorem states that it is impossible for "any number which is a power greater than the second to be written as a sum of two like powers" (Edwards, 1977, p. 2). The algebraic representation of this theorem is the following: $a^n + b^n = c^n$ has nontrivial, positive integral solutions[4] only if $n \leq 2$. This, of course, bears resemblance to the Pythagorean Theorem. Fermat never presents a proof for this theorem. "I have a truly marvelous demonstration of this proposition, which this margin is too narrow to contain" (Edwards, 1977, p.2). Even though the margin simply provided insufficient room for his proof and he never presented this "marvelous demonstration," his work shows that he was very comfortable using this particular case of Fermat's Last Theorem of when $n = 2$ and finding integers *a, b*, and *c*, which satisfy this case.

In 1643, he posed the following problem: "To find a pythagorean triangle in which the hypotenuse and the sum of the arms are squares" (Sierpiński, 2003, p. 67). Fermat wrote to Mersenne and claimed that he had found the smallest such pythagorean triangle. The triangle that Fermat found was triangle $(4565486027761, 1061652293520, 4687298610289)$. This solution was significant because obviously Fermat did not find this triangle by guessing or simply stumbling upon it (Sierpiński, 2003, p. 67). Fermat did not reveal his method of finding this triangle, but Sierpiński offers an explanation of an approach that Fermat may have used. Sierpiński explains that finding a pythagorean triangle in which the hypotenuse and the sum of the arms are squares is equivalent to finding positive rational values for x, y, u, v satisfying the following equations: $x^2 + y^2 = u^4$, $x + y = v^2$. Once these positive rational values are determined, a common denominator, *m*, can be found; then, the first equation is multiplied by m^4 and the second equation is multiplied by m^2. Because the denominator of x and y is m^2, the solution to the triples satisfying: $x^2 + y^2 = u^4$ are the integers m^2x, m^2y, and m^2u^4 (Sierpiński, 2003, p. 67-69).

Fermat's work with pythagorean triples resulted in several theorems on right-angled triangles. One theorem is that "there are no pythagorean triangles of which at least two sides are squares" (Sierpiński, 2003, p. 48- 49). An implication of that theorem is that there are no pythagorean triples, where each side within a triple is a square. Algebraically, this is represented as $a^4 + b^4 = c^4$. This is clearly, the case of Fermat's Last Theorem when $n = 4$; thus, further highlighting Fermat's efforts to unite number theory and geometry (Sierpiński, 2003, p. 55).

Fermat also sought to determine whether, given some number A, there is a pythagorean triangle with arms, whose sum is equal to A. In order to find a pythagorean triangle, whose arms sum to a given number A, A must meet the necessary and sufficient condition that A must be "divisible by at least one prime number of the form $8k \pm 1$ (Sierpiński, 2003, p. 34-35). For example, since $41 = 8(5) + 1$, there must be a pythagorean triple, where the sum of a and b is 41. If a, b, and c are such that, $a = 20$, $b = 21$, and $c = 29$, then is such a pythagorean triple satisfying this proposition.

Fermat also observed some properties of the areas of right-angle triangles, and he authored a few theorems on this topic, including the theorem, mentioned earlier as the only known number theory theorem that is accompanied with a proof of Fermat's. The proposition, whose proof was published after Fermat's death, is "the area of a right triangle cannot be a square" (Edwards, 1977, p.10-11). Algebraically, this means that there does not exist a pythagorean triple, whole numbers a, b, c satisfying $a^2 + b^2 = c^2$, such that the area of the triangle formed by this triple, $\frac{1}{2}\boldsymbol{ab}$, is a square[5] (Edwards, 1977, p. 11). Fermat essentially shows that, for this to be true, then a and b would be squares. Then, he likens this proposition to the $n = 4$ case of his last theorem, which he shows is impossible. He uses a proof by infinite descent to prove this proposition. His proof ends with "the margin is too small to enable me to give the proof completely and with all detail" (Edwards, 1977, p. 11-12; Mahoney, 1994, p. 352). The preservation of Fermat's work sure had an on-going battle with margins. Another theorem that involved areas of pythagorean triangles is "for each natural number *n* there exist *n* pythagorean triangles with different hypotenuses and the same areas" (Sierpiński, 2003, p. 37). As a result of this theorem, three triangles with different hypotenuses but with the same area exist. (Of course, this is an arbitrary example, and it could really be any *n* number of triangles.) The area of these triangles would be large because it takes large side lengths to achieve equal area but different hypotenuses. In fact, the smallest area common to three such primitive pythagorean triangles with different hypotenuses is 13123110. These triangles with the common area are $(3059, 8580, 9109)$, $(4485, 5852, 7373)$, and $(19019, 1380, 19069)$ (Sierpiński, 2003, p.37, 40). Another theorem from Fermat is the Fundamental Theorem on Right-Angled Triangles. This theorem states that every prime number in the form $4m + 1$ is the hypotenuse of one and only one primitive pythagorean triple (Vella, Vella, & Wolf, 2005, p. 237). Fermat went on to prove by infinite descent that every prime number of the form $4m + 1$ is composed of two squares and thus, is, geometrically, the hypotenuse of a pythagorean triangle (Mahoney, 1994, p. 349).

Fermat's mixing of geometry and number theory—his work with pythagorean triples and pythagorean triangles—had a significant impact on other elements of his work in number theory. Working with pythagorean triples gave Fermat plenty of experience in working with decomposition of squares. His work with pythagorean triples was the gateway to his work with decomposition of squares, in general (Mahoney, 1994, p.287). Much of his later work in number theory is concerned with the decomposition of squares—there is certainly a connection of the decomposition of squares to Fermat's Last Theorem (Mahoney, 1994, p. 303). Another significant contribution of Fermat's work with pythagorean triangles is Fermat's initial use of infinite descent. He introduced the concept of infinite descent in proving theorems, such as "no number of the form $3k - 1$ can be composed of a square and the triple of a square, or no right triangle has a square area" (Mahoney, 1994, p. 348). Fermat shows this by infinite descent by saying if such a triangle with a square area existed, then there would have to be another triangle smaller than the original that also has a square area; then, there would have to be another triangle smaller than the previous triangle with the same characteristic. He states that smaller and smaller triangles with this property would be found, "descending *ad infinitum*" and since, natural numbers are bounded below, this cannot be. Therefore, "no number of the form $3k - 1$ can be composed of a square and the triple of a square, or no right triangle has a square area" (Mahoney, 1994, p. 348). Fermat's successes and developments in number theory left most of his contemporaries unenthused.

17th Century Perception of Number Theory

If number theory were an island, Fermat would have been its only inhabitant. Although mathematicians like Mersenne and Frenicle were "number lovers," none of his peers were true number theorists (Weil, 1984, p. 51). The rest of the mathematical community showed less interest in number theory. Fermat sent problems to several mathematicians, in hopes of fostering interest in what he was so interested in. He sent some number theory problems to mathematicians in England, including Wallis. When Wallis sent rational solutions and thus, disregarding Fermat's criterion of integral solutions, Fermat rejected Wallis' solutions. This rejection did nothing but reinforce Wallis' view of the unimportance of number theory (Mahoney, 1994, p. 63). As with the Wallis incident, other mathematicians were not inclined to show much interest because of how Fermat treated them and their inquiries. Even Frenicle[6] got angry with Fermat. Fermat would send Frenicle problems, and Frenicle would ask for details, but Fermat would never send more details. At one point, Frenicle accused Fermat of sending problems that were impossible to solve (Mahoney, 1994, p. 56). Edwards makes the argument that Fermat's habit of rarely sharing methods or further explaining problems, whether or not he was aware of it, is an indication that he was a jealous, competitive, and secretive mathematician, like most of his peers (1977, p.11). Regardless if that is true, Fermat wanted his number theory to speak for itself and attract the attention and admiration of others, and he apparently did not see creating positive relationships as a way of promoting his ideas. The lack of interest in number theory was something that upset Fermat. "…his failure to convince Wallis and others of the beauty andchallenge of number theory was a source of anguish and frustration to Fermat" (Mahoney, 1994, p. 22). Toward the end of his life, Fermat, hoping to stir someone to continue his work in number theory, wrote to Huygens about "handing on the torch." Huygens, in reference to Fermat's letter, wrote to Wallis, "There is no lack of better things for us to do" (Weil, 1984, p. 119-120). No one picked up "the torch" until the 18th century, when the "rebirth" of number theory took place. This rebirth came through the works of number theorist, Leonhard Euler (1707-1783) (Kleiner, 2005, p.4; Edwards, 1977, p. 39; Weil, 1984, p. 2).

[6]Frenicle, one of Fermat's few allies, sadly never contributed anything of any significance to number theory. Fermat finally found an admirer of his work, but he was unable to contribute much of anything to Fermat's cause of promoting number theory (Mahoney, 1994, p. 340).

Looking to the Future (18th Century and Beyond)

Euler ran with this "torch." Even though he was born over forty years about Fermat's death, Euler, fascinated with Fermat's work, worked with Fermat's theorems and other propositions. Euler proved the cases of $n = 3$ and $n = 4$ of Fermat's Last Theorem. In a 1753 letter to fellow mathematician Christian Goldbach, Euler said that "the general case still seemed quite unapproachable" (Edwards, 1977, p. 59). If only Euler knew how truly "quite unapproachable" this proof really was, and that a complete proof would not be presented for over a century after his death. In 1995, nearly 350 years after Fermat introduced this theorem, Andrew Wiles, with the help of Richard Taylor proved Fermat's Last Theorem (Goldman, 1998, p. 15). Euler also worked with sum of squares, in general. He proved Euler's proposition that "every prime of the form $4n + 1$ is a sum of two squares" (Edwards, 1977, p.46). Unfortunately, Fermat's hope that his admiration for and intrigue of number theory would be shared was not fulfilled under after his death, but it was shared. Sophie Germain, a female mathematician of the late-18th century, early-19th century, like Euler, shared Fermat's interest in number theory. She discovered a result of Fermat's Last Theorem, which bears her name. Sophie Germain's Theorem relates to solutions to cases of Fermat's Last Theorem and different divisibility properties of these solutions (Edwards, 1977, p. 64). She is worth mentioning because of her significance as a woman that "[overcame] the prejudice and discrimination which have tended to exclude women from the pursuit of higher mathematics…" (Edwards, 1977, p. 61). The queen caught the attention of Germain, and she caught the attention of the mathematical community as a woman in a male-dominated field.

Conclusion

Fermat faced significant opposition to his work in number theory, opposition that he was unable to overcome during his lifetime. Firstly, most mathematicians were enthralled in a love affair with calculus. Number theory, a newcomer on the mathematics scene, had a difficult time competing for the spotlight (Weil, 1984, p. 119). Secondly, Fermat was trying to revive an idea that others had deemed as archaic. The idea of placing the constraint of only accepting integral solutions in arithmetic was unappealing to many, who saw no reason to reject rational solutions. Thirdly, even though number theory offered another dimension to geometry, many mathematicians would not entertain the idea of mixing arithmetic and geometry. They believed that arithmetic and geometry were to be two entirely separate things. Even though modern number theory was valued by few in 17th century Europe, it gained more attention and became more valuable to mathematicians in the 18th century, with Euler's emergence as a prominent number theorist. His number theory built upon Fermat's foundation, which included a relationship forged between geometry and number theory.

This forged relationship between geometry and number theory is important, as illustrated in Fermat's work, especially his extensive work with pythagorean triples, but also as illustrated by the study of Plimpton 322. Pythagorean triangles are not merely triangles, whose sides satisfy $a^2 + b^2 = c^2$. As a result of Fermat's work, pythagorean triangles almost seem like a mystical triangle. Fermat brought the study of these triangles to an entirely different level. It helps to illustrate the myriad of sources of intrigue that pythagorean triples offer to number theory because of all the unique characteristics of these triples. Also, this new relationship between geometry and numbers makes possible the study of ancient artifacts, like Plimpton 322. It has allowed for the theories of the origin of the triples on Plimpton 322 to be soundly and thoroughly developed. These theories use number theory to revisit the time of the Babylonians. If Bruins or Robson did not have the tools to discover that reciprocals and using cut-and-paste geometry could yield pythagorean triples, they would not have a theory to help them reconstruct this time in the history of mathematics. Whether or not the attempt to reconstruct the history of mathematics is acceptable or honorable is irrelevant in this paper. The fact is that number theory is a medium that can be used to visit the time of ancient mathematics. It allows for understanding of ancient civilizations' mathematics even when there is a strong cultural element embedded in their mathematics. With the Babylonians, their generation of pythagorean triples, and the discovery of Plimpton, it is clear that the combination of geometry and number theory allows for insight and formulation of probable theories.

Also, the extensions of Fermat's integration of number theory and geometry cannot be forgotten. Both square decomposition and the use of infinite descent played important roles in his career and overall contributions to number theory. Based on all of Fermat's work with pythagorean triangles and triples and his work with square decomposition and the use of infinite descent justify the rightful connection of geometry and number theory. In the 17th century, it was a matter of the mathematical community refusing to embrace or even acknowledge this connection. Regardless of attitude, the connection between the queen and geometry has been and is present. It is important that geometry courses in secondary mathematics make this connection known and take advantage of this connection and its positive attributes, which is bound to intrigue students more than if geometry is simply an interlude between algebra courses. The history of the queen is quite the tale, which offers intrigue and beauty, matching the intrigue and beauty of number theory, itself. The story of number theory also offers inspiration, in particular, inspiration to female mathematics students. Sophie Germain's success in such a male-dominated world offers hope to female students that they, too, have a place in mathematics.

Fermat was a rebel. He founded this thing called modern number theory, which bucked the trends and attitudes of the 17th century. This queen was not taken out by a rebellion. She started her own rebellion and gained a kingdom.

ENDNOTES

[1] Plimpton 322 is number 322 in the G.A. Plimpton Collection at Columbia University; this is the origin of its name (Joseph, 2000, p.115).

[2]The concept of incommensurability was deeply embedded in Greek mathematics and its roots are seen in *The Elements*. Babylonian tablets indicate that this concept has been known since 1800-1500 BC. These tablets "supposedly demonstrate knowledge of the fact that some values cannot be expressed as ratios of whole numbers" (Roskam, 2009, p. 277). The Greek discovery that the hypotenuse of a right-angled triangle (with the congruent sides equal to a length of 1) is $\sqrt{2}$, an irrational quantity, led to their understanding of incommensurability. "Prior to this inexorable discovery, the Pythagoreans viewed numbers as whole number ratios…" (Roskam, 2009, p. 277; Edwards, 1977, p. 4).

[3]Ironically, Diophantus never restricted himself to solutions with integers; he was concerned about rational numbers, in general. However, in modern terminology, "Diophantine" is practically synonymous with "integer," as in Diophantine equations (Edwards, 1977, p. 26).

[4]Negative and trivial solutions were unacceptable because they were treated with suspicion in 17th century mathematics (Edwards, 1977, p. 3).

[5]Since a and b cannot both be odd (this is a necessary condition for the formulation of a primitive pythagorean triple), $\frac{1}{2}ab$ is always an integer (Edwards, 1977, p. 11).

ACKNOWLEDGEMENT

THANK YOU TO DR. BHARATH SRIRAMAN FOR HIS MENTORSHIP, PROVIDING NUMEROUS BOOK RESOURCES AND MOTIVATING ME WITH THIS TOPIC.

REFERENCES

Buck, R.C. (1980). Sherlock Holmes in Babylon. *The American Mathematical Monthly,* 87(5), Retrieved from http://www.jstor.org/stable/2321200

Campbell, S.R., & Zazkis, R. (2006). Number theory in mathematics education: Perspectives and prospects. Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Dudley, U. (2008). *Elementary number theory*. Mineola, NY: Dover Publications, Inc.

Edwards, H.M. (1977). *Fermat's last theorem: A genetic introduction to algebraic number theory*. Springer-Verlag.

Goldman, J.R. (1998). *The queen of mathematics: A historically motivated guide to number theory*. Wellesley, MA: AK Peters.

Joseph, G.G. (2000). *The crest of the peacock: non-European roots of mathematics*. Princeton, NJ: Princeton University Press.

Kleiner, I. (2005). Fermat: The founder of modern number theory. *Mathematics Magazine*,78(1), Retrieved from http://www.jstor.org/stable/3219268

Mahoney, M.S. (1994). *The mathematical career of Pierre de Fermat.* Princeton, NJ: Princeton University Press.

Robson, E. (2001). Neither Sherlock Holmes nor Babylon: a reassessment of Plimpton 322. *Historia Mathematica*, 28, 167-206.

Robson, E. (2002). Words and pictures: New light on Plimpton 322. *The American Mathematical Monthly*, 109(2), 105-120.

Roskam, J. (2009). Book X of the elements: ordering irrationals. *The Montana Mathematics Enthusiast*, 6(1&2), 277-294

Sierpiński, W. (2003). *Pythagorean triangles.* Mineola, NY: Dover Publications, Inc.

Vella, A., Vella, D., & Wolf, J. (2005). An extension of the fundamental theorem on right-angled triangles. *The Mathematical Gazette*, 89(515), Retrieved from http://www.jstor.org/stable/3621222

Weil, A. (1984). *Number theory: an approach through history; from Hammurapi to Legendre.* Boston; Basal; Stuttgart: Birkhäuser.