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Beyond Postmodernity in Mathematics Education?

Ole Skovsmose¹

Abstract: A radical form of postmodernity is presented with reference to Nietzsche's ideas with respect to truth, knowledge, sciences, progress, democracy, and ethical values in general. Thereafter is presented Foucault's archaeology of knowledge. This brings us forward to the notion of *genealogy*, which is a defining idea for the postmodern conception of critique. However, it is emphasised that a critique can address the *generativity* of mathematical rationality by considering mathematics-based fabrications. Finally, *imagination* is presented as yet another feature of a critical enterprise. It is illustrated how such a three dimensional critical enterprise is relevant for both mathematics and mathematics education. In this way the paper suggests moving beyond the postmodern outlook.

Key-words: critique, genealogy, generativity, fabrication, imagination, postmodernity, mathematics, mathematics education.

Whilst postmodernity has brought new profoundness to critical activities, it has also formed some limitations. I find it important to address both aspects and in that way try to move beyond postmodernity. I am going to discuss this possible move with reference to mathematics and mathematics education.

The label postmodernity has been used widely with reference to new trends in architecture, art, and literature which break with modernist principles; it has been used with reference to new conditions for knowledge production; and it has been used in social theorising to refer to new social and cultural phenomena. I am going to use postmodernity as a reference to a critique of Modernity, and I will concentrate on philosophical aspects of this critique.

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Modernity itself can be associated with some general conceptions of science: that natural sciences will be able to reveal the secrets of nature, and that science in general can be both objective and neutral. An integral feature of Modernity is the recognition of the importance of science, knowledge and education, and the whole Enlightenment project makes integral these aspects of the modern outlook. Furthermore, Modernity can be associated with the emergence of ideas about democracy, freedom, and equality. Clusters of such different ideas are brought together by the notion of progress: self-improvement is a human possibility.²

Taking a closer look at the modern period we find that very different socio-political phenomena developed together with the scientific discoveries, the formulation for democratic ideals, and the preoccupations about enlightenment and progress. Colonialism and exploitation took many forms, racism as well. During Modernity one finds laborious attempts at providing a scientific underpinning of racist classifications of human beings as being more or less developed. Similar classifications were applied to languages: some were identified as being more developed and complex than others. Such ethnic and linguistic classifications were weaved together with notions of progress and development and theorised into grand discourses that could legitimate colonialism and suppression.³ Zigmunt Bauman made a gloomy addition to such observations by emphasising that even the holocaust can be seen as being made possible through discourses established within modernity.⁴ With such observations we have entered a fierce critique of Modernity, and this is what postmodernity is about.

Let me try to summarise how we are going to proceed. I will present the most radical form of postmodernity by outlining some of Nietzsche's ideas about truth, knowledge, science, progress, democracy, and ethical values in general. Then I will present an example of the archaeology of knowledge by referring to studies by Foucault. This brings us forward to the notion of *genealogy*, which is defining for the postmodern conception of critique. I will discuss genealogy with respect to mathematics education and emphasise the extreme importance of this line of critique. However, I will then emphasise that a critique addressing *fabrications* that can be related to mathematical rationality is crucial as well. Finally, I want to add *imagination* as crucial for a critical activity. By bringing

² For a discussion of the notion of progress see Bury (1955), and Nisbet (1980).

³ See, for instance, Said's study of orientalism (1979).

⁴ See Bauman (1989).

together genealogy, fabrication and imagination I try to broaden and deepen critical enterprises, and in this way to move beyond postmodernity.

1. Nietzsche's radical postmodernity

Friedrich Nietzsche confronted the official grand narratives of Modernity.⁵ He found that Modernity – although it had taken steps away from orthodox Christianity – was still lingering in a religious world-view. Nietzsche acknowledged that since Descartes all philosophers had tried “to assassinate the old concept of the soul”, and in this way philosophers have attempted to “assassinate the basic assumption of Christian doctrine.” However, Nietzsche's point is that “overtly or covertly modern philosophy is anti-Christian, although it is by no means anti-religious” (*Beyond Good and Evil*, §54). And this is the problem.

Maybe modern philosophy has assassinated the soul and other religious notions, but it maintains a religious outlook by installing new idols. The modern idols have the form of philosophical ideas through which reality becomes judged. Nietzsche's postmodernity hammers into pieces any such idols. Let us look at them one by one.

The idea of truth. According to Nietzsche there is no truth to be found anywhere.⁶ We talk, and not least through Modern philosophy, as if we are able to discover truths. The distinction between appearance and reality plays a fundamental role in the modern world view, as presented by Descartes, Galilei, Locke and many others; it was crucial as well for the formulation of Platonism which Nietzsche is also attacking. The distinction points out that behind appearance there is reality, and the claim is that true knowledge concerns this reality. But, according to Nietzsche, there is no reality behind appearance: “The ‘apparent’ world is the only one: the ‘real’ world has only been *lyingly added*...” (*Twilight of the Idols*, Reason in Philosophy, §2).⁷ The reality behind appearance

⁵ In *Ecce Homo* Nietzsche makes comments on his previous publications and about *Beyond Good and Evil* he states that it is in essence a critique of modernity (*Ecce Homo*, *Beyond Good and Evil*, §2). It could naturally be added that Nietzsche confronted much more than the Modern outlook: Christianity in general and the whole philosophical tradition since Plato.

⁶ See also Clark (1990).

⁷ In this and all other citations the italics is made by Nietzsche in the original.

is a myth. There is no reality which truth is about. Nietzsche asks: “What, then, is truth?” And he answers: “A mobile army of metaphors, metonyms, and anthropomorphism – in short, a sum of human relations which have been enhanced, transposed, and embellished poetically and rhetorically, and which after long use seem firm, canonical, and obligatory to a people: truths are illusions about which one has forgotten that this is what they are; metaphors which are worn out and without sensuous power; coins which have lost their pictures and now matter only as metal, no longer as coins.” (“On Truth and Lie in an Extra-Moral Sense”) Truth has no permanence; it is an illusion. Truth is not about anything; it is a way of talking; it is a mobile army of metaphors. Truth is just a modern idol that has substituted ideas about the existence of paradise and of an eternal life after death.

The idea of knowledge. Nietzsche introduces a radical perspectivism.⁸ There are no points outside the stream of life from where one can look at things, judge things, and formulate true statements. One is instead submerged into this stream. Nietzsche’s perspectivism destroys any hope of establishing knowledge, according to any Modern aspirations. What is called knowledge does not represent any “insight” about certain states of affairs. As a consequence, Nietzsche reaches an epistemic Darwinism: “It is improbable that our ‘knowledge’ should extend further than the strictly necessary for the preservation of life. Morphology shows us how the senses and the nerves, as well as the brain, develop in proportion to the difficulties in finding nourishment.” (1968: 272, §494) Formulated differently, one can see the brain as an organ among other organs. It is an organ necessary for our survival. The secretions produced by the brain are no more unique than the secretions produced by any other of our organs. Sweat is necessary for our survival, and so is knowledge. That we happen to call the output from the brain “knowledge” does not make it any different than any other forms of biological extracts.⁹ The problem, however, is that the notion of knowledge, through immense philosophical misinterpretation, has become reified as an idol.

The idea of science. If knowledge is not about anything, and certainly not about truth, what then to think of science, the celebrated institution of Modernity? Just a different idol. Science cannot represent any search for truth as there is no such thing to search for. What then to think of the phenomenon that is in fact

⁸ See also Hales and Welshon (2000).

⁹ Nietzsche states that the “sense for truth” will have to legitimise itself as a “means for the preservation for man, as *will to power*” (*The Will to Power*, §495). The sense for truths is useful like any other senses. It helps to ensure our survival.

referred to as science? Nietzsche simply states: “A ‘scientific’ interpretation of the world, as you understand it, might therefore be the most stupid of all possible interpretations of the world, meaning that it would be one of the poorest in meaning.” (*The Gay Science*, §373)

The idea of progress. It has been claimed that progress is a defining element in Modernity. In the essay “A Discourse on Inequality” from 1755, Rousseau emphasises that, contrary to animals, human beings have a “faculty of self-improvement”.¹⁰ This idea is axiomatic for the whole Enlightenment, which claims that knowledge and education are crucial for the common welfare and for progress in general. The *Encyclopaedia*, to which Rousseau also contributed, is an expression of the idea that knowledge should be collected and divulged broadly.¹¹ Nietzsche just shrugs his shoulders: “The last thing I promise is to improve humanity.” (*Ecce Homo*, Prologue §2, my translation) And even more explicit: “‘Progress’ is merely a modern idea, that is to say a false idea.” (*The Anti-Christ*, §4)

The idea of democracy. As progress is an illusion, one should not expect Nietzsche to think any better of democracy, which has been celebrated as part of modern progress. Nietzsche has not anything positive to say about the French Revolution, and in general he finds nothing worthwhile captured by notions like liberty, equality and fraternity. Instead Nietzsche sees democracy as a miserable expression of a herd moral: “Indeed, with the help of a religion that played along with and flattered the most sublime desires of the herd animal, we have reached the point of finding an ever more visible expression of this morality even in the political and social structures: the *democratic* movement is Christianity’s heir.” (*Beyond Good and Evil*, §202)¹² Furthermore he characterises “modern democracy ... as the decaying form of the state” (*Twilight of Idols*, §39). Nietzsche is certainly not to be found in any democratic camp.

Ethical values. According to Nietzsche, the slave morality put values upside down by celebrating the poor, powerless, suffering, deprived, sick, ugly, etc., and by nominating the noble, powerful, and beautiful as evil, cruel, lustful

¹⁰ See Rousseau (1984: 88).

¹¹ It should be noted that Rousseau not only presented the notion of self-improvement; he also developed a counter story to the general celebration of progress by commemorating the natural state of affairs.

¹² Nietzsche also observes: “The *overall degeneration of man*, right down to what social fools and flatheads call their ‘man of the future’ (their ideal!); this degeneration and diminution of man into a perfect herd animal; this bestialization of man into a dwarf animal with equal rights and claim is possible, no doubt about that!” (*Beyond Good and Evil*, §203)

insatiable, and godless (*Genealogy of Morality*, First treatise, §7). This slave morality is, according to Nietzsche, contrary to life as “life itself *in its essence* means appropriating, injuring, overpowering those who are foreign and weaker; oppression, harshness, forcing one’s own form on others, incorporation, and at the very least, at the very mildest, exploration” (*Beyond Good and Evil*, §259). Thus Nietzsche emphasises that “ ‘exploitation’ is not part of the decadent and imperfect, primitive society: it is part of the *fundamental nature* of living things, as its fundamental organic function; it is a consequence of the true will to power, which is simply a will to life” (*Beyond Good and Evil*, §259). After such claims it is not surprising that Nietzsche sees democracy as an idol; just a modernised version of slave morality.

It would be a risky business to try to assume the whole Nietzsche package including all its pre-modern post-modernity. And this is not what has been done, but Nietzsche’s radical post-modernity has inspired many.¹³ His radicalism invites for a *deconstruction* of celebrated ideals. The notion of deconstruction was formulated by Jacques Derrida in *On Grammatology* first published in 1967, and I find that this notion describes precisely what Nietzsche is suggesting as well as what many postmodern investigations are completing.¹⁴ I will use it as a general characteristic of postmodern critical endeavours, naturally acknowledging that Nietzsche talked about genealogy when identifying ideas as idols. The general claim is that behind apparently decent scientific standards, aspirations about progress, ethical values etc, one will find much less noble forces acting out. The aim of a genealogy (a deconstruction) is to reveal this reality – so different from any idealist reality that has just been “lyingly added”.

2. Foucault’s archaeology of knowledge

According to modern standards, science should not be subjective, but objective; it should not include special priorities or particular perspectives, but distance itself

¹³ See Deleuze (1986) for a careful interpretation of Nietzsche’s ideas in general and in particular between the Apollonian and Dionysian ways of life.

¹⁴ Derrida is inspired by Heidegger who uses the notion *Abbau*, for deconstruction ideas. This German word resonates with *Aufbau*, which means construction. Thus *Aufbau* and *Abbau* can be seen as an inverse processes. *Abbau* can be read as destruction, but it is better understood as an unmaking, thus Derrida emphasises: “The movements of a deconstruction do not destroy structures from the outside” (Derrida, 1974: 24)

from any interests. Furthermore it was assumed that science, due to its own intrinsic dynamic, does in fact acknowledge such demands. So, if one just lets science develop according to its intrinsic standards it will reach objectivity and neutrality. Thus science is an expression of solid epistemic standards.

Inspired by Nietzsche, one should consider the trust in science as being a variation of a religious trust, just worshipping different idols – as false as any other idols. In “Truth and Juridical Forms”, Michel Foucault refers several times to Nietzsche as an important source of inspiration. Foucault does not address natural science, but concentrates on psychology, psychiatry and related discourses. He pursues the idea that the polished surface of sciences covers a muddled situation. He presents his archaeology of knowledge in terms of a genealogy. Through detailed historical studies he demonstrates how any claim about objectivity and neutrality evaporates. What constitutes facts at a particular historical moment becomes questioned in the next moment. Facts appear time-dependent as do truth and scientific categories.¹⁵ It appears that science is developing, not in any continuous process of accumulation, but sometimes in surprising and apparently irregular ways. In the interview “Truth and Power” Foucault refers to a particular observation that illustrates his point: “In a science like medicine, for instance, up to the end of the eighteenth century one has a certain type of discourse whose gradual transformation, within a period of twenty-five or thirty years, broke not only with the ‘true’ propositions it had hitherto been possible to formulate but also, more profoundly, with the ways of speaking and seeing, the whole ensemble of practices, which served the purpose of medical knowledge.” (Foucault, 2000: 114)

One has to do with some overall and radical changes of a scientific outlook. In some places Foucault talks about an episteme as comprising a broad set of ideas, assumptions, conception, priorities that function as a scientific world view for a particular science at a particular moment. Foucault also uses the notion of discourse for such a world view. So what Foucault is referring to above is a change of a medical discourse taking place within a relatively short period. This change included changes of what were considered true propositions, and as emphasised by Foucault this change included not only a change in ways of talking; it also included changes of medical practices. As sciences become constituted through discourses, and discourses are condensing complexities of ideas, priorities, interests, etc., sciences becomes deprived of all idealised

¹⁵ It is interesting to compare this observation with Grabiner (1986), who asks if mathematical truths are time-dependent.

standards. Discourses are expression of powers, and as a consequence knowledge and power becomes intimately connected. Thus through his genealogy, Foucault deconstructed any idealised conception of science. In fact we have got close to Nietzsche's perspectivism including the observation that behind any idealised surface there exists a chaos of powers.

This dynamics becomes condensed by Foucault through the notion of "regime of truth". In the interview "Truth and Power" Foucault states: "Each society has its regime of truth, its 'general politics' of truth – that is, the types of discourse it accepts and makes function as true; the mechanisms and instances that enable one to distinguish true and false statements; the means by which each is sanctioned; the techniques and procedures accorded value in the acquisition of truth: that status of those who are charged with saying what counts as true." (Foucault, 2000: 131) So what Foucault observed with respect to medicine was a change of a regime of truths. And, as pointed out, it is possible to observe such changes with respect to many different scientific discourses. Through the notion of "regime of truths" Foucault draws directly on Nietzsche's characteristic of truth as "mobile army of metaphors, metonyms, and anthropomorphism".

Through his investigations of discourses Foucault made an important contribution to critical enterprises. He showed that what is assumed to be stable is far from being so. Truth is time-dependent; it is context-dependent; it is power-dependent; it is dependent on whatever. The notion of fact has simply become dissolved through Foucault's acid deconstruction.

3. Genealogy

Foucault's studies of genealogy have inspired many critical investigations of education and of mathematics education. Let me, however, concentrate on the later.¹⁶

Many general formulations that accompany mathematics education are in need of being deconstructed. Thus in official documents one can read, most often in a highly elaborated vocabulary, that the aim of mathematics education is to provide mathematical knowledge that can be of general interest for society and of personal relevance for the students. Such formulations assume that knowledge and

¹⁶ For Foucault-inspired discussions of education see, for instance, Popkewitz and Brennan (Eds.) (1998).

education provide an important source of welfare, both for the society and for the individual. With reference to educational practice and acknowledging the formulated aims, one finds many suggestions for improving classroom practice. This reflects another of the grand ideas of Modernity, namely that improvements and progress always have to be worked for.

Genealogies have pulled aside such decorative formulations, and there is much inspiration to be found in Foucault's work for doing so. In *Discipline and Punish*, Foucault talks about discipline, disciplinary society, and disciplinary institutions, and among them the school. With reference to the military Foucault states that "discipline increases the skill of each individual, coordinates these skills, accelerates movements, increases fire power", etc. (Foucault, 1991: 210). Discipline is a defining element in military efficiency. Similar observations, however, apply equally as well to workplaces, hospitals, prisons, and schools: "The disciplines functions increasingly as techniques for making useful individual" (Foucault, 1991: 211). Usefulness can be applied to any category of people and to any institutions in society. However, at the same time "usefulness" is defined with reference to a specific social order. Thus Foucault emphasises that the "growth of a capitalist economy gave rise to the specific modality of disciplinary power" (Foucault, 1991: 221). This power imposes a "political anatomy" in all kinds of institutions, also in schools.

To this observation one just need to add that mathematics as a school discipline also functions as a discipline in Foucault's interpretation of the word. With reference to Foucault, one could claim that mathematics education is observing the political anatomy necessary for establishing usefulness. This observation implies that the formulated aims and good intentions that accompany mathematics education are only serving a decorative purpose. They are lyingly added. Our society is organised according to the demands of the free market, and the educational institutions are docilely responding to the definition of "usefulness" as exercised by this market; mathematics education as well. Such observations bring Thomas Popkewitz to summarise and deconstruct the Modern self-understanding in the following way: "Mathematics is one of the high priests of modernity. Mathematics education carries a salvation narrative of progress into the upbringing practices of schooling. The mathematics of the school is told as a story of progress about the cultured, modern individual whose reason is bounded by the rules and standards of science and mathematics. The narrative is of the enlightened citizen who contributes to the global knowledge society" (Popkewitz, 2004: 251-252). This is a self-understanding that covers the disciplinary reality of mathematics education.

Let me try to summarise some of the very many observations addressing mathematics education as a disciplinary institution. One characterising feature of mathematics education is the many exercises that – lesson after lesson after lesson – are presented to the students. Considering general formulations about, say, leaning mathematics and creativity, the regime of exercises seems paradoxical. However, this regime can be interpreted as preparing for what I have called a prescription readiness.¹⁷ This readiness refers to a capacity of completing small tasks that are defined from outside, that should not be questioned, and that should be completed promptly and with accuracy. Such a prescription readiness makes part of the political anatomy, which defines what is useful and what is not. It can be identified as one of disciplinary features of mathematics education.

Mathematics education makes part of an extended system of tests, examinations, and labellings, which can be considered crucial for establishing the anatomy of the coming labour force. This force needs to be useful; however, there are very many forms for usefulness. For the functioning of the whole capitalist machinery of production, administration, surveillance, transport, economic transactions, etc. one needs a differentiated labour force with very different qualifications. Thus the capitalist machinery operates with a matrix of different forms of usefulness, and this makes a meticulous labelling of individuals crucial. The educational system is taking care of this. As a consequence, students leaving the education system can be located in the proper cell in this matrix of usefulness. A careful labelling of products put out at the market makes things easier for the consumers. This also applies to the labour market of the capitalist society. Among the different school disciplines, mathematics appears eager to provide labels.

Mathematics education makes part a network of power relations. Thus to understand what is taking place in the mathematical classroom cannot simply be studied this classroom. One needs to consider the whole context of schooling. On several occasions, Paola Valero has presented a diagram with a teacher-student-mathematics triangle in its centre, representing the activities in the mathematics classroom. However, she emphasises that this triangle only represents a small part of a grander network, including: youth culture, labour market, family, community, teacher education, staff, school leadership, mathematics education research, international comparisons, policy-making, academic mathematics, technological and scientific development, etc.¹⁸ All such factors, and in particular the power

¹⁷ See Skovsmose (2008)

¹⁸ See Paola Valero's figure as shown in Menghini, et al. (Eds.), (2008: 286).

relations between them, are relevant for interpreting the actual disciplinary functions of mathematics education.

Many postmodern analyses have tended to stop at this point. This would in fact be in accordance with Foucault's approach. A genealogy reveals the powers that operate behind the educational scenery. Naturally we could ask: And what then? Considering the result of the deconstruction, how to improve the educational reality? By formulating such questions we might be sliding away from a postmodern approach. Thus we have to remind ourselves of Nietzsche's remark: "The last thing I promise is to improve humanity." And this formulation can easily be translated into: The last thing a deconstruction could promise is to improve mathematics education. In fact a proper deconstruction – not assuming any of the modern ideals (or idols) including a notion like progress – must leave things as deconstructed.

Many postmodern approaches have assumed this position. I acknowledge that any form of genealogy in terms of a deconstruction is important. However, I do not want to assume that a genealogy represent the whole range of critical approaches. I want to address two more elements by addressing fabrication and imagination. This way I want to bring critical enterprises beyond a post-modern outlook.

4. Fabrication

As the notion indicates, a genealogy is working its way back in time. Through his archaeology, Foucault presented a history of "the given"; he demonstrated how a scientific discourse, a particular regime of truth, become established as a conglomerate of preconceptions. Such critical investigations are powerful, I fully acknowledge this. I want, however, to point out that there are other forms of critical investigations also relevant for mathematics education. Let me illustrate what I have in mind by referring to mathematics. An archaeology with respect to mathematical ideas will dig into their historical origin and try to reveal how knowledge-power dialectics makes part of their genealogy.

One can, however, also address the rationality of mathematics by investigating what can be accomplished through this rationality. What does it mean to bring this rationality into operation? What can be generated through this rationality? What possibilities can be established? What dangers and risks can be established as well? This brings us to consider what can be *fabricated* by means of

mathematics, and in what follows this notion has to be interpreted in a general philosophical way as referring to *homo faber*. I see questions about fabrications as being different from those addressed by a genealogy. While a genealogy concerns the past, fabrications concern the future. Thus fabrications are addressed through studies of *generativity*, which refers to what can be generated or fabricated through mathematical rationality.¹⁹ Such fabrications can have many forms, and I just provide a brief summary:²⁰

Fabricating possibilities. Mathematics makes part of a range of design processes within any technological domain, and here I use technology as referring not only to technological artefacts, like bridges, airplanes, computers, etc., but also to less tangible constructions like schemes for production, health programmes, techniques for surveillance, economic strategies, management principles, etc. In all such areas mathematics provides powerful means for identifying new possibilities. Furthermore, no natural-language formulation of possibilities operates in an equivalent way. For fabricating new possibilities within our already highly mathematised environment, mathematics is unique.

Fabricating facts. It is common knowledge that mathematics makes it possible to describe facts. To claim that mathematics fabricates facts might sound strange, but let us consider an information system through which knowledge and information becomes processed. This system is operating in terms of a computer language including several layers of formal languages. It can be considered a conglomerate of mathematical algorithms. Such a system can, for instance, configure a tax system. This system not only describes taxes; it also prescribes taxes. It condenses decisions about what people have to pay in tax. And there are information systems not only prescribing taxes, but also medications, treatments, loan conditions, working standards, etc. And also such prescriptions become life conditions to a range of people. They turn into facts.

Fabricating risks. Development of technology depends on simulations. This applies to the construction of cars, airplanes, transport systems, houses, shopping centres, robots for production, tax systems, etc. Blue-prints for such constructions have the form of mathematical models. One could assume that a mathematical

¹⁹ For a discussion of the notion of generativity, see Skovsmose (2011b).

²⁰ The following summary follows Skovsmose (2009). Mathematics-based fabrications have also been addressed in terms of mathematics in action. See, for instance, Christensen and Skovsmose (2007); Christensen, Skovsmose and Yasukawa (2009); Skovsmose (2009, 2010a, 2010b, 2010c, 2011); Skovsmose and Ravn, O. (2011); Skovsmose and Yasukawa (2009); Skovsmose, Yasukawa and Ravn (draft); Yasukawa, Skovsmose and Ravn (draft).

simulation model provides a description of the construction when completed. This is, however, only partly true. The completed construction contains a range of properties which are not anticipated by the simulation model. Many of the risks we are now facing – for instance with respect to atomic energy, economy, transport, health care – are connected to not anticipated implications of implemented technologies. And a mathematical model only anticipated things within a particular range of algorithmic rationality.

Fabricating objectivity. Turning an entity into an object of study means that we strip it for a range of properties. This process is supported by mathematics, which highlights precisely the mechanical elements of what we are studying: being a new architecture, a medical treatments, a new configuration of a production line, new security matters, etc. Mathematics helps to fabricate what becomes addressed as an object. In many cases this “objectification” makes one assume that we are addressing things in an “objective” way; but objectivity can be seen as a mathematics-based fabrication.

Fabricating life worlds. Our life-worlds are submerged in mathematics-based fabrications.²¹ If necessary our body becomes treated according to medical standards about blood pressure, cholesterol, glucose, etc. They are all calculated standards, with reference to which diagnoses are formulated, medical treatments are prescribed, and health is monitored. Standards for food quality are calculated as well: What should be considered acceptable levels of pollution is determined through statistical investigations, cost-benefit analyses, market interests, modellings of consumers’ behaviour, all condensed into well-defined numbers nominating what to call non-problematic. Very many work processes take place in highly elaborated technological environments, where automatic processes and human activities are combined into an efficient production system. The market, including the labour market, is structured according to a huge set of mathematically formulated laws and principles. Our transport systems and our means of communication are all expressions of mathematics-based fabrications. In this sense our life-worlds are fabricated.

Naturally, one can see some overlapping between a genealogy and studies of generativity, but still these studies are different. Through a genealogy one sees what has been incorporated, or *acted-into* a phenomenon, while a study of fabrications addresses what might be *acted-out* from the phenomenon. Both forms of investigation play important roles in critical investigations. We have illustrated

²¹ The notion of life-world is used by Husserl, but I am using the notion rather liberally (see Skovsmose, 2009).

generativity in terms mathematics-based fabrications. This way we have addressed mathematics rationality critically. Naturally, we can also discuss generativity with respect to mathematics education. This is a crucial feature of addressing mathematics education critically. However, we now move on to a third feature of critique.

5. Imagination

Beyond genealogy and generative studies there is one more element that I want to consider part of a critical approach, namely imagination. Critique in terms of imagination has been presented by Wright Mills (1959) in terms of sociological imagination. Through such imagination one can reveal that something being the case could be seen differently. This means that one reveals a fact as being, not a necessity but a contingency. I find that a pedagogical imagination makes part of a critical educational endeavour.

Let us assume that we want to establish democracy in the classroom. We want teacher and students to address each other on equal terms and to interaction becomes dialogical. We imagine that this facilitates new learning processes and ensure an empowerment of the students. They could develop more profound new competences in reading and writing the world, to use some formulations that have been explored by Eric Gutstein (2006). And we could expand further our imagination by exploring not only the notion of democracy but also dialogue, empowerment, and related notions like equality, and social justice.

However, one can see all such ideas as illusory by highlighting constraints that configure the educational reality. We can remind ourselves of the power-network that any classroom practice makes part of. Imaginations do not make changes possible; instead imagination could be the carrier of illusions. Inspired by Nietzsche one can see pedagogical imagination as worshipping idols. The point of the careful genealogy that has been conducted with respect to mathematics education is precisely to reveal that beneath the decorative discourse of aims and possible improvements one finds a disciplinary regime. Considering the decorative discourse, everything seems possible, while considering the disciplinary regime nothing appears possible. And certainly not to establish any genuine classroom democracy as mathematics education as a discipline is doomed to insert a prescription readiness into the students and to subject them to a meticulous labelling. The deconstruction of the classroom reality brings into the

open a grid of power relations that establishes a range of necessities which turn pedagogical imagination illusory. This is one line of argument that brings imagination in discredit.

There is, however one more line of argument discrediting imagination, drawing directly on Nietzsche's radical postmodernity. What to think of the range of the very concepts that resource imaginations such as: democracy, dialogue, empowerment, equality, social justice, etc.? According to the deconstruction of the educational reality, such notions might bring about illusions about what could be done in the classroom. According to Nietzsche, however, such ideas are themselves idols. Following he presentation in *Twilights of the Idols* they should be addresses by a hammer. Idols need to be destroyed. Thus the idols developed through Modernity represent religious ideas, although in philosophical disguise. And the ideas that resource pedagogical imaginations belong to the same family of idols. By celebrating such idols one falsely condemns reality by providing suggestions for improvements. However, according to Nietzsche, there are no norms and standards that operate from any elevated position, and certainly not from any transcendental position. There are no ideas positioned outside the stream of life. Everything makes part of a chaotic power dynamics. Within a radical postmodern approach to education there is no use of ideas stimulating a pedagogical imagination. They just represent romanticism and naivety. The falsely condemn the educational reality.

But could we get around this sharp corner, and move beyond postmodernity? Foucault (2000: 33) raises the question if one can talk about "powerless truth" and "truthless power". In other words: Could there be any truths that are not aligned with power? Could there be false powers? Or does power simply define truths? I do not clearly see Foucault's response to this question. I want, however, to answer clearly myself: truth can be powerless, and power can be truthless. If we make this assumption, we are moving away form Nietzsche's radical postmodernity. But are we heading straight into romanticism?

We have to be careful. Nietzsche knocked down any ideals as were they idols. He feverishly attacked any form of transcendental idealism. To Nietzsche any ideas are formed within the stream of life. I agree with this. So let us not try to revive any new form of transcendence. But does that put an end to any ethical principles and any conception about democracy, equality, justice? My principal point is that ethical principles and idealised notions need not be revealed or discovered. They can be constructed. *I want to propose a social constructivism with respect to ethics.* I see ethical standards and idealised notions as emerging through laborious social processes. They are human constructions, and as such

they are historical; they are temporary; they can be changed. They represent human visions, expectations, and hopes; and precisely visions, expectations, and hopes make part to the stream of life. For me imaginations cannot be discharged, as done with reference to Nietzsche, through the claim that they assume a transcendental reality. They do not. Imaginations make up part of life. They represent the claim that what is the case, it not *necessarily* the case. They try to reveal facts as contingencies. To me such revelations are important critical activities.

Let us now return to the claim that imaginations are illusions due to the very educational realities. Schools as disciplinary institutions are no simple contingencies. They make part of the whole power-network within which the educational system is caught, like a small insect in a spider's net. This power-network includes so many restrictions that one cannot assume that, say, democratic principles could be established in the classroom. This could be obstructed by the traditions of the classroom, the students' expectations, the demands for test and control, the expectations of parents, the organisations of the textbooks, the general time pressure for the teachers, etc. I agree that all such observations could very well bring us to the conclusion that establishing democracy in a particular classroom is (almost) impossible. Still, and this is my point, imaginations – also imaginations that cannot be implemented – is of critical importance. For me it is an important critical step to explore the scope of contingencies. Any particular social institution is positioned between being a necessity or an open contingency. This also applies to the education. There is no *a priori* analysis that designates every feature of this disciplinary institution as being necessary. It is, like any other institution, criss-crossed by contingencies. Naturally, seeing a difference is very different from making a difference, but this does not make imaginations superfluous. Seeing a difference is an initial critical act; making a difference might be possible.

I have been concerned about some postmodern limitations of critical activities which reduce critique to genealogy. I agree that a genealogy is an extremely important critical device. I also agree that this form of deconstruction has gained tremendous force through postmodern investigations. My position, however, is that critique has more dimensions. Genealogy is important, so are generative investigations, and so is imagination.

6. As conclusion: Beyond postmodernity

Mathematics education has been criticised though genealogical investigations. This way it has been revealed how a disciplining order has become incorporated into this form of education, and how power dynamics are operating in and behind classroom practices. However, there is no need to stop the critical investigations by such deconstructions. We can address the generativity of mathematics education in different contexts and with respect to different groups of students. What possibilities could be constructed for them? This question could be addressed not only with reference to further disciplinary initiatives, but also in terms of students' perspectives and with reference to their life-worlds. It is important that any educational practice should become recognised as a contested practice, and this brings us directly to the notion of pedagogical imagination.

I suggest that a critique of a phenomenon, referring to mathematics or to mathematics education, addresses its genealogy, its generativity, and that it becomes contrasted with imaginations. The postmodern form of critique focussed on genealogy. That gave an impressive depth to many critical studies, but it also inserted limitations. My point is to broaden the critical activities I suggest that it is possible to move beyond some limitations postmodernity has put on critical enterprises.

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Inquiry - without posing questions?

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Abstract: This article discusses what inquiry conversations could mean when learning mathematics.³ Referring to Gadamar's distinction of true and apparent questions it is discussed what it takes to be inquiring and if this attitude necessarily includes posing questions. Which qualities are expressed in inquiring questions, and what other ways of communicating may have an inquiring function in learning conversations? The intention is to develop and frame the concept of 'inquiry' in learning conversations, and this is the focus of analysis of an authentic classroom situation, where teacher and pupils are exploring the concept of 'volume'. Further, this analysis informs a discussion of listening as an important element of an inquiring learning conversation.

Keywords: teacher education, classroom communication, inquiry, listening, dialogue

Introduction

[...] the path of all knowledge leads through the question. To ask a question means to bring into the open. The openness of what is in question consists in the fact that the answer is not settled. It must still be undetermined, awaiting a decisive answer. The significance of questioning consists in revealing the questionability of what is questioned. [...] Every true question requires this openness. Without it, it is basically no more than an apparent question. We are familiar with this from the example of the pedagogical question, whose paradoxical difficulty consists in the fact that it is a question without a questioner.

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*Or from the rhetorical question, which not only has no questioner but no object.
(Gadamer, 2004, p.357)*

Teaching and learning conversations are characterised by questioning and answering. The teacher asks, and the pupils answer the teacher questions. Questions are considered to be crucial for the pupils' learning, and as such they can be seen as a *descriptive* quality of teaching as a phenomenon. There are also *normative* qualities of questions that can be considered good or bad, interesting or boring etc. In earlier studies we have been concerned with both such qualities of classroom communication and how qualities of communication may influence qualities of learning (see e.g. Alrø & Skovsmose, 2002; Johnsen-Høines, 2002).

Short sequences of communication is a descriptive quality of classroom communication. The teacher asks a question, the pupil answers the question, and the teacher evaluates the answer. Sinclair and Coulthard (1976) have described this communication pattern in terms of Initiative-Response-Feedback (IRF). This pattern is strongly positioned in the classroom, and it is easily recognized by the participants who can experience it as disciplining and authoritarian, but also as nice and comfortable as it is well-known in the context of schooling. Sometimes the IRF-pattern is broken by a more inquiring conversation style through which the participants collaborate in finding answers to questions they did not have beforehand. In such cases the communication structure becomes less predictable, as the participants in a cautious listening and dwelling manner explore ideas together. The IC-Model⁴ describes such inquiring conversations in terms of dialogue (Alrø & Skovsmose, 2002, p. 100; 2006, p. 112).

In this article we want to examine how such unpredictable conversations are initiated and unfold in the mathematics classroom. What characterizes inquiring and curious questions, who poses them, and how do they influence the direction of the learning conversation? However, inquiry may not only be about questioning. Inquiry and interrogative is not necessarily the same thing (Lindfors, 1999, p. 62).

⁴ Alrø og Skovsmose (2002) describe a dialogue as a conversation with such qualities. Through analysis of classroom communication they identify an *IC-Model* (Inquiry Cooperation Model) that includes dialogic speech acts like *getting in contact, locating, identifying, advocating, thinking aloud, challenging, and evaluating*.

What is inquiry?

Inquiry is a conversation with specific qualities. A dialogue includes equality, inquiry, unpredictability and risk taking (Alrø & Skovsmose, 2002). This means to have an open, curious, and wondering attitude towards the subject, the partner of conversation, oneself, the interaction and the relationship. "In order to be able to ask, one must want to know, and that means knowing that one does not know." (Gadamer, 2004, p.357)

According to Bakhtin utterances are addressive, which means that they include continuations and expect responses. Thus, to Bakhtin every utterance is a response to both earlier and future utterances – it is part of a communication chain. Accordingly, an utterance cannot be understood in isolation from its past and its future. Answers, however, that do not include new questions, are not dialogic. They close the conversation (Bakhtin, 1998; Johnsen-Høines, 2002). Gadamer describes such questions as "apparent questions" (2004, s. 357). Questions may be open and inquiring to possible answers, but they need not be. Other utterances including the answer might be inquiring as well in a dialogue, if they include inquiring continuations.

Thus, an inquiring approach implies an opened continuation full of curiosity and wish to understand more. However, inquiry presupposes an already existing insight into the field of inquiry – for instance as pre-understandings or underlying hypotheses. Further, for people to join an inquiry it requires an invitation and a common agenda in order to get into the potential field of interest. This agenda is about developing the subject, the community, and the interaction in progress (dialogicity).⁵

Inquiry is directed towards ongoing reflections and towards new inquiring questions. "The art of questioning is the art of questioning ever further - i.e. the art of thinking. It is called dialectic because it is the art of conducting a real dialogue" (Gadamer, 2004, p. 360). To be inquiring can mean to be experimenting, (re-)searching, wondering, trying out, anticipating. Gadamer underlines this point when he describes the hermeneutic characteristics of the question: "The essence of the question is to open up

⁵ Referring to Bakhtin's text theoretical perspective Johnsen-Høines (2002) describes, how childrens' understanding of mathematics arises from the meeting between, the conjectures between or the conflict between different understandings. Børtnes (1999, p. 24) writes: "A meaning manifests its depths when it meets and touches another and strange meaning: it seems like the beginning of a dialogue between those who conquer the one-sidedness and closedness of those meanings" [authors' translation]. Thus, to describe dialogicity means to describe movements (interaction) between meanings (utterances).

possibilities and keep them open” (p. 298). To be inquiring can be confronting established ways of thinking and ways of talking. It can be challenging.

To describe communication as inquiry is to describe how the participants aim at a common field of interest and to identify dialogical characteristics. In this article, the didactical context of organising dialogic teaching and learning activities is of special interest.

A question is directed (addressed) towards one or more answers. However, the person who asks does not necessarily have a curious aim. A question may be posed without an inquiring approach. The IRF-conversation serves as an example. If the teacher asks questions to which he has the authoritative answer himself, pupils are supposed to give minimal responses, i.e. they contribute as little as possible to the ongoing conversation (See Lemke 1990; Alrø, 1995).

Such questions can be necessary and useful into the context of learning. However, they might also unify or limit the pupils’ learning activities. That the teacher has the right answer to questions is one way of describing the quality of such interactions. That the answers are defined by the textbook would be another. That the questions might be used as training for a coming test would be a third one. The interaction might appear as a rhetorical trial practised by the pupils themselves, in which they focus on being good at drilling. Such an approach, however, can obstruct a subject-oriented curiosity. Thus, the IRF-pattern does not support empowerment and educating pupils as independant and critical reflective citizens.

To ask questions can appear to be fairly authoritarian. Questions legitimise authoritative ways of communicating, when it requires a continuation in line with the intention of the question (Johnsen-Høines, 2002, p. 87). Questions can be used as tool for oppressing or disciplining, e.g. when the person to answer does not understand or has no possibility to follow the ”rules of the game” (Streitlien, 2009). They may not know the answer, or they may not understand what is expected from them. To feel free from such oppression can mean to question the questions. It requires an inquiring stance and practice of communication in terms of a curious, investigative, challenging and critical attitude to critically reflect different kinds of questions.⁶

⁶ For a discussion of being critically reflective towards authoritarian texts, see Johnsen-Høines (2002, p. 201–207).

The situation we refer to in what follows takes place in a classroom where student teachers do practice teaching⁷ in a fifth grade mathematics classroom. The student teachers participate together with teacher trainers and didacticians in a school-industry partnership. The aim of this partnership is for the students to learn mathematics and mathematics in use when pendling between different contexts of school and industry (Johnsen-Høines, 2010). The group of student teachers referred to in the situation below collaborates with a grocery shop. They collect data for their theses focussing on language and learning. The conversation below is part of the student teachers' data. We analyse the conversation in order to get an insight into the collaborative interaction taking place between student teacher and pupils when dealing with the subject of 'volume'. The focus of analysis is the function of questioning in the interaction including intention, context, and potentials for learning mathematics. A special interest is paid to utterances of inquiry.

The analytical approach refers to pragmatics and the study of language in use (Austin, 1962; Searle, 1969; Wunderlich, 1975) that operates with a broad understanding of language as a combination of words, body and voice. The analysis considers the use of language in the conversational context and how the participants create meaning of the conversation. This qualitative approach starts from what is actually being said and done. As the use of language is the data of analysis, this is what is quoted and referred to as documentation when interpreting what is going on in the conversation in the classroom context. In this way data can be explicated and interpretations be challenged from other research perspectives.

***"Underneath the desk" – an example from 'Real-life Education'*⁸**

Student teachers teach mathematics close to practice in a 5th grade mathematics class. They collaborate with a grocery shop situated in the neighbourhood of the school. The pupils are going to learn about trade and economic life in the community. They are going to apply mathematics to real-life situations and they are going to learn mathematics from real-life situations. The important thing is that learning mathematics is closely related to using mathematics. The student teachers plan their own practice teaching by also

⁷The notion *practice teaching* is in this chapter referring to the practicum that is included as part of the teacher training study.

⁸ In Norwegian *praksisnær undervisning*. The schools are situated in Fjell, one of the three municipalities participating in the initiative 'Real-life Education'.

reflecting what such a concept could involve. Together with teacher trainers and didacticians they want to study the communitive conditions for learning mathematics. They study how the mathematics conversations develop when the pupils act in the shop and when they act in the classroom. They have chosen *volume* as their mathematical subject. They prepare themselves by discussing their own insight in the subject, and by discussing the coming presentation of the subject in the classroom. What would they want the pupils to learn and how would they arrange the learning environment. For instance they do not find it sufficient for the pupils to know that volume is "length times width times height". The teacher trainer stresses that it might be difficult to teach 'volume' in that class. It has not been taught before and the pupils have been working only a little with the concept of 'area'. But the student teachers feel positively challenged by the situation. They discuss how the pupils would get insight into the subject of 'area' by formulating this as "how much space would it take..." Further, so they argue, they have previously been working with space by referring to the measure of capacity. They have had pupils find out how much water could be contained in different kinds of bowls, and the pupils have been measuring and making their guesses. For instance they have used litre as the unit of measure, not cubic decimetre. They would also use milk cartons. "How much water could be contained?" would be another formulation of the problem than "How many milk cartons could be contained?" In such ways the students want to stimulate the pupils' movements between different ways of thinking.

"How much space would it take?" is a didactical grip to develop a concept of length, area and volume, and to develop an insight into the process of measuring. The pupils are Norwegian children from the western part of the country. They have a good basis for understanding the concept of area through their experience with ferry traffic. They know how many cars could be placed at the deck, and they are able to estimate whether there is space enough for them to go or not.

The student teachers enter a discussion with the teacher trainer and the didacticians about approaching the subject of volume in this respect. Further, they want the pupils to get a meta level insight as regards what it would mean to measure. "It is necessary but insufficient that they become able to measure length times width times

height”, the student teachers emphasize both before and after the teaching course. ”How much space would it take?” indicates an inquiring approach.

In what follows we present a classroom situation where the pupils work on presented exercises. The student teacher⁹ catches sight of Karl (a pupil) who is sitting under his desk. Jonas (another pupil) is standing next to the desk. The student teacher arrives at the desk and addresses Karl¹⁰:

ST: How are you doing? Did you lose anything?

Karl: Not at all. We finished the exercises.

ST: OK, can I see them?

Karl: Please, show them, will you? [addressing Jonas] I am a little busy.

ST: What keeps you busy, then?

Karl: I am figuring out how many pizza boxes could be placed underneath the desk.¹¹

Jonas: Yes, we pretend the desk to be the shop, and then we just need the pizzas inside. Are we aloud to try this? We finished the other things [exercises].

ST: How exciting. Please, go on.

The student teacher initiates the conversation from a pedagogical and organisational perspective. The pupils are going to do the exercises they are supposed to do, and the student teacher is the classroom manager. If we interpret her question from this perspective it is not inquiring if Karl has lost anything. It would rather be a way of indicating that Karl is not supposed to sit underneath the desk. Thus, her question can be seen as an attempt to make the pupils work and behave. She is supposed to make sure that the pupils work in a fruitful way. However, a good reason to sit *underneath* instead of *at* the desk, would be to have lost something that has to be found. Karl’s answer indicates that he has understood the implicit managing meaning of the teacher question. He explains that they have finished the exercises they were supposed to do – and therefore he

9 There is only one student teacher participating in the quoted excerpt.

10 The transcript is translated from Norwegian.

11 Karl talks about a small scale pizza box of those they have seen in the shop. Making small scale boxes is one of their previous activities in the project.

would have a valid reason to sit underneath the table. He is "a little busy" and that is why he asks Jonas to present their work. This may be understood as a little unpolite way of neglecting the student teacher's request of showing her the exercises, as if he is doing something much more important. The student teacher's question is backwards directed towards activities the pupils have already been doing, but Karl's answer is forwards directed. He may be much more occupied with the new idea they are just about to figure out. However, Karl shows his good intentions by asking Jonas to show the exercise as he is not sitting underneath the desk.

Karl initiates a new direction of the conversation by telling that he is busy. The student teacher follows up: "What keeps you busy, then?" She does not seem to notice Karl's kind of unpolite rejection of a teacher enquiry, and she does not insist in controlling the exercises. On the contrary, she seems to be genuinely interested in Karl's upcoming activity. We cannot hear it from her tone of voice, but from the continuation of the conversation it seems obvious to interpret her question as inquiring, i.e. she asks in order to get to know, not in order to controll. The student teacher praises ("how exciting") the pupils and encourages ("please, go on") them to continue their inquiry.

Karl's continuation may not be caused by the comment of the student teacher at all. His argumentation and the follow up from Jonas indicate that the two of them continue almost without further reflection of the question. Jonas asks the student teacher if they are "allowed" to go on with their activity, and the student teacher accepts. Again, the question has a managing function in the classroom context. The boys seem to be well aware that they participate in, and at the same time break with, this classroom management. "Are we aloud to try this? We finished the other things [exercises]". They use the opportunity to tell the student teacher about the alternative activity they cope with, and simultaneously they accept the authority and role of the student teacher. The argument for being "allowed to" is that they have already finished the things they were supposed to do.

The pupils have initiated a conversation about their inquiry cooperation. They have entered a landscape of inquiry (Alrø & Skovsmose, 2002; Skovsmose, 2001) in which they themselves have defined their route into the classroom subject of 'volume'. One might say that through the pizza box exercise, Karl and Jonas have taken ownership

of their learning process. They are so preoccupied with the idea that they want to carry on. It looks like willing hands make light work. They almost play a game in which they "pretend the desk to be the shop". They keep inquiring "how many pizza boxes could be placed underneath the desk", a question they would not be able to answer beforehand. They may not need further argumentation for the reason to find out how many pizza boxes "can be placed underneath the desk". They have just decided for themselves to find out. Maybe they are well aware that they consider the concept of volume in different ways, and that they have taken charge of "How much space would it take?" by using different units of measuring. Their activity might generate a continuing didactical question at a more superior level: How would the activity of those boys come to function as a reference for continuing inquiry?

The boys decide to find out how much space it would take to stack the pizza boxes underneath the desk. However, this is not the game of playing shopkeepers. They construct different units to express the size of space and how many boxes it takes to fill out the space. This indicates that the pupils have adopted the goal of the teaching course. They seriously try to get insight in: "How much space would it take?"

The student teacher has left the inquiry to the students, and Karl is still sitting underneath the table. He does the measuring and Jonas does the registration:

Karl: It is 83 cm down here.

Jonas: What do you mean 'down'? Here in the front?

Karl: No, the other way around. On this side it is 83.

Jonas: [nods and writes down]

Jonas wants to know what is being measured. He questions Karl's result of 83 cm by asking: "What do you mean 'down'? Here in the front?" This makes Karl specify what he has been measuring so that Jonas can put it down. They measure breadth, length and height in order to estimate how many units (pizza boxes) can be contained underneath the desk.

The student teacher approaches the desk when the pupils are about to calculate further. Karl is clearly impressed by the size of the number he gets:

Karl: Oh! It might contain... eh... six five three six two, that is a lot. How much is it?

ST: Yes, take a look. Sixtyfivethousandthreehundredandsixtytwo, this... this is certainly a big number. What have you been calculating right now, then?

Karl: How many pizza boxes could be placed if we had a shop underneath the desk.

ST: Yes, there would be space for a whole lot of pizza, wouldn't it?

Karl: Maybe we could calculate some more things?

ST: Just go for it.

Karl reads the result as "six five three six two" and then he asks: "How much is it?" He seems to be impressed by the size of the number, and he probably asks in order to know how big it is. The answer of the student teacher "Sixtyfivethousandthreehundredandsixtytwo" might give Karl a better idea of the size of the number. Maybe this is what he is getting at. Through the question: "How much is it?" he seeks information, and the answer can be stored together with other information about the size of numbers. This could have continuing questions incorporated that can be productive for further inquiry.

The student teacher acknowledges the big number, although it would not really make sense to imagine that many boxes underneath the desk. Instead, she wants to know *what* they have actually calculated. Her question: "What have you been calculating right now, then?" can be heard as a controlling classroom manager voice. However, in this context we may rather interpret the question as an inquiry into something she does not know, yet. This interpretation is supported by the following comment: "Yes, there would be space for a whole lot of pizza, wouldn't it?" The student teacher paraphrases their wording. She confirms and acknowledges the pupils work, which can be very important for their ongoing activity. The big number of pizza boxes might support new reflections on volume, which may be noticed in their use of language. They wonder if they are allowed to continue their examination of volume as they address the teacher manager

role: "Maybe we could calculate some more things?" The student teacher encourages them by saying: "Just go for it," and Karl returns to Jonas.

- Karl: Shouldn't we take the boxes for toilet paper as well? This was fun!
[another pupil made containers for toilet paper the day before]
- Jonas: Then we can find out how many boxes for toilet paper can be placed in the shop. Then we have to see how big is the box, right?
- Karl: [flicks through his note book] I've got it here. I solved it last lesson.

Karl's question may be understood as an invitation: "Shouldn't we take the boxes for toilet paper as well?" He emphasizes how he enjoys what they are doing by adding: "This was fun!" Jonas gives a more explicit formulation to what they are about to do and that they first have to define "how big is the box". Jonas follows up on Karl's invitation and refers to the shop game they have invented when working with volume: "how many boxes for toilet paper can be placed". Further, they refer to previous activities where they have constructed other kinds of boxes for the shop game. Karl can even use his calculations from an earlier lesson.

The student teacher shows up and intervenes in the conversation:

- ST: What do you expect to find out now, then?
- Jonas: How many boxes for toilet paper we can place underneath the desk.
- ST: Yes, that is true. But I wonder, if you would be able to place less or more boxes for toilet paper than for pizza underneath your desk. What do you think about that?
- Karl: We can have less boxes for toilet paper.
- ST: OK... how can you be so sure?
- Karl: Because the boxes for toilet paper take much more space... don't they?
[Addressing Jonas]
- Jonas: Sure they do.
- ST: It is going to be very exciting to see if you are right.

The pupils go on by measuring new "things" in order to find out "how much space would it take". "What do you expect to find out now, then?", the student teacher asks, and adds "What do you think about that?" She poses *forwards oriented inquiring questions* in order to get to know what they are about to do. A backwards oriented question like 'what did you find out, then?' could easily be heard as a control of the pupils' behavior rather than as a curious and wondering interest.

The student teacher supports the pupils' initiative to compare boxes of different sizes. She wonders if they "would be able to place less or more boxes for toilet paper than for pizza" underneath their desk. This comparison is a challenge for further inquiry. But the task is initiated from the pupils themselves, so Karl answers without hesitating: "We can have less boxes for toilet paper." The student teacher does not evaluate the answer. She accepts and challenges it: "OK... how can you be so sure?" Again, Karl has a ready answer to the question and he includes Jonas by seeking his confirmation. The student teacher still does not evaluate the answer, but invites for them to examine "if they are right".

The conversation above illustrates pupils' curious and wondering inquiry in mathematics education. The students examine and compare how many pizza boxes respectively boxes for toilet paper could be placed underneath their desk. The idea of "how much space would it take" and a variation of units help them to work seriously on the concept of volume. They seem to take *ownership* to their activity, which is confirmed and supported by the student teacher. They have an inquiring attitude to the subject, to each other and to self, that seems to support their learning processes. Through collaborative participation in dialogue they express, examine and challenge their findings and so they seem to collectively learn.

The excerpt also shows that a conversation can be put together by several conversations with different intentions. An inquiry conversation and a classroom managing conversation can be identified simultaneously. They influence each other and are both of importance to the classroom context. The student teacher handles the management perspective, but she also contributes to the inquiry conversation of the pupils. The pupils on the other hand preserve the management conversation (Johnsen-

Høines, 2002), and still both of them take the initiative and responsibility for the inquiry as they listen, observe and act together.

The analysis indicates how inquiry in conversation is closely related to listening. In what follows we want to elaborate on this assumption in a broader theoretical perspective.

Authentic and non-authentic inquiry

Most didactical approaches are concerned with childrens' ability to ask questions. For instance children can be encouraged to pose wh-questions when participating in different activities. The teacher is supposed to evaluate the quality of such questions. However, it might be difficult for pupils to generate questions of real interest when simultaneously trying to satisfy a teacher request for good and right questions. The pupils would probably focus on the questioning technique instead of the subject content. Inquiring questions come from within, Lindfors claims (1999, s. 56). They presuppose a motivation and intention in learning. Thus, inquiring questions are part of what Rogers would call significant learning (Rogers, 1969). Or as Gadamer puts it: "To reach an understanding in a dialogue is not merely a matter of putting oneself forward and successfully asserting one's own point of view, but being transformed into a communion in which we do not remain what we were" (2004, p. 371).

Inquiry can emerge when pupils take ownership of the learning process like in the excerpt above, or when they consider themselves to be intentionally participating in the learning process. A teacher trainer and participant in the project 'Real-life Education' expresses it this way: "Only when the pupil defines himself as intentionally participating in the learning process he will experience the need for learning"¹²

Conversations may involve a lot of questions but this does not guarantee inquiry in terms of an examining and wondering attitude, or that the participants intend to know more. Lindfors (1999, p. 64) prefers to talk about two kinds of inquiry. One is *information seeking* about questions to which you have no answer, yet, but that can be answered in terms of concrete facts. The other is *wondering* that opens up for co-wondering and new inquiry, but not in order to come up with a solution to the question.

¹² The quotation is taken from an unpublished text by Olav Vårdal: *Praksisnær undervisning, eit komplementært læringsfellesskap*, 2008.

Both kinds of inquiry aim at learning, but as information seeking aims at getting clarification and harmony, wondering aims at keeping the examination in the open (Lindfors 1999, p. 134–135). The latter includes conflict, risk taking and examination, and this may be a challenging experience. Both information seeking and wondering are inquiry activities but they include different qualities of learning. Maybe the combination of the two can work as a mixture of support and challenge. Thus, in the didactical context we have analysed in this article, we find both of these inquiring attitudes.

Inquiring questions cannot be followed by evaluating comments like in the IRF-structure or in the construction of wh-questions. It has no meaning to evaluate for instance the wondering question of Jonas: "What do you mean 'down'? Here in the front?" Neither when Karl questions the big number: "How much is it?" Such questions cannot reasonably be followed by answers like: 'Yes, very well' or 'No, please try again'.

Lindfors (1999, p. 51–52) introduces the concepts of authentic and non-authentic questioning activities in the classroom. *Authentic* questioning activities means collaborative wondering and information seeking. The participants keep an open attitude when they collaborate in order to learn together. Staged examinations, as for instance when pupils have to formulate specific wh-questions, are *non-authentic* inquiry activities. They include no wondering, and the activities remain instrumental (Mellin-Olsen, 1984; 1987). Pupils' inquiry, however are not always expressed in terms of wh-questions. That is the reason why they cannot easily be identified in the classroom (Lindfors, 1999, p. 53).

Inquiring utterances are tentative, dwelling and trying out. This is the case for some hypothetical utterances like: "Could it be the case that ...? What if ...? How about ...? Could we try to ...? How come that ...?" or tentative utterances like when Karl presents his idea: "Shouldn't we take the boxes for toilet paper as well?"

Thus, inquiring utterances might be formed as questions, but they could also be formulated differently: "I imagine that ... Let us try to ... If it is so, then ... One might see it this way ... How strange that ...". In other words "inquiry" or "interrogative" are not identical concepts (Lindfors, 1999, p. 62). Inquiry includes an invitation, like when the student teacher says: "I wonder, if you would be able to place less or more boxes for toilet paper than for pizza". In the same way the final comment of the student teacher, "It

is going to be very exciting to see if you are right”, can be interpreted as an invitation and a support to the pupils’ inquiry.

Dialogic listening

An inquiring attitude can include ”dialogic listening”. This concept was introduced by Stewart & Logan (1999, p. 226–227) as an alternative to active listening. Active listening is directed towards one party of the conversation in order to address what this person understands, whereas dialogic listening is directed towards both parties and their mutual understanding. Dialogic listening means to be open and inquiring to both parties in order for them to co-create meanings (Stewart & Logan, 1999, p. 227). In the excerpt above we identify such a common process of meaning production. The boys try to figure out together how to operate with volume. All through this process they support each other and they get support from the teacher as well.

Davis (1996) does not use the concept of ’dialogic listening’ in his research on teacher roles in mathematics education. But he reflects similar ideas when distinguishing between evaluative, interpretive and hermeneutic listening.

In sum, then, evaluative listening is an uncritical taking in of information that is out there, interpretive listening involves an awareness that one is projecting onto one’s understandings particular biases that are in here, and hermeneutic listening is a participation in the unfolding of possibilities through collective action.
(Davis, 1996, p. 118)

Like dialogic listening hermeneutic listening does not focus on one party of the conversation. The hermeneutic listener is rather interested in the relationship and what is happening between the parties: ”[T]he tone of these conversations was not what-I-think; what-you-think, but more toward what-we-think” (Davis, 1996, p. 118).

Stewart and Logan (1999) describe dialogic listening as a playful process with certain qualities in the party attitude: modesty, humility, confidence and acknowledgement towards the perspectives of others and of self. Dialogic listening is a collaborative process through which the parties aim at getting to know. This takes a lot of concentration and ability of being present.

Dialogue presupposes a focus on a subject and an intention of getting to know. Thus, dialogic listening is directed towards this intention. It gives an important basis for further inquiry among the parties. Utterances can be characterised as listening acts if they reflect what has been said or what is going to be examined further. They would relate to future conversations. Thus, both information seeking and wondering utterances can be listening acts.

Dialogic listening takes an open and curious attitude and a number of communicative competences. First of all, the parties have to encourage each other to tell more, to elaborate, and to explain points of view. The use of metaphors can be of special interest because they contribute to reveal understandings and give new meaning to others. Stewart & Logan (1999, p. 229) claim that dialogic listening is like going by tandem; you are not supposed to take the lead all the time. By examining this metaphor you might reach a better understanding of dialogic listening. Stewart and Logan also use a turning lathe as a metaphor for dialogic listening; you sit in front of each other, you slap the clay on the lathe and form the object from both sides. In this way you make a common product. During the process you can remove or add more clay. Sometimes the product will be ugly and insignificant. Sometimes it will be beautiful and unique. It is not possible to foresee the result. In the classroom conversation above there is no use of metaphors, but we have identified a playful approach in terms of a virtual shop with virtual pizza boxes and toilet paper. This could have a similar function as it develops during the interaction between the boys.

Another aspect of dialogic listening is reflecting back what has been communicated verbally and non-verbally in the conversation. By paraphrasing it is possible to encourage the other to continue his or her reflections. Stewart & Logan (1999, p. 230) introduce the concept of "paraphrase plus", which means for the parties to add something to the paraphrase, e.g. a question or a questioning tone through which they challenge or shed light on other dimensions of the utterance in order to encourage further reflections. This can be seen in the excerpt above where the student teacher paraphrases the big number that Karl has found by a confirming comment: "Sixtyfivethousandthreehundredandsixtytwo, this is certainly a big number." In what follows she questions the big number: "What have you been calculating right now?" The

concept of dialogic listening may be specified through communicative competences like being curiously examining, information seeking and wondering (see Alrø & Skovsmose, 2002; Johnsen-Høines, 2002).

Closing remarks

In this article we have pursued the concept of 'inquiry' in the mathematics classroom. We have discussed inquiry as a quality of developing a dialogic learning community. This community is basically established through verbal and non-verbal information seeking and wondering interaction. Such interaction pays less attention to what students have already thought and done. It is more an inquiry into what they could be about to think or do and of which they are not so certain. Thus, inquiry takes place in the Zone of Proximal Development – in the relationship between what is already known and what is not known, yet, but can be achieved by support from others (Vygotsky, 1978; Lindfors, 1999). The excerpt illustrates this and further it gives the opportunity to reflect the role of listening as an important dimension of inquiry.

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Demographics of self-selected participants in mathematical tracks in Swedish Upper Secondary School

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Abstract: This study investigates the three demographic factors gender, geographical origin and parents educational level among self-selected students in mathematical tracks in Swedish Upper Secondary School (n=147). Comparisons are made, through statistical tests of significance, with corresponding student distributions on the closely related Natural Science program (n = 1528). Data is obtained from Statistics Sweden. Results confirm that females are under-represented in mathematical tracks ($p < 0,01$) and that parents of students in mathematical tracks have higher educational level than parents of students in the comparison group ($p < 0,05$). Results do not show that students with foreign background are under-represented in mathematical tracks, but raise questions about under-representation of Swedish students. Possible implications of these findings for these and similar programs are discussed.

Keywords: mathematical track; demographics; Sweden, self-selection; gifted

Introduction

In the search for educational equity the representation of different groups of students regarded as gifted has been studied since the rise of gifted education. Gender, students' ethnic origin and socio-economic background have been, and still are, three of the main measures used (see e.g. Bianco, Harris, Garrison-Wade & Leech, 2011; McBee, 2010; Yoon & Gentry,

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2009; Preckel, Goetz, Pekrun & Kleine, 2008; Wallace & Eriksson, 2006; Freeman, 2003; Donovan & Cross, 2002; Sarouphim, 2002; Borland & Wright, 2000; Feldhusen & Jarwan, 2000; Ford, 1998). These studies often identify a lack of equality in the representation or enrollment of different student groups in gifted programs.

According to the Swedish law (Swedish statute-book, 2010) all students should have equal access to educational opportunities in the school system. Thus lately there has been a focus on the demographic representation of students in “cutting-edge programs” (Swedish National Agency for Education, 2010a, 2011a). These cutting-edge programs, which were established nationally in 2009, are the Swedish version of gifted programs and the only form of gifted education that has been initiated by the Swedish government. The cutting-edge programs have been introduced in various subjects, including mathematics, for Upper Secondary School (Swedish statute-book, 2008). However, so far no evaluation of demographic factors of students in these gifted programs has used comparison groups, nor have clear accounts for the representations of different demographic factors in subject-specific programs been given.

During the last quarter of a century there have been a few Upper Secondary Schools in Sweden that have offered the national mathematics courses at higher pace and with special depth, and that have offered extension courses as well as university courses in mathematics (Mattsson & Bengmark, 2010). For many years these programs, addressing self-selected gifted students in mathematics, have gained very little attention in education policy, although these programs have offered, and still offer, similar activities as the newly established cutting-edge programs in mathematics. The major difference is that all the cutting-edge programs have national recruitment (see *ibid.* for a more detailed discussion about the benefits of national recruitments). Hence, studying these earlier, long-running, gifted

mathematics programs could give valuable insight when developing the cutting-edge programs.

From a national perspective, there is a need for research on the demographic representations for gifted students that takes into consideration relevant comparison groups, the subject specificity of the program, as well as the earlier programs for gifted students. Such studies will contribute to a more well-established discussion about the representations in these programs. This study aims to fill this gap for gifted programs in mathematics in the Swedish Upper Secondary School. Moreover, in Sweden, in gifted programs in mathematics it is in principle self-selection among students with high overall grades. Thus the findings will contribute to the international discussion about the relationship between civic culture and representations in gifted programs in the specific subject of mathematics.

The study is done by comparing representations of the students' gender, geographical origin and parents' educational level in purposively selected gifted programs in mathematics with the representations of the corresponding factors in comparison groups.

Mathematical tracks, cutting-edge programs and cutting-edge programs in mathematics

The gifted programs in mathematics in Upper Secondary School, including the cutting-edge programs in mathematics, cannot be compared with the special schools for gifted mathematics students such as the Gymnasiums in Hungary, the Kolmogorov's schools in Russia (Vogeli, 1997), the Center for Talented Youth at John Hopkins University in the U.S. (<http://cty.jhu.edu/about/index.html>), or the Mofet schools in Israel (<http://www.mofet-il.org/884/>; Ratner, 2008), where the subject of mathematics permeates many activities in these schools at large. Instead, the Swedish students in the gifted mathematics programs follow the national Natural Science Program, (Naturvetenskapsprogrammet, NV), which is

entered by some 10 000 students every year. As all the other national programs at the Swedish Upper Secondary School, the NV is a three year program. However, the NV is a theoretically intensive program, including more advanced studies in chemistry, biology, physics and mathematics as compared to the other national programs in Upper Secondary School. Thus, students in gifted programs in mathematics also need to take fairly advanced studies in all of the subjects of Natural Sciences. Gifted mathematics students can belong to different student groups during the other classes, but from the first year on they take their mathematics lessons in special groups. It is hard for a student to join this special mathematics group if he or she has not participated from the very start of the first year. Thus, the Swedish gifted mathematics programs could be seen as high-ability tracks (Feldhusen & Moon, 2004; Fiedler, Lange & Winebrenner, 2002) in mathematics.

In this article, ‘mathematical tracks’ will denote these earlier as well as newly-established Swedish gifted programs in mathematics. When reference is made only to the newly-established *national* gifted programs in mathematics, the term ‘cutting-edge programs in mathematics’ is used. By ‘cutting-edge programs’, on the other hand, we refer to the cutting-edge programs in *all* subjects areas (Mathematics, Natural Sciences, Humanities, and Social Studies) in Sweden. In the following the expression ‘gifted program’ is used as an umbrella term which includes different kinds of gifted programs.

Literature overview

Gender

Studies from the U.S., where gifted programs have been running for many years, give evidence of females being under-represented in accelerated gifted programs in mathematics (Fox & Soller, 2007). One major reason for this is that the highest achieving females score lower than the highest performing males on the mathematics part of tests like SAT-M, ACT

and EXPLORE, and these scores often act as the basis for the identification of giftedness (see e.g. *ibid.*, Olszewski-Kubilus & Lee, 2011). Even though there are studies indicating that the gender gap, that has been noted for many years (Lubinski & Benbow, 1992), has decreased over the years, from 12:1 to 3:1 males per female scoring over 700 points on the SAT-M test (Brody & Mills, 2005), males still outperform females on mathematic tests at very advanced levels (Olszewski-Kubilus & Lee, 2011; Lacampagne, Campbell, Herzig, Damarin & Vogt, 2005).

From Russia there are also reports of a large under-representation of females since less than 20% of the participants in schools for gifted mathematics and science students are women (Vogeli, 1997). Further, in Korea there is evidence of severe under-representation of females in schools for gifted mathematics students (Lee & Sriraman, 2012). There are indications of female under-representation in gifted mathematics programs in China (Vogeli, 1997) too. As in the US, research indicates that Chinese male students perform better in mathematics than do female students. A recent study shows that the ratio between male and female in the top five percent Chinese high-school seniors taking the College Entrance Examination was 2,9:1 (Tsui, 2007). There are no gifted programs in the UK (Freeman, 2004). Still, in recent years, there has been increased provision for the gifted in the UK, and the British Government Department for Education and Skills has devised a project designed to identify gifted students (*ibid.*). Using the tests specifically designed for gifted students, top 5%-10% of the cohort, no significant gender difference in mathematical performance was found among nine or thirteen year old students that were selected as gifted mathematics students by their teachers (*ibid.*). This result differ from the achievement trend in English nation-wide public examinations in mathematics, as males seem to outperform females at the higher levels on these tests (Hargreaves, Homer & Swinnerton, 2008).

In Sweden, males' and females' performance are in line with each other at the higher grade levels on the national tests in mathematics for grade nine (Swedish National Agency for Education, 2010b, 2011b, 2011c) as well as on all the national course tests in mathematics for Upper Secondary School (Swedish National Agency for Education, 2010c, 2011d, <http://www.edusci.umu.se/np-pb/np/resultat/>). Also, females get slightly higher grades in mathematics than do males in ninth grade as well as in Upper Secondary School (Swedish National Agency for Education, 2011e, 2011f). Moreover, there are on average about as many males as females in the Swedish cutting-edge programs (Swedish National Agency for Education, 2011a). However, when looking at data in detail it becomes clear that the gender distribution in the cutting-edge programs in mathematics indicates an under-representation of gifted mathematics female students also in Sweden, since only 33% (in 2010) and 26% (in 2011) of the first year students in these programs were females (Swedish National Agency for Education, 2010a, 2011a). Still, these data have not been analyzed in comparison with data on gender distributions in different comparison groups.

Origin

In international research many different terms are used to denote the genetic, cultural and/or geographical origin, of an individual, with 'ethnicity' and 'race' being two of the most common (see e.g. the terminology used in McBee, 2010; Yoon & Gentry, 2009; Ford & Grantham, 2003; Robinson, 2003, and Donovan & Cross, 2002). The definitions of origin are not always clear and terms are sometimes used interchangeably. This makes it hard to make comparisons between different studies focusing on students' origin. Still, a brief presentation of different research findings gives a general idea of the research field.

Researchers emphasize the importance of, but also the difficulty in, reaching ethnic or racial equality in gifted education (Borland & Wright, 2000). Repeated studies from

the U.S. show that certain minority groups such as Hispanic, African American and Native American students are, and have been, under-represented in gifted programs for a long time (Yoon & Gentry, 2009; Ford, 2003). The same studies also show that Asian or Pacific Islander and White students have been over-represented during this time. Other research indicates that ethnic minority groups are under-represented in gifted programs in Australia (Garvis, 2006), at advanced level of studies in mathematics in Israel (Mulat & Arcavi, 2009), and in the ranks of the gifted and talented in New Zealand (Reid, 2006).

The Swedish vital statistics speak in terms of geographic background in which the country of birth of an individual, or the country of birth of the parents of an individual, forms the basis of the ascription of an individual to a specific geographical origin (Statistics Sweden, 2010). Following this convention, ‘origin’ in this study refers to geographic origin.

In Sweden, reports (Swedish National Agency for Education, 2010a, 2011a) focus on the geographical origin of students in cutting-edge programs. However, none of the Swedish reports present the representations of origin in mathematical tracks separately. Moreover, as findings lack any comparison between different student groups, the results tell us very little about the conditions in the Swedish cutting-edge programs.

Parents’ educational level

Research has shown a correlation between socio-economic status (SES) and representation in gifted programs in mathematics as well as achievement in the subject. For students in gifted programs (see e.g. McBee, 2006; Borland & Wright, 2000) as well as for students in ordinary classes (Preckel, Goetz, Pekrun & Kleine, 2008) the correlation between high family SES and high representations among the ones considered mathematically gifted are confirmed. There are, however, many ways of choosing socio-economic factors (see e.g. Nam & Powers, 1983;

Erikson, Goldthorpe & Portocarero, 1979) which make up the measure of SES. Parents' academic background, parents' title of occupation, full- versus part-time occupation, and family income are commonly used socio-economic factors in educational studies. In studies connected to equal value in schools in Sweden, where the education is free of charge, it is the first two factors mentioned that are most often used to make up a measure of class or social group (Tallberg-Broman, Rubinstein-Reich & Hägerström, 2002).

In this study the 'parents' educational level' is used as the measure for SES. This measure was chosen because the register on parents' level of education is more reliable than the register of parents' occupation, and on the fact that students with well educated parents are the ones who generally do the best in the Swedish school system (*ibid.*) Also, parents' experience and knowledge of the educational system are said to influence the culture of their children's up-bringing, by highly valuing education (*ibid.*). Moreover, numerous studies from the US show that the talent search participants in mathematics at the CTY have very well-educated parents (Brody & Mills, 2005). Parents' educational level has also been selected as the most important factor when evaluating the SES among Swedish cutting-edge program participants (Swedish National Agency for Education, 2011a). Drawing from the results in this evaluation the Swedish National Agency for Education also states that there is an over-representation of students with highly educated parents in the cutting-edge programs (Swedish National Agency for Education, 2010a, 2011a).

Method

Comparison groups

In this study, since participation in the Swedish mathematical tracks in Upper Secondary School is often tied to participation in the National Natural Science Program (NV) (Swedish statute-book, 2008), data on representations in mathematical tracks are compared with data

on demographic representations in the NV. Since different regions show different distributions with respect to geographical origin and educational level among the citizens, and has created “a school structured by class and ethnicity” (Tallberg-Broman, Rubinstein-Reich & Hägerström, 2002, p 26), we chose to define one comparison group as all the students studying in the NV parallel to the students in mathematical tracks in each selected school with a mathematical track.

In order to give further reference levels for data in this study, representations of gender and geographical origin are also presented for two other comparison groups as well, including *all* students ($n=45,213$) in the NV in Sweden, and Swedish citizens ($n=520,307$) in the corresponding age group. In addition, the educational level of the parents of students in mathematical tracks is compared with data for all Swedish citizens ($n_{\text{men}}=1,238,448$ and $n_{\text{women}}=1,207,580$) in the corresponding age group.

Sampling

The target population of this study is the mathematical tracks in the Swedish Upper Secondary Schools. Since there is no comprehensive list of schools in the target population it was not possible to select schools randomly. Instead, schools were selected from the list of 31 schools that, in 2009, applied for starting cutting-edge programs in mathematics. From that list of applicants five schools were purposively selected. These schools were considered the most appropriate for the sample since they either:

- (1) Offer a national recruiting cutting-edge program in mathematics, or
- (2) Have offered another mathematical track for more than ten years.

Students who studied in mathematical tracks were identified in each selected school.

In the spring of 2010, this included 158 students spread over 11 classes, of which five classes were from the first year, three from the second year and three from the third year of Upper Secondary School. The comparison group of students who studied in the parallel NV included 1235 students. Lists of personal identity numbers for the students in the mathematical tracks as well as for the students in the parallel NV were collected in order to obtain data from Statistics Sweden.

Data collection

On request, Statistics Sweden compiled data on mathematics, as well as comparison group students' gender, geographical origin and parents' geographical origin, and parents' educational level by linking and matching files from the *Total Population Register* (TPR) and the *Swedish Register of Education* (SRE). Data on education refer to parents' highest educational level reached in January of the year the students entered their first-year studies in Upper Secondary School.

Data was gathered on distribution of gender and geographical origin for the total population of students in NV in Sweden in 2010 and the Swedish citizens at ages 16-19 years in 2009, from registers at Swedish National Agency for Education (Swedish National Agency for Education, 2011g) and Statistics Sweden (Statistics Sweden, 2010) respectively. Data on the educational level of the Swedish citizens between 40 and 59 years old was attained from Statistics Sweden (Statistics Sweden, 2011). It is reasonable to assume that this group corresponds to the age group of parents for youths at the ages 16-19 years old.

Non-response

Of the 158 students in the selected mathematical tracks, data on 147 students (93,0%) was acquired from Statistics Sweden. Non-response for eleven students is due to a student whose co-ordination number are not included in the national registers, students who asked to be excluded from the study (n=6, four men and two women from four different schools) and

students who belong to the mathematical tracks but whose information was not available when the data collection from Statistics Sweden was done (n=4). However, all 158 students are included in the results for gender distribution.

Of the 1235 in the comparison group of students in the NV in the sample schools, data on 1227 (99,4%) was acquired from Statistics Sweden. Non-response for eight students is due to protected identity (n=6) and wrong personal identification numbers (n=2) on the list that was sent to Statistics Sweden. In the latter cases the gender of the students is still known.

Data analysis

By considering data as randomly selected in time, one-sided tests of significance ($p=0,05$) (see e.g. Milton & Arnold, 2003) were made when comparing proportions for students in the mathematical tracks and in the comparison group of NV students in the sample schools. Since multiple tests that are made on one and the same data affect the total level of significance of the study, only three hypotheses, emerging from the literature, were tested:

- (1) Female students are under-represented in mathematical tracks as compared to in the comparison group.
- (2) Students with foreign background are under-represented in mathematical tracks as compared to in the comparison group.
- (3) Parents of students in mathematical tracks are more highly educated than parents of students in the comparison group.

Data is also analyzed using simple comparison of proportions for students in mathematical tracks with proportions in the larger comparison groups.

Measures

The measures considered in this study are defined as follows.

Gender

Gender means biological sex.

Parent

Parent is defined as biological parent or adoptive parents alternatively.

Origin

The categorization of geographical origin follows the classification of Statistics Sweden in the TPR. Students with 'foreign backgrounds' are defined as persons who are foreign born or Swedish-born with two foreign-born parents. A student that is 'foreign-born' is born abroad to a mother and a father, neither of them registered in Sweden at the time of the child's birth. Hence, students with 'Swedish background' are born in Sweden to at least one parent who was born in Sweden. Students with 'unknown origin' (n=2 in the mathematical track group and n=12 in the comparison group of NV students in the sample schools and) are also included here.

Educational level

Parents' educational level is based on the Swedish educational nomenclature (Statistics Sweden, 2000) and is aggregated into three groups – more than three years education in Upper Secondary School or higher education, less than or equal to three years education in Upper Secondary School and unknown. The 'maximum educational level' for the parents is defined by the parent with the highest educational level. This definition is founded on the idea that the longer educational experience of one of the parents affects a child's motivation for educational studies in a positive way more than the shorter experience of the educational system of two parents does.

Limitations

This study is based on a purposive sample of schools and thus the results cannot be generalized in a simple way to be applied to the whole target group of Swedish mathematical tracks. The strength of the results from this study lies in use of the comparison groups. This allows for a well-founded discussion, of e.g. the highlighted question of uneven recruitment in mathematical tracks (see e.g. Swedish National Agency for Education 2010a, 2011b), now lacking in Swedish educational studies.

The findings give a snapshot of the demographics of students studying in mathematical tracks today. A more valid knowledge of the demographics among students in mathematical tracks will be gained by repeating this study in the future and comparing findings over time.

The reliability of this study relies on the reliability of the data in the national registers, which can be assumed to be quite high. However, some factors negatively influence the reliability. ‘Parent’ is defined as *biological parents* or *adoptive parents* alternatively. This means that this study is built on data on parents who might not live with their children. This might influence the validity in results about the co-variance of parents’ educational level and students’ representation in the mathematical track. Moreover, there might be incorrect interpretations in the registers of the educational level for parents who have studied abroad.

Results and discussion

Gender

Table 1 shows the representation of females for the different groups. The test of significance confirms that females are under-represented in mathematical tracks in comparison with the NV group in the sample schools ($p < 0.01$). Data in Table 1 indicates that the under-representation of females also applies in comparison with the gender distribution in the NV-program nationally as well as with the Swedish population between 16 and 19 years old.

Table 1. Representation of females in different groups.

Group of (and number of) individuals	Percentage (number) of females
Students in mathematical tracks (n=158)	29,7 (47)
Students in the comparison group of NV students in the sample schools (n=1229)	44,9 (552)
Students in the NV in Sweden (n=45,213)	48,6 (21,967)
Individuals in the Swedish population, 16-19 years old (n=520,307)	48,6

Since 2001 students in the NV have had the possibility to choose a specialization in mathematics and computer science. This specialization, offered all over Sweden, offers students more mathematics courses during their Upper Secondary School studies as compared to other specializations in the NV. However, there is no university course included in this specialization. Since 2001 less than 23% (n=3,177) of students in this specialization have been female, and during the last six years at most 17% of students have been female (Swedish National Agency for Education, 2011g). Thus, in comparison with the representation in this program, the mathematical tracks have recruited females to a higher degree.

The under-representation of females in mathematics is also visible at under graduate level as well as at graduate level. This stand in contrast to most other subjects at university level in Sweden (Brandell & Staberg, 2008). In the academic year of 2009/2010, only 31,2% (n=10,900) of students studying mathematics at university level were female (Swedish National Agency for Higher Education & Statistics Sweden, 2011a). Only 24,5% of the

students at advanced level were female (*ibid.*). Moreover, among the mathematics graduate students who began their studies in 2001 to 2010, on average 23,5% (n=179) were female (Swedish National Agency for Higher Education & Statistics Sweden, 2011b).

Thus, for years the under-representation of females in studies specializing in mathematics is seen from the beginning of Upper Secondary School and on, although national test scores and school grades show that females are performing just as well as males in mathematics. This low participation of females in advanced mathematics studies could partly be due to the gifted females' competence beliefs and self concepts, as they show less self-confidence, more anxiety, lower interest and motivation in mathematics as compared to gifted males (Heller & Ziegler, 1996/7; Preckel, Goetz, Pekrun & Kleine, 2008). Moreover, it is probably influenced by the socialization practices relating to gender-role stereotypes (see e.g. Hyde & Lindberg, 2007; Preckel, Goetz, Pekrun & Kleine, 2008), such as the stereotypical image of mathematics as a male domain (Lacampagne, Campbell, Herzig, Damarin & Vogt, 2007). Research has shown that students in the theoretical programs in the Swedish Upper Secondary School perceive mathematics as a male domain (Brandell & Staberg, 2008). The strongest belief in this view is shown by males in the NV. Moreover, Swedish Upper Secondary School students believe that males, more than females, need mathematics in order to get good jobs and for their adult life (*ibid.*). Considering that individuals entering Upper Secondary School are at a sensitive age as they are forming their identity, we could probably not expect gifted female students to overlook prevailing socializing practices and gender-linked academic motivation or the fact that females are under-represented in different mathematics studies from Upper Secondary School on, when selecting programs, especially the NV, for future studies. Thus, for many females, enrolling into mathematical tracks might be a huge step to take, especially if they have to move in order to join the program.

Moreover, accelerated mathematics studies are an essential feature of the studies in all mathematical tracks in this study. Research has shown that, for those who were identified as mathematically gifted, females were less likely than males to accelerate their study of mathematics (Fox & Soller, 2007). This raises the question of whether we can argue that the very construction of the mathematical tracks with national recruitment really fulfills the demand (Swedish statute-book, 2010) for programs which are equally accessible regardless of gender.

In order to reduce the influence of stereotypical views of mathematics and mathematicians, and perhaps increase the female representation in mathematical tracks in Sweden, we need to foster more advantageous attitudes towards mathematics for gifted female mathematics students. This work needs to start at an early age as research shows that older students hold more strongly gendered views than younger (Brandell & Staberg, 2008). Moreover, interventions should also address environmental factors such as teachers and parents (Preckel, Goetz, Pekrun & Kleine, 2008). Thus, in Sweden it might be helpful to support gifted females at a much earlier age, in local gifted programs not only connected to the NV, and in less accelerated mathematics activities.

Origin

In Table 2 the representation of individuals with foreign background is presented for different groups. Since data show that the representation of students with foreign background is slightly higher in the mathematical tracks as compared with the comparison NV group in the sample schools, students with foreign background are not under-represented in mathematical tracks.

Table 2. Representations of individuals with foreign background in different groups.

Group of (and number of) individuals	Percentage (number) of individuals with foreign background
Students in mathematical tracks (n=147)	32,7 (48)
Students in the comparison group of NV students in the sample schools (n=1227)	29,2 (358)
Students in the NV in Sweden (n=45,213)	21,7
Individuals in the Swedish population, 16-19 years old (n=520,307)	16,6

Data from Table 2 also show that we cannot claim that students with foreign background are underrepresented when compared with all the students in the NV nationally nor when compared to the Swedish population between 16 and 19 years. Rather, the data raises questions about the under-representation of students with Swedish background in mathematical tracks.

In Sweden, schools with gifted programs are a fairly new phenomenon and many people consider these schools as “elite schools”. Thus, a first interpretation might be that the extreme expression of egalitarianism that has influenced the Swedish society for so long (Persson, Joswig & Balogh, 2000) influences the non-participation of students with Swedish background in gifted programs. However, a report (Swedish National Agency for Education, 2011a) indicates that 85% of the participants in the national cutting-edge programs in all subjects are of Swedish background. This reflects quite well the representation in the corresponding group of 16-19 year olds in the Swedish population, indicating that students

with Swedish background do not reject special tracks for gifted students per se. Thus, this study raises culturally-related questions about the reasons why students with Swedish background choose not to participate in mathematical tracks as compared to several other subject tracks in the cutting-edge programs.

In this study it is mainly students from Europe (the Nordic Countries excluded), especially from former Eastern European countries, that contribute to the relatively high representation of students with foreign background in mathematical tracks. Further studies are needed to clarify whether this is a trend that will hold over time, and, if so, if this is due to a relatively long tradition of gifted programs and gifted schools in the Eastern Europe (see e.g. Vogeli, 1997; Persson, Joswig & Balogh, 2000).

Parents' educational level

Table 3 presents the maximum educational level for the parents of students in the mathematical tracks as well as for parents of students in the comparison group. The test of significance shows that the distribution of parents with a maximum educational level of more than three years of studies in Upper Secondary School or higher education is higher among parents of students in mathematical tracks than among parents of students in the comparison NV group in the sample schools ($p < 0,05$). Non-response due to lack of information about the educational level of some individuals does not affect the study result.

Table 4 (in the Appendix) shows the educational level of mothers as well as of fathers of students in the mathematical track and in the comparison group respectively. Data indicate that there is a clear distinction in educational levels for mothers as well as for fathers in the group of mathematical tracked students as compared to the corresponding groups in the sample schools. Further comparison with the distribution of the educational level of the Swedish citizens between the age of 40 and 59 shows an even larger distinction.

Table 3. Distribution of maximum educational level of parents of students in the mathematical track and the comparison group respectively.

Maximum educational level of parents	Percentage (number of students in mathematical tracks (n=147)	Percentage (number of students in the sample schools (n=1227)
More than three years of studies in Upper Secondary School or higher education	85,7 (126)	78,6 (965)
Less than or equal to three years of studies in Upper Secondary School 6-19 years old (n=520,307)	13,6 (20)	20,9 (257)
Unknown	0,7 (1)	0,4 (5)

The evaluation of the cutting-edge programs in all subjects show that 68% of the mothers, and 58% of the fathers, of the participating students have more than three years of studies in Upper Secondary School or higher education (Swedish National Agency for Education, 2011a). These numbers could be compared to 76,9% and 68,7% respectively for parents of students in mathematical tracks in this study (see Table 4).

The great covariance between a high academic parental educational background and advanced studies in mathematics is also noted among the first year graduate students in

mathematics. Statistics that show the average parental educational background of Swedish graduate students who began their studies between the year of 2004/2005 and 2008/2009 show that the portion of highly educated parents (corresponding to at least three years of Upper Secondary School Studies) is the highest (65%) for graduate students in mathematics as compared to graduate students in other national research fields (Humanities/ Religion, 57%; Medicine, 58%; Natural Sciences, 51%; Social Sciences; Forest/Agricultural Research, 51%; Engineering Sciences, 53%). Thus, in Sweden, it seems that a high parental education is more important for students taking educational paths in advanced mathematics than for students in other advanced subjects.

The strong connection between advanced Upper Secondary School studies in mathematics and studies in the NV is convenient for the organization of mathematical tracks. Nevertheless, it might be one of the reasons for the strongly uneven representation of students with a lower parental educational background in comparison with the educational background of the Swedish citizens between 40 and 59 years old. Thus, one way to open up for the participation of more students with a lower parental educational background in mathematical tracks might be to make it possible to participate in the track while still taking other Upper Secondary School programs than the NV. Still, since this study shows that the distribution of parents with more than three years of studies in Upper Secondary School or higher education is higher among parents of students in mathematical tracks than among parents of students in comparison group of the NV in the sample schools, findings from this study suggests that there must be other reasons for the under representation of students of less educated parents. Further research is needed in order to find out more about the reasons why high parental education is especially shown by students in tracks in the subject of mathematics.

Conclusions

Results from this study confirm that, in mathematical tracks in Sweden, females are under-represented and questions are raised about a possible under-representation of Swedish born students. These results are specific for the subject of mathematics, since they differ clearly from the average distributions of gender as well as of origin taken over cutting-edge programs in all subjects (Swedish National Agency for Education, 2010a, 2011a). Findings from this study also confirm that students with well educated parents are over-represented in mathematical tracks.

The reason for uneven representations in gifted programs are often ascribed to biased conceptions of giftedness, identification procedures and teacher nominations (Bianco, Harris, Garrison-Wade & Leech, 2011; McBee, 2006; Borland & Wright, 2000). However, under-representation of different groups in the Swedish mathematical tracks cannot be ascribed to inequalities in an identification process since it is, in principle, self-selection among students with high overall grades (Swedish National Agency for Education, 2011a). Instead, the findings stress the importance of finding out more about the attitudes towards and the interest in mathematics that influence the students' socializing practice and that affect some students' choice to study and other students' choice not to study in *mathematical* tracks. Thus, the results of this study emphasize the discussion of the influence of the civic culture on the representation of students in gifted programs (as seen in e.g. Preckel, Goetz, Pekrun & Kleine, 2008; Hyde & Lindberg, 2007; Freeman, 2004 and Borland & Wright, 2000).

Finally, findings also suggest that, in Sweden, we need to continue to develop and complement the gifted education in mathematics in order to avoid the current uneven recruitment.

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Appendix:

Table 4. Educational level of mothers and fathers of individuals in different groups.

	Percentage (number) of mothers/fathers of students in the mathematical tracks		Percentage (number) of mothers/fathers of students in the sample schools		Percentage (number) of individuals among the Swedish population 40-59 years old	
	Mother (n=147)	Father (n=147)	Mother (n=1227)	Father (n=1227)	Women (n=1,207,580)	Men (n=1,238,448)
More than three years of studies in Upper Secondary School or higher education	76,9	68,7	63,5	61,5	38,3	31,5
Less than or equal to three years of studies in Upper Secondary School	21,8	25,2	35,0	33,0	60,8	67,4
Unknown	1,4	6,1	1,5	5,5	0,9	1,1

Mattsson

Loss of Dimension in the History of Calculus and in Student Reasoning

Robert Ely¹

Abstract: Research indicates that calculus students often imagine objects losing a dimension entirely when a limit is taken, and that this image serves as an obstacle to their understanding of the fundamental theorem of calculus. Similar imagery, in the form of “indivisibles”, was similarly unsupportive of the development of the fundamental theorem in the mid-1600s, unlike the more powerful subsequent imagery of infinitesimals. This parallel between student issues and historical issues suggests several implications for how to provide students with imagery that is more productive for understanding the fundamental theorem, such as the imagery of infinitesimals or the more modern quantitative limits approach, which relies heavily on quantitative reasoning.

Keywords: Calculus; infinitesimals; history of Calculus; Fundamental Theorem of Calculus

Introduction

Recently several colleagues told me they think that the fundamental theorem of calculus is not something that undergraduate students can really make sense of when they take a course in calculus, although it may snap into focus later in their education¹. The implication was that we should not bother spending very much time teaching it. This is a bit too cynical for me. I don't

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think it's too much to ask that students come out of a calculus class understanding the *fundamental* idea of calculus, why there is a deep connection between integrals and derivatives. Nonetheless, research has shown that there are some common mental images that students use to make sense of derivatives and integrals, images that can serve as obstacles to their learning of the fundamental theorem of calculus.

The purpose of this article is to explore one of the most common of these mental images: the loss of dimension. First I describe this mental image and some of its manifestations in student reasoning, and discuss its limitations for promoting understanding of the fundamental theorem of calculus. Then I discuss some historical occurrences of pre-calculus ideas that also involve the loss of dimension, in the form of “indivisibles,” and how these indivisibles were likewise limited in promoting the understanding of the fundamental theorem of calculus. I describe how the approach of infinitesimals, which replaced indivisibles, led to the successful conceptualization of calculus. This suggests that such imagery of infinitesimals might be useful for calculus students as well. Finally, I discuss the more modern approach of focusing just on limits of quantities, and why this is also more successful than approaches that involve the loss of dimension.

It is important to clarify what I mean by “mental image” in this article. One way to interpret a mental image is loosely as Vinner does, as a set of mental pictures that a person uses when considering a mathematical term like “limit”—pictures that may not necessarily, if somehow externalized, satisfy the mathematical definition of the term (e.g. Tall & Vinner 1981, Vinner 1989, 1991, Vinner & Dreyfus, 1989). While this notion of mental image captures the spirit of the idea I will be using, a Piagetian interpretation of mental image clarifies a few aspects of this. Piaget generally speaks of a mental image as expressing an anticipated outcome of an action taken on an object (Piaget 1967). These actions and objects can certainly be products of

reflective abstraction, and so may be many steps removed from actual physical actions and perceived objects, but nonetheless the idea of mental image is ultimately grounded in physical action and attention (Thompson 1994). This means that a mental image (a) need not be entirely visual, (b) need not directly correspond with “real-world” objects or operations, and (c) need not directly correspond with things that have mathematical definitions. The Piagetian interpretation of mental image helps us keep in mind how such an image functions in the learning, rather than simply noting how it may or may not differ from standard mathematical treatments or definitions. Here this is particularly important, because it is not always clear if the mental images I am discussing can be seen to directly conflict with mathematical definitions. Furthermore, these mental images are results of imagined limit processes or infinite decompositions, so they are not going to be changed by any amount of checking against any real-world objects they might be seen to portray.

Dimension loss, or “collapse,” in student reasoning about calculus

Recent research shows that one of the most common mental images that students employ when conceptualizing limits, and calculus concepts like derivatives and integrals that rely on limits, is one of dimension loss (Oehrtman 2009; Thompson 1994). Oehrtman calls this the “collapse metaphor,” in which an object is seen as becoming smaller and smaller in one of its dimensions, and when the limit is reached it *becomes* an object whose dimension is one less than it was before (2009). When the limit is taken, one of the object’s dimensions collapses and is lost. To illustrate this mental image, I draw quite a bit upon on Oehrtman’s study (2009), which details it in a variety of contexts with examples from undergraduate students. Students use this

metaphor when making sense of Riemann sums and definite integrals, derivatives, and the fundamental theorem.

A classic example of the collapse metaphor can be seen when students discuss a definite (Riemann) integral. Ten of Oehrtman's 25 undergraduate participants used the collapse metaphor in this context, seeing the integral as a sum of "infinitely many one-dimensional vertical lines over the interval $[a, b]$ extending from the x -axis to a height $f(x)$ and produced by a collapse of two-dimensional rectangles from the Riemann sum as their widths became zero" (p. 413).

Oehrtman found that many of these students also bring this collapse metaphor to bear in the context of the fundamental theorem of calculus. For instance, students were asked why the derivative of the formula for the volume of a sphere, $V = \frac{4}{3}\pi R^3$, with respect to the radius r , is equal to the surface area of the sphere, $SA = 4\pi R^2$. The students "described thin concentric spherical shells with 'the last shell of the sphere' getting thinner and thinner and eventually becoming the 'last sphere's surface area'" (ibid, p. 413). Then later the students described the definite integral in the equation

$$\frac{4}{3}\pi R^3 = \int_0^R 4\pi r^2 dr$$

as representing the limit of the sum of all these shells as they became thinner and eventually became two-dimensional surfaces. Thus, "the volume is just adding up the surface area of small spheres" (p. 413). Here we see the loss of dimension that occurs when students use the mental image of dimension loss when they imagine the result of a limit.

Constraints and affordances of the image of dimension loss in student reasoning

This metaphor seems to be a very natural and visually appealing way of thinking about what happens to objects in a limit. It does not necessarily serve as a significant obstacle for students who use it to conceptualize derivatives (Oehrtman 2009). Where it hinders the understanding is with understanding the relationship between the derivative and the definite integral, without which there is no calculus. This is reflected in the first part of the fundamental theorem of calculus (which is what I will be referring to when I say “fundamental theorem”).

In a teaching experiment focusing on the fundamental theorem, Thompson (1994) asked students to explain why the instantaneous rate of change (with respect to height) of the volume of water filling a container is the same as the water’s top surface area at that height (Figure 1). One student responded that the volume of the thin top slice would *become* the area of the top surface when the thickness went to zero in the limit. This is an example of dimensional collapse. According to Thompson, the basic problem with this reasoning is that the student is not seeing the connection between an increment of volume and an increment of height. The increment of volume is being viewed as only an area, having no height. The student’s mental image “could be described formally as

$$\lim_{\Delta h \rightarrow 0} V(h + \Delta h) - V(h) = A(h)$$

which would have meant that

$$\lim_{\Delta h \rightarrow 0} \frac{V(h + \Delta h) - V(h)}{\Delta h} = \lim_{\Delta h \rightarrow 0} \frac{A(h)}{\Delta h}$$

an equality I cannot interpret” (Thompson 1994, pp. 263-4).

Figure 1—Student's diagram for identifying amount of change in volume as water rises in height within a conical storage tank, from Thompson (1994, p. 259). As $\Delta h \rightarrow 0$, some students treated the volume of the top region as becoming the area of the top surface.

The student using this metaphor of collapse is thus not attending to the *ratio* of an increment of volume to an increment of height. The height dimension is lost when the limit occurs, and thus so is the information about relationship between an increment of volume and an increment of height. For the student, when the limit occurs, the resulting quantity $A(h)$ is no longer a *rate* at all.

When the students who use the collapse metaphor for Riemann sums conceptualize the fundamental theorem of calculus, the same kind of problem can occur. If a thin rectangular area simply *becomes* a line segment when the limit is taken, then the relationship between the rectangle's area and its width is lost (see Figure 2). Therefore the height of the function $f(x)$ is not seen as the *rate* at which the area under the curve changes as x changes. This metaphor is thus an obstacle to understanding the fundamental theorem, because it obscures the fact that the value of $f(x)$ is a derivative, a rate of change, of the area under the curve. In these ways, the collapse metaphor can preempt or hinder investigation into the ratio structures that are crucial to the fundamental theorem of calculus (Oehrtman 2009).

Figure 2—One rectangle in a Riemann sum. As $\Delta x \rightarrow 0$ the rectangle might be seen to collapse to the line segment with height $f(x_0)$, obscuring the idea that this height is the *rate* of change of the rectangle's area $[f(x_0) \times \Delta x]$ with respect to its width Δx .

Historical dimension loss: indivisible techniques

In the historical development of geometry, techniques that involve the loss of a dimension in a figure have been very prominent, in the form of the technique of indivisibles. Indivisibilist techniques were used rather widely in ancient Greek mathematics, not as a method of proof but as a heuristic tool for solving problems (see Knorr 1996 for a good summary).² The technique was rediscovered in Europe in the early 17th century, and gained prominence as a method of solving area and volume problems particularly through Bonaventura Cavalieri's 1635 work *Geometria indivisibilibus continuorum nova quadam ratione promota* (*Geometry developed by a new method through the indivisibles of the continua*). The technique of indivisibles uses lower-dimensional pieces to reason about a figure, and it invokes the same kind of mental imagery that the collapse metaphor invokes. The technique is similarly limited in allowing for the conceptualization of the fundamental theorem of calculus. In fact, I argue later that the abandonment of indivisibles in favor of infinitesimals was crucial to the development of the fundamental theorem.

To use Cavalieri's method of indivisibles, imagine two plane figures ABC and A'B'C' bounded by two parallel lines EO and BC (see Figure 3). Now move the top line down until it meets BC, keeping it always parallel to BC. "The intersections of this moving plane, or fluent, and the figure ABC, which are produced in the overall motion, taken all together, I call: *all the lines of the figure ABC*," taken with "transit" perpendicular to BC (Cavalieri 1635, book 2, pp. 8-9). If each of these lines of figure ABC (say, LM) is as long as its corresponding line of figure A'B'C' (L'M'), then the two figures have the same area. More generally, if the lines of one

² Archimedes, for instance, used this technique extensively and brilliantly in his *Method*, a work found in a written-over palimpsest in 1906. The rediscovery and very recent reconstruction of the text is an amazing story, and can be read about in *The Archimedes Codex* (Netz & Noel 2007), among other places.

figure are all in a given ratio to the lines of the other, then the two figures' areas are in the same ratio.

Figure 3—Cavalieri's principle, adapted from Mancosu (1996, p. 41)

Cavalieri states this method for solids also and applies it to a wide variety of figures. For instance, through a series of comparisons of indivisibles of several figures, one can quickly conclude that a cone takes up a third of the volume of the cylinder that contains it. The principle marked a significant step toward the development of integral calculus, and it is still commonly used today, although it is no longer stated in such a way that requires us to consider a figure as being comprised of “all its lines.”

Cavalieri's technique of indivisibles uses the idea of dimensional collapse by treating the two-dimensional figure as being entirely comprised of its one-dimensional line segments. These segments are “indivisible” slices of the figure because each has no width and cannot be further divided in the dimension of thickness. The figure is thus precisely the same as what you get when you conceptualize a definite integral using the collapse metaphor: an area comprised of infinitely many one-dimensional line segments. Although this similarity is striking, there is a key difference between the collapse metaphor and the imagery associated with indivisibles. In the collapse metaphor a lower-dimensional object comes about as a result of an envisioned limit process in which a dimension is *lost*, whereas for Cavalieri the indivisible slices are not the result of a limit process and never really were two-dimensional.

Historical issues with dimension loss

In 1643, E. Torricelli famously adapted the technique of indivisibles to find the volume of his “Trumpet,” a figure often now called Gabriel’s Horn. In Figure 5 this trumpet is the infinitely long solid created by rotating a hyperbola around its asymptote AB, from some point E upward, together with cylinder FEDC. Torricelli decomposes the solid into all of its indivisible nested shells, two-dimensional cylindrical surfaces (e.g. POMN). Then he shows that each such surface has the same area as a corresponding disc (e.g. NL) whose radius is the same as the hyperbola’s diameter AS. All of these discs comprise the cylinder PCIH, so the volume of the trumpet is the same as the volume of this cylinder (e.g. $\pi(AS)^2$).

Figure 5—Torricelli’s trumpet, from Mancosu (1996, p. 133)

Torricelli’s result was impressive not only because it shows an infinitely extended solid to have finite volume, but also because it uses *curved* indivisibles to show it. One of the most bizarre aspects of this demonstration is the last indivisible pairing, in which the infinite axis AB “is equal to” the disc AH (Torricelli, 1644, v. 1, p. 194). Not only is there already a collapse of

dimension by viewing the original figure as comprised of its indivisibles, but the last indivisible sustains yet another dimensional collapse and as a one-dimensional line is seen as “equal” (in area?) to a two-dimensional disc.³ This paradoxical conclusion is not the only one of its sort—for instance, Torricelli’s teacher Galileo had already shown that, in using two-dimensional indivisible slices to find the volume of a particular bowl-like figure, the last indivisible degenerates to a one-dimensional circle which is equal to a zero-dimensional point (Galileo 1634).

Note also the strange fact that although each indivisible slice of Torricelli’s trumpet has finite area according to Torricelli’s slicing method, if you sliced the trumpet in vertical planar slices instead you would get one slice with infinite area, namely the slice containing the axis AB. This last point indicates a danger with indivisibles—one slicing method produces correct results while another does not, and it is not clear that there is an a priori way to distinguish between the two. Torricelli was aware of this kind of problem with indivisibles, and used a diagram like the one in Figure 6 to demonstrate it. Suppose we have two right triangles with different area, Triangle 1 (BDA) and Triangle 2 (BDC), as shown. For each point P on side BD there corresponds a vertical indivisible slice JI of Triangle 1 and a vertical indivisible slice KL of Triangle 2. Thus all the slices of the Triangle 1 are in one-to-one correspondence with those of Triangle 2, yet the two triangles have different areas.

³ It is noteworthy that some of Oehrtman’s students also used a collapse metaphor to make sense of Torricelli’s trumpet, albeit in a different manner; see Oehrtman 2009, pp. 412-13.

Figure 6—Torricelli's demonstration of problems with correspondences of indivisibles when determining areas

Cavalieri solved this problem by pointing out that his principle only works if the slicings of the two figures are done with the same “transit.” The idea is that the indivisibles must have the same *density* in both figures. Indeed, Cavalieri likens the notion to the threads in a cloth: if Triangle 1 and 2 are cut from the same cloth, then the first might have 100 threads and the second 200 (Cavalieri 1966). So any one-to-one correspondence between threads must leave 100 threads of Triangle 2 unaccounted for.

Some of the issues with indivisibles are certainly solved by restricting oneself to cases in which the density of indivisibles is the same. But this also indicates that we are doing something a bit unnatural. It is an odd thing to conveniently lose a dimension by using indivisibles, only to have to pay attention to what is happening again in this dimension by reference to density or transit. Indeed, this issue led Torricelli to develop a theory in which indivisibles have a certain thickness (*spissitudo*) to them (Mancosu 1996), thus in a nebulous way returning some quality of the lost dimension back to the indivisibles. Thus, almost immediately after indivisibles made a splash in the 1630s, there was an effort to restore information about the collapsed dimension in order to avoid some of these pitfalls. Only when a method came along that retained the powerful technique of dividing something into infinitely many pieces, without losing the thickness dimension of these pieces, that the full power of calculus became available. This was the method of infinitesimals.

The infinitesimal calculus

Although many 17th century mathematicians found tangent lines to and areas under a variety of curves, most scholars officially credit the development of “Calculus” to Isaac Newton and Gottfried Wilhelm Leibniz (independently). The reason is that these two mathematicians created systems that (a) centrally recognize the relationship between derivatives and integrals, that is the fundamental theorem of calculus, and (b) provide a systematic method for determining derivatives of many types of curves, and hence integrals (e.g. Whiteside 1960). Newton’s method of fluxions and fluents is the earlier method (1665-6), but was not widely circulated until it was in a later re-imagined form in *Principia Mathematica* (1687). Leibniz’ infinitesimal calculus was developed in the 1670s and circulated in the early 1680s, and it is its notation we generally use today.

There are differences in mental imagery between the two systems, but the image of the infinitesimal is central to both. A more thorough account of Newton and Leibniz’ treatments of the fundamental theorem of calculus can be found in Bressoud (2011); here I focus only on the aspects that are important to my point.

The crucial idea for Leibniz is that any curve is a “polygon with infinitely many [infinitesimal] sides,” so on infinitely small intervals the curve is straight (Kleiner 2001, p. 146). Thus, for example, an instantaneous rate of change of two quantities is the ratio dy/dx of an infinitesimal difference in one quantity to an infinitesimal difference in the other. This means that an area under a curve can be seen as an infinite collection of infinitesimal rectangles (well, trapezoids, but which quickly can be seen as having equivalent areas as rectangles). Thus in “ $\int y dx$ ”, the $\int$ is a big S for “summa” (sum) of all of the rectangles’ area, each being the product of a height y and width dx (the difference between two infinitely close x values). Now the question

can be asked: for an infinitesimal change in x , i.e. dx , how much does the area under the curve change? An infinitesimal rectangle is accrued, with area ydx , where y is the height of the curve (Figure 7). Thus the rate of change of the area accumulating under the curve as x increases an infinitesimal amount, is ydx/dx , or simply y , the curve's height at that point.

Figure 7—The Fundamental Theorem of Calculus in terms of infinitesimals: for an infinitesimal increment of area accrued, the rate of change of this area with respect to the infinitesimal increment dx is just y , the curve's height

In this version of the fundamental theorem, notice how important it is to retain the dimension of thickness of the rectangle, even though it is infinitesimal. The *ratio* of the (rectangle's area)/(rectangle's width) is only meaningful if the infinitely small rectangle still *has* a width, and hence an area. If the rectangle were instead seen as collapsing to an area-less line segment, this quotient would become meaningless, so the fundamental theorem would remain unavailable.

Newton's understanding of the fundamental theorem of calculus is more dynamic than Leibniz', and is often treated in terms of velocities, yet the key idea is similar: "to recognize the ordinate as the rate of change of the area" (Bressoud 2011, p. 104). As x increases, the area under the curve accumulates at a rate given by the height y of the curve. Thus the area changes incrementally as x changes incrementally, and the ratio of the two is the curve's height. Newton often treats such increments as infinitesimal (Whiteside 1960), although they can also be seen as finite quantities that would vanish simultaneously if shrunk; the key idea is that the ratio between them is maintained and neither of them collapses.

In this way the fundamental theorem of calculus was unavailable using indivisibles, but became accessible using the image of infinitesimals. And it was because of this that the infinitesimal calculus became The Calculus in the subsequent century, transforming mathematics and science in its widespread application.

What is an infinitesimal?

The imagery of infinitesimals made the fundamental theorem of calculus available in ways that indivisibles and the imagery of dimension loss could not. This suggests that the imagery of infinitesimals may also offer students a way to surpass the collapse metaphor they often use, in order to make sense of the fundamental theorem. Although I contend that it is very promising to employ the imagery of infinitesimals with students, infinitesimals bring with them their own issues that were problematic historically and could serve as obstacles for students. Other works have addressed these philosophical issues in more detail, which I do not attempt here (e.g. Mancosu 1996, Knobloch 1994, Bell 2008, Russell 1914). The purpose of this section is to summarize these issues and how they may function for students.

The most important issue is that it is not quite clear what an infinitesimal quantity is; what does it mean for a quantity to be infinitely small, smaller than any finite magnitude, but not be zero? To address this, note that an “infinitesimal” width is precisely one that is not subject to what we now call the Archimedean Axiom. This axiom states that it is always possible to find a finite multiple of one magnitude which will exceed a given other magnitude. An infinitesimal length cannot satisfy the axiom, because it can be multiplied infinitely many times to still comprise a finite, not infinite, length. The rigorous work to construct infinitesimals by denying this axiom is technical and relatively recent (e.g. Robinson 1966). A calculus student who uses

the image of infinitesimals might be uneasy about this property of infinitesimals, and is probably not going to study up on the formal construction of infinitesimals in order to allay his concerns. There are two related kinds of conception that are plausibly important for such a student to have in order to overcome this potential obstacle to working with the image of infinitesimals.

First, such a student must be able to imagine an infinitesimal as a convenient formal object, a “useful fiction,” as Leibniz says, not as a thing that exists in a tangible way, while yet not having the machinery for its formal creation. This kind of thing may not be a significant obstacle for students; by the time they are in calculus they have often worked with mathematical objects, such as negative and imaginary numbers, in a formal way without worrying about their real-world referents and without the machinery to understand the formal construction of these objects.

Second, if some of the expected properties of finite numbers, such as the Archimedean Axiom, do not work with infinitesimals, then there must be some compensating idea of what the rules *are* when working with infinitesimals. This is one of the issues that Berkeley famously criticizes about the infinitesimal calculus, in which one is seemingly allowed to treat a rectangle width as finite at one moment and zero at another, depending on which happens to be more convenient (Berkeley 1734). The way to address this issue is to do precisely what Leibniz did by developing a plausible list of rules for working with infinitesimal (and infinite) numbers or magnitudes: e.g., (a) a finite length times an infinitesimal length is always infinitesimal, and (b) when comparing two finite magnitudes, any infinitesimal parts are negligible, and many more (Leibniz 1684). Such rules are sensible enough, and there is evidence that students can even develop them themselves by pursuing the implications of their own conceptions about infinitesimals (Ely 2010).

A more modern calculus image: quantitative limits

I have already made the case that the image of infinitesimals promotes the understanding of the fundamental theorem while the image of dimensional collapse is an obstacle to this understanding. Although this supports the case that calculus might be taught using infinitesimals, most instructors are probably not planning to do such a thing.⁴ Luckily the image of the infinitesimal is not the only one that is available that helps with the understanding of the fundamental theorem, even though it was the image that, historically, led to this understanding. Another alternative is to have an image of the rectangles vanishing entirely when the limit is taken in a Riemann sum. In other words, in the limit when the number of rectangles goes to infinity, the rectangles disappear and there only remains the envisioned area under the curve as one cohesive region. For the sake of this paper, I will refer to this as the quantitative limits approach, for reasons that will hopefully become clear.

To create the notion of a definite integral using this image, one must rely heavily on quantitative reasoning. Take for the moment a function that is increasing over some interval, and partition the interval into some finite number of equal subintervals. On such a partition, the right-hand sum overestimates the area and the left-hand sum underestimates it. But we note that the area between the right- and left-hand sums, shown by the lightly shaded regions of Figure 8, approaches 0 in size as our partition gets finer (as $\Delta x \rightarrow 0$). Thus we can say that the right-hand sums approach the same *value* that the left-hand sums approach as $\Delta x \rightarrow 0$. So since the area under the curve is trapped between these, its value must be this same numerical value too.

Figure 8—The area between the right- and left-hand Riemann sums for a given function for two different partitions of the domain. This area is seen telescoped into a single strip to the right of each graph, to illustrate that this region diminishes in area as the partition becomes finer.

⁴ In (Ely & Boester 2010) we discuss the merit of such an “infinitesimals” approach.

All of this we can say without a particular image for what happens to each rectangle in the limit. In other words, we can remain agnostic about the limit *object* that is attained as our rectangle widths Δx approach 0, and we only need to understand what limit value, or *quantity* of area, is attained by $\sum_{i=1}^n f(x_i^*) \Delta x^*$ when $\Delta x \rightarrow 0$ (and $n \rightarrow \infty$ accordingly). At each step, $\sum_{i=1}^n f(x_i^*) \Delta x$ is a quantity of area, and so is its limit. We need not worry about the qualities of the object that this limit value corresponds to. What we do need is a satisfactory way of determining the numerical limit of a sequence or set of quantities.

To conceptualize the first part of the fundamental theorem with this quantitative limits image, one may ask about the ratio of the change in accumulated area to the change in x , and what the limit of this ratio is as the change in x goes to 0 (Figure 2). As this Δx goes to 0, the object becomes more and more like a rectangle (well, trapezoid) divided by its width, so in the limit this ratio can be imagined to approach the height of the function at the given point x . This bit of reasoning can be done without the collapse metaphor and without infinitesimals. As in the case of the definite integral, it just requires imagery for understanding the numerical limit of a sequence of quantities.

* Supposing the interval over which one wants to find the area is partitioned into equal subintervals of size Δx , with each x_i being, say, the right-hand endpoint of the i^{th} subinterval.

This imagery might be best provided by the metaphor of *approximation*. This metaphor involves (a) an unknown value, (b) a set of known values that approximate this unknown value, and (c) the amount of error between each approximation and the unknown value, with the idea that this error value can be made as small as one likes by taking a good enough approximation (Oehrtman 2009). In the case of the fundamental theorem rectangle there is the unknown value (the limit ratio of $A(x)/\Delta x$), the approximating value (here, $(f(x)\Delta x)/\Delta x$ for x on a small interval $[x, \Delta x]$), and the idea that the error quantity between the two can be made as small as one needs by taking a good enough approximation (i.e., making the width Δx small enough).

This metaphor is particularly compatible with the quantitative limits approach, in which no rectangle is imagined as the final resulting graphical *object* of such a limit process, because the approximation metaphor focuses the attention instead on the *quantities*: the approximating and error quantities, and thus, the limit quantity. And indeed, students readily and resiliently appeal to the approximation metaphor, which can be productively and deliberately built upon for understanding the formal definition of limit (Oehrtman 2009; Oehrtman, Swinyard, Martin, Hart-Weber, & Roh 2011).

While it makes sense that the quantitative limits approach can promote productive conceptualization of the fundamental theorem of calculus, there may also be a cognitive tension that arises with this approach. This tension is based on the fact that the approach specifies images for the objects before the limit is taken, but does not necessarily specify an image for the *resulting* object of a limit process. For example, with this approach, when the limit is reached for a definite integral, the rectangles disappear. Another example is the fundamental theorem: when the limit is reached, there is no accumulation rectangle that remains, nor does the rectangle degenerate to a line segment, which would be the collapse metaphor. And similarly, when the

limit is taken in a derivative, there is no differential triangle that remains, nor does the triangle degenerate to a point, which would be the collapse metaphor. In all of these cases the quantitative limits approach offers no clear image for the final result of the object that shrinks in the limit process; there is only the quantity that is reached in the limit, and a broad image of what the quantity measures, such as the total area under the curve.

According to a few theoretical perspectives this may conflict with our tendency to imagine a resulting state for a limit process. For instance, Lakoff & Nuñez claim that learners apply the Basic Metaphor for Infinity (BMI) to limit processes, which entails the envisioning of a final resultant state of the limit (2000). Indeed, they claim that all mathematical objects involving the infinite employ this metaphor in their cognitive construction. If this is the case, then students will imagine some sort of final result of what happens to each rectangle in a Riemann sum, by the fact that the BMI is used when the limit is envisioned. If the quantitative limits approach presents them with no such final result, they may fill in such a result themselves. The most obvious choice might be to imagine that the rectangles degenerate into line segments—that is, the collapse metaphor. And I have already outlined concerns with this metaphor.

Whether this issue actually arises or not with the quantitative limits approach is a testable hypothesis, not just one for speculation: Do students who use the quantitative limits approach often find themselves resorting to the collapse metaphor as an image? Indeed, although there is evidence that the collapse metaphor can cause some problems with the fundamental theorem, might a student's successful understanding of the quantitative limits approach supersede these problems, even if they do at times use the collapse metaphor?

Conclusion

I have argued that both the quantitative limits approach and the infinitesimals approach allow for successful conceptualization of the fundamental theorem of calculus (and of derivatives and integrals) in ways that approaches involving dimensional collapse do not. It is not our purpose here to thoroughly explore the affordances and limitations of using either of these types of images when learning, or teaching, calculus concepts (a brief point/counterpoint related to this can be found in Ely & Boester 2010). Either way, the understanding of the fundamental theorem relies on the coordination of the change in area accumulated under a curve with the change in the underlying variable, an idea that is obscured by images of dimension loss or collapse. Patrick Thompson, Marilyn Carlson, and their colleagues have shown how this understanding is grounded in students' prior experience with covariation of quantities, experience that ideally should begin long before these students step into a calculus classroom (Carlson, Persson, & Smith, 2003; Oehrtman, Carlson, & Thompson, 2008; Thompson & Silverman, 2008).

It is important to note that the images of dimension loss, such as the collapse metaphor, can have some successful affordances too and should not be viewed as failed reasoning on the part of the student. These images may well arise as a result of prior instruction, and certainly any particular student may not use such an image consistently or exclusively across all relevant calculus contexts (Oehrtman 2009). The collapse metaphor will likely be common in any calculus class that focuses on conceptual, and not just procedural, knowledge of the important ideas of calculus. As a result, the instructor should take its affordances and limitations seriously and know how to furnish other imagery that will be more helpful, particularly for understanding the fundamental theorem of calculus.

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ⁱ Editorial Comment: The Fundamental Theorem of Calculus (FTC) can be viewed as an instance of “institutionalized” mathematical knowledge. In other words, even though everyone (i.e., mathematicians) know what it is, there is no uniformity in its portrayal in textbooks. It has been argued that its portrayal in textbooks is the “victim” of didactical transposition. Readers interested in this line of thought or “What is really the Fundamental Theorem of Calculus?” are referred to the work of Jablonka and Kliniska (2012) in the Montana Mathematics Enthusiast Monograph Series.

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Investigating engineers' needs as a part of designing a professional development program for engineers who are to become mathematics teachers

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1. Introduction

The average age for teachers in mathematics and physics in Norway is about 60 (Sørensen, 2003) and few students taking a masters degree in a science subject choose a career as a teacher (KD, 2006). This implies that there may be a critical lack of science teachers in schools in the near future (Næss, 2002). At the same time, rather many graduated engineers – who have worked as engineers sometimes several years – want to contribute to the educational setting by taking part time jobs as teachers; lector-II (Vassnes, 2008). To combine this supply and demand for qualified teachers in mathematics, a professional development program in mathematics for engineers is planned. Focusing on this education, a research project is planned. It shall be a design research which initially will include a design of the tailored program in cooperation with the practitioners. This will be followed by an investigation of the engineers who take part in the program: How their mathematical knowledge develops as a result of completing the education.

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The present paper has three aims: First, to outline the theoretical perspectives on which the planned design research project is based, along with general components to be considered when the engineers taking part in the program act as objectives. Second, to present the data collected to find out more about experiences and needs that engineers may have in a mathematics teaching position. Third, to discuss how findings in the incoming data material may influence the plans for the complementing studies.

The research question formulated is: What are the opinions, advantages and needs of an engineer in order to become a mathematics teacher, and how may these be implemented in a design research of a professional development program?

2. The design and theoretical framework

The theoretical framework of the planned main study is by Koenen Gravemeijer called developmental research (1994). He goes back to Freudenthal (1988), who meant that thought experiments are important in educational development. The developer envisions the teaching and learning process and after carrying it out tries to find evidence to see if the expectations he had were right or wrong. Based on practical experience, new thought experiments create an iteration of development and research. Gravemeijer claims that this cyclic process is at the centre of Freudenthal's concept of developmental research (1994, p. 449). Developmental research is similar to what is alternatively called design research. Eric Wittman is one researcher who sees advantages by regarding mathematics education as a design science (1995). He emphasizes that an important element is building theory related to the design and then making the empirical investigations.

In our main study, researchers and practitioners will work closely together to develop the courses to be offered in the program. The design of the courses tailored for engineers is an important part, see Figure 1.

Figure 1: Phases in the design research

The cooperation between researchers and practitioners is intended to be a composition of Wagner's clinical partnership and co-learning agreement (Wagner, 1997), and is guided by the particular type of students involved. The reason for this adjusted framework is an earlier experienced difficulty with collecting realistic data among engineers (Rensaa, in preparation). The close cooperation between researchers and practitioners in the design research may contribute to building a bridge between the research and the teaching practice (Czarnocha & Prabhu, 2004).

It is not relevant to mix the engineers with other students at our university. This is both because the designed courses need to be tailored to the particular setting that the engineers are in, and because earlier experience with such a mix has been somewhat diversified (Rensaa, 2009a; 2009b). The tool for analysis of data – both in the preliminary investigations of the present paper and in the main study – will be mathematical competencies (Niss, 2003b). As a main task they have to consider and answer the question ‘What does it mean to master mathematics?’ To do this, a competency based approach is adopted, defined in the following way:

Mathematical competence then means the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role (Niss, 2003b, p. 7).

Altogether eight competencies are defined, in which the first group focuses on the ability to ask and answer questions in mathematics: Competencies in thinking mathematically, posing and solving problems, modelling and reasoning mathematically. The second group of competencies is about the

ability to deal with mathematical language and tools: Representing entities, handling symbols and formalism, communicating and aids/tools. In our analysis of the engineers building of new knowledge in mathematics, all these competencies will be relevant. The competencies are, in themselves, behavioral with focus on what to *do*. In some of the data collection this will also include what the engineers are expected to do.

The engineers to be attending the planned development program will typically be expected to have valuable experiences from their work as engineers in how mathematics is used in practical settings. In the present research, an investigation is generated in order to put light on which experiences some engineers themselves emphasize as their advantages. These experiences are advantages that should be utilized in a teaching position. They represent the benefits of being a teacher as an engineer, thus they need to be encouraged in the professional development program.

The program will encourage the participants partly to continue their work as engineers and partly to work as teachers after completing the studies. Thus, the professional experiences and practical illustrations may be updated and referred to contemporary with the teaching.

However, the transition from working as an engineer to being a teacher in mathematics is not straight forward. Many engineers have a 'workman approach' to mathematics (Kümmerer, 2001). They view mathematics as a set of procedures to be used, without seeing the necessity of deeper understanding, to produce the right answer. This implies a need of complementing studies that take into consideration the particular background of the engineers, utilize the experiences they have and expand their knowledge of mathematical topics, but at the same time aim at changing apprehensions that do not hold when a subject is to be taught. The transition from knowing mathematics for use to knowing mathematics for teaching is rather difficult and needs to be problematized. It is an example of *didactic divide* between disciplinary and pedagogical knowledge (Ball & Bass, 2000; Bergsten & Grevholm, 2004). Additionally, there are didactical and pedagogical aspects about being a teacher that a development program needs to take into consideration. The Danish KOM project (Niss,

2003b) lists six mathematics teacher specific competencies, all being relevant to an engineer in a teaching position: Curriculum competency, teaching competency, uncovering of learning competency, assessment competency, professional development competency and collaboration competency. The first four competencies are expected to be rather ‘new’ to an engineer, but depending on work content they may be part of experiences acquired by the engineers. Professional development competency is one that may evolve throughout the tailored educational program, while collaboration competency may very well be familiar to engineers.

In a study of engineers’ progress throughout the complementing studies, their previous knowledge will influence their study. It is important that this aspect is included in the design of the program because meaningful learning as defined by Ausubel (1968) is learning that connects to what is already known. New knowledge needs to be assimilated within the existing knowledge structure to implement modification of prior knowledge and thereby meaningful learning. Thus, it is relevant to carry out some preliminary studies to identify aspects about engineers’ experiences that are relevant for inclusion in the design of the program. Results from such preliminary studies may also help in preparing to the manner in which to collect data about previous knowledge of the engineers that will take part in the main study. Adler et al. states in their survey of research in mathematics teacher education (Adler, Ball, Krainer, Lin, & Novotna, 2005) that one of a number of fields that are underrepresented in this research, is teachers’ learning to directly address inequalities and diversities in the teaching with respect to mathematical background. Engineers will probably connect their mathematical knowledge and competencies to applications differently from students within the basic mathematics teacher education. Closer investigation of this is interesting.

3. Data and analysis

The data of the present study was collected by interviewing four engineers. The aim was to investigate more closely which needs engineers may have in order to become mathematics teachers.

The interviews were done to get input to the design of the professional development program, and are by no means claimed to provide general results. However, experiences and needs brought to bear during this study represent most certainly not isolated cases.

Criteria for selecting the interviewees were:

- Completed bachelor engineering education at least 5 years ago; so as to have a distance from the subjects.
- Work experience as an engineer; to have knowledge about practice.
- Some type of relation to teaching; in order to express opinions about the theme.
- Both sexes represented.

Based on this, two bachelor engineers and two master's engineers were chosen with help from teachers that have educated engineers for many years and are familiar with students' careers afterwards. However, since the development program in its overall form is to be designed to meet the bachelor engineers' needs – with an adjusted scheme for master engineers to join the program at a certain stage – the main part of the analysis and discussion deals with the bachelor engineers' responses. Some interesting statements and comments of the master engineers are however included. In the discussion to follow, all interviewees are made anonymous by given fictitious names and vague descriptions of their professional activity. I did not introduce the plans for the professional development education or any teaching themes to the engineers ahead of each interview, so as not to influence the answers. I wanted to grasp the engineers' immediate reactions and thereby their unaffected point of views. Concepts used in the questions were not explained or defined, just referred to and meant to be understood in everyday, social terms. Particularly, the word 'kompetanse' was used in more than one question. The public understanding of this word is more likely to be translated to 'skill', not the strictly defined word 'competency'. Thus, with a few exceptions, I will refer to it as skill.

The interviews were semi-structured with a planned schedule, but allowing other questions and comments to be brought about. However, I followed a main framework of themes for each of the four interviews:

- Initially about taking a part-time mathematics teacher position and asking if they regard themselves as being qualified for this - at the moment.
- Advantages of an engineer as teacher in mathematics.
- Views about mathematics, both own skills and what students in schools should learn. This involved two multiple choice questions listing alternatives like ‘understanding of mathematical relations and logical structures’, ‘interest’, ‘understanding of how mathematics is used in work and daily life’, ‘calculation skills’, ‘ability to use the correct formulas and methods’, ‘ability to translate practical problems to mathematics and solve them’, and finally the alternative ‘other abilities’ where the interviewee could give his or her own description.
- Which skills in mathematics and didactics the engineers think they lack.
- Teacher specific competencies. This was given as a table where each competency was listed and was to be graded as large, good, some or no own possession of, in addition no opinion. When referring to this particular table in the interviews to follow, ‘kompetanse’ is exceptionally translated to ‘competency’. This is since the categories of the table are based directly on the ones given by Niss (2003a). As referred to earlier, he presents six teacher specific competencies, but only five are referred to in the table. Development of professional competency is left out as only one of the interviewees is attending a teacher education program.

Each interview was ended by a question about interest in attending a complementing study to obtain the number of study points in mathematics that is required in a teaching position in school.

Interview with Sophie; a bachelor engineer.

Sophie has long experience as bachelor engineer but has not taken any steps towards teaching practice. She knows a lot about the profession from members in her family. The first question she is asked is if she presently would consider taking a part-time mathematics teaching position and if she regards herself as being qualified for this:

No, I haven't enough knowledge in the subject to do so. I've seen what it takes, and by that I bring with me big demands as to what a teacher in mathematics needs to know. This is also why I don't regard myself as qualified for such a job.

This is a reflective answer, showing that Sophie really has thought about the demands that rest upon a teacher's shoulder. She emphasizes lack of knowledge in the subject itself as a factor that makes it problematic for her to take a teaching position. This is elaborated further when she is asked about which mathematical skills she believes she lacks:

My theoretical background is too sparse, it dates from so many years back that I have forgotten about it. I need to brush up on it. Additionally, I have seen from helping my son in upper primary school with his homework that the mathematics has changed. They learn the subject differently from how I did at that age. Things are in a way inverted.

This last statement is most interesting. Sophie has probably grasped the problem oriented approach – that is rather usual in schools today – as an ‘inverted’ way of learning mathematics. NCTM defines this as “engaging in a task for which the solution method is not known in advance” (2000, p. 51). By starting with a problem and seeking ways to solve it, relevant mathematical arguments are introduced. This may surely be apprehended as a reversed order compared to how mathematics topics often were presented some thirty years ago. Back then theory and methods were introduced first and used in practical setting as a conclusion. Being aware of the changes, Sophie indicates that she has reflected on problem solving abilities which is an important mathematics didactics perspective. It is, however, notable that this feels somewhat unfamiliar to Sophie. Engineers are usually dealing with open, practical problems in their work, problems for which they seek solution

methods. Sophie does not directly relate this experience to the problem solving strategies in school subjects. And she does not stress this experience as a resource that an engineer has in a mathematics teaching position. Her answer to the questions about which advantages and contributions an engineer brings into the classroom, is that: “One can relate the subject to reality, tell about the usefulness and practical use of mathematics and put the subject into another professional connection”. The last statement indicates that Sophie regards the mathematics teaching situation as one where concepts can be brought into the practical world to illustrate how they may be used. The transition from a real world problem to a mathematical problem – which is the task for an engineer – seems not to be included. A conclusion to this is that Sophie somewhat separates her engineering experiences from what she reflects upon as being relevant in a teaching position. The connection is not as tight as her underlining of practical use known to an engineer, indicates. This is an aspect that needs to be taken into consideration in the professional development program. The interplay between the real world and the mathematical world needs to be emphasized to make engineers conscious about the back and forth transition and engineers’ advantages in this respect. Particularly, with reference to Sophie’s answer, the performance of active modelling as mathematising practical engineering problems needs to be problematized. This is an important part of the modelling competency (Niss, 2003a)

Sophie’s answer to the multiple choice question about own skills in mathematics underlines her own view about having practical experiences. She has chosen ‘understanding of how mathematics is used in working and daily life’ and ‘being able to translate practical problems to mathematics and solve them’ as her two most important skills. However, only the last option about being able to translate is chosen when she is to select the two most important skills that a student in school should hold. Additionally to this translation knowledge, Sophie emphasizes ‘understanding of mathematical relations and logical structures’ as important to students. This shows that she regards herself as one who possesses only by parts the competencies that are important for a student to

have. Her earlier statement referring to the lack of theoretical background and her background being dated, explains this. With respect to Sophie's comment on the earlier question about mathematics in schools today being inverted, her marks in the multiple choice question indicate that she actually has some knowledge about 'inverted situations'. This is in the meaning of inverted situations as situations where one has a (practical) problem and seeks to solve it by referring to mathematical arguments. Earlier, she expressed a lack of knowledge about this, but in the multiple choice question she actually indicates possession of this, all the same. She probably does not see the connection between engineering competencies and mathematical competencies. A professional development program need to make engineers like Sophie conscious about how engineering competencies may be translated to be used in a mathematics teaching situation.

In the table where Sophie is to grade her teacher specific competencies, a rather realistic view is revealed. The only competency she grades as having good possession of is cooperation. This competency she has probably gained through her work as engineer. Her teaching competency is marked out as 'some', while she has checked off both teaching and uncovering of learning competencies with a cross on the border line between 'some' and 'no' possession of. As she has not completed any type of teacher preparation programs, this is most pragmatic. It is probably due to her own rather rigorous interpretation of what it means to have a competency in each of the teacher specific abilities. As the word 'competency' not was defined ahead of the interview, Sophie was free to put whatever requirements she wanted into the word. But obviously she has rather high demands. The reason for this is probably what she has observed through her family members working as teachers.

Interview with Robert; a bachelor engineer

Robert is the second bachelor engineer under interview. His experience as an engineer is not as long as Sophie's, but he completed his education more than 5 years ago and has worked as an engineer

since then. Recently, he started a new job and parts of this job involve supervision of students when doing experiments in a laboratory. Thus, he is somewhat engaged in teaching.

Initially he claims, however, not to have considered taking a mathematics teacher position at any moment. The reason he gives for this, is that he has experienced on a number of occasions, how unmotivated children in primary and secondary schools can be. Still, he classifies himself as being qualified for such a position, with enough knowledge in mathematics to teach and ability to explain mathematical topics to teenagers. Some youths among his acquaintances, he explains, have told him that he is good in explaining mathematics to them. Robert gives an example of this:

A girl complained to me about not understanding her teacher's way of solving of a second order equation. The teacher had moved a number 4 from the left side of the equal sign to the right hand side, and then suddenly changed the sign of the 4. The girl could not understand *why*. I showed her how the teacher actually had subtracted 4 on both sides of the equal sign, and the result came out as minus 4 on the right hand side. This girl did suddenly understand what it was all about!

The statement is given to deepen his answer 'yes' to the question about being qualified to teach mathematics in school. He continues this argumentation when explaining which advantages an engineer has in a teaching situation: "We have learned a lot of mathematics. We have seen how important the subject is, and in how many places one needs to use mathematics". In general terms, Robert highlights experiences from practical use as the main advantage of an engineer as teacher, and he signalizes that he has knowledge within the subject. To him, this is enough to be a qualified teacher. In the multiple choice question regarding his views about mathematics, his emphasis of practical use is enhanced by his selection of 'knowledge of how mathematics is used in working and daily life' and 'knowledge of how to translate practical problems to mathematics and solve them' as the two most important mathematical skills that he holds. To Robert, however, these abilities seem to be the vital ones also in a learning situation. When I ask him to mark the two most important

approaches to mathematics that students need in school in order to learn the subject, he chose similar alternatives; how mathematics is used in working life, and translation from text problems to mathematics problems. Both alternatives – repeated twice in Roberts answers – may be recognized within the competency of mathematical modelling that is given by Niss (2003a) as “being able to analyze and build mathematical models concerning other areas” (p. 183). However, Robert seems to be rather tool oriented in his approach to the problems since he has not chosen the first alternative about understanding of mathematical relations and logical structures neither as a skill that he possesses himself nor as one of the two most important skills that a student needs to have. To Robert, understanding the ‘why’ is less important than understanding of the ‘how’. Robert is not the only engineer asserting this view (Kümmerer, 2001; Bergqvist, 2006), and this needs to be problematized most sincerely in the professional development program.

In the final part of the interview, dealing with didactical skills needed for a mathematics teacher, Robert continues in a self confident manner. When I ask him which skills – both mathematically and didactically – he thinks that he lacks in order to teach, his answer is “In the subject; none. In didactics; probably some”. In the table where he is to grade his curriculum, teaching, uncovering of learning, assessment and cooperation competencies, all but the first are classified as ‘good’. Curriculum competency is marked out as having some possession of. When returning the marked table to me, he explains that through his children’s school attendance – where he has taken part in a number of Parents Committees – and by having apprentices during his engineering profession, he has gained experience within most of the competencies. This is continued with a detailed story about how he on several occasions has had to motivate and encourage apprentices to do a better job. Once again, Robert’s self-confidence is brought to bear. He believes he possesses most teacher specific competencies to a good degree, despite never having practiced as an ordinary teacher or having completed any teacher preparation programs. Similar to Sophie’s case, this may be explained by Robert’s interpretation of the word ‘competency’. In lay terms it may be both vague

and inaccurate. And in contradiction to Sophie's interpretation, Robert has obviously no strict definition in mind. Thus, his grading of his own competency as 'good' to Robert may just mean that he knows about it and has personal experiences of it. He does for instance probably not regard uncovering of learning of mathematics as difficult, despite research saying something else. This competency includes interpretation and analysis of students' learning, their notions, beliefs and attitudes in addition to identifying each student's development (Niss, 2003a). This is surely not a competency that is picked up by taking part in Parents Committees. When instructing apprentices at an industrial establishment, some experiences in uncovering learning and development may be gained, but these are not with respect to mathematics. They are probably of a more practical nature.

The two masters' engineers

The two masters' engineers, Lisa and Paul, are both experienced engineers but with somewhat different priorities. Paul has earlier been teaching at an engineering education for a decade but has, in addition, several years of experience from the engineering profession – which he has returned to lately. Lisa on the other hand, started off her career by working as an engineer, but has changed her professional life to teaching both at secondary and tertiary levels. She teaches both mathematics and science subjects. Recently she has started a teacher credential program which in minor parts includes studies in mathematics didactics. Lisa claims not to have any inadequacy when it comes to skills in mathematics, and Paul claims to see himself as highly qualified to teach the subject at a tertiary level, but they have somewhat different views about what it takes actually to *be* a teacher in school. Lisa has reflected on this part rather carefully:

I have come to realize that there is a great challenge in knowing how to explain things and what one actually needs to explain. Things that seem so obvious to me that they are indisputable are stated in the mathematics didactics literature to be difficult. For instance, the calculation of integrals is to me a routine matter, often just easy and evident when you have the experience. But then, when I have to explain *why* we do it in this way and what the arguments are behind

using integrals to calculate areas, then it becomes difficult. Thus, I have come to realize that in teaching it is not enough to know the pure mathematics which I do, there is also the challenge in the aspect for the need of explanation.

Paul – on the other hand – emphasizes that he as part of his education has completed so many mathematics subjects or math related subjects that he is well qualified to teach at the secondary level. However, when I ask him which skills he believes he lacks, he does admit that since his own education is dates so many years back, some of the modern mathematics subjects are missing. This he has experienced through helping his children with their homework:

But it is an easy subject to teach, we have group works very often at home. I probably have some shortages when it comes to didactical skills, but as I see it the most important thing is to be *engaged* in your teaching.

The two rather different views about teaching of mathematics put forward by Lisa and Paul show the importance of the engineers' own experiences when it comes to realizing which needs they have in a teaching position. While Lisa is attaining a teacher credential program that requires her to think through difficult problems within teaching, Paul has his practical experiences from the domestic arena and during teaching of special subjects in an engineering education. At least in his formal teaching, it is most likely that if he has referred to mathematics it has been to readymade tools or equations that often are used to solve specific practical problems. Thus, his comprehension of what is required when teaching mathematics is probably to have a tool oriented approach.

4. Discussions and implications for the main study

The shortage of teaching professionals in mathematics and physics in schools may partly be solved by recruiting teachers from other science and engineering fields. Engineers as teachers is an example of this. They may be included in what literature defines as STEM (Science, Technology, Engineering and Mathematics) career changers, even if they only take part-time positions as

teachers in schools. All the interviewees and the literature points out that career changers bring a wealth of experiences, both personally and professionally, to the classroom (Grier & Johnston, 2009). Chamber (2002) lists a number of additional positive contributions from this type of teachers: Collegial support, good communicational skills, management of multiple projects, strong work ethic, more tolerance to diversity and more willingness to adopt student-centered methodologies. But as the present study suggests, the engineers could benefit from completing a professional development program in mathematics before starting teaching. At least two challenges have crystallized through the answers in the interviews, challenges that to our earlier experiences from engineering education most certainly are not outstanding for the engineers interviewed. These challenges need to be taken into consideration when designing the program:

- The engineers' apprehension of their own abilities and what it takes to teach, both in mathematical and didactical terms.
- The engineers' apprehension of how to connect mathematics and practical use outside schools.

The two bachelors engineers, Sophie and Robert, represent in a way two extremes when it comes to apprehension of own skills and what is required to be a mathematics teacher. While Sophie signalizes having fairly little confidence about her own knowledge – both lacking theoretical foundation and most teacher specific competencies – Robert gives the opposite impression. His self-confidence may in a classroom situation make him not hesitate to praise the usefulness of mathematics in engineering and provide the students with lots of examples. This may motivate students in schools. However, the drawback with having self-confidence like Roberts' is that he may not become conscious of his own limitations. Most engineers do unmotivated realize that they lack of didactical competencies, since this is absent from their education. Some knowledge may be gained through their work as engineers, and the experience in collaborative work may be rather extensive, but engineers do see the need for didactical training. The challenge is, however, to make these engineers also to realize their limitations when it comes to competencies in and

comprehension of, mathematics. Engineers have a rather broad knowledge of mathematical topics with reference to the list of themes that are dealt with – sometimes rather superficially – in mathematics courses in their own education. Thus, if they just look at the content of the mathematics subjects in schools, they will recognize most topics and thereby may be misled to believe that they are very qualified in the subject. Robert clearly has this opinion, and both masters' engineers also express this. Paul admits that he lacks some of the modern mathematics courses, but he does not see this or his limited didactics education as a problem. Lisa says that she has no shortfalls in mathematics in order to teach, but she is aware of the didactical needs. None of the four engineers do, however, seems to regard their own comprehensions of the mathematics *subject* as a challenge. Kümmerer (2001) states that many engineers have a workman approach to the subject, where following a receipt of rules produce the right answer to a problem. Bergqvist (2006) has revealed what she calls a tradition within the engineering education when interviewing a senior lecturer in a master engineering program: These students are prepared to learn a lot of algorithms in mathematics without much time for reflection. But a tool oriented approach is not enough in a teaching position. Problem solving, logical reasoning and concept understanding are all abilities of great importance. Among the interviewees, only one has marked off 'understanding of mathematical relations and logical structures' as an ability that she possesses herself; the masters engineer Lisa. The other three have not regarded this as one of the two most important skills that they hold. Thus, both Robert and Paul are examples of engineers who regard themselves as being qualified to teach mathematics even if they have gaps in their understanding of mathematics relations and logical structures. This is a major challenge that occurs ahead of the running of the complementing studies, and may result in the non-attendance of the program by some engineers. They do not realize themselves that they are in need of developing their mathematical competencies. A way to meet this challenge is use of publicity; getting professional journals to write about what is required. In this publicity it is important to emphasize the advantages that

engineer as a mathematics teacher can bring to the classroom in order to promote career change, but at the same time focus on additional mathematical competencies that an engineer lacks in order to become a qualified mathematics teacher.

In contrast particularly to Robert, Sophie has far less confidence in her own abilities to start teaching mathematics in primary or secondary school. She has – through observing family members’ work – seen that it takes more than her ‘engineering mathematics competence’ to teach in schools and in this way she appears far more realistic in her answers. Too realistic, in a way. For professionals like Sophie it may be a problem that their expectations as to what is needed to be prepared for teaching in mathematics, are so vast that they hesitate in entering both complementing studies and a teaching position.

Many engineers do, however, have confidence in their own abilities that are not as extreme as Robert’s or Sophie’s, but lies somewhere in between. The masters’ engineers Lisa and Paul, are examples of this. Indeed they both regard themselves as possessing mathematical skills enough to be teachers in schools, but they do also realize that they have some shortfalls. Lisa sees the need for learning mathematics didactics while Paul lacks some of the modern mathematics subjects. Both have confidence in their own skills, but see the need for learning more. Grier and Johnston (2009) writes about STEM career changers that “Their maturity brings confidence in their abilities to manage a classroom, understand student learning in the context of school culture, and to access content knowledge” (2009, p. 57). The professional development program needs to challenge these beliefs about own abilities and make the engineers conscious that they need to further develop their knowledge in mathematics and also with respect to didactics. This may both be done by introducing mathematics courses that usually are new to an engineer and by challenging the engineers within the topics in which they regard themselves to have knowledge in. For instance, a number theory course could initially be offered. This is a subject that engineers with a degree from some years back have not studied, and it appears to be an easy subject that in reality is both challenging and

demanding. Moreover, it is a subject that a teacher needs to have knowledge in. In calculus – which an engineer may think she is sufficiently qualified in just by looking at the themes – a challenge would for instance be to let him or her introduce the derivative to a group of students. The concept may very well be motivated by practical illustrations, but is not easy in theoretical terms: Students have problems understanding the meaning and definition of the derivative (Orton, 1983).

Both Sophie and Robert emphasize knowledge of practical use of mathematics as a main advantage an engineer has when teaching the subject and this is supported by statements from Lisa and Paul. But Lisa is the only one that refers to an example that may illustrate this usefulness. She mentions that when introducing the derivative to students, an engineer may easily come up with practical examples of areas where it is used. This is the only example referred to during all four interviews. In light of the setting, this is not surprising. The engineers were not given any time to prepare themselves for the questions and almost no information was given ahead of the interview. They were just asked there and then to answer some questions about mathematics and teaching of mathematics. The intention was to get immediate reactions reflecting the current situation. To the questions about advantages that an engineer has, the answers have consequently become more of a general nature. Still, since only one of the interviewed engineers is working as a mathematics teacher – and the other more distant to the mathematics content in school subjects – a prepared interview would perhaps not have given concrete suggestions anyway. Nevertheless, the answers do indicate that practicing engineers may not have immediate suggestions of illustrations and examples from their work. The professional development program needs to encourage this part rather profoundly. Even if engineers often and in different connections use mathematics as part of their work, it is not immediately given that they are able to identify this use. Thus, it becomes important to provide them with means for recognizing this. This is an aim of mathematical archaeology; “to make explicit the actual use of mathematics hidden in social structures and routines” (Skovsmose,

1994, p. 95). Engineers can be made aware of the use of mathematics through examples, but perhaps even more importantly through group work discussions about their own work.

Sophie's remark about an engineer's ability to place mathematics into another professional connection is noteworthy. The 'traditional' way of relating mathematics to the life outside schools is to conclude the teaching of a theme or a concept by illustrating how it may come to be of use in a practical situation. In a problem solving situation, the opposite way of thinking is initiated. One starts with a practical, often open-ended problem, and seeks ways to solve it. The necessity of deriving mathematical methods to solve the problem is then enforced. For experienced engineers who finished their education a number of years back, a problem solving aspect may be 'new' with reference to mathematics teaching. In an engineering job, however, this way of working – starting with a problem and searching for ways to solve it – is most certainly the usual approach. Thus, engineers need to be made aware of how to translate their experiences from problem solving as engineers to the classroom situation. Lecturers in the professional development program need to take into consideration that engineers may not directly see how their professional working methods are useful in a teaching situation. Their references to practical examples are important, but additionally their experiences from methods of their work as engineers are relevant. Students in schools can organize their work with the subject in a similar manner in the classroom. Sophie's statement indicates that she has not thought about this as an advantage, and she is most certainly not alone in this respect. If the main reference to how things are done in the classroom is from own schooling, one may have rather limited teaching aspects in mind. The complementing studies need to 'open the eyes' of some engineers.

To the last question in the interview about whether the engineers are interested in attending complementing studies in mathematics, they all answers 'yes'. This answer is somewhat inconsistent at least with Robert's and Paul's earlier statements saying that they regard themselves as qualified for a mathematics position in school. Their answers may just be given to please me as

the interviewer. But it may also be that the engineers – throughout the interview – have come to realize that they could benefit from completing a professional development program before starting to teach. The challenge is to reach other engineers with the same message, but still emphasize their valuable professional traits to motivate a career change.

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From designing to implementing mathematical tasks: Investigating the changes in the nature of the T-shirt task

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Introduction

From looking at research literature it is possible to see that research on design, implementation and analysis of mathematical tasks is an actual theme: there is a special issue of the Journal of Mathematics Teacher Education (2007) with Anne Watson, John Mason and Orit Zaslavsky as editors (Watson and Mason, 2007), a book published by Clarke, Grevholm and Millman (2009) concerning “Tasks in primary mathematics teacher education” and under ICME 11 in Mexico (2008) the title of one of the Topic Study Groups was “Research and development in task design and analysis”. In addition several substantial research projects conducted in the United States focus on this issue. For example the QUASAR project (Quantitative Understanding: Amplifying Student Achievement and Reasoning), involving a group of researchers (Stein, Smith, Henningsen & Silver, 2000), aimed at improving mathematics instruction for students by emphasising thinking, reasoning, problem solving and the communication of mathematical ideas. One of the central aspects of their research was to focus on the use of instructional tasks in project classroom and they proposed the elaboration of “the mathematical tasks framework” where the kinds of thinking needed to solve tasks were referred to as “cognitive demands”. They reported on observations concerning the change of cognitive demands during a lesson where “a task that starts out challenging ... might not induce the high-level thinking and reasoning that was intended as the students actually go about working on it” (Stein et al., 2009, p.xviii). This aspect is also address by Artigue (1994) arguing that it might be tempting to implement too quickly development products arising from research into products for teaching. She characterises the processes related to the transmission of products from didactic engineering in terms of distortions and she emphasises the distinction between the activities of conducting research and of engaging in teaching. My aim, in this article, is to follow Artigue’s argumentation and to investigate, trace and characterise the distortions of a specific mathematical task (the T-shirt task) from its design by a group of didacticians at University of Adger (UiA) to its implementation by two different teachers. This research is situated in a larger research project conducted at (UiA), the *Teaching Better Mathematics* project (TBM).

The structure of the article is as follows: First I present central aspects of the TBM project and emphasise the theoretical constructs of didactical aim and pedagogical means. I also introduce the methodological approach adopted in the project. Then I turn to an example and explain how the T-shirt task was designed by didacticians at UiA and how it was implemented by a teacher from primary school and by a teacher from lower secondary school. Finally, I discuss the results and present implications for further collaboration between didacticians and researchers.

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Central aspects of the TBM project

Co-learning agreement with teachers

The aims of the TBM project are reflected in the title: *Teaching Better Mathematics*. First it promotes to develop better understanding and competency in mathematics for pupils in schools (Better Mathematics), and second to explore and develop better teaching approaches (Teaching Better) as a means to achieve the first aim. The nature of the project is developmental research and we collaborate with in-service teachers from 4 kindergarten, 6 primary and lower secondary schools and 3 upper secondary schools. Our collaboration with teachers is organised around workshops, approximately 4 – 5 per year, and school visits during which a didactician get the opportunity to observe the nature of the impact the project had on the participating schools. In the project we see teachers and didacticians as working together as co-learners (Wagner, 1997). This implies that both teachers and didacticians are engaged in action and reflection, and by working together, each has the opportunity to develop further understandings of the world of the other and of his/her own world. In addition, we are all, teachers and researchers from the university, engaged in researching and developing our own practice and this is the reason why I use the term “didacticians” for the group of researchers at UiA, since we consider all participants are researchers (Berg, in press). Thereby through our collaboration with teachers we, as didacticians, might learn about the teachers’ teaching practice in schools and, at the same time, about our own research practice. Central to the project activity is the design of mathematical tasks, by the didacticians at the university, as a means for engaging collaboratively with the teachers in mathematics. This engagement takes place between teachers and didacticians during the workshops regularly organised at UiA, and between teachers and pupils in the classrooms during their teaching practice. Teachers often take mathematical tasks which were presented during the workshops and adapt them to their own teaching practice (Berg, 2010).

Theoretical basis of the TBM project

The theoretical basis of the TBM project is elaborated from two fundamental ideas: community of practice as developed by Wenger (1998) and the idea of inquiry (Jaworski, 2010). By using the notion of *community of practice* I can refer to the activities in which we, teachers and didacticians, are engaged, and are mainly related to learning, teaching and didacting. In addition, the idea of *inquiry* enable us to consider a *community of inquiry* within which we use inquiry as a tool in all aspects of our practice and aim at developing “inquiry as a way of being” in practice (Jaworski, 2004). The idea of inquiry refers to asking questions and recognising problems, seeking for answers and solutions, and at the same time wondering, exploring and investigating the activity we are engaged into while looking critically at what we do and what we find. In the TBM project inquiry is addressed at three different levels: at the first level, inquiry in mathematical tasks in relation to pupils’ mathematical learning in classrooms, at the second level, inquiry in the developmental process of planning for the classroom and exploring ways of developing better learning environments for pupils in mathematics, and at the third level, inquiry in the research process of systematically exploring the developmental processes involved in the two previous levels (Jaworski, 2007). The first level refers to the collaboration in classrooms between a teacher and his/her pupils, the second and the third levels address the collaboration between teachers and didacticians/researchers.

Theoretical basis of the research

My aim with this article is to conceptualise the collaboration between didacticians at the university and the teachers working in schools in terms of Activity Theory (Engeström, 1999). Central to this approach is the idea of *activity*, a term which has a precise meaning. According to Leont'ev

... in a society, humans do not simply find external conditions to which they must adapt their activity. Rather these social conditions bear with them the motives and goals of their activity, its means and modes. In a word, society produces the activity of the individuals it forms. (Leont'ev, 1979, pp. 47-48)

Thereby the key idea is that human activity is motivated by the sociocultural and historical processes of society and comprises mediated goal-directed actions. Furthermore, Leont'ev proposes a three-level explanation of the idea of activity: First human activity is always stimulated by a motive. Second, human activity is rooted in actions which enable the subject to realise her activity through achieving the goal of each action. Finally, each action is realised through operations which are dependant of the conditions associated with the action. In the TBM project, our activity, as didacticians, is to work collaboratively with teachers on developing mathematics and the teaching of mathematics in order to learn more about the role played by our community of inquiry and inquiry processes both in relation to teachers' teaching practice and to our own research practice. Concerning our actions and goals, the design of mathematical tasks for workshops is one of the actions our group is engaged with. As a consequence, the workshops conducted by the didacticians from UiA are understood as one of the outcomes of our activity system at UiA. Similarly, I consider the different schools where the teachers are working as activity systems where the teachers' teaching practice is one of the outcomes of their activity system. There is no space here to go into details concerning Activity Theory (see Roth & Lee, 2007; Virkkunen & Kuutti, 2000) but I will concentrate on the processes behind the design of a particular task by the didacticians and the implementation of this task by two mathematics teachers, one from primary and one from lower secondary school.

However, I consider that Activity Theory addresses "activity" *in general terms* and as a means to point to the specificity of our activity systems, which is addressing the collaboration between didacticians and teachers with focus on mathematical learning and teaching, I propose to introduce the ideas of *didactical aim* and *pedagogical means* in order to articulate the goal of our actions in a more precise way. Didactical aim refers to "the choice of a particular area or knowledge target within a subject-matter" (Berg, 2009, p.100), while pedagogical means refers to a task "to use in order to address the chosen didactical aim" (p.100). For example, a didactical aim could be "to investigate Pythagoras theorem" and it is possible to find many different tasks used as pedagogical means to achieve this didactical aim. Furthermore, I consider that "...by presenting a particular task within a specific social setting, a didactician creates a *mathematical environment* whose characteristics depends *both* on the mathematical task and on the social setting." (Berg, 2009, p.103).

In this article I consider three different mathematical environments: the first one refers to the workshop during which the T-shirt task was presented to the teachers. Here the social setting refers to the group of didacticians working collaboratively together with the teachers while engaging in the T-shirt task. The second and the third mathematical environments refer to the way the task has been implemented in the teaching practice of each of the two teachers. In that case the social setting refers to the teacher together with his/her pupils working on the T-shirt task. In each of these mathematical environments, I explain what the didactical aim and the chosen pedagogical means are and I follow the processes behind the design of the task and its implementation in terms of change and modification in the didactical aim and pedagogical means. I argue that the introduction of these theoretical constructs enables me to develop

central concepts of Activity Theory further and to articulate the collaboration between the teachers and ourselves in a more accurate and focused way. In this article I consider the processes behind the design and implementation of the T-shirt task (Figure 1). The T-shirt task was previously introduced during a seminar at UiA concerning theoretical perspectives in mathematics teaching and learning through an article concerning socio-mathematical norms (Tatsis & Koleza, 2008).

The T-shirt task was designed as an imaginary phone call where one person had to explain to another the design of a logo to be reproduced on a T-shirt (Figure 1). In this context the nature of the questions the other person may ask, as a means to reproduce the logo of the T-shirt in an accurate way, was important. The team of didacticians at UiA had decided to present the T-shirt task as a means to engage and explore what communication in mathematics might mean.

Figure 1: The T-shirt task

Methodology

The methodological approach adopted in the TBM project is developmental research. According to Goodchild (2008), developmental research is characterised by a cycle between a *development cycle* and a *research cycle*. The research cycle refers to a cycle between global theories and local theories and, in the TBM project, global theory refers to community of practice (Wenger, 1998) and co-learning agreement (Wagner, 1997), while local theory refers to community of inquiry (Jaworski, 2007). Furthermore, a development cycle consists of a cyclical process between *thought experiment* and *practical experiment*. In other words, the notion of thought experiment refers to the preparation of the workshops where we collaborate with teachers, while practical experiment refers to the actual realisation of these. Feedback from participants informs the next step of thought experiment. In the previous section, I argued for introducing Activity Theory as a means to articulate the collaboration between the teachers and our group of didacticians. Thereby, in this article, the local theory refers to Activity Theory with the ideas of didactical aim and pedagogical means as a further elaboration of the theoretical framework. I consider that this theoretical approach allows for a better understanding of what thought experiment (preparation of the workshops or of teaching period) and practical experiment (realisation of the workshops or of the teaching period) mean in the context of this research.

The research setting

In this article I report episodes taken from a meeting in November 2008 at UiA during which didacticians were engaged in preparing for the next workshop and discussing a specific task (the T-shirt), from the presentation of the task during the workshop, and from interviews with the two teachers before the implementation of the task in December 2008 and May 2009. Within the TBM project we collected data consisting of video recording of all workshops we organised with the teachers. Furthermore all classroom observations and interviews with teachers are either video recorded or audio recorded. In addition, we keep field notes and e-mail correspondence as part of our data collection. In the following sections I present the first,

the second and the third mathematical environments and trace the way the T-shirt task has been modified and adapted by the teachers using the central ideas of didactical aim and pedagogical means.

The first mathematical environment: preparation and realisation of the workshop in December 2008

The theme of the workshop in December 2008 was “Communication in mathematics”. This theme has been chosen among several suggestions sent to us by the teachers. Before the workshop we had several meetings where we prepared for the workshop and discussed how to address this theme. During one of the first meetings, in November 2008, we agreed on the fact that it was important to contextualise the discussion around the idea of communication in mathematics by relating it to a specific task. In addition one of the didactician emphasised that:

”...in communicating mathematics, questions are a far more effective way of communicating than telling. In order to make sense of mathematical knowledge, pupils need to take the responsibility for exploring which means questioning the teacher, questioning others. The fundamental aspect about communication is *questioning*” (Didacticain 1, TBM meeting 261108)

Thereby we decided that the idea of questioning was crucial to communication in mathematics and we see both issues as deeply rooted in our inquiry-based approach to mathematics and mathematics teaching (Berg, in press; Jaworski, 2006; Wells, 1999). I consider these meetings, while preparing for the workshop, as part of our “thought experiment” (Gravemeijer, 1994) since we were envisaging possible ways of teachers’ engagement. Looking at these preliminary meetings it is possible to identify both our didactical aim and pedagogical means: Communication in mathematics was the chosen theme for the next workshop and, among several tasks, we choose the T-shirt task (see Figure 1) since we considered that it had the potential to stimulate the participants in engaging in reflecting about the issue of “questioning”. During our discussion we mentioned the fact that one of the possible elaborations of this task could be to introduce a coordinate system instead for the grid on which the logo is drawn (see Figure 1). These ideas were presented to the teachers during the plenary session in the workshop from December 2008 where the theme was Communication in mathematics and the title of the workshop was “To ask good questions in mathematics”. In addition, we initiated a discussion related to the meaning of inquiry in mathematics and to the importance of inquiry as an approach to mathematics and to teaching mathematics. The T-shirt task was introduced as a context for addressing the following questions: How can we ask “good” questions? Can we, as teachers, learn how to formulate questions which stimulate students? After the plenary session, we all divided in different groups, each one consisting of teachers from the same level (primary, lower secondary or upper secondary), and one of us, didacticians. Looking at the preparation and realisation of the workshop where the T-shirt task was introduced it is possible to extract the main aspects of this social setting using the constructs of didactical aim and pedagogical means where “communication in mathematics” was the didactical aim and the T-shirt task was chosen as the pedagogical means. The main elements from the first mathematical environment are summarised in Table 1.

Implementing the T-shirt task during the workshop	Mathematical task: Didactical aim	Mathematical task: Pedagogical means
Social setting: Didacticians and teachers Workshop, December 2008	Communication in mathematics	The T-shirt task

Table 1: Central aspects of the first mathematical environment

In the previous section I explained the nature of the first mathematical environment where the didacticians chose to focus on “communication in mathematics” as the didactical aim for the next workshop. In the next section I present the results of the analysis of the way the T-shirt task was implemented by two teachers and thereby I offer a characterisation of the second and the third mathematical environments by explaining their didactical aims and pedagogical means.

The second and the third mathematical environments: implementation in primary and lower secondary school

As mentioned before, the rational for investigating the implementation of the T-shirt task is that shortly after the workshop a teacher from primary school invited me to come and observe her teaching since she wanted to implement the T-shirt task in her class. The same happened during spring 2009 with a teacher from lower secondary school. The first teacher (Kari) is teaching in primary school, grade 6 (age 10 to 11), while the second teacher (Rikard) is teaching in lower secondary school, grade 8 (age 13 to 14). In addition to classroom observations I asked the teachers if it would be possible to interview them both before and after their teaching period. My aim was to investigate the reasons why they found this particular task interesting and worthwhile to implement in their respective classes. In addition I wanted to follow in which ways they modified and adapted it.

Following Kari in her class

As mentioned I had the possibility to interview Kari before observing her teaching period. My aim was to ask her the reasons why she selected the T-shirt task as an interesting task for her class. Furthermore I was interested in the way she had modified and adapted the task to her pupils:

”Yes, it [the task] captured me, and then, when I started to think that it could be about coordinate system, then, then I thought that this is a task I will use.... I relate it [the task] to my teaching and to what I do on that grade.... Pupils will need to use their mathematical language, they can talk about circles, triangles, and several concepts I would like them to have.”

It seems that Kari became interested in the T-shirt task when she realised that it could be related to the learning of coordinate system. Furthermore she explained that this was the focus of her teaching on that grade and she considered this task as a good opportunity for her pupils to develop their understanding of the use of coordinate system further. In addition it seems that she saw the opportunity for pupils to engage with the task while using accurate mathematical terminology, like “circles” and “triangles”. Even though Kari recognised the importance of mathematical communication and the use of accurate terms, my interpretation of her reflections, as presented during the interview before class, is that the main focus or didactical aim for her teaching period was on the use of coordinate system. One important aspect here is the introduction of the idea of “coordinate system”. Looking back to the way the T-shirt task was introduced during the workshop (Figure 1), the logo of the T-shirt was drawn on a grid, not on a coordinate system. Thereby, by removing the grid and introducing a

coordinate system, Kari was able to relate the task to her teaching. During the classroom observation, I had the opportunity to follow how Kari organised her teaching lesson by first offering a repetition of the main aspects concerning coordinate system (for example, what does “origin” mean, how to write the coordinates of a point, what does the first number represent, the abscissa, and what does the second number represent, the ordinate). Then she introduced a task consisting of several logos, starting with a logo with only straight lines and located in positive y-values, and finally ending with the same logo as the T-shirt task. The pupils were sitting in pairs and the challenge consisted of describing to each other the logo, which was drawn on a coordinate system, and using only the coordinates of the different points. Elsewhere I address in more detail the elaboration, realisation, and the challenges Kari met during her teaching period (Berg, 2010, in preparation). The main aspects of Kari’s teaching period are summarised in Table 2.

Implementing the T-shirt task in grade 6	Mathematical task: Didactical aim	Mathematical task: Pedagogical means
Social setting: Kari and her pupils (grade 6) Teaching period, December 2008	Coordinate system	The T-shirt task (communication)

Table 2: Central aspects of the second mathematical environment

Following Rikard in his class

As mentioned previously, I was also invited in May 2009 by Rikard as he wanted to implement the T-shirt task in his class. Similarly to Kari’s observation, I asked Rikard the possibility to interview him before observing his teaching. Again my aim was to follow the processes behind his interest for the T-shirt task and to trace the way he modified and adapted it to his class. During the interview before his teaching period Rikard explained the rationale for choosing this task:

”From the curriculum, there are first of all two aspects which I would like to have as goals for my teaching, it is the use of coordinate system, and the second is the introduction of functions... And you can say, what I want to emphasise is communication, I would like the pupils to have an understanding of how one communicates in mathematics.”

Again it seems that Rikard became interested in the T-shirt task as he saw the possibility to relate it to the curriculum and more specifically to the use of coordinate system. In addition the curriculum for grade 8 introduces the notion of “function” and links it to the use of coordinate system. Thereby, similarly to Kari’s arguments, the first aspect Rikard emphasised was the opportunity to develop pupils’ understanding of the notion of “coordinate system” further and to establish a link to “function”. At the same time, he seemed to recognise the importance of how one communicates in mathematics and therefore the T-shirt task would be suitable for implementation in grade 8. Here again, Rikard modified the T-shirt task by introducing a coordinate system instead for the grid from the original presentation of the task. By following Rikard in his class I was able to observe the way he introduced the task to his pupils. First of all, he presented a repetition of the main elements of a coordinate system, emphasising the fact that when writing the coordinates of different points, the first value corresponds to the x-coordinate (abscissa), while the second value corresponds to the y-coordinate (ordinate). Then Rikard organised his lesson as follows: first he introduced the logo from the T-shirt task *without* any coordinate system or grid behind it. He asked two pupils to come in the front of the class and one of them had to describe the logo to the other.

The rest of the class was following the discussion and Rikard asked them to be sensitive to the terms the two pupils used in their description of the logo. In the second phase, Rikard introduced the logo of the T-shirt task *with* a coordinate system behind it and chose two other pupils for describing the logo. According to Rikard, the aim of organising the teaching period in that way was to emphasise the usefulness of a coordinate system when describing a logo or a figure. From Rikard's explanation, it seems that even though the main focus during his teaching period was on the usefulness and advantages of using a coordinate system, he also emphasised the use of accurate mathematical terms during the description of the logos. Elsewhere I address in more detail the elaboration, realisation, and the challenges Rikard met during his teaching period (Berg, 2010, in preparation). The main aspects of Rikard's teaching period are summarised in Table 3.

Implementing the T-shirt task in grade 8	Mathematical task: Didactical aim	Mathematical task: Pedagogical means
Social setting: Rikard and his pupils (grade 8) Teaching period, May 2009	Coordinate system	The T-shirt task (communication)

Table 3: Central aspects of the third mathematical environment

In the previous section I described and analysed the way the T-shirt task was implemented both in primary and in lower secondary school, and identifying in each cases the didactical aims and pedagogical means. In the next section I discuss the findings and propose implications for further collaboration between didacticicians and researchers.

Discussion and conclusion

From the analysis of Kari's and Rikard's teaching session, as presented above, it seems that they both decided to implement the T-shirt task in their own teaching practice since they saw the possibility to address the use of coordinate system as a topic which is relevant both in grade 6 and 8. Thereby the rationale for choosing this particular task is rooted in the role played by the curriculum in the organisation of the teachers' teaching practice. Using terminology from Activity Theory and referring to the teachers' activity system, it is possible to consider the curriculum as one of the rules the teachers follow while preparing for their practice. Results from the analysis of the implementation of the T-shirt task both in Kari's and in Rikard's teaching practice show that both of them chose "coordinate system" as the didactical aim for their lesson and the T-shirt task as chosen as a pedagogical means for achieving this aim. In order to address the use of coordinate system, the teachers modified and adapted the task in the following way: in Kari's class, the task consisted of several logos, all of them drawn on a coordinate system and thereby offering the possibility to identify each point through its coordinates. She modified and adapted the T-shirt task by designing several logos where the first one consisted of straight lines only and where it was drawn in the positive y-values. Finally she presented the same logo as in the T-shirt task by the end of her lesson. The issue of communication in mathematics was addressed through the challenges Kari met during the lesson (Berg, 2010): pupils were sitting in pairs and the first one tried to give a description of the logo to the other but some of the pupils used gestures (pointing to the figure) in order to help their friend in drawing the logo.

Looking at Rikard's class, even though his didactical aim was the use and advantages of introducing a coordinate system, it seems that he also tried to emphasise communication in mathematics as he organised the class as follows: two pupils were responsible for reproducing

the logo (one is describing the other is drawing the logo) while the other pupils were asked to observe which mathematical terms the two pupils used in describing the logo. Thereby my interpretation of classroom observations is that there was more emphasise on communication in mathematics in Rikard's class than in Kari's class. Comparing the three mathematical environments in terms of didactical aim and pedagogical means I argue that it is possible to observe an inversion between the first environment, during which the T-shirt task was introduced during the workshop in December 2008, and the second and third environments, during which the T-shirt was implemented in Kari's and Rikard's class. Looking back to the way the T-shirt task was designed and its implementation both in primary and lower secondary school, it is possible to observe that from a focus on communication in mathematics, in the first environment, the task has been modified and adapted in order to focus on the use of coordinate system. This inversion is illustrated in Table 4 where the main aspects of the three mathematical environments are summarised.

Comparing the implementation of the T-shirt	Mathematical task: Didactical aim	Mathematical task: Pedagogical means
First mathematical environment: Didacticians and teachers Workshop, December 2008	Communication mathematics	The T-shirt task (coordinate system)
Second and third mathematical environment: Kari's and Rikard's teaching period	Coordinate system	The T-shirt task (communication)

Table 4: Inversion of didactical aim and pedagogical means between the first and the two others mathematical environments

I argue that by following this research approach I am in a position to establish a link between theory, here using Activity Theory which has been developed further with the ideas of didactical aim and pedagogical means, and the teachers' teaching practice. This approach allows me to study and analyze the one of the outcomes from the didacticians' activity system (here the workshop during which the T-shirt task was presented to the teachers) and to follow the processes behind its implementation in the teachers' activity system (here the teaching period of the two teachers). By developing awareness of these processes, I consider that we, as researchers, get insights into the way teachers prepare their teaching and more generally "by working together, each might learn something about the world of the other" (Wagner, 1997, p.16). I consider that the recognition of the importance of the rules, using an Activity Theory terminology, within the teachers' activity system is of crucial importance while engaging in collaboration with teachers.

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Berg

Foundations for success in mathematical competitions: A study of best praxis in lower secondary schools in Norway

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Abstract: The purpose of this study is to investigate the following questions: What factors leads to success in mathematics, and how can these success factors and qualities be described? Will the teacher's education and pedagogical praxis have an impact on good learning results? We report results from a case-control association study on among high achievement classes in mathematics in Norway. The data were collected from matched pairs of schools, paired on the basis of location and socioeconomic status. The questionnaire was first distributed to teachers in 38 Norwegian secondary schools at grade 9, which have had repeated success in the annual KappAbel competition in mathematics. Subsequently, 38 teachers at schools without success were contacted, and answered the same questionnaire. The main findings of the study are the following: The formal academic competence of the teacher is the best predictor for good results. Moreover, the pedagogical profile is reason oriented, where students are challenged to evaluate and substantiate their arguments, and spreadsheet is used for exploration and computation.

Keywords: KappAbel; Teacher competence; Pedagogical profiling

1. Introduction

The teacher

Hanushek and Rivkin (2006) has given an overview of international research literature which shows that there is a big difference between teachers with regard to what effect they have on students learning. However, little is known from existing high-quality research about what effective teachers *do* to generate greater gains in student learning (National Mathematics Advisory Panel 2008, p. xxi). What active ingredients characterize a good teacher, is still a question with no clear answer, and the research literature conclude that it is difficult to

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associate the quality differences to the objective characteristics of the teachers. Some new research (Clotfelter, Ladd and Vigdor, 2010), however, have found significant and positive effects on learning results related to teachers' professional skills and their *site* of education. Further research is needed to identify and more carefully define the skills and practices underlying the differences in teachers' effectiveness.

In mathematics there has been an extensive discussion about the significance of teacher subject matter knowledge and students success (Hill & Ball, 2005). But even if there is an agreement that mathematical content knowledge is a precondition to be able to teach mathematics, studies that examined the influences of teachers' subject matter knowledge on student result have produced mixed findings (Hattie 2009). Ahn and Choi (2004) conducted a meta-analysis based on 27 primary studies of mathematics achievements in order to examine the relationship between teachers' subject matter knowledge and student learning. They found a very low effect size between knowing mathematics and student outcomes (effect size $d=0.12$). These results suggest that subject matter knowledge, as currently transmitted to teachers-in-training by colleges of education, is not very useful in the elementary school classroom. It may also be argued that it is probable that subject matter knowledge do have a positive impact on teaching up to some level of basic competence, but less so after that (Hattie, 2009; Monk, 1994).

On the other hand, it is well documented in the research literature an effect of teacher verbal and cognitive ability on student achievement. Every study that has included a valid measure of teacher verbal or cognitive ability has found that it accounts for more variance in student achievement than any other measured characteristic of teachers (Greenwald, Hedges and Lane, 1996; Ferguson and Ladd, 1996; Kain and Singleton, 1996). Greenwald, Hedges and Laine even point out that the rational ability of the teacher may be more powerful than teacher training.

Concerning the teacher's education within problem solving, Hattie underline (2009, p. 210):

The teacher characteristic with the most positive effect on student's performance was specialist training in heuristic methods (effect size $d=0.71$). These methods include, for example, Pólya's (1945) four phases of: (1) understanding the problem, (2) obtain a plan of the solution, (3) carry out the plan, and (4) examine the solution obtained.

Therefore, teachers' educational level and skills are included in this study based on an assumption that this may affect the academic priorities and methodological choices. Some previous studies show that teachers' educational background may be important for students' academic achievement, and we need to know more on how this is linked with teachers' praxis theories and teaching.

The pedagogy

The general pedagogical praxis orientation may be categorized in different ways. Hattie (2009) uses the concepts of teacher as *activator* and the teacher as *facilitator*, where the terms stand for different roles in the management of education. The teacher as "facilitator" is more facilitating the activities, in contrast to the teacher who actively participates directly in it to convey an educational content. Lie et al. (1997) refers to a similar analysis of the teaching

practices in science and mathematics by two different models, "*Teaching 1* and *Teaching 2*". In the Norwegian section of SITES study (Ottestad 2008), funded by the Norwegian Ministry of Education and Research, a more normative term is applied, such as "the traditionally oriented teacher" in contrast to "teacher oriented towards lifelong learning".

	Praxis Description I	Praxis Description II
Lie, 1997	Teaching 1: is working with the project methods, group work, use of ICT	Teaching 2: Traditionally, teacher-controlled teaching methods
Ottestad, 2008	Lifelong learning: Group work. Cooperative learning and problem-based learning. Students have an active role in identifying issues, as well as the way one should solve the tasks. The teacher typically takes the role of launch pad in the learning processes	Traditional orientation: orientation toward knowledge and achievement as measured by traditional means (tests, exams). The teacher typically takes the role of instructor and evaluator. Students follow instructions and work with assigned tasks.
Hattie, 2009	Facilitator: Problem-based learning, project methods, the Internet supported learning, computer games and simulations.	Activator: Teacher-directed teaching methods. The teacher actively participates in teaching, giving direct instructions on effort, learning and behavior.

Table 1

These three ways to describe different teaching practices shows several similarities as summarized in Table 1, and reminds us of the well-known division of the teacher-centered versus student-centered teaching. However, this is an over-simplified subdivision, since teacher-controlled teaching can be dialogic and student active learning can be authoritarian

The various praxis theories have been associated with different learning results. Lie et al. (1997, p.203) describes the "Teaching 1" seems clearly negative impact on mathematics achievement, while instruction 2, traditional teacher-controlled education, is clearly the best outcome:

It is for us a paradox that the ways of working which is highly recommended for the time, project work, group work and use of IT, appears to be linked to the weak performance in mathematics.

Hattie (2009) also emphasizes the teacher's major influence on students' learning results. This applies in cases where the teacher actively participates in teaching, giving direct instructions on effort, learning and behavior. Nordahl (2005) points out that this is consistent with the conclusions of the PISA reports, where it is expressed that the somewhat weaker results in the Norwegian school system can be linked to the teachers too much has been supervisors and in the little stand forth as leaders. It is further emphasized that the student activation has been more important than structural and technical requirements related to learning. This means that the practice description (II) listed in the right column of Table 1, achieve the best learning results. In the report "*Time for heavy lifting*" (Kjærnsli et al., 2007) states:

There is a clear tendency for a strong emphasis on students' exploration of ideas is linked to low achievement. This message is fairly clear, although these results for

many may come unexpectedly. These results imply at least a powerful provocation for those who have argued for the importance of as many as possible "degrees of freedom" in practical work.

In the SITES – study (Law, 2008; Ottestad, 2008), however, researchers believe that the pedagogical orientation of schools and education authorities should work towards greater degrees of freedom. Under the heading "*Education for the 21st century*," teachers should mainly be facilitators, while students have an active role in efforts to identify interesting problems and select methods to resolve these. It is reported that the learning results of these methods provide a relatively small but positive increase in student's inquiry skills (including information-handling, problem-solving and self-directed learning skills) and the ability to cooperate. But is unclear what kind of knowledge in mathematics and science this gives, which are the subjects this survey reviews.

The more specific pedagogy theory related to mathematics is traditionally categorized explicitly as *reason*-oriented versus *rule*-oriented approaches. People are in general used to speak about meaningful learning, and Skemp (1976) introduced the well known discern between *relational* understanding versus *instrumental* understanding. This issue has later been considered by means of different terminologies in the literature: *conceptual* knowledge versus *procedural* knowledge (Hiebert, 1986), *analytical* thought processes versus *pseudo-analytical* thought processes (Vinner, 1997) and *creative* reasoning versus *imitative* reasoning (Lithner, 2008). It will be of great interest to look for differences in learning outcome between these two kinds of pedagogical approaches.

The technology

In mathematics, Norwegian teachers are encouraged to use Internet and a variety of software: dynamic geometry and symbolic calculation software, spreadsheet, etc. Most schools and teachers are still characterized by patchy uncoordinated provision and use, and for the time being the impact by IT on learning outcome in mathematics appears to be unclear and contradictory (Balanskat, Blamire and Kefala, 2006). In OECD countries (OECD 2004) there is detected a positive association between the length of time of IT use and students' performance in PISA mathematics tests. But in a meta-analysis investigating different methods for teaching in the secondary-algebra classroom, Haas (2005) found no effects ($d=0.07$) of technology-aided instruction using computer software applications and/or hand-held calculators. However, in an earlier meta-study Hembree and Dessart (1986) found that the *pedagogical* use of calculators in precollege mathematics education improved student's basic skills both in completing exercises and problem solving.

As a consequence The US-National Mathematics Advisory Panel (2008, p. xxiv) concludes that the available research is insufficient for identifying the factors that influence the effectiveness of instructional software under conventional circumstances. One of the arguments often met is that IT impacts on competency development - like team work and higher order thinking skills - are activities that are not yet recognized by the education systems with ways of assessing them. Since our study is based on results from a competition, and hence not a traditional evaluation system, this kind of approach may provide new input to the ongoing discussions.

The KappAbel contest

The choice of the "best practice" classes was done on the basis of the national Scandinavian KappAbel mathematics contest for ninth grades (see: http://www.kappabel.com/index_eng.html). The overall aims of the competition are (1) to influence the students' beliefs and attitudes towards mathematics and (2) to influence the development of school mathematics. The name "KappAbel" is first and foremost about being capable (Norwegian: kapabel). In addition, the name is meant to honour the Norwegian mathematician Niels Henrik Abel (1802-1829).

The KappAbel contest rewards problem solving of relatively open-ended tasks and other type of skills than just routine exercises in mathematics. It is based on the following ideas:

1. The whole class collaborates in two introduction rounds and hands in joint solutions
2. The class is doing a project work with a given theme.
3. Each class is represented by a team of four students, two boys and two girls, in the national semi-final and final.

First the participating classes within each of the 19 Norwegian counties (fylker) compete in the two introductory rounds of the competition, which are held locally. In Norway one class from each county qualifies. The local winners succeed onto the semi-final, and prepare a project that will make 1/3 of the ruling in the semi-final. The topic of the project is given by KappAbel. In 2007/08 the theme was "Mathematics and animals" When they meet for the semi-final, the students present the results of their project work in a report, a log book and at an exhibition. The three best classes meet the next day for the national final.

Around 20 % of the Norwegian schools participate in KappAbel. In the school year 2007/08 574 classes from 243 schools joined, which means that more than 10 000 students were involved in round 1 and 2. These first parts of the contest are based on teamwork performance by the whole class, not by individual students.

In the present study we address these research questions:

1. What are the common features found between teachers who belong to a school who repeatedly achieve a high learning performance in mathematics?
2. What characterizes their pedagogical praxis?

The study is both a description of some best praxis in the field of mathematics education, and the analysis of the "active ingredients" that make them be of such excellence.

2. Methodology and data

Research design

Case-control studies provide a research method for investigating factors that may cause or prevent success (Schlesselman, 1982). Basically the method involves the comparison of cases with a group of controls. The comparison is aimed at discovering factors that may differ in the two groups and explain occurrence of success. In the KappAbel study, we apply a comparative design from stratified data, with strata defined by the 19 Norwegian counties

(fylker). The data-set was increased by starting with two cases in each of the counties. In each county the sample of two cases were matched with controls from the same socioeconomic background. The data must be considered as a strategic sample, based on overall coverage of the country.

Cases comprised the local record of winning classes in the year 2008 from each of the 19 counties. Cases belonging to schools with no previous top results were excluded, and hence in each county we selected the two best classes from schools with repeated top results. From each county we define a top class in 2008 as a “best practice class in mathematics” if its school earlier has been among the 5 best in the local KappAbel competition (it started in year 2000). This is to exclude classes where the result is dominated by one or two very clever students. If the school has not been on the top 5 list earlier, we test the next one by the same procedure, and so on.

Controls should be comparable and similar to cases and were obtained by the principle of matched sampling. Matching involved the pairing of one control to each case by selecting a near neighbour to each school above that enrol students with the same socioeconomic status (SES). Schools with top results in the previous three years of the local competition were excluded. By this sampling strategy it is possible to eliminate some of the effects from social and geographical variables. In most circumstances, a matched design results in a modest improvement in efficiency in detecting an association (Schlesselman, 1982 p. 116).

Design of the questionnaire

The actual approach of the study aims to measure what kind of activity and output that leads to skills in mathematics based on self-reported perceptions of the teacher. The questionnaire had a descriptively purposes where one wants to describe teachers activity in education. It also has an analytical objective where one looks for relationships between teachers' backgrounds, educational qualifications, pedagogical practices and how IT is a priority.

Thematically, the questionnaire can be grouped into five different sets of questions:

1. Teachers' education, teaching experience, and IT skills
2. Teachers' prioritization of educational activities
3. The uses of software in use at the math education, and how
4. Teachers' opinion about IT in relation to pupils' learning performance
5. The teachers' attitudes to mathematics and its educational goals

The questionnaire contained closed questions and opens a possibility to write comments at the end. A part of the survey maps the affective conditions, perceptions and attitudes to mathematics and mathematics teaching. In this context it is used Likert-scales where teachers are asked to take a position on questions and statements by checking one of six options.

A 98 item self-report questionnaire was designed to explore background variables and perceptions of competence. The study and its purpose were described on a separate page in the questionnaire according to standards prescribed by the Norwegian Data Inspectorate, and the questionnaire was sent to the mathematics teacher in the best practice classes. These teachers also had helpful local knowledge about a matching neighbour school to go on with. The same questionnaire was used with the KappAbel teacher, and the mathematics teacher at the comparing school. For their contribution, the teacher was offered a book as a personal gift.

The sample (N=38+38) included 4 classes from grade 9 from each of the 19 Norwegian cantons. It was easy to establish contact with the 38 case-teachers, since this was an interview with “winners”. The subsequent 38 teachers had to be followed up by SMS and telephone. Three of them refused to answer the questionnaire, and a substitute in the same neighborhood had to be selected. The final response rate was 100% (N=76). Forty-two percent of the teachers were women both in the Kappabel group and the control group. Data related to scores in round 1 and 2 of the Kappabel competitions were also collected. For the Kappabel group these were known for each class, but for the control group we had to use the mean value obtained in the canton where the school belonged.

Statistical analyses

The questionnaire contains a number of individual variables. Some of these are meant to function separately, but many of them are part of a collective variable, and these aggregate variables represent values of a *construct*. By this the number of variables in the questionnaire were reduced to more fundamental constructs based on the logical content of the question and reliability testing, with the number of items from 4 to 6, and Cronbach alpha reliability scores value of 0.60 or above to assess statistical quality of the construct.

The matching of data should be accompanied by a statistical analysis that corresponds to the matched design. The data samples from the winning class and its appropriate social neighbour were compared by a paired test (paired Wilcoxon nonparametric test). By this we may infer the difference in use of i.e. IT in a “best practice class in mathematics” compared to the baseline. The null-hypotheses are that there is no difference.

It is also natural to be able to provide a measure of the size differences between two groups. What constitutes a “big” difference for a particular variable depends on how widely spread it is in the material as a whole. A usual way is to define the differences as standardized differences, also called effect size: How big the difference is compared to a standard deviation (King and Minium, 2008 p. 258).³ We calculated the effect size and report this measure (*d*-value) together with ordinary p-values.

3. Results

The data were collected from matched pairs of schools, paired on the basis of location and SES. Thus, the paired Wilcoxon test was applied to compare the means of the variables between the Kappabel and the control classes. In Table 2 we report all significant results found for single variables. The research questions were further operationalized and analyzed

³ It remains controversial as to whether the correlation between case and control should be taken into account when calculating an effect size estimate for matched pair designs. Some think so (e.g., Rosnow & Rosenthal, 1996 and 2009), but others don't (Dunlop, et al. 1996). We calculated the effect size by not taking the correlation into account, and report this measure or *d*-value, together with ordinary p-values. The estimator for effect size is given by the difference between the means of the case- and the control group, divided by the pooled sample standard deviation, and with equal sample size it is found as the square root of the average of the squared sample standard deviation:

$$\hat{d} = \frac{Mean_{Case} - Mean_{Control}}{SD_{pooled}} = \frac{Mean_{Case} - Mean_{Control}}{\sqrt{(SD_{Case}^2 + SD_{Control}^2) / 2}}$$

through five main constructs as also listed in Table 2, with Cronbach alpha reliability scores of 0.60 or above.

As can be seen in Table 2, more teachers in the Kappable classrooms studied mathematics in universities rather than in colleges ($M = 1.61$ and 1.11 , respectively, $p < .0001$). Kappable classes were more often engaged in hypothesis testing than the control ones ($M = 3.34$ and 2.79 , $p = .056$); more Kappable students are using ICT for research, exploration and calculation than control students ($M = 4.13$ and 3.50 , respectively, $p = .01$); Kappable classes use portfolio for reporting progress of students' projects more than the control classes ($M = 2.71$ and 1.71 , respectively, $p = .002$); students in Kappable classes, more than their peers in the control classes, are encouraged to evaluate their strategies of solving math problems ($M = 4.55$ and 4.11 , respectively, $p = .038$); and more Kappable students are using spreadsheets than students in the control classes ($M = 4.13$ and 3.79 , respectively, $p = .036$).

Importantly, more Kappable teachers adhered to a reason-based understanding ($M = 4.03$ vs. 3.70 , $p = .024$) while more of the control teachers adhered to a rule-based instrumental understanding approach to teaching mathematics.

Comparison of the means of specific questionnaire items pertaining to the use of a variety of IT tools, yielded no significant differences between the groups, except for spreadsheets and portfolios and overall IT use. The general and social use of Internet does not show any difference in the material. The Kappable teachers themselves, however, use digital mathematical Internet resources somewhat more than their peers ($M=3.68$ vs. $M=3.24$, $p = .062$). More of the control teachers adhered to use IT to stimulate students to figure out ways to solve problems without help from the teacher, and to student collaboration via Internet. Last, Kappable and control classes did not differ significantly from each other on such variables as use of calculators, engagement in projects and – most importantly – in their math grades as measured by the traditional mid-term evaluation.

Table 2: Wilcoxon paired test between Kappable ($N=38$) and Control Classes ($N =38$). Column three shows the accompanying d-values (effect size). A positive d-value indicates a higher score in the Kappable group than in the control group, and a negative d-value indicates the opposite.

VARIABLE	Statistical test	Effect size
Organizing teaching:	ns	
The teacher:		
Teaching experience, mathematics	0.051	0.52
Mathematics education, credit points	0.044*	0.34
Formal degree (Teachers College/ Univ. Bachelor/ Univ. Master)	0.003**	0.82
College/ University	<0.0001**	1.21
Content and activities:		
Students plan and test hypotheses	0.056	0.56
Challenge students to evaluate and substantiate their strategies	0.038*	0.53
Use of digital tools for exploration and computation	0.010*	0.71
Reform based ©	0.29	0.24
Traditional ©	0.54	-0.07
Reason oriented ©	0.024*	0.53
Rule oriented ©	0.83	-0.13

IT usage:		
Spreadsheet	0.036*	0.45
Digital portfolio to hand in exercises	0.002**	0.77
Overall use of IT ©	0.025*	0.50
Use of IT stimulate students to figure out ways to solve problems without help from the teacher	0.016*	-0.46
Students can to a greater extent help each other through collaborating over the Internet	0.031*	-0.44

© = Constructs combining single variables on the basis of the logical content of the variables and reliability control.

* $p < .05$; ** $p < .01$

These analyses reveal that teachers' level of education makes a significant difference when the success in carrying out math projects in problem solving is the learning criterion, coupled by the flagship of the practiced open-ended pedagogy: Exploration and hypothesis testing.

4. Discussion and conclusion

The intention of the present study was to look at the diversity between school classes that were found to be doing very well in applied math competition and comparable average achievement classes. Information was collected from teachers of the two groups of classes.

The most important influence on individual differences in teacher effectiveness is teachers' general intellectual ability as documented by the formal academic competence, followed by experience and subject matter and content knowledge. The initial academic competence of the teacher is the best predictor for good results. In our material subject matter knowledge of the teacher is also observed to be significantly related to student achievement, a result that is consistent with Hill, Rowan and Ball (2005) and Falch and Naper (2008). More teachers in the Kappable classes have a university rather than college grade which means that they have been exposed to a full load of math studies and are likely to have a better mastery of that subject, and the teaching is reason-based. Moreover, these teachers view IT as a tool for exploration at the expense of using it as a teaching device. Our results show that a subject specific tool like spreadsheet is more in use in the Kappable, best practice classes, than in the control ones. In other words, it is not IT *per se* but more reasoning-based pedagogical student activity, for which IT is used, that makes the difference.

For the time being, spreadsheet seems to be the only discipline specific digital tool that makes a significant positive effect for lower secondary school mathematics. We may therefore conclude that there are good reasons that spreadsheets should be considered as a useful tool in developing students' fast, accurate, and effortless performance on computation, freeing working memory so that attention can be directed to the more complicated aspects of a problem. Spreadsheets are characterized as a tool that are a free, open and flexible resources that allow for exploratory activities in the mathematics subject and can help promote understanding (Goos et al., 2005; Haspekian, 2005; Fuglestad, 2007; Erfjord & Hundeland,

2007). It seems that the best practice teachers acknowledge this. If other digital tools will have the same qualitative impact within school mathematics is yet to be seen. Dynamic geometry tools are slightly more used among high performance classes, but programs of this type seems not to be so familiar and valuable that reaching the necessary program understanding pays off in mathematical understanding. The impact of IT on mathematics and teaching in general has been heavily dependent on the political objectives related to IT. However, research should not focus on IT alone, but include wider didactic topics such as subject specific innovations and find instruments to capture and detect this kind of sustainable results and processes.

Regarding the overall methodological approach, the research findings in this paper are assumed to be reasonably valid, although here might be some bias in the selection of control classes. A case-control study is in general considered to have some limitations in relation to a regular population based study (Schlesselman, 1982), and our results must be handled with some care. One of the further reservations of the present study is that it is mainly based on teachers' *beliefs* of what is going on in the classroom. The research literature in mathematics education points to an often observed inconsistency between teachers' beliefs as expressed in interviews and questionnaires, and their actual praxis in the classrooms (Thompson, 1992; Raymond, 1997; Beswick, 2005). Since our data collection relates to a particular class at grad 9, this may possibly increase the validity of the questionnaire. But further elaboration and real classroom observations are needed to verify the results.

Teachers have to be encouraged to become active shapers of the reasoning and learning process. This requires a professional environment and culture that allows teachers to do so. Training programs should be more adapted to subject specific needs of teachers that can serve the learning of mathematics.

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University Mathematics Teachers' Views on the Required Reasoning in Calculus Exams

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Abstract: Students often use imitative reasoning, i.e. copy algorithms or recall facts, when solving mathematical tasks. Research shows that this type of imitative reasoning might weaken the students' understanding of the underlying mathematical concepts. In a previous study, the author classified tasks from 16 final exams from introductory calculus courses at Swedish universities. The results showed that it was possible to pass 15 of the exams, and solve most of the tasks, using imitative reasoning. This study examines the teachers' views on the reasoning that students are expected to perform during their own and others mathematics exams. The results indicate that the exams demand mostly imitative reasoning since the teachers think that the exams otherwise would be too difficult and lead to too low passing rates.

Keywords: reasoning; creative vs. imitative; Calculus; University Calculus courses; Swedish exams

Introduction

The purpose of this study is to better understand university teachers' rationale when they create calculus exams, especially concerning the reasoning that the students are expected to perform in order to pass the exams. Earlier research indicates that students often use imitative reasoning when they solve mathematical tasks (Schoenfeld, 1991; Tall, 1996; Palm, 2002; Lithner, 2003). Imitative reasoning is a type of reasoning that is founded on copying task solutions, for example by looking at a textbook example or by remembering an algorithm or an answer. The students seem to choose¹ imitative reasoning even when the tasks require creative reasoning, i.e. during problem solving when imitative reasoning is not a successful method (the concepts of “imitative” and “creative” reasoning are thoroughly defined in Section *The Reasoning Framework*). The use of algorithms

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saves time for the reasoner and minimizes the risk for miscalculations, since the strategy implementation only consists of carrying out trivial calculations. Thus using algorithms is in it self not a sign of lack of understanding, but several researchers have shown how students that work with algorithms seem to focus solely on remembering the steps, and some argue that this focus weakens the students' understanding of the underlying mathematics (e.g. Leinwand, 1994; McNeal, 1995) and that it might eventually limit their resources when it comes to other parts of mathematics, e.g. problem solving (Lithner, 2004). This is an important observation because it might be one of the reasons for students' general difficulties when learning mathematics.

An important question related to this situation is *why* the students so often choose imitative reasoning instead of creative reasoning. Several possible explanations are indicated by research. Hiebert (2003) states that students learn what they are given the *opportunity to learn*. He argues that the students' learning is connected to the activities and processes they are engaged in. It is therefore important to examine the different types of reasoning that the students perform during their studies. In a previous study (henceforth referred to as 'the classification study') more than 200 items from 16 calculus exams produced at 4 different universities were analysed and classified (Bergqvist, 2007). The results showed that only one of the exams required the students to perform creative reasoning in order to pass the exam.

The general question in this study is therefore: *Why are introductory level calculus exams designed the way they are, with respect to required reasoning?* This question is examined through interviews with the teachers that constructed the exams analysed in the classification study (Bergqvist, 2007). The same conceptual framework (Lithner, 2008) used to classify the items is used in this study to analyze the teachers' statements.

Background

Students that are never engaged (by the teacher) in practising creative reasoning are not given the opportunity to learn (Hiebert, 2003) creative reasoning. Students also seem to follow a “minimal effort” principle, i.e. they choose processes that are as short and easy as possible (Vinner, 1997). This principle indicates that the students would rather choose a method that rapidly provides an answer than a method that demands more complicated analytic thinking. It is therefore crucial to examine to what extent the students encounter creative reasoning in e.g. *textbooks*, *teachers' practice*, and *tests*. Firstly, this section presents research concerning these three parts of the students' environment. Secondly, the research framework (Lithner, 2008) is described, and thirdly, the results from the classification study (Bergqvist, 2007) are presented since it is the basis for the present study. The section is concluded with a presentation of the purpose and research questions of the present study, now possible to formulate based on the results from the classification study and using concepts from the framework.

Textbooks

There are several reasons to believe that the textbooks have a major influence on the students learning of mathematics. The Swedish students—both at upper secondary school and at the university—seem to spend a large part of the time they study mathematics on solving textbook exercises (e.g. Johansson & Emanuelsson, 1997). The Swedish TIMSS 2003 report also shows that Swedish teachers seem to use the textbook as main foundation for lessons to a larger extent than teachers from other countries (Swedish National Agency of Education, 2004). The same study notes that Swedish students, especially when compared to students from other countries, work independently (often with the textbook) during a large part of the lessons. Furthermore, Lithner (2004) shows that it is possible to solve about 70 % of the exercises in a common calculus (university) textbook using imitative reasoning. All these results and circumstances imply that the textbooks play a prominent role in the students' learning environment and that it focus on imitative reasoning.

Teachers' practice

The teachers' practice, especially what they do during lectures, is another factor that affects the students' learning. Teachers also often argue that relational instruction is more time-consuming than instrumental instruction (Hiebert & Carpenter, 1992; Skemp, 1978). There are however empirical studies that challenge this assumption, e.g. Pesek and Kirshner (2000). One way for the teachers to give the students the opportunity to learn creative reasoning could be to 'simulate' creativity during lectures. Bergqvist and Lithner (2005) show that creative reasoning is only simulated by the teacher to a small extent.

Vinner (1997) argues that teachers may encourage students to use analytical behaviour by letting them encounter tasks that are not solvable through pseudo-analytical behaviour. This is similar to giving the students the opportunity to learn creative reasoning by trying to solve tasks that demand creative reasoning. Vinner comments, however, that giving such tasks in regular exams will often lead to students raising the 'fairness issue,' which teachers try to avoid as much as possible. This limits the possible situations in which students can be compelled to use analytical behaviour (Vinner, 1997).

Tests

Several studies show that assessment in general influence the way students study (Kane, Crooks, & Cohen, 1999). Palm, Boesen, and Lithner (2005) examine teacher-made tests and Swedish national tests for upper secondary school. The focus of the study was to classify the test tasks according to what kind of reasoning that is required of the students in order to solve the tasks. Palm et al. (2005) showed that the national tests require the students to use creative reasoning to a much higher extent (around 50 %) than the teacher made tests (between 7 % and 24 % depending on study programme and course) in order to get a passing grade. The results from these studies indicate that upper secondary school students are not required to perform creative reasoning to any crucial extent, since the national test does not determine the students' grades to any higher extent than the teacher-made

tests. This result is in alignment with other studies indicating that teacher-made tests mostly seem to assess some kind of low-level thinking. One example is an analysis of 8800 teacher-made test questions, showing that 80 % of the tasks were at what is called the “knowledge-level” (Fleming & Chambers, 1983). Senk, Beckmann, and Thompson (1997) classify a task as *skill* if the “solution requires applying a well-known algorithm” (p. 193) and the task does not require translation between different representations. This definition of skill has many obvious similarities with Lithner’s definition of algorithmic reasoning (see the following section). Senk et al. (1997) report that the emphasis on skill varied significantly across the analysed tests—from 4 % to 80 % with a mean of 36 %. The authors also classified items as requiring reasoning if they required “justification, explanation, or proof.” Their analysis showed that, in average, 5 % of the test items demanded reasoning (varying from 0 % to 15 %). Senk et al. (1997) also report that most of the analysed test items tested low-level thinking, meaning that they either tested the students’ previous knowledge, or tested new knowledge possible to answer in one or two steps.

This presentation of research on textbooks, teachers' practice, and tests, suggests that students use imitative reasoning since that is what they are given the opportunity to learn.

The Reasoning Framework

Lithner (2008) has developed a conceptual research framework that specifically considers and defines different types of mathematical reasoning. In the framework, *reasoning* is defined as “the line of thought adopted to produce assertions and reach conclusions in task solving” (p. 3). Reasoning is therefore *any* way of thinking that concerns task solving—it does not have to be based on formal deductive logic—and can denote even as simple procedures as recalling facts. Lithner (2008, p. 257) points out that with this definition the line of thought is called reasoning “even if it is incorrect as long as there is some kinds of sensible (to the reasoner) reasons backing it.” The framework defines two basic reasoning types; *imitative reasoning* and *creative mathematically founded reason-*

ing. The classification study (Bergqvist, 2007) uses two types of imitative reasoning, *algorithmic reasoning* and *memorized reasoning*, and two types of creative mathematically founded reasoning, *local* and *global creative mathematically founded reasoning* (Figure 1). These four types of reasoning are defined shortly below and are presented more thoroughly in the classification study and in the framework itself.

Figure 1: Overview of reasoning types in the framework used in this study

Imitative reasoning

Imitative reasoning (IR) is a type of reasoning based on copying task solutions. For example, a student can solve a task using imitative reasoning by remembering a fact or by using an example in the textbook as a guide. The two main types of imitative reasoning are *memorized* and *algorithmic* reasoning.

The reasoning in a task solution is denoted ***memorized reasoning*** (MR) if it is founded on recalling a complete answer by memory and it is implemented only by writing down the answer.

The reasoning in a task solution is denoted ***algorithmic reasoning*** (AR) if it is founded on remembering a set of rules that will guarantee that a definite solution can be reached, and consists of car-

rying out trivial (to the reasoner) calculations or actions by following the set of rules.

Creative mathematically founded reasoning

Lithner (2008) uses the concept of creative mathematically founded reasoning (from now on abbreviated 'creative reasoning' or CR) to characterize the creation of new and reasonably well-founded task solutions. To be classified as CR the reasoning has to be new (to the reasoner) and, in order to be mathematically founded, to be supported by arguments anchored in intrinsic mathematical properties of the components involved and that these arguments motivate why the conclusions are true or plausible. If a task is almost completely solvable with IR and requires CR only in a very local modification of e.g. the used algorithm, the task is said to require *local creative reasoning* (LCR). If a task has no solution that is globally based on IR and therefore demands CR all the way, it is said to require *global creative reasoning* (GCR).

The classification study

The purpose of the classification study was to examine the reasoning that Swedish university students in mathematics have to perform in order to solve exam tasks and pass exams. The study aimed at answering two research questions:

- *In what ways can students solve exam tasks using imitative and creative reasoning?*
- *To what extent is it possible for the students to solve exam tasks using imitative reasoning based on surface properties of the tasks?*

The results were obtained through analysis and classification of more than 200 items from 16 calculus exams produced at 4 different Swedish universities. These exams each completely determined the students' grades at the courses that they were given. The grades were *failed*, *passed*, and *passed with distinction*. The students at all courses had a maximum time available of approximately 6 hours. This setting varies between universities, courses, and teachers, but is quite common in Sweden.

A task is only possible to solve using imitative reasoning (IR) if the students have had a chance to learn and familiarise themselves with the assignment given in the task and its solution. Whether a task demands creative reasoning (CR) of the students or not is therefore directly connected to what type of tasks the students have practiced solving before the exam. Thus a task was classified based on what the students have had a chance to practice and what is familiar to the students. The concept of familiarity is connected to imitative reasoning and is presented more in detail in Bergqvist (2007).

The analysis of the classification showed that more than 65 % of the tasks were solvable by imitative reasoning (IR), and all exams except one were possible to pass without using creative reasoning (CR). The qualitative analysis shows that all task solutions demanding local creative reasoning (LCR) consist of a familiar global algorithm that has to be adjusted in some small way. The fact that the global algorithms in these tasks are familiar gives the students a reasonable chance of solving at least parts of the given tasks, even without using CR. It is in fact possible for the students to completely (IR tasks), or to a large extent (LCR tasks), solve 91 % of the tasks without using CR since only 9 % of the tasks demand global creative reasoning (GCR). Using IR and LCR, i.e. basically memorized solutions or solutions based on a familiar global algorithm, it is possible to get the grade passed with distinction on all exams except one.

Purpose and questions

The purpose of this study is to better understand university teachers' rationale when they create calculus exams, especially concerning the reasoning that the students are expected to perform in order to pass the exams. The aim is to answer the general question (GQ): *Why are introductory level calculus exams designed the way they are, with respect to required reasoning?* The study therefore examines why the exams analysed in the classification study (Bergqvist, 2007) were con-

structured with the particular proportion of types of tasks that they have but also how the teachers who constructed the exams view exam construction in general and the results of the classification study. In order to examine the possible answers to the general question, this study aims at answering the following research questions.

Research question 1: *What factors affect university mathematics teachers when they construct exams?* (RQ1)

Research question 2: *In what ways, and to what extent, is the aspect of required reasoning considered by university mathematics teachers when they construct exams?* (RQ2)

Research question 3: *What are university mathematics teachers' views on the existing reasoning requirements in exams?* (RQ3)

Method

Choice of method

The research questions concern the teachers' rationale for constructing exam tasks so the answers are sought through interviews with teachers. The interview situation is especially appropriate since it concerns difficult concepts and it provides the interviewees with possibilities to ask questions and the interviewer with possibilities to clarify concepts in case of any misunderstandings. The interview outline is semi-structured, with pre-defined questions, but allowing deviations with an aim to explore statements by the teachers that are especially relevant, e.g. as described in Kvale (1996).

Collection of data

The interviewees

The eight teachers whose exams were classified during the previous study (Bergqvist 2007) were selected to participate in this interview study, and six of them accepted. The two remaining teachers declined due to a heavy work-load. The six teachers are in some sense selected randomly, since the exams originally were chosen by randomly selecting three Swedish universities, and adding the author's own, and then studying all introductory calculus exams produced at these universities during one academic year (2003/2004). The group is too small to completely represent all Swedish university teachers in mathematics, but the teachers work at four universities of different size, location, and age. Five of the six teachers in the study are senior lecturers and have Ph. D. degrees in mathematics. The sixth teacher is an instructor which implies an educational level equivalent to a master in mathematics. The teachers' ages vary between 30 and 60 years. Five of the teachers were male, one was female.

The Interview

To facilitate communication of previous results and of concepts, the interviews use a type of stimulated recall (Calderhead, 1981). From the beginning of each interview the interviewer and the interviewee has an exam constructed by the interviewee at hand. The interviewee is invited to exemplify any statements with tasks from the exam if appropriate. Later on during the interview the classification of each task in the exam is presented to the interviewee, and any discussion of concepts and classification can be given concrete form through examples from the teacher's own exam. The method of using existing data from the teachers' practices (in this case their exams) is inspired by Speer (2005), who argues that some previously reported inconsistencies between teachers' professed and attributed beliefs might be a consequence of a lack of shared understanding.

Each interview starts with a brief introduction concerning the study and is then divided into four parts, each described below. The interview outline was piloted on two teachers, not included in the final version of the study, and the outline was adjusted according to the resulting experiences. Each interview question is based on at least one research question (see each part of the interview). Each interview took approximately one hour.

Part One. The teacher is asked about his/her views on exam construction, how he/she constructs exams and exam tasks, and about choices, difficulties, goals and practical circumstances. Two examples of practical circumstances were given to all teachers: “e.g. the *time* or the *students*”. All interview questions are connected to the first research question (RQ1: *What factors affect university mathematics teachers when they construct exams?*) and some, that specifically consider the exam tasks, are also connected to the second research question (RQ2: *In what ways, and to what extent, is the aspect of required reasoning considered by university mathematics teachers when they construct exams?*). The questions are posed without any specific vocabulary or concepts being introduced, so that the teacher him-/herself can make the judgement on what is important in this context. The teacher is in the beginning of this part explicitly invited to use his/her own exam to exemplify any statements.

Part Two. Three concepts regarding the requirement of reasoning are introduced: *memorized reasoning*, *algorithmic reasoning* and *creative reasoning*. This is done by letting the teacher read a few pages of text that described these different types of mathematical reasoning, including relevant examples from the teacher’s own exam. The difference between local and global creative reasoning is mentioned but not elaborated on. The purpose of introducing some specific concepts is to create a basis for common understanding of a vocabulary that enables communication between the interviewer and the interviewees. The teacher is then asked if he/she takes the required reasoning of the

tasks into consideration during exam construction, a question that is directly related to the second research question (RQ2).

Part Three. The teacher is first informed of the classification results for one of his/her own exams (Bergqvist, 2007) and is then asked about his/her opinion of these results, e.g. why the exam had the particular distribution and if he/she is pleased with the distribution. The interview questions are all directly connected to the second and/or third research questions (RQ2 and RQ3: *What are university mathematics teachers' views on the existing reasoning requirements in exams?*).

Part Four. The teacher is informed of the general results from the classification study (see Section *The classification study*). The interview questions all concern the teachers' views on the existing reasoning requirements (RQ3); the teachers are e.g. asked about his/her opinions of these results, what he/she thinks are possible reasons for them, and whether the teacher thought that the aspect of reasoning is important or not.

Method of Analysis

Main analysis

The main analysis includes the transcriptions of all interviews and are performed in several steps according to the following description, similar to suggestions by e.g. Glesne (1999). The transcribed interviews are divided into sections based on 1) the subject treated in each section and 2) which research question that is most closely related to that subject. Each section is headlined using a description of the subject (1). For example, if the teacher talks about *when* she constructs the exams this will be seen as connected to RQ1 (*What factors affect university mathematics teachers when they construct exams?*) and therefore headlined with "When the exam is constructed (RQ1)". For each particular RQ there will be several different headlines. All sections (from all the teachers)

are then grouped according to their headlines and the teachers' opinions in the grouped sections summarised. The summaries are then used to answer each of the three RQs and the general question. If some section treats a subject not related to the RQs and therefore cannot be given a headline that is connected to any of the RQs, that section will be treated separately after the rest of the analysis was done. There will probably be very few such sections since the interview is semi-structured and each interview question is connected to at least one RQ.

Additional method of analysis for RQ1

The first research question (RQ1) is “*What factors affect university mathematics teachers when they construct exams?*” The answers to this question may concern anything from practical conditions to opinions and intentions and there are probably many factors. To enable a better overview of the factors that affect the teachers the factors are grouped. Each group will consist of one or several connected factors where each connection is motivated by direct argumentation and/or by several explicit teacher statements. Direct argumentation means here for example that the factor *student group* is connected to the exam's *degree of difficulty*, with the argument that the students' previous knowledge will affect how difficult the exam is to them. Motivation via the teachers' statements could instead be that these two factors are connected since several teachers said that when they teach a low-performing student group they, more or less subconsciously, make the exam less difficult. The factor that is most strongly connected to the others in the group and/or most relevant to the purpose of this study is chosen to represent the group and is called the *main factor*.

Analysis

Answers to RQ1

The first research question is “What factors affect university mathematics teachers when they construct exams?” (RQ1). A lot of different factors were mentioned by the teachers: time, students'

proficiency and previous knowledge, course content, tradition, degree of difficulty, task type, etc.

The analysis resulted in four groups of factors, and accordingly four main factors.

There are a few weak connections ignored in this structure. An example of a weak connection is that two teachers think that the degree of difficulty of the exams is linked to the factor tradition via the examiners at their department. This connection is ignored since the examiners are part of a system present at only one department and the connection is not mentioned by any of the other teachers.

Main factor 1: Time

The first group consists of only one factor and that is *time*. This factor is mentioned to all teachers when the question is asked (as an example) and all teachers confirms explicitly that time plays a prominent role in the exam construction process. The teachers do not specifically connect time, or the lack of time, to any other of the factors mentioned in the study. Two of the teachers do not feel that they lack time considering the way the exams look today. Both of them, however, have the opinion that the type of examination most common today, the written exam, is a product of a general lack of time. Several of the alternative examination types, e.g. oral exams, demand more time, which the teachers simply do not have. The four other teachers view the situation in different ways. The first states that time is a factor in exam construction but he does not specify how. The second states that consequence of the lack of time is that he does not spend as much time constructing exams as he thinks that he should. The third teacher believes that the access of time admittedly affects the process, but that constructing exams takes whatever time the teacher has at hand, and that sometimes the exam will not get any better no matter how much time he or she spend working on it. The fourth teacher does at first not seem to think that time is a factor, but later in the interview, during the discussion on alternative forms of examination, he states that he would choose to administer an oral exam instead of a written if he did not lack the time.

Main factor 2: Degree of difficulty

The second group consists of four factors, three tightly connected and one more loosely related. The first three factors are the *student group*, the *degree of difficulty*, and a *high passing rate*. The last factor is *whether the exam is a make-up exam or a regular exam*. The connections between factors in this group are the following. A group of high performing students will have a higher passing rate than a group of low performing students. If the degree of difficulty of the exam is relatively low, the passing rate will be relatively high. If the students lack certain previous knowledge, a task might be more difficult for them than for other students. The first three factors are therefore directly connected. The last factor is connected to the student group, since most high performing students do not take a make-up exam and the student group taking a make-up exam is therefore in average probably lower performing than a group taking a regular exam. The factor chosen to represent the whole group of factors is ***degree of difficulty*** since it is a property not only of exams but also of tasks and therefore is most closely connected to the focus of the study. Also, all teachers mention the degree of difficulty as an affecting factor. The first three factors will be more extensively discussed below since the teachers did so during the interview.

The teachers all agree that the composition of the *student group* is a factor that affects the exams. A couple of teachers state explicitly that there is a direct link between the students' proficiency and the exam's degree of difficulty.

“Of course, if you have very poor students at a course then... (...) I don't do it deliberately, it's not as if I try to construct an easier exam, but I still believe that... you're affected by it.” (Teacher D)

The students' previous knowledge is a property of the student group and is mentioned by five of the teachers. Two of them mention that when students lack certain proficiencies, it limits the degree of difficulty possible to set for the exam. One adds that it would not help to change the form of the examination, since a lot of other circumstances, e.g. the extent of the contents that is included and the need for a high passing rate, also limit what is possible to test. The three other teachers discuss the issue of students' previous knowledge in connection to the distribution of tasks of different reasoning types in their own and other teachers' exams. They all express the view that the students' previous knowledge, or rather their lack of such, is directly connected to the amount of creative reasoning (CR) that is reasonable to test at an exam (or even to work with during teaching). One of these teachers suspect that it would be possible to test a higher amount of CR at a course in Linear Algebra because it is not based on the students' presumed previous knowledge to the same extent as the Calculus courses. The teachers' views on this connection are further described in the section related to the second research question (RQ2).

All teachers mention in different contexts that the *degree of difficulty* of the tasks is something that they consider to some extent during exam construction. Four of the teachers say that it is often, if not always, difficult to produce (or choose) tasks with a reasonable degree of difficulty. Some specific examples mentioned by the teachers are: the first time a teacher is teaching a course; if the teacher tries to make the tasks interesting and/or fun; when a course is graded with more than two grade levels²; and when the course is more advanced. One of these four teachers adds that even though the curriculum is intended to point out what the students are supposed to master at the end of the course, it is very difficult to use the curriculum in order to set a reasonable difficulty level.

Two teachers point to the need for a *high passing rate* as a factor influencing exam construction.

Two teachers remark that they usually do not spend as much time constructing a make-up exam as they do with a regular exam. One of them saves the “really interesting” tasks for the ordinary exams since most of the capable students do not take the make-up exam (they already passed the regular exam). The other teacher simply states—without explaining how—that this factor affects him during exam construction.

Main factor 3: Task type

Group three consists of four factors with the factor *task type* as main factor. Directly connected to task type are the factors *textbook*, *previous exams*, and *tradition*. The task type, i.e. whether a task is a standard task or not, is according to several of the teachers determined by what type of tasks the students have encountered before. Previous encounters are connected to the textbook and to some extent also to previous exams. (The subject matter presented in the textbook is not focused here, only the *type* of tasks in the textbook.) One teacher points to the tradition within the engineering students courses, where mostly tasks solvable by well-known algorithms (task type) are expected. Another teacher mentions previous exams as a part of the tradition in mathematics. The factor tradition is therefore connected to both task type and previous exams. The teachers' wish to vary the contents of the exams within its traditional form is directly connected to previous exams and variation is therefore also a factor included in this group. *Task type* was chosen to be the main factor in this group since it is, according to the analysis of RQ2, directly connected to required reasoning in tasks.

All the teachers consider *task type*, whether a task is a standard task or if it somehow requires more from the students, when constructing exams. There is a separate terminology for almost every teacher in the study, but when the teachers are asked to specify what they mean it turns out that it is possible to connect their definitions to the definitions in the framework via the concept of familiari-

ty (see Section framework). The connection is more thoroughly analysed in relation to RQ3 and is presented in the sub-section *Common understanding of concepts*.

The *textbook* is mentioned by three of the teachers in the study and one of them suggests a connection between the type of tasks in the textbook and in the exam. He is of the opinion that the textbook gives a frame for what the students have been working with during the course and therefore indirectly affects the exam. Three teachers feel that it is difficult to produce an interesting, and preferably enjoyable, exam that is not a copy of *previous exams*, which at the same time does not differ too much from the older exams. Two other teachers state that previous exams from the course affect the design and contents of the exam and, according to one of them, also the difficulty level.

Four teachers mention *tradition* as a factor. One of the teachers explicitly states that the tradition within teaching mathematics is very strong, and both he and another teacher mention that the typical written exam, containing 6--8 tasks and being presented to the students at the end of a course, is a product of this tradition. The third teacher mentions another type of tradition, the engineering students' strong traditions that are inherited from one age group to the next. He states that it strongly affects the teaching and the examination of these students. Concerning this culture among the engineering students (studying to be Masters of engineering), this teacher says that:

“[These students] are in some way prepared to learn a lot of algorithms. It is a part of the tradition. (...) [They] are supposed to squeeze in massive amount of credits into a small number of years, there isn't much time for reflection (...) [they] adapt, that's understandable. At the

same time as you try to get at it, you feel that the whole system is working against you then.” (Teacher C)

The fourth teacher declares that the existence of an examiner at his department naturally influences the form and degree of difficulty of the exams. This can be regarded as a type of formalised tradition affecting the exams.

Main factor 4: Task content domain

The last group consists of three factors: the *task content domain*, the *course content*, and the *textbook*. The connection between these three factors is the following: the content domains of the tasks in an exam are directly connected to the course contents, which usually are presented in the textbook (in this case it is the subject matter present in the textbook that is focused, not the type of tasks included). The main factor is chosen to be ***task content domain*** since it too represents a property of the exam tasks.

Three teachers say that the *content domains* (e.g. differentiation, related rates, or limits) of the exam tasks should be a representative selection of the course contents. Three of the teachers point out that the *course itself* naturally affects the exam's structure and contents, e.g. via the syllabus. The *textbook* is mentioned by three of the teachers in the study and two of them mention aspects connected to the content of the course. One states that he uses the textbook as a starting point for exam construction. The other says that even though the textbook influences the content of the exams, the tasks in the textbook are often too easy and uncomplicated in comparison to the exam tasks. All teachers mention the *course content* in at least one of the themes: the exam construction process, the practical considerations, or the task properties, although then as the factor task content domain.

According to the analysis there are more than ten different factors, each affecting at least two of the teachers in the study, and connected either to the exam construction process, to the task properties, or to practical considerations. The analysis further shows that there are four main factors that each represents a group of factors: the *time*, the *task type*, the *task content domain*, and the task's or exam's *degree of difficulty*.

Answers to RQ2

Here the answer to research question 2, *In what ways, and to what extent, is the aspect of required reasoning considered by university mathematics teachers when they construct exams?*, is presented.

All teachers state during the first part of the interview (before the concepts of imitative and creative reasoning has been introduced to them) that they consider task type, i.e. whether a task is a standard task or not, during exam construction (see the concept of *task type* introduced in the Section Answers to RQ1). Their descriptions of standard tasks differ, but can in every case be connected to the aspect of required reasoning and the connections are in all cases but one explicitly confirmed by the teachers themselves (see section Answers to RQ3, last sub-section). These connections imply that the teachers do consider the aspect of required reasoning when they construct exams.

Five teachers agree that the distribution of tasks of different reasoning types varies, and should vary, with the level of the course. They all think that the proportion of tasks demanding creative reasoning (CR) should be larger in courses on higher level mathematics, but have different ways of motivating their opinion. One teacher says for example that on the lowest university level, the students are supposed to learn calculation skills, and that if they want to “learn mathematics” they should keep on studying. Some teachers also guess that the distribution would be different on courses on the same level but with different mathematical contents. Three of the teachers think that

there are a smaller percentage of algorithmic reasoning (AR) tasks in a linear algebra exam than in these calculus exams, and one teacher points to Geometry as a subject where the part CR would be particularly high. All four teachers believe that it is the lower demand on the students' previous knowledge in these courses that makes it possible for the teacher to work with more CR during teaching and assessing.

All the teachers also say, more or less explicitly, that tasks where the students themselves are supposed to construct a part of the solution, a common example is a short proof, usually are more difficult for the students than the tasks they solve using well-known algorithms. Translated into the terminology of the reasoning framework this means that the teachers view tasks that demand creative reasoning (CR) as more difficult than tasks solvable with algorithmic reasoning (AR).

Answers to RQ3

Research question 3 reads: *What are university mathematics teachers' views on the existing reasoning requirements in exams?* In order to draw reasonable conclusions of the teachers' statements, it is important to determine if their views, as they are expressed in the interviews, actually concern the same properties of the tasks that the theoretical framework defines. The first sub-section below therefore discusses if there is a common understanding between interviewer and interviewees of the concepts involved. That discussion is concluded in the fourth sub-section after more relevant information is presented in sub-section two and three. It would also be difficult to have a fruitful discussion with the teachers concerning the existing reasoning requirements in their exams if they saw the classification results as unreasonable. The second sub-section therefore takes a closer look at the teachers' views of the classification of the tasks in their own exam. The third sub-section focuses on the teachers' views on the general results from the classification study. As mentioned, the fourth sub-section concludes the discussion on the common understanding.

Common understanding of concepts

Five of the teachers explicitly say that they think that the aspect of required reasoning is important and relevant. All six teachers use an informal and personal terminology (presented before the formal concepts are introduced by the interviewer) to describe task properties, and for every teacher this terminology is possible to link to the definitions of algorithmic and creative reasoning (AR and CR). The connection between the teachers' personal vocabulary and the formal terminology is in most cases based on the notion of familiarity of answers and algorithms connected to imitative reasoning. Furthermore, all teachers but one (Teacher B) did explicitly say that the presented definitions (in part 2 of the interview) mainly coincided with what they intended to describe with their own terminology. The specific case of teacher B is presented in the end of this sub-section.

AR tasks

The teachers' personal terminology describing AR tasks are "standard tasks", "routine tasks", "typical tasks", "template tasks" and "tasks without a twist." During the interviews the teachers explained what they meant by these terms, and their explanations were very similar to the definitions of AR tasks. The similarities were particularly obvious when the teachers' descriptions were compared to the classification tool and the concept of familiar tasks (Bergqvist, 2007; Lithner, 2008). One teacher for example said that: "Most of the tasks are typical routine tasks, standard tasks, that you should be able to solve if you've solved most of the tasks in the textbook." (Teacher E) When this teacher later in the interview was introduced to the definitions of the reasoning types, he pointed out that AR tasks were the same tasks as the ones he called routine tasks. Another example is one teacher that describes the concept of "template tasks" as "tasks that are more or less a copy of tasks that the students have solved before" (Teacher C). The teachers generally defined different types of tasks very concretely and in direct connection to what the students have done during the course.

CR tasks

The teachers' descriptions of non-routine tasks were possible to connect to CR tasks in a similar way. Teacher E describes as follows:

“You can always complicate one [task], of course, it's very easy, you just have to change a formulation, and the task is all of a sudden significantly more difficult. Add an arbitrariness, or calculate a task backwards, or... almost whatever.”

Typical examples of the teachers' personal terminology for CR tasks are “tasks with a twist,” “tasks that contain something tricky or unexpected,” and “types of tasks that the students have not seen before.” This is in direct alignment with the definition of CR tasks.

MR tasks

Memorized reasoning (MR) is not defined or commented upon by the teachers to the same extent as algorithmic and creative reasoning. All of the exams discussed with the teachers during the interviews contain MR tasks according to the classification, so the type is used by the teachers. Only a couple of the teachers, however, mention or describe this type of tasks.

The first teacher points out that even if a student has memorized something, e.g. a proof of a theorem, the student's presentation at the exam might reveal information on what the student really understands. This opinion is not shared by all of the other teachers. During other parts of the interview (not in connection to the definitions) one teacher said that the memorized reasoning (MR) tasks are practically uninteresting, since they only measure the students' capacity of memorising information. Another teacher thinks that memorising definitions and theorems definitely has its place in mathematics, the same way that memorising vocabulary is a part of learning a foreign language. In

spite of this opinion he does not think that MR tasks measure anything more than what the students have memorized.

Teacher B

The only teacher that did not say that his own view of task properties coincided with the presented definitions explained that he does not specifically think in those terms. Compared to the other teachers, he is also the one most hesitant to use the definitions after they were presented. It is, however, apparent from his description of tasks and how he constructs tasks that he in many cases does not want the students to use AR. He says about task construction: “That is also something that I try to think about, that you [the student] won't be able to just take a formula from the textbook and just insert...” (Teacher B) He says that he wants the students to learn the line of reasoning behind the solutions, and that he often constructs tasks that are similar to the textbook tasks but includes small changes. According to him these changes do not “really” make the tasks more difficult, but he also acknowledges that they are harder to solve for the students. His strive for the students to perform local creative reasoning corresponds with the result from the classification study where he is the only teacher who constructed an exam that is not possible to pass using only imitative reasoning.

Teacher B also notes that a certain type of tasks might first belong to one category and later on to another. He takes the introduction of a task in keeping with “What does it mean that a function f is differentiable in a point?” as an example. He describes how he introduced such a task in his exams one year, and then let it be included in each exam for a few years to follow. The task was difficult for the students the first year and their results were poor, but after a few years the students seemed to know how to answer the task. He suggested that the student at first needed some type of creative reasoning (CR) to solve the task, but that in later exams it was solved by memorized or algorithmic reasoning (MR or AR). This is in alignment with the classification of AR and MR tasks: when the

students have encountered a task several times, it becomes familiar and therefore CR is no longer necessary in order to solve it. Teacher B is discussed again in the following sub-section and the question of common understanding of concepts is therefore concluded in a summary in the end of the section Answers to RQ3.

The teachers' views on the classification

All teachers in the study accept the classification of the tasks in their own exam, two of them straight away, and four after some discussion. The discussions concerned for example why a particular related rates task was classified as local creative reasoning and not algorithmic reasoning. In this case the reason was that the specific real world situation was new to the students and they had to perform creative reasoning when they created a mathematical model. In all cases but one the teacher agreed with the classification after the discussion, and in that single case the task was in fact misclassified. This implies that the teachers have embraced the definitions and it supports the reliability of their opinions of the definitions and the classifications of their exam tasks.

All teachers expressed that they were satisfied with the distribution of tasks of different reasoning types in their own exam. An opinion presented by several teachers in different contexts, and by two teachers in this specific context, is that it is not reasonable, or maybe not even fair to the students, to believe that the students will be able to solve tasks requiring creative reasoning during the exam of this course. One teacher says for example that it is not possible for the students to be creative if they do not master the basics first. Another teacher states, based on the opinion that the students' previous knowledge has deteriorated, that:

”We can't demand from the students that they, in addition to learning to handle the basic concepts, also manage to put them in context, at least not as soon as at the exam.” (Teacher E)

This is in line with the fact that most of the teachers believe that there could, and maybe should, be a larger percentage of creative reasoning demanded by students taking higher level courses and also on courses that demand less previous knowledge by the students.

Four of the teachers said that tasks solvable with memorized and algorithmic reasoning (MR and AR) are important and should not be regarded as less valuable. Two of those teachers were of the opinion that approximately half of the points on the exams should be MR or AR. A third teacher said that even though it is not explicitly discussed among the teachers, it is clear to him that they try to measure calculation skills in the introductory calculus exams. The teacher later confirms that his concept *calculation skills* is directly connected to AR. He also believed that the percentage of AR in this course is increasing over the years, and that this increase is a sign of the teachers adapting to the deterioration of the students' previous knowledge.

The teachers' views on the results from the previous study

The five teachers whose exams were possible to pass using only IR were not surprised by the results from the classification study and also viewed the results as reasonable. One of them said:

”If I were to guess, then 16 of 16 exams would be like that, where you can get more than 50 % of the points with those two categories [MR and AR] So it is definitely not a surprise that it is such a large percentage.” (Teacher A)

When the teachers were asked what they think is the reason for the resulting distribution of tasks of different types, their answers varied to some extent. One teacher suggested that what the students primarily need is to learn mathematical tools that will help them with the upcoming courses. Another stressed that it is important that the students are able to implement algorithms. Some of the teachers meant that over the last ten years the percentage of algorithmic reasoning in the exams in

this type of course has increased, and that this is due to the fact that for each year the students have poorer and poorer previous knowledge. Four of the teachers agreed that the main reason for the distribution is that a larger proportion of CR tasks would make the exams too difficult, which in turn would result in too few students getting passing grades.

Teacher B again

The teacher whose exam was not possible to pass without creative reasoning (CR), Teacher B, was the one most surprised by the results of the classification of the teachers' own exams. He was a bit puzzled since his opinion was that it should be possible to pass the exams using only IR, but he still did not think that his own exam was too difficult. Based on a quote presented earlier (in Section *Teachers' views on the aspect of reasoning*) he however said: "That is also something that I try to think about, that you [the student] won't be able to just take a formula from the textbook and just insert..." (Teacher B) This could indicate that his conception of imitative reasoning differs somewhat from the definition used by the researcher; that imitative reasoning tasks are very familiar to the students and that the students have solved more or less identical tasks several times. It is possible that what he consider to be imitative reasoning (e.g. in his own exam) would be defined as local creative reasoning (LCR) in a formal classification (which is what his exam requires according to the classification study). The difference between these two concepts is indeed small and probably the most difficult to grasp in such short time as during the interview.

Summary of common understanding of concepts

There is a common understanding between interviewer and all interviewees of the basic concepts *imitative* (IR) and *creative reasoning* (CR), even if the finer details concerning the difference between algorithmic (AR) and local creative reasoning (LCR) might be lost in such a short interview. The presentation of the concepts in the second part of the interview was indeed focused on memorized, algorithmic, and creative reasoning, without going into detail considering local and global

creative reasoning, something that might be the cause of the difference in views. For five of the teachers the difference between AR and LCR does not affect the classification of their exam: the students can pass it using only IR (memorized and algorithmic reasoning). For one teacher (Teacher B) the results might be more confusing since his exam requires the students to solve IR tasks *and* some LCR tasks in order to pass. If the difference between these concepts is small, it is not surprising that he is more hesitant than the other teachers to accept the suggested terminology. Teacher B's statements concerning the difference between CR and IR in general however suggests that these two basic concepts are clear to him. On the other hand, Teacher B also noted that a CR task could change into an IR task when it became familiar to the student, which implies that he has grasped the basic idea of the difference between imitative and creative reasoning.

Discussion

The main purpose of this study is to find answers to the general question: *Why are the exams designed the way they are, with respect to required reasoning?* In order to answer this question, three research questions were formulated and the first four sub-sections below focus on answering each of these three research questions and the general question. How the teachers connect task difficulty to required reasoning, and the consequences thereof, is discussed in a separate sub-section after that. The last sub-section focuses on the teachers' views of the students' competences.

What factors affect university mathematics teachers when they construct exams? (RQ1)

The analysis shows that there are four main factors that affect the teachers, and they each represents a group of factors: the *time*, the *task type*, the *task content domain*, and the task's or exam's *degree of difficulty*. Time and student group was suggested as possible factors to all the teachers and all of them agreed that time was an important factor. They did not specifically connect this particular factor to any of the other ones. The three other main factors mentioned above represent the proper-

ties of tasks that the teachers say they consider and all these were connected to several other factors according to the grouping described in the analysis.

In what ways, and to what extent, is the aspect of required reasoning considered by the teachers when they construct exams? (RQ2)

According to the teachers' statements, they consider the aspect of required reasoning during exam construction. They verbalise the aspect through descriptions of properties and types of tasks that they choose to include, or not include, in their exams. An important result is that these teachers see an obvious connection between the type of reasoning required of a task and its degree of difficulty. They strongly feel that creative reasoning is more difficult than imitative reasoning for the students, even though some teachers comment that AR tasks can be difficult as well. This result points to a strong link between two of the main factors in the analysis of RQ1: *task type* and *degree of difficulty*.

What are the teachers' views on required reasoning in exams? (RQ3)

The analysis shows that there is a common understanding between the interviewer and the interviewees concerning the basic concepts imitative and creative reasoning. The teachers generally accept the definitions and also the classifications presented to them, and they view the aspect of required reasoning as important. They also feel that the main result from the previous study, that it is possible to pass most of the exams using only imitative reasoning, is reasonable. Some of the teachers meant that over the last ten years the percentage of algorithmic reasoning in the exams in this type of course has increased, and that this is due to the fact that for each year the students have poorer and poorer previous knowledge. A larger percentage of tasks demanding creative reasoning would therefore result in too difficult exams and too low passing rates. Most of the teachers seemed to feel, however, that the aspect of required reasoning is both relevant and important, and that it could and should be demanded to a higher extent in e.g. courses in more advanced mathematics.

Why are the exams designed the way they are? (GQ)

The analyses of the three research questions show that the teachers are in general quite content with the situation as it is. During exam construction, they primarily focus on the tasks' content domain and degree of difficulty. Since the teachers generally regard a task demanding creative reasoning as more difficult than a task solvable with imitative reasoning, they use the tasks' types to regulate the difficulty. The lack of time, both during exam construction and during teaching, in connection to the students' poor previous knowledge makes it unreasonable to the teachers to demand that the students master creative reasoning in order to pass the exams. The teachers' referral to the students' poorer previous knowledge over the years seems to be based on their personal experiences, but is also supported by several local and national studies (Höskoleverket, 1999; Boo & Bylund, 2003; Petterson, 2003; Brandell, 2004). Four teachers suggested (Answers to RQ2) that a lower demand on the students' previous knowledge in some courses (e.g. linear algebra) makes it possible for the teacher to work with more CR during teaching and assessing during these courses. According to the teachers it can therefore be more problematic to require the students to perform CR in the courses focused on in this study (introductory calculus courses) than in introductory courses in general.

The described situation supplies a possible answer to the general question of why the exams are designed the way they are, with respect to required reasoning: *The exams demand mostly imitative reasoning since the teachers believe that they otherwise would, under the current circumstances, be too difficult and too difficult exams would lead to too low passing rates.*

Task difficulty and required reasoning

The teachers use the task type (the required reasoning) as a tool for regulating the tasks' degree of difficulty, and they believe that tasks demanding creative reasoning (CR) are more difficult than tasks solvable with imitative reasoning (IR). A relevant and important question is therefore if CR tasks really *are* more difficult than IR tasks. To answer this question it is necessary to more specifi-

cally define the term *difficulty*. A task's difficulty is not a constant property since a task that is difficult to solve for a student may be easy to solve for a teacher. Task difficulty therefore has to be determined in relation to the solver. There are many task properties that may affect a task's degree of difficulty, e.g. the task's complexity and its wording. It is therefore not easy to describe task traits that completely determine a task's degree of difficulty. In the following discussion, an exam task's degree of difficulty is therefore determined by the success rate for the students participating in the course. Although the success rate is a blunt and partly unstable instrument, it provides a useful definition in this context.

In a study on the relation between required reasoning in test tasks and the reasoning actually used by students, Boesen, Lithner, and Palm (2010) noted that the success rate in the students' solution attempts was higher for tasks classified as having high relatedness to textbook tasks (cf. IR tasks). Combined with the teachers' statements this makes it reasonable to believe that CR tasks generally are more difficult for the university students than IR tasks. One of the teachers in the present study had the following comment on whether CR is more difficult or not:

"It's probably not more difficult for everybody. It depends on how good you are perhaps. For the weaker students it is definitely more difficult, for the good students it might be easier. Because, I mean... they don't have to learn that much, they can do it anyway. Because if you solve that one [points at a definite integral task] well, then you have to learn the method of partial fractions and so on, you know, and that isn't something you figure out if you haven't seen it [before]." (Teacher B)

A CR task can be uncomplicated and simply put, and the solution might be based on resources that are well established with the solver. So the obvious question follows: do CR tasks *have to* be more difficult than IR tasks? Are the CR tasks more difficult due to some inherent property of creative

reasoning, or due to circumstances connected to the situations in which the students meet the different types of reasoning?

The CR tasks that the students encounter in tests and textbooks are often difficult for them to solve. The two task properties—demanding CR and being difficult—often appear simultaneously. If CR tasks do not *have* to be difficult, then one reason for this coincident could be that it is easier to construct difficult CR tasks than difficult IR tasks. A related circumstance is that most of the uncomplicated and easy tasks in the beginning of exams and textbook chapters are solvable with IR. This could result in students never practising creativity in connection to fairly simple task settings. As a consequence, the students would not develop the competences necessary to perform CR and would not consider CR as a first choice option during task solving. Another reason that CR is more difficult than IR could simply be that the students do not practice CR. If easy tasks are often solvable with IR, and difficult tasks often demands CR, than this could also be a reason that the teachers use required reasoning to regulate the task difficulty. The teachers, and the students, might simply be used to CR tasks being difficult.

A way of changing this situation could be to let the students become more familiar with encountering unfamiliar tasks. Since a particular task is solvable with imitative reasoning if it is familiar to the students, they cannot practise using CR simply by becoming familiar with more types of tasks. The situation of *trying to solve an unfamiliar task using CR* could however become familiar in itself. Vinner (1997) argues similarly that teachers may encourage students to use analytical behaviour by letting them encounter tasks that are not solvable through pseudo-analytical solution processes. Hiebert (2003) points to the concept *opportunity to learn* and argues that what the students learn is connected to what activities and processes they engage in. Engaging the students in solving

unfamiliar tasks using CR can be seen as giving the students the opportunity to learn creative reasoning.

The teachers' views of the students' competences

That the teachers use the task type (the required reasoning) as a tool for regulating the tasks' degree of difficulty is also relevant in light of how mathematics is structured in several international frameworks for school mathematics. One example is NCTM (2000), often called the Standards, that describes mathematics using two dimensions: *content standards* and *process standards*. According to the Standards, mathematics should be seen as having these two dimensions, and task type is part of the process dimension. When designing tests and exams, it is possible to consider a third dimension: the tasks' degree of difficulty. That the teachers use the tasks' required reasoning as a tool to regulate the tasks' degree of difficulty, reduces this three dimensional setting to an more simple setting with only two dimensions, content and difficulty/process, at least when reasoning is the process considered.

The two task properties that the teachers mainly consider, task content domain and degree of difficulty, concern relatively 'concrete' aspects of the tasks. The task content domain is directly visible in the formulation of a task; it concerns e.g. integration or drawing graphs. The degree of difficulty is obvious from the students' results on the exam and can partly be predicted by experienced teachers. Also, the task type is determined by the extent to which the students have previously encountered very similar tasks. Since the teachers use task type to regulate degree of difficulty, this makes the degree of difficulty even more concrete for the teachers. A possibility is that the teachers primarily focus on content and difficulty *because* they are, or can be seen as, more concrete properties of the tasks than other aspects of the process dimension.

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Notes

- 1 In this context the word choose does not necessarily mean that the students make a conscious and well considered selection between methods, but just as well that they have a sub-conscious preference for certain types of procedures.
- 2 Then usually *failed*, 3, 4, and 5 instead of *failed*, *passed*, and *passed with distinction*.

Values in effective mathematics lessons in Sweden: what do they tell us?

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Abstract: This study aims to examine values in effective mathematics lessons in Sweden from the perspectives of students in different groups and their teachers. By using methods with lesson observations, student focus group interviews and teacher interviews, it shows that instructional explanation and classroom atmosphere with quietness are shared-values of students and their teachers. The findings propose some crucial issues which related to how mathematics teaching could be adjusted to different students' learning conditions and whether it needs more instructional explanation in mathematics teaching in Sweden.

Keywords: value, effective teaching and learning, mathematics teaching, explanation, Sweden

1. Introduction

The effective teaching and learning of school mathematics is one of the major objectives of mathematics education. Researchers have drawn on an extensive range of intellectual resources to address how effective mathematic teaching and learning can be implemented, which are demonstrated in several dominant approaches in the history of mathematics

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education research. The cognitive approach has a long tradition, until 1990s, it was still prominent with the typical vocabularies of *mental schemes*, *misconceptions*, and *cognitive conflicts* (Sfard, Forman & Kieran, 2001). And the subsequent language seems to be given way to the affective approach with *beliefs*, *attitudes* and *emotions* (Zan, Brown, Evans & Hannula, 2006). Both cognitive and affective approach deepens our understanding of how we can better facilitate mathematics teaching and learning. However, the insights from cognitive perspective and the knowledge we have gained from the affective perspective appear to lead to inconclusive attributes to what constitute effective mathematics teaching and learning (Lerman, 2001; Cai, 2007).

With the increasing interest in the socio-cultural nature of mathematics education, researchers are becoming increasingly aware of the significant role that the socio-cultural milieu plays in the teaching and learning of mathematics (De Corte & Verschaffel, 2007). And there is a growing agreement that there may not be a universal trait for effective mathematics teaching and learning within different cultural contexts, therefore, it is expected through a focus on the socio-cultural construct of values to further contribute to our sifting, clarifying and understanding of the different grains of constituents of effective mathematics education (Bishop, Seah & Chin, 2003; Seah, 2007).

This present study is part of a large scale collaborative project (NOTE 1 HERE) which adopts the socio-cultural perspective, to investigate how effective mathematics teaching and learning might be facilitated through an understanding of what teachers and students value in different social contexts. This paper reports the preliminary findings from Sweden, namely, what are valued in effective mathematics lessons from the Swedish students and their teachers' perspectives.

2. Theoretical background

There are two major issues in this paper. One is about effective mathematics teaching and learning, and the other is about value related to mathematics education. This section offers a general overview of various studies related to these two issues and acts as a point of departure for this study.

2.1 Effective mathematics teaching and learning

During the last decades, the body of research on effective mathematics teaching and learning has grown exponential. These studies focus on characteristics on different levels in mathematics education or some specific kind of intervention in the mathematics educational process. Common to these studies is their main interest in what works in mathematics education. Empirically these studies search for the relationship between these relevant characteristics at the different levels or the intervention, and some dependent variables that often labeled as “effectiveness criteria” or “output measures” (Muir, 2008). Many of these studies focus on the effects on solely one effectiveness criterion. However, despite the broad range of research, the very notion of “effectiveness” remains to be an elusive concept (Seah, 2007). Recently Slavin and his colleagues draw a similar conclusion from a series of comprehensive synthesis on the research on the achievement outcomes of different types of approaches to improving elementary and secondary mathematics, that is, the most striking finding from the review is that the evidence supports various instructional process strategies (Slavin & Lake, 2008; Slavin, Lake & Groff, 2009).

Research into the actions of teachers and their interactions with students is particularly useful as observed features of effective teaching practice. An influential study

in this regard was conducted with 90 primary school teachers and more than 2,000 students in the United Kingdom in the late 1990s by Askew, Brown, Rhodes, Johnson, and Wiliam (1997), which is widely cited in the literature (Muir, 2008). The findings from that study seem interesting that relatively high achievement gains were not necessarily related to specific teaching styles, but were associated with teachers who had ‘connectionist’ orientations (as opposed to ‘transmission’ or ‘discovery’ orientations), focused on students’ mathematical learning (rather than on provision of pleasant classroom experiences), provided a challenging curriculum (rather than a comforting experience), and held high expectations of initially low-attaining students (Seah, 2007).

Many of the features listed above reflect the value of meaningful and constructive classroom interactions between teacher and students, and perhaps also amongst students. Meanwhile, the various international comparative studies generally arrive at similar conclusions that effective teaching is more about responding to and valuing the socio-cultural aspect of the learning environment than it is about adopting particular teaching methods. For example, Hollingsworth, Lokan and McCrae’s (2003) analysis of the Third International Mathematics and Science Survey (TIMSS) 1999 Video Study data revealed that successful teaching of mathematics in schools has not been found to be associated with any one teaching method, and in fact, it was evident that amongst the high achieving countries a variety of teaching methods had been employed. Based on insights and suggestions from the previous studies, instead of defining the effectiveness in our study, we assume that the effective mathematics teaching and learning is reflective of a group of values identified.

2.2 Values related to mathematics education

Important theorists in a variety of fields have emphasized the importance of people's value priorities in understanding and predicting attitudinal and behavioral decisions (Rohan, 2000). In mathematics education, Bishop (1998) claimed that dealing with issues of democracy in mathematics education clearly requires engaging with values. In light of the findings from TIMSS and the debates on mathematics curriculum standards in the UK, Macnab (2000) argued that standards of attainment in school mathematics are closely connected to belief systems regarding value and purpose; those systems do not always collectively offer a credible and coherent vision for mathematics education which can be effectively implemented in school classrooms; and that this coherence of vision is what to a large extent characterizes the higher performing TIMSS countries.

Meanwhile, researcher argued that it is difficult to identify values. For this, some concepts such as "good" and "bad" are necessary (Swadener & Soedjadi, 1988). According to Fraenkel, a value is a concept or an idea that was considered as by someone in life, and values are ideas about the worth of thinking, they are concepts, abstractions (1977; Cited in Dede, 2006). Jablonka and Keitel (2006) see values as the principles, standards and qualities explicitly or implicitly considered worthwhile or desirable by the participants of a distinct social practice. In fact, mathematics education is designated as social practices where the teaching and learning of mathematics actually occur, and it is deeply rooted in its particular culture. In this sense, cultural values in mathematics education cannot be removed from the environment with which the values are held. Therefore, in our study, what the Swedish students and their teachers value in effective mathematics lessons are what we look for.

Values related to mathematics education operate at different levels. Bishop (1998)

classifies values in mathematics lessons into three different types, mathematical values (e.g. *control, progress*), mathematics educational values (e.g. *practice, multiple representations*) and general educational values (e.g. *respect, honesty*). Seah (2007) added on to this categories through identify the relevance of organizational or institutional values (e.g. *professional development, numeracy*) and personality value (e.g. *clarity, organization*). These categories will be used in the data analysis of this study.

3. Research aim

Before the year 1994, the Swedish schools was centralized and governed in detail on the national level. After two decades of decentralization and deregulation, the Swedish school system is now a goal-based system with a high degree of local responsibility. At the present, one of the most closely followed educational issues in Sweden is centred on pupils who find it difficult to reach educational attainment objectives. There is especial focus on students who do not manage to achieve the pass levels in the three core subjects (Swedish, English and mathematics) and who, for this reason, do not satisfy the requirements for admission to upper-secondary education. Mathematics, one of the three core subjects, stands out and presents the principal concern. For approximately 13% of year nine students in Sweden, mathematics has been the main obstacle to future studies (Skolverket, 2007). This is a major worry not only for students, parents and teachers, but also for politicians and decision-makers. In the election-debates in Sweden in 2006, attention was frequently focused on education-related issues and, in particular, the problem presented by the large number of students who do not manage to satisfy the requirements for admission to upper-secondary education (Sjöberg & Nyroos, 2009). According to the declaration of human rights with the intention “the school for everyone”

in the Swedish school system, teachers have an obligation to teach each student in accordance with his/her own prerequisites and previous knowledge in order to promote further learning. An underlying idea is that each human being is teachable even despite any type of categorized disability, and this assumes that the teaching is adjusted to the individual's conditions for learning. However, this aim still remains to be fulfilled (Eriksson, 2008). This study aims to make up for this gap, through focusing on an understanding of values of students in different groups and their teachers as well as the values differences they negotiate. Students in the regular group have the regular mathematics teaching and learning, and students in the special group have special needs in mathematics teaching and learning. It is designed to answer the following research questions: What are valued in effective mathematics lessons by students in the regular group? What are valued in effective mathematics lessons by students in the special group? What are valued in effective mathematics lessons by their teachers?

4. Methodology

4.1 Participants

Two mathematics teachers and their students respectively in grade 7 and 8, in the same school, participated in the present study. The school was relatively large and had high prestige, located in a city in the northern part of Sweden. Both teachers are female and experienced, with teaching years of 12 and 24, respectively. The grade 7 class was a regular group and the grade 8 class was a special group. These two classes had been continually observed for one year by one of the authors.

4.2 Data collection

Data collection included lesson observations, student focus group interviews and teacher

interviews. The data collected as a result of the lesson observation sessions represent valuing in process, and the lessons observed provided a basis for teachers and students to reflect on and discuss. Students were encouraged to take notes of the moments when they feel that they are learning mathematics particularly well in the observed lesson. After each of the observations, interview sessions were held with the student focus group and followed by a session with each of the teacher participants. Structured interview questions were respectively based on students' recall of the moments of effectiveness and drawn upon each teacher's reflective thoughts relating to the lessons observed.

Interview questions for the students included the general ones like "What should an effective mathematics lesson look like?", and the specific ones like "Why do you think this represents a moment of effectiveness?" Interview questions for the teachers included the general ones like "In your opinion, what should an effective mathematics lesson look like?", and the specific ones like "Here are some episodes mentioned by your students, which are regarded as moments of effectiveness. Have a look through. Which episodes surprise you? Would you like to explain why?" The interviewer tried to be careful not to lead the students and teachers to any answer but to encourage them to express their thoughts freely. The student interview lasted about 30-40 minutes and the teacher interviews about 50-60 minutes. The interviews were performed in Swedish, audio-recorded and transcribed into English. Six students in three different levels in the regular group were chosen by their teacher according to their achievements and genders to participate in the interview session, while all seven students in the special group were interviewed. All students had the permission from their parents.

The data collection was guided by the ethical rules formulated by the Swedish

Research Council, concerning information, consent to participate, scientific use of information and confidentiality (Swedish Research Council, 2006).

4.3 Data analysis

The data was analyzed in several phases. Firstly, every sentence which appeared to be valued was recorded. Secondly, different words and expressions connected to the different values were recorded. Thirdly, values were named. For examples, when the interviewer asked *“Was there any one of you that got an experience that ‘I’m learning mathematics particularly well’ during this lesson?”*, one student said that *“I understood it better now when she had her lecture”*. We recognized that the teacher’s instructional explanation was valued by the student, then, taking the references from the cross-checking with experiences from other participating regions (or countries), the name was given so as to clarify what it means. Here, it was identified as valuing “explanation”. Furthermore, in the transcripts, different students had the same value, but expressed it in a different way. For example, students said that *“Perhaps that I got to hear how others thought about these problems. And I also answered a question myself. I feel that I already knew it, but it’s still good to hear it more when she explains something in front of the whole class”*. We analyzed it as the same value “explanation”.

By using this way, values were identified, labeled and categorized. To describe the value priorities within different students, the frequencies of the values was calculated in the student interview transcripts. Two researchers analyzed the data independently. Disagreements concerning the analysis were negotiated until joint agreement was established. Validity of research findings was enhanced through triangulation of data sources.

5. Results

5.1 Values in effective mathematics lessons: students' perspective

A total of 36 counts of 7 different values from 25 descriptions were identified as values in effective mathematics lessons, which represented the perspective from students in the regular group. And a total of 24 counts of 6 different values from 22 descriptions were identified as values in effective mathematics lessons, which represented the perspective from students in the special group. Table 1 summarizes the results.

Table 1: Values in effective mathematics lessons: students' perspective

Students from regular group	Students from special group
36 counts of 7 values from 25 descriptions	24 counts of 6 values from 22 descriptions
personalized help [ME]	explanation [ME]
explanation [ME]	independence [ME]
quietness [IO]	relaxation [IO]
collaboration [ME]	quietness [IO]
sharing [ME]	fun [ME]
strictness [P]	personalized help [ME]
concentration [IO]	
ME=mathematics educational value	
IO=institutional/organizational value	
P= (teacher) personality value	

Amongst the values identified from students in the regular group, there are 4 mathematics educational values, 2 institutional values, and 1 (teacher) personality value. The values include *personalized help*, *explanation*, *quietness*, *collaboration*, *sharing*, *strictness*, and *concentration*. This may allow us to depict what an effective mathematics

lesson in Sweden might look like from the perspective of students in the regular group. With reference to the three most cited values, an effective mathematics lesson is likely to be one which contains instructional *explanation* presented by the teacher, and keeps the classroom atmosphere of *quietness*. And the effective mathematics lesson would mostly likely embrace *personalized help*. Of course, this is not to suggest that all these values operated in any one effective mathematics lesson.

Amongst the values identified from students in the special group, there are 4 mathematics educational values, 2 institutional values. The values include *explanation*, *independence*, *relaxation*, *quietness*, *fun*, and *personalized help*. This allows us to depict what an effective mathematics lesson in Sweden might look like from the perspective of students in the special group. With reference to the three most cited values, an effective mathematics lessons is likely to be one in which students have the *independence* on their work, and the classroom atmosphere of *relaxation* would be remained. And the effective mathematics lesson would mostly likely contain instructional *explanation* presented by the teacher.

5.2 Values in effective mathematics lessons: teachers' perspective

A total of 8 different values from 10 descriptions were identified as values in effective mathematics lessons, which represented the perspective from the teacher who teaches the regular group. And a total of 6 values from 13 descriptions were identified as values in effective mathematics lessons, which represented the perspective from the teacher who teaches the special group. Table 2 summarizes the results.

Table 2: Values in effective mathematics lessons: teachers' perspective

Teacher from regular group	Teacher from special group
8 values from 10 descriptions	6 values from 13 descriptions
explanation [ME]	interests[ME]
whole-class interaction [ME]	communication [ME]
quietness [IO]	visualization [ME]
communication [ME]	quietness [IO]
group work [IO]	explanation [ME]
experiment [ME]	authenticity [ME]
hands-on [ME]	
outdoor learning[ME]	
ME=mathematics educational value	
IO=institutional/organizational value	

These values are listed according to the order of appearance in the interviews, and the order of the appearance of which might potentially reflect the value priorities. With reference to the first three cited values of the teacher who teaches the regular group, an effective mathematics lessons is mostly likely to be one which contain instructional *explanation*. And it would capitalize on *whole-class interaction*, while also keeping the classroom atmosphere of *quietness*. Likewise, from the perspective of the teacher who teaches the special group, an effective mathematics lessons is mostly likely to be one in which students' *interests* would be aroused. And it would embrace *communication*, while also capitalizing on *visualization* to eliminate the potential learning difficulties.

6. Conclusion and discussion

6.1 Differences and similarities of values of students in the different groups

The results show that there are differences and similarities of values of students in the regular group and special group.

In general, the values of students in the special group are more divergent, which can be concluded from table 1. 7 values identified from 36 counts compared to 6 values identified from 24 counts, which may suggest that values are more shared by students in the regular group than in the special group. Next, there are specific value differences, for example, *collaboration*, *sharing*, *strictness*, and *concentration* in the regular group are not mentioned in the special group, while *relaxation*, *independence*, and *fun* in the special group are not evident in the regular group. Among those different values in the two groups, there are two pairs of opposite values, *collaboration* and *sharing* as opposed to *independence*, *concentration* as opposed to *independence*. These striking differences of values of students in different groups may reflect their different learning experience and learning requirements, hidden behind these important differences may also reflect something about the origins and role of values for effective mathematics learning for different individuals. This finding is crucial, in the sense that, it gives more evidences for that, to meet the human rights in teaching, we have to abandon the idea of using the same teaching methods for all students of the same age (Eriksson, 2008). And it reminds us that we should be more sensitive to different students' prerequisites and learning needs, to facilitate the implementation of mathematics teaching adjusted to students in different groups. As the findings show, for students in different groups, it might need to consider opposite methods. This situation is also urgent for the mathematics education in special groups or in special schools, where the main requirements are to arrange teaching and

learning with regard to those students' specific conditions.

There are three common values of students in both groups, that is, *personalized help*, *explanation*, and *quietness*. Especially, the value *explanation* are highly valued by both groups, listed respectively as the second one in the regular group and the first one in the special group. This may reflect that teacher's instructional *explanation* is very important to facilitate effective mathematics learning for students in both groups. This finding is supported by some related studies, for example, to examine some of the classroom processes that may be responsible for the stellar mathematical performance among Asian children compared to U.S. children, Perry (2000) studied the differences in the frequency and type of mathematical explanations during lessons observed in 80 U.S., 40 Chinese, and 40 Japanese 1st- and 5th-grade classrooms, and found that explanations occurred more frequently in the Japanese and Chinese classrooms than in U.S. classrooms.

6.2 The shared-values of students and teachers

The results suggest that there are some important shared-values of students and their teachers. For the students in the regular group and their teacher, there are the shared-values of *quietness* and *explanation*. For the students in the special group and their teacher, there are the same shared-values of *quietness* and *explanation*. These shared-values identified may contribute to our understanding how effective mathematics teaching and learning might work.

However, according to the different sources related to this finding, it seems there is a paradoxical situation. On the one hand, explanations seem to be a large and natural part of our cognitive lives, not necessary to mention it in the classroom teaching (Wittwer & Renkl, 2008), thus, it is understandable that clear and detailed *explanation* are a

shared-value of students in different group and their teachers in effective mathematics lessons. But on the other hand, according to a longitudinal classroom observation study conducted by one of the authors, preliminary findings show that the pattern of mathematics teaching in Sweden seems not based on explanation, on the contrary, it is mainly based on students' individual work, their work on mathematics textbooks (Peng, 2010). Logically, these two opposite observations cannot be seen to be simultaneously valid. One interpretation could be explained from the limitation of the methodology in this study. Since the qualitative methods adopted in this study, might not lead to the generalization to the pattern of mathematic teaching in the whole Sweden. Another interpretation could be that, indeed there lacks of enough instructional explanation in mathematics lessons or good explanations which should be adapted to the learner's knowledge prerequisites, and should focus on concepts and principles, and should be integrated into the learner's ongoing cognitive activities as research suggested (Wittwer & Renkl, 2008). To understand which interpretation is more reasonable, it's expected to have further investigation on this important issue.

6.3 Differences and similarities of values of the teachers

There are three common values of the two teachers who taught students in different group, namely, *explanation*, *quietness*, and *communication*. *Communication* are valued by both teachers, which is consistent with the point of view in growing number of researchers have been interested in studying the learning of mathematics as a collective enterprise in socio-cultural contexts, rather than as a process occurring only within an individual mind (Inaqaki, Hatano & Morita, 1998). Thus, it is understandable that teachers value that in effective mathematics lessons students' mathematical ideas develop through their

communicative practices or dialogical interactions with other members of the classroom.

There are value differences between the two teachers. For example, *whole-class interaction*, *group work*, *experiment*, *hands-on* and *outdoor learning* by the teacher who taught the regular group are not mentioned by the teacher who taught the special group, while *interests*, *visualization*, and *authenticity* valued by the teacher who taught the special group are not evident in the values of the teacher who taught the regular group. Among the different values of both teachers, there is only one institutional value *group work*, and the dominance of mathematics educational values of both teachers reflects the nature of values of showing variety of form in mathematics education. However, are the different mathematics educational values of teachers is a testimony to the role that the teacher plays in making different professional choices to facilitate effective mathematics learning according to different students' learning needs? The answer for this question needs to be confirmed by including more samples.

7. Concluding comments

This study had sought to examine values in effective mathematics lessons in Sweden from the perspectives of students and their teachers. The preliminary findings reveal that both the teachers and the students share some commonalities in what they both value in the shaping of effective mathematics lesson. These include instructional *explanation* and the classroom atmosphere with *quietness*. It helps us understand better how the values attribute the effectiveness in mathematics teaching and learning in Sweden. More importantly, the findings in this study propose some crucial issues which related to how mathematics teaching could be adjusted to different students' learning conditions and

whether it need more instructional explanation in mathematics teaching in Sweden. It is expected those issues will inform school districts of improved mathematics teaching practices.

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Note1

The project is called *The Third Wave: Regional Study of Values in Effective Mathematics Education*, which involves 12 places across 4 continents, designed to investigate how effective mathematics teaching and learning might be facilitated through an understanding of values of teachers and students across different nations/culture, and how they negotiate value differences to optimize their interactions, leading by Dr. Wee Tiong Seah, Monash University.

The role of theory when studying epistemological characterizations of mathematics lecture(r)s

Magnus Österholm¹

Introduction and background

The study presented in this paper is a contribution to the scientific discussion about the role and use of theory in mathematics education research. In particular, focus is here on the use of and comparison between different types of theories and frameworks, which is discussed primarily through the example of an empirical study examining what types of messages about mathematics are conveyed in lectures. The main purpose of this paper is to examine how different types of theories and frameworks can affect different parts of the research process.

The role of theory in research

In research, the use of theory is an important and frequently discussed issue. When publishing research reports, it is most often a demand that you should relate to a theory. This centrality of theory is also evident from how different researchers describe the relationship between research and theory. Silver and Herbst (2005) discuss a general function of theory when they place it in the centre of the scholarship triangle, which consists of research, problems, and practice, where theory functions as a connection between all three parts of the triangle. Lester (2005, p. 458) describes four general purposes of using a research framework (later I discuss relationships between similar notions such as theory and framework); to give structure to a research study, that a framework is always needed for

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data to make sense, to come further than common sense, and in order not to be limited to finding answers to local problems. Also Niss (2006, 2007) describes different general purposes and functions of a theory, for example to predict or explain phenomena, to organize observations and interpretations into a coherent whole, and to give a methodology for empirical studies.

The descriptions above are from a general perspective, but if we think more specifically about the different parts of a research study it could be of interest to, as Bergsten (2008, p. 190) does, ask how “a theoretical basis adopted for a study influence the nature of the purpose, questions, methods, evidence, conclusions, and implications of the study”. This highlights the potential of a theory to affect all parts of a research study. However, for a specific study it might be that some choices in some parts of the study are not based on a given theory and there are also “numerous cases where [...] the research is carried out without really involving the theory which is being invoked” (Niss, 2006, p. 9).

The diversity of theories

If theory does, or should, play a central role in all or many parts of a research study, how should we handle the situation when different studies that focus on the same phenomenon use different types of theories? One problem can then be to build new research on previous research if different theories are used, since:

Taking isolated research results at face value, without relating them to the conditions and constraints of the research processes behind them, provides no criteria or bases for relating them to other seemingly contradictory or similar results. (Bergsten, 2008, p. 189)

To relate results to the theory that has been used in the process of producing these results is therefore an important aspect. However, it is not necessarily the case that research based on different theories, which can even be partially contradictory, produces results that are impossible to combine in some way, for example if the contradictory parts of the theories are not relevant for the studies in question. As previously mentioned, it can also be that the parts of specific research studies that are compared are not dependant on the given theory. This relationship is also highlighted by Prediger’s (2008, pp. 284-

285) conclusions from a comparison between researchers' use of different theories: "the theoretical base alone did not completely predetermine their conceptualizations", showing that "research practices and theoretical bases of course are strongly connected, but it would be a misleading simplification to propose a direct causal or deterministic connection". It is therefore not enough to focus on the comparison of theories in themselves, whether (partially) contradictory or (fully) compatible theories are used, but it is important to become more specific regarding if and how theory affects different parts of the research process. This issue is examined in the present study.

Regarding how to relate different theories to each other, Bikner-Ahsbals and Prediger (2006) describe four different ways to do this; unify, integrate, network, and compete/compare. To unify refers to the creation of one theory from several theories that deal with the same phenomenon, while to integrate refers to the combination of different, but compatible perspectives. Networking refers to a strategy that includes for example both comparing and integrating theories, and demands cooperation between different research groups that represent different research cultures. The present study focuses mainly on compete/compare, "by treating the same set of data from the basis of different theories, similarities and differences of theories can be specified although the diversity is respected" (p. 55). There are several examples of studies that have compared different theories regarding how they affect some part(s) of the research process; Prediger (2008) examines the effect on the formulation of research questions, while several other studies examine the analysis and interpretation of a given set of data from different theoretical perspectives (Bergsten, 2008; Bergsten & Jablonka, 2009; Even & Schwarz, 2003; Gellert, 2008), often highlighting the effect of theory on several aspects/parts of the research process. All these comparative studies show clear differences in the descriptions and interpretations that are done based on different theories. The authors also discuss more detailed relationships between theory and some part of the research process.

Bergsten and Jablonka (2009) note that from observing the same situation, described through transcripts of student dialogues, the analyses based on two different theories have actually not used

the same data, partly because focus is on different parts of the transcripts and partly because one theory demands more data than what is present in the transcripts. Although much of the method, regarding the collection and presentation of data, was given beforehand as a common starting point for both theories, this example shows an effect of theory on methodology in a more intricate manner. However, to say that the theory is *affecting* the methodology might not be suitable, but perhaps methodology should be seen as *a part of* the theory, which for example Radford (2008) is suggesting. More discussion about what constitutes a theory is given later.

When Gellert (2008, p. 221) compares the use of two different theories, he finds that interaction patterns are described as “emerging” in one theory and as “reflections of macro-social structures” in the other theory, and he remarks that “emergence and structure are quite opposite concepts”. Despite this apparent contradiction between the theories, Gellert suggests a way of coordinating the results originating from the use of these two theories. He creates a combination of the two perspectives that could be seen as having “improved the usefulness and accuracy of the interpretation” (p. 223), but leaves it up to the reader to judge if this is actually the case. In line with the work of Gellert, one way to look at the combining of studies that focus on the same phenomenon but use different theories could then be to view the use of different theories as looking at the same thing from different angles or perspectives. The use of different theories is then not seen as a problem of having or creating potential contradictions, but that the use of different theories will complement each other. This is in line with Bikner-Ashbahr and Prediger (2006, p. 54) who do not see diversity of theories as a defect but as richness and as showing the character of the subject regarding its complexity; “we should not aim to reduce this complexity. Instead, the richness gained should be better exploited”. In similar fashion, Even and Schwarz (2003, p. 309) note that a classroom situation is too complex to be understood from only one perspective, and that a more complete understanding requires the use of different perspectives.

Above, the potential benefits of seeing the same thing from different perspectives are highlighted, but is it actually “the same thing” that is being examined so that different theories could be said to

complement each other? Similarly, are two theories contradictory when they include or produce contradictory statements about “the same thing”? For example, just because two theories claim to be about learning, are they then comparable in the sense that you can see them as potentially contradictory, or are the theories actually focusing on different things and just using the same word for these things? This problem is discussed by Rodriguez, Bosch, and Gascón (2008), regarding the difficulty to say that theories deal with the same questions since they see each theory as defining and formulating their own problems without there being an easy way to “translate” between theories. In their study they examine the notion of metacognition, which is a concept created in one theory and they see a difficulty in translating it to another theory. Instead of focusing directly on the concept, they examine the original practical problem that created the need for the notion of metacognition in the theory in question, and then focus on this problem when trying to translate to another theory. Based on this example, the authors argue that all comparisons between theories need to be done via the type of problematic situations or issues that were the origin to the specific content of a theory. Perhaps this could be seen as looking for a least common denominator for theories, regarding the type of event, situation or phenomenon that can actually, to some extent, be seen as “the same thing” seen from different perspectives. This idea shifts focus from looking at differences to looking at potential commonalities regarding relationships between different theories. In the present study, both similarities and differences between theories are discussed in relation to empirical data.

The concept of theory

Theory has so far been discussed without giving an explicit definition of this notion, which sometimes seems possible to do, thus basing the discussion on some type of common understanding of this notion, perhaps created through common properties of examples of constructs labeled as theory. However, the existing “absence of clear definitions of ‘theory’ and ‘theoretical’ in many publications that invoke these terms” (Niss, 2007, p. 1308) becomes problematic if we wish to go deeper in our awareness of the use

of theory in research. Therefore, we also need to discuss what we mean by 'theory' and relate this meaning to comparative studies of theories in use.

There are many concepts in use that seem to relate to some aspect of theory in research; "Model. Construct. Theory. Paradigm. Framework. To some, these words have vastly distinct meanings, while to others they are separated by shades of grey" (Mewborn, 2005, p. 1). Words like approach and perspective can also be added to the list, and also that some of the words can be combined, for example in 'theoretical framework'. It is neither possible nor necessary to discuss and define all these words here. Instead, my discussion is limited to the two notions that seem most commonly referred to in discussions about theory in research; theory and framework. The works of Niss (2006) and Radford (2008) are utilized regarding the notion of theory while the work of Lester (2005) is utilized regarding the notion of framework and to some extent also regarding the notion of theory.

Niss (2006) defines a theory as consisting of an organized network of *concepts* and *claims*, where the concepts are linked in a connected hierarchy and where claims are either fundamental (i.e. of axiom type) or derived from the fundamental claims. Radford (2008) defines theory as consisting of a system of *basic principles* (P), a *methodology* (M), and a set of *paradigmatic research questions* (Q). There are many similarities and overlaps between these two definitions. *Basic principles* can be regarded as *claims* of a certain kind, which also *methodology* can since Niss describes that theories can function as methodology. *Concepts* are not mentioned as one separate part of a theory by Radford, but P is "characterized by its hierarchical structure and the ensuing meaning of its key concepts" (Radford, 2008, p. 320), highlighting the similarity with the definition by Niss.

The set of *paradigmatic research questions* is not found in the definition of theory by Niss (2006), and he does not discuss the relationship between research questions and theory. Radford (2008) describes Q as "templates or schemas that generate specific questions" (p. 320) and not as a fixed set of questions. His reason for including Q in the definition of theory is that "because they [theories] emerge

as responses to particular problems, they bear the imprint of the initial questions that they sought to answer” (p. 321).

In this paper, I choose to define theory in accordance with Radford (2008). This choice is made because Radford describes a structure that is easier, compared to the one given by Niss, to relate to when comparing the use of different theories, which is also the focus of the article by Radford.

Mason and Waywood (1996) describe two aspects of the use of theory; as foreground or background theory, which can be related to that it is not only theory that is explicitly referred to that affects research. Both these aspects can be labeled as theory also when using the definition by Radford (2008, p. 320) since the system of basic principles “includes implicit views and explicit statements”. The notion of approach, defined by Bergsten (2008, p. 192) as “a more informal inclination by the researcher to interpret an observed commonsensical problem”, can therefore also be included in the notion of theory as defined by Radford.

The relationship between research questions and theory was highlighted through the definition of theory by Radford, and is also an important issue in Lester’s (2005) discussions about different types of frameworks, in particular regarding theoretical and conceptual frameworks. For a theoretical framework, there is reliance on a formal theory and research questions “would be rephrased in terms of the formal theory that has been chosen” (p. 458), while:

A conceptual framework is an argument that the concepts chosen for investigation, and any anticipated relationships among them, will be appropriate and useful given the research problem under investigation (Lester, 2005, p. 460).

The use of theory is not limited to the theoretical framework, since conceptual frameworks “may be based on different theories” (Lester, 2005, p. 460). The difference between the two types of frameworks seems to be the ordering of theory and research questions; in a theoretical framework the theory decides what types of questions can be asked while in a conceptual framework the questions

affect the choice of theory. Frameworks in general could then be described as different ways of doing research, in particular regarding different ways of using theory, while theory (as defined by Niss and Radford) can be seen as a type of structure of concepts and statements. This view of frameworks is somewhat different than what is presented by Bergsten (2008), when he discusses the role of different types of frameworks in the process of analyzing data, also referring to Lester when utilizing the notion of framework. Bergsten focuses on the multitude of sources/theories as a central aspect of conceptual framework, for example when discussing one specific study; “the research framework [is] conceptual, since the study uses theoretical concepts from various sources rather than one overarching theory” (p. 193). I do not see the number of sources/theories as the main difference between the two types of frameworks because it becomes problematic what constitutes *one* theory or *several* theories and, as argued above, I interpret Lester’s descriptions of different frameworks as primarily be about different ways of relating theory and research questions.

Lester does not define the notion of theory, and the general definition of framework seems quite similar to the definition of theory used in this paper; “a research framework is a basic structure of the ideas (i.e., abstractions and relationships) that serve as the basis for a phenomenon that is to be investigated” (Lester, 2005, p. 458). The relationship between framework and theory is therefore not easy to sort out. A more in-depth analysis of this relationship is not done here, but I choose to characterize the different types of frameworks based on the different relationships between research questions and theory as described above.

Purpose and structure of the present paper

The main purpose of this paper is to examine how different types of theories and frameworks can affect different parts of the research process, when using the notions of framework and theory as described earlier. More specifically, the discussions in this paper focus on two aspects:

- The utilization of different types of frameworks and different types of theories.
- Relationships between different parts of the research process and different parts of theories.

The aim is not to discuss the use of theories only from a “theoretical perspective” but the discussions also focus on the content and structure of a specific empirical study; about epistemological characterizations of mathematics lectures (this topic is discussed more in the next section). In relation to this specific research topic, a purpose is also to examine if/how it seems possible to unify or integrate different theories used when studying this topic, and not only to compare them. I have examined some aspects of both comparison and integration of theories related to this research topic before (Österholm, 2009), but then without relating to a specific empirical study and without a more in-depth view of the notion of theory.

Thus, in order to discuss the use of theory in research, an empirical study is used as a starting point. This empirical study is described in the next section, as a study in itself including descriptions of background, purpose, method, analysis of results, and conclusions, and thereby becoming somewhat separated from the overarching purpose of the present paper. However, the description of the empirical study is not as elaborate as could be expected if the study was reported independently, partly in order not to make this paper as a whole too long and partly in order to create a better overview of the empirical study for the discussion of the use of theory. Some parts of the empirical study is therefore affected by and adjusted to the overarching purpose of discussing the role of theory in research. The empirical study can therefore be seen as somewhat “artificial”. The same empirical study has been described elsewhere (Österholm, 2010), where some parts of it are more elaborately described, but also where the focus is different; on exploring and discussing the methodology of the study.

Similarly as Prediger (2008), a type of “classroom problem” or “issue of classroom practice” is taken as the starting point for the discussions of the use of theory in research. In contrast to the study of Prediger, research questions are then not discussed from the perspective of some given theories, but the first step is here the use of a conceptual framework in relation to the given problem. Thereafter, the method of the empirical study is described, followed by the results and conclusions. The

conclusions are the end of the separate empirical study, and thereafter the focus is redirected to discussing the issue of use of theory in different ways in relation to the presented empirical study.

Firstly, in relation to the given conclusions from the empirical study, the need to “add more theory” is discussed, as a means to be able to interpret the conclusions in a broader perspective. At this point, the use of two different theories is discussed, in particular regarding a comparison between them based on the definition of theory by Radford (2008) and in relation to different parts of the research process.

Finally, the focus is directed to the differences between conceptual and theoretical frameworks by analyzing if/how some part of the empirical study would have been different if a conceptual framework had not been the starting point, but if one of the two theories introduced in relation to the interpretation of the conclusions had been used together with the given classroom problem as a starting point for the whole empirical study, that is, if a theoretical framework had been used. Besides being able to compare the use of different frameworks, this utilization of different theories also allows for an analysis of relationships between parts of the research process and parts of theories, in particular since these theories are introduced at a late stage in the process but then also related to other parts of the research process.

The empirical study

Introduction

The teaching activities in a mathematics classroom can be very different, depending on the content, the teacher, the students, available materials, the classroom environment, etc. The subject of mathematics can have very different character depending on these properties of teaching activities, that is, that properties of the teaching situation can produce different pictures or characterizations of mathematics. The focus in this empirical study is on this type of issue of classroom practice, regarding how certain properties of teaching can give different pictures of mathematics.

Others have studied this type of issue, for example regarding what properties of mathematics “is conveyed by mathematics textbooks” (Raman, 2004, p. 389) where a comparison between different

textbooks showed “conflicting messages regarding the status and purpose of mathematical definitions”. Shield (1998, p. 516) also examines textbooks, in order to “identify the types of messages about mathematics [...] inherent in their presentations”, where messages in textbooks is shown to differ compared to “the intent of recent reports and syllabuses” (p. 521).

Ikonomou, Kladrimidou, Sakonidis, and Tzekaki (1999, p. 170) focus their analysis on properties of mathematics lessons regarding “epistemological features” of the lessons, by examining how the teacher handles “the nature, meaning and definition of concepts, the validation procedures and the functionality of theorems”. Their conclusions focus on a comparison with mathematics as a discipline, and they see large differences, for example that activities in the classroom are reduced to procedures and that central notions are not separated (such as definition and theorem).

Although using different notions, I see the given examples of previous studies as focusing on the same kind of phenomenon, regarding a characterization of some properties of teaching that can be interpreted as messages about mathematics. Similar to the examples of other studies, the purpose of the present empirical study is to examine the types of messages about mathematics inherent in, or conveyed by, properties of teaching.

In order to be able to focus on this issue in a research study, we need to discuss in more detail what can be meant by pictures of, or messages about, mathematics, and also to limit the study to focus on some aspect/property of teaching. In addition, it is also needed to discuss how to examine certain properties of teaching as messages about mathematics, in order to plan and carry out an empirical study focusing on this issue. Thus, there is a need to discuss these central aspects and concepts, which can be done through a conceptual framework, in order to have a structure for conceptualizing and designing the research study (Lester, 2005).

A conceptual framework

Messages about mathematics can be related to the basic philosophical issues of epistemology and ontology, that is, the nature of knowledge and the existence/location of (mathematical) entities respectively. The discussions are here limited to epistemological aspects, through two questions regarding the nature of knowledge; what knowledge is and how knowledge is acquired, which are shortly referred to as questions about properties of *knowledge* and *knowing* respectively.

There are many aspects of epistemology, and many types of epistemologies, in relation to mathematics and mathematics education (Sierpiska & Lerman, 1996), so there is a need to limit the focus of the present study, which is done by focusing on one central aspect in relation to each of the two mentioned questions about knowledge and knowing. In relation to *knowledge*, I relate to a central distinction in mathematics education regarding two different types of knowledge; conceptual and procedural knowledge (Hiebert, 1986). In relation to *knowing*, I relate to properties of argumentation (Toulmin, 1958), regarding the process of drawing conclusions.

A next step is to relate these two aspects of epistemology to properties of teaching. Among the examples of similar studies done before, which examine messages about mathematics inherent in some properties of the teaching situation, some have focused on epistemological aspects; regarding properties of textbooks (Raman, 2004) and regarding properties of classroom activities (Ikonomou et al., 1999). For the present empirical study, I choose to focus on properties of oral communication, because it is a very central aspect of teaching and it is more focused than to examine classroom activities more generally. I also choose to limit the study to focus on one-way communication; lectures, in order to reduce the complexity somewhat by not including aspects of dialogue or discussions among several persons.

Thus, the focus of the present empirical study is properties of oral communication that could be said to convey something about epistemological aspects of mathematics. The communication can include explicit statements about knowledge and knowing in mathematics, but also other properties of

communication can represent certain aspects of knowledge or knowing. Since the most common activity in a lecture probably is to present mathematics and not to talk *about* mathematics (from an epistemological perspective), the present empirical study focuses not on explicit statements about epistemology.

The *types of statements* used in a lecture could highlight the epistemological aspect regarding the nature of *knowledge*. In relation to conceptual and procedural knowledge, I therefore separate two main types of statements; statements about the use of mathematical objects, labeled *use-statements* (related to procedural knowledge) and statements about properties of mathematical objects, labeled *object-statements* (related to conceptual knowledge). Here I choose to use the more general notion 'mathematical object', which can refer to concepts as well as procedures. The difference between the two types of statements is therefore that they describe either properties of objects or the use of such objects, which is seen as a central aspect regarding the difference between conceptual and procedural knowledge. For example, the statement "The derivative of $\ln x$ is one over x " is an object-statement while the statement "When you take the derivative of $\ln x$ you get one over x " is a use-statement.

As mentioned before, the *types of argumentations* could highlight the epistemological aspect regarding the nature of *knowing*. For the limited purpose of this empirical study, I choose to use a simplified version of the elaborate structure of argumentation presented by Toulmin (1958). This simplified structure consists of a conclusion that is drawn (or a *claim*, using Toulmin's vocabulary) together with statement(s) used as argument for this conclusion (or *data*, using Toulmin's vocabulary). The words or wordings used to make explicit the argumentative relationship between statements (e.g. words such as 'therefore' and 'since') are here labeled *connect-words*. These words or wordings are of great importance when locating arguments, and focus can be put on these types of words in the analysis of types of argumentations.

When focusing on the *types* of statements and the *types* of argumentations, focus is not on the mathematical content of the statements or the argumentations, and the purpose of the analysis in this empirical study is not didactical, in the sense that the focus is not on aspects of teaching and learning the mathematical content nor on the teaching and learning of argumentation or proving. Instead, the analysis of types of statements and types of argumentations in lectures is used in order to draw conclusions about what is conveyed *about* mathematics, in particular regarding epistemological aspects.

In summary, the conceptual framework used in this empirical study focuses on epistemological properties of oral communication in mathematics lectures by discussing the following concepts and relationships between concepts:

- Aspects of epistemology, regarding knowledge and knowing.
- How epistemological aspects can be conveyed through properties of oral communication, either explicitly (through direct statements about epistemological aspects) or more implicitly.
- Aspects of knowledge, regarding conceptual and procedural knowledge, in relation to the types of statements used in oral communication.
- Aspects of knowing, regarding argumentation, in relation to the use of connect-words.

Purpose

As noted in the introduction, the overarching purpose of this empirical study is to examine the types of messages about mathematics inherent in, or conveyed by, properties in teaching. This purpose has now been further specified; to examine properties of oral communication that could be said to convey something about epistemological aspects of mathematics. That is, the purpose is to perform a type of epistemological characterization of mathematics lectures. Through the conceptual framework, this purpose has also been specified regarding different aspects of epistemology (the four points at the end of the last section).

The length of this paper is limited by not including results and analysis of the epistemological aspect of knowing, thus only including the aspect of knowledge, regarding the different types of statements; use- and object-statements. However, since the aspects of knowledge and knowing are integrated in the analysis of data, both aspects are related to in the description of method.

Thus, the purpose of the present empirical study is to compare different lectures regarding how they, through properties of statements in the oral presentations, convey something about epistemological aspects of mathematics. In relation to this purpose, the following research questions are in focus:

- Is there a tendency to use different types of statements in different lectures?
- How can potential differences between the lectures be characterized?

Method

Two mathematics lectures at university level are analyzed in this empirical study. The lectures have different lecturers and different mathematical content. One lecture is part of a course in calculus and this particular lecture is about improper integrals, while the other lecture is part of a course in statistics for natural scientists and this particular lecture is about some examples of discrete probability distributions. Both lectures are approximately 45 minutes long. Only the lecturers' activity is analyzed, in order to focus on one type of discourse; the one used in lecturing, and not in for example dialogue. The lectures were recorded with audio and video, but students' statements are not audible in the recordings and the camera is always focusing on the lecturer's activity at the whiteboard.

The analysis in this paper focuses on the lecturers' auditory communication, and the lectures were transcribed from the audio recording, but using the video recording in case of doubt in the process of transcription and in case of unclear references in the lecturers' statements (e.g. referring to "this" or "that" when pointing to something on the whiteboard).

In order to create a clear structure in the analysis, statements from the lectures are analyzed in several steps. The first step in the analysis is to mark use- and object-statements, and also connect-words in the transcription. Each coherent section of the transcription is then extracted from the transcription, for further analysis. A coherent section refers to a set of statements that are connected through the use of connect-words. Such a section can for example be only one conclusion together with an argument, as in the following example from the lecture in calculus, where the connect-words are in italics: “The derivative of $\ln x$ is one over x , which is larger than zero, *which means that* it grows all the time”. Note that there is actually a linguistic ambiguity about exactly what the word ‘which’ in ‘which means’ refers to, but logically all information given before the conclusion is needed, and this full statement is therefore regarded as the argument. A section can also include several argumentations, as in the following example from the lecture in statistics, where the connect-words are in italics:

And you can show that the expected value is one over p . This can be seen as. Yes if we imagine that for example p is zero point two. Then this *means that* we will succeed on average every fifth time. And this also *means that* the expected value then becomes one over zero point two, which is five. *So* it is exactly that we will have to do on average five tries in this case.

A next step in the analysis, which is mostly relevant for sections that do not consist of a single argumentation, is to extract the relevant statements from the excerpt and arrange them in a structured manner, which for the latest example can be done in the following way, where the connect-words are in italics:

1. You can show that the expected value is one over p .
2. p is zero point two.
3. *Means that:* We will succeed on average every fifth time.
4. *Means that:* The expected value is one over zero point two, which is five.
5. *So:* We have to do on average five tries.

From this structure it is easier to analyze how the statements are related according to the connect-words, although the analysis has to include some considerations to what is reasonable, as was done in

the previous example about the derivative, since it is not always clear exactly what is referred to as being the argument for the conclusion. In such situations, the logically necessary statements previously stated are listed as included in the argument. In the example above, we see that line 3 is only based on line 2 as an argument, while line 4 cannot only be based on the previous line, although the exact same type of connect-words are used. The result of this type of analysis is then summarized in a three column table with a conclusion, the argument(s) for this conclusion, and the connect-words used in the argumentation. From the example above, one line in the table thus becomes:

We will succeed on average every fifth time	P is zero point two	Means that
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Based on the content of the produced tables, each lecture can then be characterized through examining the different types of statements (use- and object-statements); quantitatively regarding which type of statement is most commonly used and qualitatively regarding the formulation of statements of different kinds.

Results

Through utilization of the described method, the produced table for each lecture consists of 39 lines for the calculus lecture and 43 lines for the statistics lecture. Each line in the table corresponds to one argumentation, which consists of a conclusion, the statement(s) used as argument(s) for the conclusion, and the connect-words.

When studying how common the different types of statements are in the two lectures, a clear difference between these lectures appears: In the calculus lecture, use-statements appear as a conclusion on four lines in the table (10 %) and as an argument on two lines (5 %), while in the statistics lecture, use-statements appear as a conclusion on 22 lines (51 %) and as an argument on 18 lines (42 %). Thus, in the calculus lecture object-statements are most common, while in the statistics lecture the two types of statements are about equally common.

Besides this quantitative characterization of the use of different types of statements, it can also be noted that there are many examples of statements of one type that can easily be reformulated in order to turn it into a statement of the other type. For example, in the statistics lecture there is the use-statement “if you add a constant to all values of the function, this will not change the variation”, which can be reformulated into a statement saying that a property (the variation) of two functions is the same (i.e. an object-statement). An example of the opposite type of reformulation is taken from the calculus lecture, where there is the object-statement “(the graph of) one over x has a similar appearance (as the graph of one over x squared)”, which can be reformulated into “if we sketch the graph of one over x , the result is similar as when we sketch the graph of one over x squared” (i.e. a use-statement).

Conclusions

The study of use- and object-statements shows a potential difference between the two lectures regarding some epistemological aspects. However, since many statements seem easy to reformulate into the other type of statement, there is some arbitrariness, but not necessarily randomness, regarding what type of statement is used. These observations highlight the questions if or how these properties of communication can be seen as primarily tied to the individual, to the mathematical content, to the type of course, or to other aspects of the situation. These questions seem possible to examine in more detail using the type of method for data analysis presented in this empirical study, but for a larger set of data. Thus, it would be interesting to compare epistemological characterizations of different settings (not only lectures) for the same person and of communication of different persons (in particular to include also students) within one setting.

The use of different theories

Let us now return to the overarching purpose of this paper and discuss properties of the empirical study in relation to the use of theory. In the conclusions in the empirical study it is questioned whether properties in the discourse could be seen as tied to the individual, to the mathematical content, to the type of course, or to other aspects of the situation. A possible way of examining this question is also described, as a continuation of the use of the conceptual framework, through more extensive empirical

studies regarding variations in properties of communication for different persons and different situations. However, you could also see it as necessary to relate to a theory for the interpretation of the results from the data analysis, for example in order to make it possible to relate to other kind of studies that are examining similar kind of data or phenomena. This aspect of *interpreting* the results through a theory can also be seen as a way of *explaining* the results by relating to a theory that describes more general aspects of the observed situation or phenomenon. Two different theories are here introduced as examples of the use of theory for interpreting and explaining the empirical results.

Introducing theory as a means to explain empirical results

It is here not possible to give full descriptions of two theories. Therefore, I relate to two theories that are believed to be familiar enough for the reader to make the description and discussion comprehensible. The two theories are here labeled as *cognitive theory* and *discourse analytic theory*. From the short description of these theories given here it is not self-evident that these can actually be labeled as theories, according to the definition by Radford (2008). The label of each of these two theories does not refer to a specific theory in the sense that I can refer to a specific article or book that describes the theory, but refers to something that is common in several different specific theories. Perhaps each of the two theories could be better characterized as an *approach* (Bergsten, 2008) or as a *perspective* (Lester, 2005) – I return to the discussion of this issue.

A very short description of the foundation of each theory is given below, together with an interpretation of the conclusions in the empirical study. Thereafter, the use of these two theories is discussed, including a comparison based on Radford's (2008) defining components of a theory.

The cognitive theory focuses on mental objects and processes of individuals. Regarding epistemological properties of communication, focus can then be put on individuals' epistemological beliefs. In addition, a lecture can be described using the notions of sender and receiver. The lecturer's beliefs can be seen as a cause for how s/he presents the mathematics, for example that epistemological beliefs are a basis for how it is argued that one knows something. How the lecturer presents the mathematics is then

influencing how the students think about mathematics, including epistemological aspects. In this theory, focus is on cause and effect, where the study of properties of communication in the lecture can be relevant both as a sign of the lecturer's beliefs and also as a potential cause for students' beliefs.

This focus on cause and effect seems common in educational research about beliefs, for example when studying teachers' *attributed* beliefs (Speer, 2005) or when *explaining* students' differences in performance through differences in beliefs (Schommer, 1990).

Based on the cognitive theory, the empirical study can be said to characterize *the lecturer*, since focus is on properties of individuals. The observed differences regarding properties of communication can then be explained by that the lecturers have different epistemological beliefs, or at least that different beliefs are active in the two lectures.

The discourse analytic theory focuses on the discourse in itself as a central process, and not seeing the discourse as a product of other, more basic, (mental) processes. A lecture can then be described using the notions of participation and enculturation in a discourse community. The lecturer's statements are not seen as a reflection of some cognitive structure, but as being constitutive themselves, as a part of the social situation (Edwards, 1993). The study of properties of communication in the lecture can then be relevant both for characterizing the discourse community and also by seeing the lecture as a part of students' enculturation (including becoming familiar with the discourse somehow related to epistemological aspects of mathematics).

Based on the discourse analytic theory, the empirical study can be said to characterize *the lecture*, since focus is on properties of the situation. The observed differences regarding properties of discourse can then be explained by that different discourse communities are observed in the two lectures.

Looking back at the empirical study and the use of the two theories, the following summary can be given.

1. Issue of classroom practice: Different messages about mathematics inherent in, or conveyed by, properties of teaching.

2. Purpose of research study: Examine epistemological characterizations of oral communication in mathematics lectures.
3. Conclusion: Difference between two lectures regarding the epistemological characterization.
4. Interpretation/explanation:
 - (a) Alternative 1, cognitive theory: Properties of the lecturer, difference regarding beliefs.
 - (b) Alternative 2, discourse analytic theory: Properties of the lecture, difference regarding discourse communities.

Above, the use of the two theories seems only to affect point 4, since these theories were introduced at that stage. However, we need also to examine if and how other parts of the research process need to be changed or reinterpreted in some way due to the introduction of a new theory.

The effect of theory on different parts of the research process

As a starting point, the two theories are compared based on the definition of theory by Radford (2008), which relates to aspects of the research process, and here focus is on specific aspects of the presented empirical study.

Regarding *basic principles*, both theories include the concepts of mental object/process and communication/discourse, but a main difference between them is regarding where and how these concepts are located in the hierarchical structure of the system. While the cognitive theory regards properties of communication as an effect of mental processes, the discourse analytic theory begins “not by questioning the existence or ontology of underlying cognitive representations but by questioning their epistemological basis in discourse” (Edwards, 1993, p. 218).

Regarding *methodology*, the same empirical data, analyses and results in the empirical study are relevant for both theories. This observation is specific for the present empirical study and does not

mean that the same observation would always be made when using these two theories. In the next section, relationships between theory and methodology are discussed more.

Regarding *research questions*, the same purpose and questions are used, and are of interest, for both theories. However, in the process of interpreting/explaining the results it became clear that ‘epistemological characterization’ refers to different things in the two theories, whether characterizing the lecturer or the lecture. This difference regarding research questions can be seen as stemming from the differences regarding basic principles, which also Radford (2008, p. 322) highlights when noting that “research questions must be clearly stated within the conceptual apparatus of the theory”.

When applying the two theories to the empirical study, some differences between the two theories have been observed, in particular regarding basic principles that create different interpretations/explanations of results and also a reinterpretation of the purpose. However, there are also many similarities in the use of these two theories, which is evident in particular since it was possible to introduce the theories at such a late stage of the research process, when the theories could utilize the same empirical data, analyses and results. Thus, although some basic differences have been noted between the theories, the same type of empirical study is relevant for both theories.

Another similarity in the use of the two theories is that, as noted when introducing them, both function as explanation of the observed differences between lectures. That is, the question of what is “causing” these differences is not an empirical question but the question is answered within each theory.

However, this question is only answered at a general level, and both theories have a common interest in a type of follow-up question that is not answered within the theories but that demands empirical research; regarding the “limits” or “borders” for variations in the properties of communication. For the cognitive theory, this issue can be described as dealing with how general or context dependent epistemological beliefs are (see Limón, 2006), for example if it is reasonable to talk about epistemological beliefs for mathematics in general or if the beliefs are different for different areas of mathematics. For the discourse analytic theory, this issue can be described as dealing with the borders

and overlaps between different communities (see Skott, 2009), for example if it is reasonable to talk about a discourse for mathematics in general, regarding epistemology, or if different areas of mathematics create their own discourse communities about epistemological aspects.

The “follow-up question” discussed above is quite similar to the question asked at the end of the empirical study, before the two theories were introduced, regarding what aspects of the observed situation can be tied to the different properties of communication. Although the empirical study described here is somewhat artificial and partly adjusted to suit the overarching purpose of the present paper, this similarity of questions of interest at least shows a potential for such a similarity to exist. Based on these similarities between the uses of the two theories it is suggested that, for the specific topic of study in the present empirical study, the main difference between the two theories is a choice of wording when describing the observed phenomenon. It seems possible to “translate” descriptions within one theory into descriptions within the other theory, by noting what type of issue or phenomenon is being referred to in the languages of the theories – similarly as suggested by Rodriguez et al. (2008) in their comparison of theories. It should be noted, again, that this discussion focuses on the use of theories in relation to one specific empirical study, and does not deal with a general comparison of theories regarding their differences, similarities or compatibility. Therefore, it is not necessarily the case that the same relationship between these theories would be evident in relation to some other empirical study. This fact is also one important point I try to make; that it is more fruitful, and even necessary, to discuss the use of theories in relation to specific empirical studies in order to be able to examine if and how different theories, and different parts of theories, can affect different parts of the research process.

The process of choosing a theory to use

If the similarities between the uses of the two theories in the present empirical study are so great that the main difference is a choice of words, do we need to use a theory at all, or if we do, on what basis do we choose a theory?

Regarding the choice between the cognitive theory and the discourse analytic theory, this could be based on a personal conviction about how to view the more general issue of the relationship between mental processes and discourse. In this case, a *choice* between theories is perhaps not made, but the specific theory used serves as a means to make explicit the personal conviction. Even if not made explicit, these types of personal convictions can be labeled as theory. However, it can sometimes be of interest to separate these more general types of theories when discussing the role of theory in research, for example by labeling “a more informal inclination by the researcher to interpret an observed commonsensical problem” as an approach (Bergsten, 2008, p. 192) or by labeling “the viewpoint the researcher chooses to use to conceptualize and conduct the research” as a perspective (Lester, 2005, p. 458).

There is also the option of not choosing any of the two (or other similar) theories in relation to the empirical study. As previously described, questions triggered by the empirical results could be pursued without adding a new theory. As mentioned, the same types of questions are also of interest within the two theories, which might even more question whether it is relevant to add a new theory. Another alternative could then be to use the results from continued empirical studies as a basis for choosing a theory, regarding which theory seems more *useful*. For example, if further studies show that the epistemological characterizations are more tied to the individual than to the situation; that the variation is larger between individuals in the same situation than between situations for the same individuals, a theory locating the source of variation at the individual level could be more suitable. In this way, a theory is not seen as something generally applicable but the use of theory is adjusted according to specific needs and circumstances, which is in line with the notion of conceptual framework (Lester, 2005).

So far, after using a conceptual framework in the empirical study, two theories were introduced late in the research process. It was then examined how the use of these theories can affect different parts of the research process, in particular the interpretation/explanation of empirical results, the methodology, and different aspects of research questions (including reinterpretation of existing ones and possible

future questions of interest). These discussions about the use of theories in research can now be expanded and deepened by also discussing the use of theoretical frameworks, which is done in the next section by examining the possibility to introduce the two theories at an earlier stage in the research process.

The use of conceptual and theoretical frameworks

As discussed in the introduction of this paper, in relation to Lester's (2005) description of different frameworks, a main difference between theoretical and conceptual frameworks is regarding the relationship between research questions and theory. A simplified way of describing this relationship for a theoretical framework is that the theory comes first and that the research questions follows from, or are to some extent included in, the theory. For a conceptual framework on the other hand, the research questions "decide" what type of theory to use.

Based on the specific empirical study described in the present paper, when using a conceptual framework, it can be noted that it is not as simple as to say that research questions and theory are "separated" from each other and that one "affects" the other. In the empirical study, the starting point was a certain question, and based on this question there was a need to relate to and use certain theories. However, the original question was then also made more specific through the use of these theories. More generally, a research question cannot exist in a theoretical vacuum, but we could perhaps distinguish different types of questions based on how closely connected they are to a theory. For example, what Prediger (2008) labels as issues of classroom practice and Bergsten (2008) labels as observed commonsensical problems could be seen as questions formulated based on more intuitive grounds. However, these types of questions are not entirely non-theoretical; in relation to the issue of classroom practice given by Prediger, Radford (2008) states that the described issue would be difficult to relate to if using a certain theory, and Bergsten notes the use of an approach (i.e. a more informal, implicit type of theory) when interpreting the observed commonsensical problem. The types of questions that are more intuitively formulated need to be made more precise in order to function as

research questions, in particular that central concepts need to be defined and relationships between them need to be described, which are central parts of a theory.

Thus, for a conceptual framework, there is a kind of dual relationship between research questions and theory, but where, as Lester (2005) highlights, focus is always on arguments for choosing relevant theories in relation to the topic of study. However, looking at the empirical study described in this paper, it can be noted that the arguments for using a certain theory focus on the need for a certain *type* of theory, for example a theory dealing with aspects of epistemology in mathematics or a theory dealing with possible explanations of observed results. The choice of a specific theory of the needed kind was then perhaps affected more by, for example, personal viewpoints or convictions (as discussed also in the previous section). Thus, even if focus is on arguments for the relevance of theories used in relation to the topic of study, there can be a complex relationship between theories (including the more informal, implicit types) and research questions.

Instead of using a conceptual framework in the empirical study presented in this paper, what could the effect be of using a theoretical framework? That is, we decide to use one theory beforehand, and let this theory affect all aspects of the research process, in relation to a topic of study. In the following, a comparison is made between the use of the cognitive theory and the discourse analytic theory, when using them as part of a theoretical framework in relation to the same issue of classroom practice used before; to examine the types of messages about mathematics inherent in, or conveyed by, properties in teaching, including the limitation to focus on aspects of communication in teaching and epistemological aspects of messages about mathematics. Of course, this thought experiment is rather artificial, in particular since the described topic of study might not be equally relevant for both theories, but this limitation is seen as relevant in order to highlight some aspects of the potential use of these theories. Since I do not present any new empirical studies using these theories as part of theoretical frameworks, I limit the discussion to research questions and methodology.

Regarding the formulation of research questions in relation to the topic of study, it has already been noted that the questions given in the empirical study are relevant to both theories but that they were interpreted somewhat differently. Therefore, the arguments for studying these questions would be different when using the two theories and the specific wording of the questions might also be different. For example, the cognitive theory could formulate the questions more specifically about relationships between epistemological beliefs and properties of discourse, while the discourse analytic theory could formulate the questions more specifically about examining potential similarities and differences between different communities.

Regarding methodology, the discourse analytic theory focuses on discourse in itself in communities, and therefore focuses on the use of data from genuine situations in these communities, and not on creating “artificial” situations for data collection. The recording of lectures represents data of such genuine situations. The cognitive theory does not focus on discourse, but can be said to regard recordings as the method for being able to observe, indirectly, aspects of mental objects and processes. Therefore, for this theory there is openness for different types of data that could make it possible to give an, indirect, picture of the mental. To only rely on recordings of lectures might then also be seen as a limitation, since you do not know if it is epistemological beliefs that are the main cause of observed differences. Instead, the cognitive theory could utilize a study with two parts; something similar as has been done in the present empirical study together with a questionnaire or interview that in a more direct manner could examine epistemological beliefs. These two parts of data could then be related to each other in order to examine if/how epistemological beliefs affect the properties of discourse.

Thus, studies using the two theories as part of theoretical frameworks could have been quite different, although starting with the same topic of study. These differences could make it difficult to notice the potential similarities between the studies that have been noticed when starting with a conceptual framework and applying the two theories later in the research process. Therefore, one argument *against* using theoretical frameworks can be that potential similarities between studies focusing on the

same kind of phenomenon or topic become more hidden. However, you could also argue in similar manner *for* using theoretical frameworks. If several studies use the same theory, possibly examining somewhat different phenomena, this could simplify the process of relating results from these studies to each other and to see the results from one study in a broader perspective. For continued studies about the role of theory in research it then becomes important to be aware of these different aspects of the use of theory, in particular that it is central to not only compare different theories but to compare the uses of different theories in specific empirical studies.

Conclusions

In relation to the given purpose of this paper, some central results of the discussions are here summarized regarding four aspects of the use of theory in research: (1) The utilization of different types of frameworks, (2) The utilization of different types of theories, (3) Relationships between different parts of the research process and different parts of theories, and (4) Potential unification/integration of theories in relation to the specific research topic of the empirical study.

Regarding the different types of frameworks, they seem to have different pros and cons in relation to if/how we can relate results from different studies to each other in a fruitful manner. Theoretical frameworks can simplify the comparison of results from studies that have used the same theory while conceptual frameworks can simplify the comparison of studies using (partly) different theories.

Some more specific properties of conceptual frameworks have here been problemized, in particular regarding the primacy of research questions when choosing theories to use, where focus is on the relevance of the theories for a specific study. A complexity has been highlighted in the discussions of the issue regarding relationships between research questions and theories. In particular, different types of questions have been discussed, from more informal/intuitive interpretations of observations to more specified research questions, and that the development of research questions occur in a dual interaction with theories, where also different types of theories are active.

Thus, different types of theories have been discussed based on how they are used; some theories are of a more informal type that are perhaps most often not referred to explicitly as theories while other theories are explicitly referred to and used in a study. Regarding the more informal types of theories, authors label these types of theories in different ways, for example as approaches (Bergsten, 2008) or perspectives (Lester, 2005), which both are described as something not part of a framework. Other authors distinguish between background and foreground theories (Mason & Waywood, 1996). In this paper, the relationships between these types of theories have not been discussed in depth, but there is a need to create a more coherent structure of these types of theories, including utilization of explicit definitions of central notions such as theory and framework. Such a structure would be beneficial for the continued work on comparing the use of different (types of) theories in order to increase our awareness of the role of theory in research.

Important when comparing different theories is to not only focus on how they are used but also how they are chosen. The argumentation for choosing suitable theories is central for conceptual frameworks, which creates a greater potential to also make explicit how the theories are used. For theoretical frameworks, there might be a greater risk that focus is on stating what theory is used and neither on why or exactly how. However, regardless of the kind of framework used, it can never be taken for granted that all parts of a theory have affected all parts of the research process.

Several examples have been given showing that it is more fruitful and also necessary to discuss the use of theories in specific empirical studies and not only to compare the descriptions of different theories. The example in the present paper shows that there can be empirical studies that are very similar although based on theories that can be seen as very different, and partially contradictory. This way of comparing theories seems to become more common in mathematics education research, which is shown by several studies referred to in this paper.

Finally, I turn to the use of theories in relation to the specific topic of epistemological characterizations of mathematical discourse. Based on a conceptual framework, using “smaller” theories, the addition of two “larger” theories at the end of the research process did not add much to the empirical study. For each theory a type of explanation of the empirical results was added, and there was also a reinterpretation of the original purpose. These additions and changes were mostly a matter of wording and suggestions for further studies were similar for both theories and also similar to the suggestions given before the theories were introduced. The need to add any of these two theories is therefore challenged. However, when instead using these two theories as part of a theoretical framework, the empirical studies based on the two theories could have been more different, although then perhaps hiding potential similarities between the uses of these theories and possibly creating results of more limited usefulness. Thus, for continued studies of this topic it is suggested to continue using the conceptual framework and continue the critical examination of possible future needs to add other theories, of any size or type.

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Evaluating Mersenne Primes Using a Single Quadrant Expanding Square

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Abstract

By forming a table of sequential odd numbers using a single quadrant expanding square pattern it was observed that multiple Mersenne primes fall into the first column. We prove that the Mersenne primes which fall into the first column will be of the form $p \equiv 3 \pmod{4}$, where p is the Mersenne prime exponent. Focusing the search of primes to those which fall into the first column of this square may be a means to increase the speed at which large primes are discovered.

Keywords: Algorithms; Mersenne Primes; Primes

Introduction

A Mersenne prime is a prime of the form $2^p - 1$. For example, the first Mersenne primes are 2, 3, 7, 31, and 127 which corresponds to p values of 1, 2, 3, 5, and 7. Currently only 47 Mersenne primes have been discovered.¹ The first attempt to compile the primes was performed in the 17th century by the French scholar Marin Mersenne. The search for these primes intensified with the advent of digital computing. As with all primes, as the numbers become larger, the primes become increasingly more remote, making an exhaustive search labor intensive. The aim of this paper is to explore a potential means to limit the set of the eligible numbers to those most likely to be a Mersenne Prime. To this end, sequential odd numbers are arranged in the expanding square pattern described below.

Single Quadrant Expanding Square Pattern

This is defined as expansion of a square limited to one quadrant (Figure 1).

Figure 1 - Single Quadrant Expanding Square

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With odd numbers placed along the lines of expansion a limitless grid of numbers is formed. This will be referred to as an Odd Number Single Quadrant Expanding Square, ONSQES, and is demonstrated in Figure 2 (Mersenne primes are bold). The numbers in the first column will be referred to as First Column Odd Number Single Quadrant Expanding Square Integers, FCONSQUESI.

1	3	17	19	49	51	97	99
7	5	15	21	47	53	95	101
9	11	13	23	45	55	93	103
31	29	27	25	43	57	91	105
33	35	37	39	41	59	89	107
71	69	67	65	63	61	87	109
73	75	77	79	81	83	85	111
127	125	123	121	119	117	115	113

Figure 2 – 8 x 8 Odd Number Single Quadrant Expanding Square (ONSQES)

Theorem: *Mersenne primes which occur in the first column of the ONSQES will be of the form $p \equiv 3 \pmod{4}$, where p is the Mersenne Prime exponent.*

Proof: Given that the numbers of interest in the first column of the ONSQES follow the equation:

$R = 2^{\frac{p-1}{2}}$, where R is the row number of the ONSQES and p is the Mersenne prime exponent.

By applying the Theorem of Quadratic Residue and using Legendre symbols this equation becomes $\left(\frac{R}{p}\right) = \left(\frac{2^{\frac{p-1}{2}}}{p}\right)$. So when the statement $\frac{2^{\frac{p-1}{2}}}{p} = -1$ is true, the

statement $p \equiv 3 \pmod{4}$ will also be true. To that end, observe that the expression $\frac{p-1}{2}$ is at all times either even or odd.

If $\frac{p-1}{2}$ is even, then $\frac{2^{\frac{p-1}{2}}}{p} = \left(\frac{2}{p}\right)^{\frac{p-1}{2}} = 1$

If $\frac{p-1}{2}$ is odd, then $\frac{2^{\frac{p-1}{2}}}{p} = -1$.

Therefore, $p \equiv 3 \pmod{4}$

Yet for the case $\frac{2^{\frac{p-1}{2}}}{p} = -1$, the Law of Quadratic Reciprocity states:

$$\left(\frac{2}{p}\right)^{\frac{p-1}{2}} = \left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod{8} \\ -1 & \text{if } p \equiv \pm 5 \pmod{8} \end{cases}$$

Thus there now exists $p \equiv 3 \pmod{4}$ and $p \equiv \pm 5 \pmod{8}$.

This creates two possibilities:

Case 1: $p \equiv 3 \pmod{4}$ and $p \equiv -5 \pmod{8} \equiv 3 \pmod{8}$

$$p \equiv 3 \pmod{4} = p \equiv 3, 7 \pmod{8}$$

so $p \equiv 3 \pmod{4}$ becomes $p \equiv 3 \pmod{8}$

Case 2: $p \equiv 3 \pmod{4}$ and $p \equiv 5 \pmod{8} \equiv 5 \pmod{8}$

This cannot occur because

$$p \equiv 3 \pmod{4} \equiv 3, 7 \pmod{8} \text{ and then } p \not\equiv 5 \pmod{8}$$

Thus to achieve $\frac{2^{\frac{p-1}{2}}}{p} = -1$, p must be of the form $p \equiv 3 \pmod{4}$

Remarks

Fundamentally there are two ways to increase the speed of the search for Mersenne primes. One is through increased speed of assessment of the individual candidates, such as with faster processors or with more efficient verification of the individual candidates. The second is by decreasing the number of candidates to be considered. The theorem described above is meant to address the latter.

Figure 2 lists the p values of the Mersenne primes discovered at the time of writing.¹ In this table the primes which conform to $p \equiv 3 \pmod{4}$ are placed in bold. Only 19 of the 47 known primes follow this form. While not capable of capturing all of the Mersenne prime numbers, these primes are significant in that they arise from a much smaller set of integers.

2	61	2281	21701	859433	24036583
3	89	3217	23209	1257787	25964951
5	107	4253	44497	1398269	30402457
7	127	4423	86243	2976221	32582657
13	521	9689	110503	3021377	37156667
17	607	9941	132049	6972593	42643801
19	1279	11213	216091	13466917	43112609
31	2203	19937	756839	20996011	

Figure 2 - List of the p values for all known Mersenne Prime

A summary article by Schroeder discusses the distribution of Mersenne primes.² When the sequence of Mersenne primes are graphed as $\log_2 p$ the slope is 0.59. The same graphic can be applied to the set of FCONSQESI Mersenne primes which yields a slope of 1.28 (Figure 3). This increased rate of growth is compatible with the smaller number Mersenne primes that fall in the FCONSQESI set.

Figure3 – The set of FCONSQESI Mersenne primes compared to that of all Mersenne primes

To consider the discordance of the sizes of the set, consider the number of odd integers less than the FCONSQESI for a given row. From this FCONSQESI set, consider the set conforms to $R = 2^{\frac{p-1}{2}}$ with integer solutions for p. Figure 4 compares the sizes of these sets. In general $\{2^{\frac{p-1}{2}} \text{ FCONSQESI}\}$ is many orders of magnitude smaller than $\{\text{Odd}\}$. A PYTHON code for the FCONSQESI is described in the Appendix. Thus limiting the search for Mersenne primes to those which fall into the first column of the ONSQES may be a means to increase the rate at which very large prime numbers are found.

Figure4 – The size of the sets less than the FCONSQESI for a given row.

In conclusion, arranging integers as described by the ONSQES provides a novel method to analyze the distribution of Mersenne primes. Further more the finding that the Mersenne primes which fall in the first column take the form of $p \equiv 3 \pmod{4}$ is just one application of this technique to the search for Mersenne primes. While the overall utility of the approach remains to be determined, the future seems promising.

References

1. http://en.wikipedia.org/wiki/Mersenne_prime. Accessed May 22,2012.
2. Manfred R. Schroeder. "Where Is the Next Mersenne Prime Hiding?" The Mathematical Intelligencer. VOL. 5, NO. 3. p31-33.

Appendix

PYTHON code for FCONSQESI

```
#!/usr/bin/python
"""
This PYTHON program provides the numbers necessary to form the
sets describe in the above paper. A primality test can be
applied to them to verify which are primes. As written below,
this program describes rows 4 to 512.

A sample printout is:
row, Mersenne_prime, p-value
4 31 5.0
6 71 6.16992500144
8 127 7.0
10 199 7.64385618977
...

"""
import math
FCONSQESI_2_rows_up = 7
#since we will start on row 4, this is for row 2
print "row, Mersenne_prime, p_value"
for row in range(4,513,2):#calc even rows 4 to 512
    FCONSQESI=FCONSQESI_2_rows_up +2+((row // 2)*16)-10
    #use integral division above since row always even
    p_value = 1 + 2 * math.log(row) / math.log(2)
    print row, FCONSQESI, p_value
    FCONSQESI_two_rows_above = FCONSQESI
exit()
```