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Prospective Teachers' Interactive Visualization and Affect in Mathematical Problem-Solving

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Abstract: Research on technology-assisted teaching and learning has identified several families of factors that contribute to the effective integration of such tools. Focusing on one such family, affective factors, this article reports on a qualitative study of 30 prospective secondary school mathematics teachers designed to acquire insight into the affect associated with the visualization of geometric loci using GeoGebra. Affect as a representational system was the approach adopted to gain insight into how the use of dynamic geometry applications impacted students' affective pathways. The data suggests that affect is related to motivation through goals and self-concept. Basic instrumental knowledge and the application of modeling to generate interactive images, along with the use of analogical visualization, played a role in local affect and prospective teachers' use of visualization.

Keywords: problem-solving strategies, visual thinking, interactive learning, drawing, diagrams, teacher training, visual representations, reasoning, GeoGebra.

1. Experimental conditions and research questions addressed

At present, the predominant lines of research on problem-solving aim to identify underlying assumptions and critical issues, and raise questions about the acquisition of problem-solving strategies, metacognition, and beliefs and dispositions associated with problem-solvers' affect and development (Schoenfeld, 1992; Lester and Kehle, 2003). Problem-solving expertise is assumed to evolve multi-dimensionally (mathematically, metacognitively, affectively) and involve the holistic co-development of content, problem-solving strategies, higher-order thinking and affect, all to varying degrees (English & Sriraman, 2010). This expertise must, however, be set in a specific context.

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Future research should therefore address the question of how prospective teachers' expertise can be holistically developed.

The research described here was conducted with a group of 30 Spanish mathematics undergraduates. These future teachers took courses in advanced mathematics in differential and Riemannian geometry, but worked very little with the classical geometry they would later be teaching. They were accustomed to solving mathematical problems with specific software, mainly in areas such as symbolic calculation or dynamic geometry, but were not necessarily prepared to use these tools as future teachers. Research on teaching in technological contexts (Tapan, 2006) has shown that students are un- or ill-acquainted with mathematics teaching, i.e., they are unaware of how to convey mathematical notions in classroom environments and find it difficult to use software in learning situations. Hence the need to specifically include the classroom use of software in teacher training.

This paper addresses certain understudied areas in problem-solving such as visualization and affect, with a view to developing discipline awareness and integrating crucial elements for mathematics education in teacher training. As defined by Mason (1998), teachers' professional development is regarded here as development of attention and awareness. The teacher's role is to create conditions in which students' attention shifts to events and facts of which they were previously unaware. Viewed in those terms, teaching itself can be seen as a path for personal development.

The main aim of this essay is to explain that in a dynamic geometry environment, visualization is related to the viewer's affective state. The construction and use of imagery of any kind in mathematical problem-solving constitute a research challenge because of the difficulty of identifying these processes in the individual. The visual imagery used in mathematics is often personal in nature, related not only to conceptual knowledge and belief systems, but laden with affect (Goldin, 2000; Gómez-Chacón, 2000b; Presmeg, 1997). These very personal aspects are what may enhance or constrain mathematical problem-solving (Aspinwall, Shaw, and Presmeg, 1997; Presmeg, 1997), however, and as such should be analyzed, especially in technological contexts.

Gianquinto (2007) and Rodd (2010) contend that visualization is "epistemic and emotional". Giaquinto suggests that visual experience and imagining can trigger belief-forming dispositions leading to the acquisition of geometrical beliefs that constitute knowledge. According to Rodd (2010), the nature of belief-forming dispositions is not confined to perception, but incorporates the results of affect (or emotion-perception relationships). Hence, the belief-forming dispositions that underlie geometric visualization are affect-laden.

The present study on teaching geometric loci using GeoGebra forms part of a broader project involving the design, development and implementation of multimedia learning scenarios for mathematics students and teachers[2]. The solution of geometric locus problems using GeoGebra was chosen as the object of study because a review of the literature revealed that very little research has been conducted on teaching that aspect of geometry. A recent paper (Botana, 2002) on computational geometry reviewed current approaches to the generation of geometric loci with dynamic geometry systems and compared computerized algebraic systems to dynamic symbolic objects. However, it did not address the educational add-ons needed by teachers. Several authors have compared

[2] Complutense University of Madrid Research Vice-Presidency Projects PIMCD-UCM-463-2007, PIMCD-UCM-200-2009; and PIMCD-UCM-115-2010

the visual (and sometimes misleading) solutions generated by dynamic geometry systems to the exact solutions obtained using symbolic computational tools (Botana, Abánades and Escribano, 2011). The approximate solution problem affects all dynamic geometry systems, due to the numerical nature of the calculations performed. The GeoGebra team has been working on improving this feature as part of the GSoC[3] project. In the meantime, however, external tools must be used to obtain accurate solutions[4].

This article specifically explores the role of technological environments in the development of students' competence as geometricians and future teachers. More precisely, it focuses on the relationship between technology and visual thinking in problem-solving, seeking to build an understanding about the affect (emotions, values and beliefs) associated with visualization processes in geometric loci using GeoGebra. The questions posed are: how does affect impact visual thinking through dynamic geometry software (GeoGebra)? and how does interactive visualization impact affect in learning mathematics? The difficulties encountered in training students to build strategic knowledge for the classroom use of technology, which weaken personal problem-solving, are also explained.

The rest of the paper is organized as follows. A description of the scientific theory underlying the research is followed by a presentation of the training and research methodology used. A subsequent section discusses the results of all the analyses, including tentative answers to the questions formulated above. A final section addresses the preliminary conclusions of the study and suggestions for future research.

2. Theoretical considerations

[3] http://www.geogebra.org/trac/wiki/Gsoc2010

[4] http://nash.sip.ucm.es/LAD/LADucation4ggb/

Different theoretical approaches to the analysis of visualization and representation have been adopted in mathematics education research. In this study the analysis of the psychological (cognitive and affective) processes involved in working with (internal and external) representations when reasoning and solving problems requires a holistic definition of the term visualization. Arcavi's proposal (Arcavi, 2003: 217) has consequently been adopted: *"the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings"*.

Analysis of those two complementary elements, image typology and use of visualization, was conducted as per Presmeg (2006) and Guzmán (2002). In Presmeg's approach, images are described both as functional distinctions between types of imagery and as products (concrete imagery ("picture in the mind"), kinesthetic imagery, dynamic imagery, memory images of formula, pattern imagery). In Guzman they are categorized from the standpoint of conceptualization, the use of visualization as a reference and its role in mathematization, and the heuristic function of images in problem-solving (isomorphic visualization, homeomorphic visualization, analogical visualization and diagrammatic visualization[5]). This final category was the basis adopted in this paper for addressing the handling of tools in problem-solving and research and the precise

[5] *Isomorphic visualization:* the objects may correspond "exactly" to the representations. *Homeomorphic visualization:* inter-relationships among some of the elements afford an acceptable simulation of the relationships between abstract objects They serve as a guide for the imagination. *Analogical visualization:* the objects at hand are replaced by that are analogously inter-related. Modeling process. *Diagrammatic visualization:* mental objects and their inter-relationships in connection with aspects of interest are merely represented by diagrams that constitute a useful aid to thinking processes. (Guzmán, 2002).

distinction between the *iconic* and *heuristic function of images* (Duval, 1999; Souto and Gómez-Chacón, 2011) to analyze students' performance. The *heuristic function* was found to be related to *visual methods* (Presmeg, 1985) and cognitive aspects as part of visualization: the effect of basic knowledge, the processes involved in reasoning mediated by geometrical and spatial concepts and the role of imagery based on analogical visualization that connects two domains of experience and helps in the modeling process.

The reference framework used to study affective processes has been described by a number of authors (DeBellis and Goldin, 1997 & 2006; Goldin, 2000; Gómez-Chacón, 2000 and 2011), who suggest that local affect and meta-affect (affect about affect) are also intricately involved in mathematical thinking. Goldin (2000: 211) contends that affect has a representational function and that the affective pathway exchanges information with cognitive systems. According to Goldin, the potential for affective pathways are at least in part built into the individual. Both these claims were substantiated by the present data. For these reasons, while social and cultural conditions are discussed, the focus is on the individual and any local or global affect evinced in mathematical problem-solving in the classroom or by interviewees. This aspect of students' problem-solving was researched in terms of the model summarized in Figure 1 and used in prior studies (Goldin, 2000: 213; Gómez-Chacón, 2000b: 109-130; Presmeg and Banderas-Cañas, 2001: 292), but adapted to technological environments.

Affective pathway 1 (enabling problem-solving): curiosity →puzzlement→ bewilderment →encouragement→ pleasure →elation →satisfaction →global structures of affect (specific representational schemata, general self-concept structures, values and beliefs)

Affective pathway 2 (constraining or hindering problem-solving): curiosity → puzzlement → bewilderment → frustration → anxiety → fear/despair → global structures of affect (general self-concept structures, hate and rejection of mathematics and technology-aided mathematics)

Figure 1. Emotions and meta-affective aspects in problem-solving

This idealized model illustrates how local affect might influence the heuristic applied by a problem solver. This model was used in individual case studies because it provides insight into how visual processes, emotions and cognitive strategies interact. It also helps detect mental blocks and emotional instability where confusion and perceived threat are significant, generating high anxiety levels, and therefore conditioning visual thinking and attitudes. Here, emotions are not mere concomitants of cognition, but are intertwined with and inseparable from it. Most importantly, they are bound up with the individual's self-image and self-concept and the global affective dimension where purpose, beliefs and goals have a substantial impact.

3. Training and the research methodology used

The qualitative research methodology used consisted of observation during participation in student training and output analysis sessions as well as semi-structured interviews (video-recording). The procedure used in data collection was student problem-solving, along with two questionnaires: one on beliefs and emotions about visual reasoning and the other on emotions and technology (one was filled in at the beginning of the study and the other after each problem was solved). All screen and audio activity on the students' computers was recorded with CamStudio software, with which video-based information on problem-solving with GeoGebra could be generated. Consequently, at least four data sources were available for each student.

Six non-routine geometric locus problems were posed, to be solved using GeoGebra during the training session. Most of the problems were posed on an analytical register (Table 1: for a fuller description see Gómez-Chacón and Escribano, 2011).

Finding the solutions to the problems called for following a sequence of visualization, technical, deductive and analytical steps.

Table 1: Geometric locus problems

PROBLEM	
Problem 1: find the equation for the locus formed by the barycenter of a triangle ABC, where A = (0, 4), B = (4, 0) and C is a point on circle $x^2 + y^2 + 4x = 0$.	*Level: basic* *Geometric locus:* the wording of the problem determines the steps to be followed.
Problem 2: assume a variable line r that cuts through the origin O. Take point P to be the point where line r intersects with line Y=3. Draw line AP from point A = (3,0), and the line perpendicular to AP, s. Find the locus of the intersection points Q between lines r and s, when r is shifted.	*Level: medium* *Geometric locus:* in this problem, the difficulty is to correctly define a variable line. That done, the rest is fairly straightforward. The instructions for using GeoGebra are stated explicitly in the problem.
Problem 3: assume a triangle ABC and a point P. Project P on the sides of the triangle: Q1, Q2, Q3. Are Q1, Q2 and Q3 on the same line? Define the locus for points P when Q1, Q2 and Q3 are aligned.	*Level: medium – advanced* *Geometric locus:* the locus cannot be drawn with the "locus" tool in GeoGebra, because it is non-parametric. There is no mover point.
Problem 4: the top of a 5-meter ladder rests against a vertical wall, and the bottom on the ground. Define the locus generated by midpoint M of the ladder when it slips and falls to the ground. Define the locus for any other point on the ladder.	*Level: medium – advanced* *Geometric locus:* the problem does not give explicit instructions on the steps to follow. The situation is realistic and readily understood, but translation to GeoGebra is not obvious. An ancillary object is needed.
Problem 5: find the locus of points such that the ratio of their distances to points A = (2, -3) and B = (3, -2) is 5/3. Identify the geometric object formed.	*Level: Advanced* *Geometric locus:* the problem is simple using paper and pencil. The difficulty lies in expressing "distance" in GeoGebra.
Problem 6: find the equation for the locus of point P such that the sum of the distances to the axes equals the square of the distance to the origin. Identify the geometric object formed.	*Level: Advanced* *Geometric locus:* the problem is simple using paper and pencil. The difficulty lies in expressing "distance" in GeoGebra.

Geometric locus training was conducted in three two-hour sessions. Prior to the exercise, the students attended several sessions on how to use GeoGebra software, and were asked to solve problems involving geometric constructions.

In the two first sessions, the students were required to solve the problems individually in accordance with a proposed problem-solving procedure that included the steps involved, an explanation of the difficulties that might arise, and a comparison of paper and pencil and computer approaches to solving the problems. Students were also asked to describe and record their emotions, feelings and mental blocks when solving problems.

The third session was devoted to common approaches and the difficulties arising when endeavouring to solve the problems. A preliminary analysis of the results from the preceding sessions was available during this session.

The problem-solving results required a more thorough study of the subjects' cognitive and instrumental understanding of geometric loci. This was achieved with semi-structured interviews conducted with nine group volunteers. The interviews were divided into two parts: task-based questions about the problems, asking respondents to explain their methodologies and a series of questions designed to elicit emotions, visual and analytical reasoning, and visualization and instrumental difficulties.

A model questionnaire proposed by Di Martino and Zan (2003) was adapted for this study to identify subjects' belief systems regarding visualization and computers to study their global affect and determine whether the same belief can elicit different emotions from different individuals. In this study, students were asked to give their opinion of a belief and choose the emotion (like/ dislike) they associated with it, e.g.:

Table 2: Example of items of student questionnaire on beliefs and emotions

- Visual reasoning is central to mathematical problem solving. - Visual reasoning is not central to mathematical problem solving. Give reasons and examples. How do you feel about having to use problem representations or visual imagery? I like it. I don't like it. I'm indifferent. …..Explain the reasons for your feelings.

A second questionnaire, drawn up specifically for the present study, was completed at the end of each problem. The main questions were:

Table 3: Student questionnaire on the interaction between cognition and affect

Please answer the following questions after solving the problem: 1. Was this problem easy or difficult? Why? 2. What did you find most difficult? 3. Do you usually use drawings when you solve problems? When? 4. Were you able to visualize the problem without a drawing? 5. Describe your emotional reactions, your feelings and specify whether you got stuck when doing the problem with pencil and paper or with a computer. 6. If you had to describe the pathway of your emotional reactions to solving the problem, which of these routes describes you best? If you do not identify with either, please describe your own pathway. *Affective pathway 1* (enabling problem-solving): curiosity →puzzlement→ bewilderment →encouragement→ pleasure →elation →satisfaction →global structures of affect (specific representational schemata, general self-concept structures, values and beliefs). *Affective pathway 2* (constraining or hindering problem-solving): curiosity → puzzlement → bewilderment → frustration → anxiety → fear/despair → global structures of affect (general self-concept structures, hate or rejection of mathematics and technology-aided mathematics). 7. Now specify whether any of the aforementioned emotions were related to problem visualization or representation and the exact part of the problem concerned.

The protocols and interviewee data were analyzed for their relationship to affect as a representational system and the aspects described in section two.

4. Findings

The results shown here attempt to answer the concerns formulated in the introduction. The affective pathways reported for each problem consistently showed: a) the effect of subjects' beliefs and goals on the preference and use of visual thought/knowledge in computerized environments; b) that students proved to have a poor command of the tools, especially the locus tool; c) that notwithstanding, beliefs on the potential of GeoGebra helped them maintain productive affective pathways. As a qualitative study, the aim here was to describe the findings in detail. Consequently, the cases that best exemplified the results that were consistent across the entire group (30 students) and the nine volunteers were chosen and characterized by: gender, mathematical achievement, visual style, beliefs about computer learning, computer emotion, beliefs about visual thinking, feelings about visualization processes and global affect.

4.1. Beliefs about visual reasoning and emotion typologies

The data showed that all students believed that visual thinking is essential to solving mathematical problems. However, different emotions were associated with this belief. Initially, these emotions toward the object were: like (77%), dislike (10%), indifference (13%). The reasons given to justify these emotions were: a) pleasure in knowing that expertise can be attained (30% of the students)[6]; b) pleasure when progress is made in the schematization process and a smooth conceptual form is constructed (35%); c) pleasure and enjoyment afforded by the generation of in-depth learning and the control over that process (40%); d) pleasure and enjoyment associated with the entertaining and intuitive aspects of mathematical knowledge (20%); e) indifference about visualization (13%); f) dislike or displeasure when visualization is more cognitively demanding (10%).

[6] Some students cited several reasons.

A similar response was received when the beliefs explored related to the use of dynamic geometry software as an aid to understanding and visualizing the geometric locus idea. All the students claimed to find it useful and 80% expressed positive emotions based on its reliability, speedy execution and potential to develop their intuition and spatial vision. They added that the tool helped them surmount mental blocks and enhanced their confidence and motivation. As future teachers they stressed that GeoGebra could favour not only visual thinking, but help maintain a productive affective pathway. They indicated that working with the tool induced positive beliefs towards mathematics itself and their own capacity and willingness to engage in mathematics learning (self-concept as a mathematical learner).

4.2. Cognitive and instrumental difficulties: student's geometric constructions with GeoGebra

This section describes the solution typologies for the six problems.

Typology 1: static constructions (discrete treatment). In this typology, the students used GeoGebra as a glorified blackboard (Pea, 1985), but none of its dynamic features. They repeated the constructions for a number of points. To draw the geometric locus, they used the "5-point conic" tool. This underuse of potential appeared in problems 1 and 4.

Typology 2: incorrect definition of the construction. The students solved the problem (imprecisely), but with solutions that implied that the GeoGebra tools were unusable. The "locus" tool can only be used if the defining points are correctly determined (they may not be free points). Adopting this approach, at best the students could build a partially valid construction, but since the GeoGebra tools couldn't be used, no algebraic answer was obtained.

This typology appeared in problems 2 and 4. In problem 2, the sheaf of lines had to be defined by a point on an ancillary object such as a line, and not as a free point. Otherwise, the approximate visual solution obtained was unusable with GeoGebra. The students concerned were absolutely convinced that their solution was right and wholly unaware of any flaw in the solution.

The difficulty in problem 4 was to define a point that was not the mid-point. The locus tool could not be used for a free point on the ladder.

Typology 3: incorrect use of elements. For example, in problems 1, 2, 4 and 6, some students used the "slider" tool to move the "mover point". They realized that the "mover point" had to be controlled, which is what the slider is for. In GeoGebra, however, the slider is a scalar and can't be used with the locus tool.[7]

Problem 2 is a case in point. Some students defined the sheaf of lines as the lines passing through the origin on a point in the circle, and this point in the circle was moved with the slider. For example, student 9 said: "This problem is similar to the one before it. I built the construction while reading the problem. The hardest step was to construct the variable line. First, I thought I'd use a slider for the slope of the line passing through the origin, but that way I never got a vertical line, so I used the slider as in the preceding problem to build point C that revolves around the origin, and then to build the line connecting C and O. After that, I just followed the instructions in the problem, and I was very careful about the way I named the elements" (student 9, problem 2).

[7] http://www.geogebra.org/help/docues/topics/746.html

Typology 4: failure to use the locus tool. Here, the construction was correct, but the student did not use the locus tool. To use it, the point that projects the locus (tracer) must be distinguished from the point that moves the construction (mover). The mover must be a point on an object. Some students were apparently unable to make that distinction, which prevented them from using the tool correctly.

This misunderstanding arose in problems 1, 2, 3 and 4. Student 8 exemplifies this type of reasoning: “The first thing I had to do was find the center and radius of the circle to draw, to complete the square in the equation: $(x+2)^2 + y^2 = 4$. Therefore, point C is in a circle with a center at (-2, 0) and a radius of 2. (I didn’t actually need this because in GeoGebra I could enter the equation directly and draw the circle). Now, to solve the problem I had to know what a barycenter was. I took point C on the circle (creating an angular slider so the point would run along the entire circumference of the circle) and drew the triangle ABC. I calculated the triangle barycenter (I drew the medians as dashed green lines to make it easier to see that G is the barycenter). Using animation to project point G gave me the locus. Since the locus was a circle, I was able to solve the equation by finding three points, G1, G2, G3, and activating the “circle through three points” tool. Then I entered the data in GeoGebra: (x-0.66) 2 + y-1.34) 2 = 0.44" (student 8, problem 1).

4.3. Maintaining productive affective pathways

As noted in the preceding paragraph, the belief that visual thinking is essential to problem-solving and that dynamic geometry systems constitute a visualization aid, particularly in geometric locus studies, was widely extended across the study group. That belief enabled students to maintain a positive self-concept as mathematics learners in a technological context and to follow positive affective pathways with respect to each problem, despite their negative feelings at certain stages along the way and their initial lack of interest in and motivation for computer-aided mathematics.

A comparison of the affective pathways reported by the students revealed: a) concurrence between the use of visualization typologies and associated emotion; b) that the availability of and subsequent decision to use GeoGebra was often instrumental in maintaining a productive affective pathway. This section addresses three examples, in two of which the affective pathway remained productive and one in which it did not. It discusses the determinants for positive global affect and positive self-concept as mathematical learners. The key characteristics of the case studies are given in Table 4.

Table 4: Three case studies: characteristics

Case	**Gender**	**Mathematical achievement**	**Visual style**	**Beliefs about computer learning**	**Feelings about computers**	**Beliefs about visual thinking**	**Feelings about visualization**	**Global affect**
Student 19	Male	High	Visualizing student	Positive	Likes	Positive	Likes	Positive self-concept
Student 20	Female	Average	Non-visualizing student	Positive	Dislikes	Positive	Dislikes	Positive self-concept
Student 6	Female	Low	Style not clear	Positive	Dislikes	Positive	Likes	Negative self-concept

Problem 4 (Table 1) was chosen for this analysis. The students' affective pathways for this problem are given in Table 5.

Student 19 is a visualizer. In the interview he said that the pleasure he derives from visualization is closely associated with the mathematics view. He regards visual reasoning as essential to problem-solving to monitor and generate in-depth learning, to contribute to the intuitive dimension of knowledge and to form mental images.

When he was asked whether his feelings were related to visualization and problem-solving and to specify the parts of the problem where they were, he replied: "curiosity predominated in visualization. Since the problem was interesting and seemed to be different from the usual conic problems, I was keen on finding the solution. I had a major mental block when it came to representing the problem and later, as I sought a strategy. I was unable to define a good strategy to find the answer. I was puzzled long enough to leave the problem unsolved and try again later. When I visualized the problem in a different way, I found a strategy: construct a circle with radius 5 to represent the ladder and another smaller circle to represent the point in question. When I reached that stage, I felt confident, happy and satisfied" (student 19).

Student 20 is a non-visualizing thinker with positive beliefs about the importance of visual reasoning. However, she claimed that her preference for visualization depends on the problem and that she normally found visualization difficult. It was easier for her to visualize "real life" than more theoretical problems (the difference between problems 4 and 5, for instance).

Her motivation and emotional reactions to the use of computers were not positive, although she claimed to have discovered the advantages of GeoGebra and found its environment friendly. She also found that working with GeoGebra afforded greater assurance than manual problem-solving because the solution is dynamically visible. Convincing trainees such as student 20 that mathematical learning is important to teaching their future high school students helps them keep a positive self-concept, even if they don't always feel confident in problem-solving situations (Table 5).

Student 6's visual thinking style could not be clearly identified. She expressed a belief in the importance of positive visual reasoning ("because visual reasoning helps gain a better understanding of the problem and consequently the solution"). This confirmed a liking for visualization and representation because it made it easier to understand the problem and she found formalization helpful. She added, however, that she felt insecure applying technological software to mathematics, although she believed GeoGebra, specifically, to be useful. In her own words, "I don't like it and never will. I feel a little nervous and insecure, not because of GeoGebra but because computers intimidate me because I don't understand them completely. But when I managed to represent the problem with GeoGebra, I felt more satisfied with the result than when I solved it with paper and pencil". Although student-6's pathway was essentially negative in problem 4, she persisted until she found the solution. In some cases students were unaware of their mistakes and misunderstandings, however.

GeoGebra can be used to solve problem 4, although an average student cannot be expected to build the entire construction from scratch. The visual and instrumental challenge is to deploy the sliding segment, and that calls for an auxiliary circle (which may be concealed to simulate the effect of the ladder). The point in the ladder must be chosen with care to use the locus tool. Just any "point in segment" will not do; the "middle point" tool or a more sophisticated construction must be used.

While none of the three students applied the "locus" command, student 19 used the visual power of the technology to gain a better mathematical understanding of the problem. That inspired a change in context which facilitated notion and property applications. He used GeoGebra as a genuine mathematical modeling tool. He did not solve the problem with the geometric locus command, however, even though he came up with the right answer by modeling. A comparison of this student's pathways in the six problems revealed that the interaction between visual reasoning and negative feelings arose around the identification of interactive representation strategies and the formulation of certain representations in which the identification of parametric variations plays a role. This student's command of the use of concrete, kinesthetic and analogical images was very helpful and contributed to his global affect and his positive overall self-concept when engaging in computer-aided mathematics.

An analysis of the relationship between these three students' affective pathways (Table 5) and their cognitive visualization shows that visualization - negative feelings interactions stem essentially from students' lack of familiarity with the tools and want of resources in their search for computer-transferable analogical images and their switch from a paper and pencil to a computer environment in their interpretation of the mathematical object.

Behavior such as exhibited by student 6 denotes a need to include construction with locus tools in teacher training. Although no general methodology is in place, any geometric problem that aims to determine locus must be carefully analyzed. This calls for identifying three categories of geometric elements in such problems: fixed (position, length, dimension); mobile (position, length, variable points); and constant (length, dimension).

The data also revealed the relationship between beliefs, goals and emotional pathways. The analysis of student 20's responses showed that while she had no inclination to use computers, the importance she attached to mathematics and IT in specific objectives and the structuring of her overall objective kept her on a productive affective pathway (McCulloch, 2011). Student 20's solution to problem 5 (Table 1), for instance, constitutes a good example of a productive pathway: despite negative feelings:, she maintained a positive mathematical self-concept, which she reported when she explained her global affect. (Her self-reported pathway in problem 5 was: curiosity →confusion /frustration → desperation → puzzlement → satisfaction → a negative mathematical self-concept in terms of technology for problem 5, but a positive global affect regarding computer use in solving the six problems). Questions designed to elicit the reasons for her positive mathematical self-concept in terms of technology showed that objectives, purposes and beliefs were clearly interrelated. Her own words were: "I think that computers, not only the GeoGebra program, are an excellent tool for anyone studying mathematics. Nowadays, the two are closely linked: everyone who studies mathematics needs a computer at some point… mathematics is linked to computers and specifically to software like GeoGebra (if you want to teach high school mathematics, for instance. I at least am trying to learn more to be a math teacher) (student 20)".

Table 5: Affective pathways and visual cognitive processes reported for this problem by three students

Problem 4	**COGNITIVE-EMOTIONAL PROCESS**	
Student 19 Own pathway	Curiosity	Reading and understanding problem
	Confusion	Drawing (patterns and lines/figure) Analytical
	Puzzlement. Mental block	(Search for mental image) (specific figure/illustration and dynamic image)
	Confidence	Search for mental image
	Perseverance-motivation	Search for mental image
	Excitement and hope	Physical manipulation - kinetics Kinesthetic learning Mental image Identification mathematical object
	Confidence	Technological manipulation with the computer Representing circle radius (specific illustrations)
	Confidence, joy	Interactive image generation, slider (analogical)
	Joy and happiness	Interactive image generation, slider (analogical)
	Perceived beauty	Specific illustration with interactivity (analogical)
	Satisfaction	Analytical-visual Memorized formulaic typology
	GLOBAL AFFECT	Positive self-concept
Student 20 Own pathway	Curiosity	Problem reading
	Frustration	Global visualization of problem Pictorial image
	Confusion	Search for mental image Inability to visualize the ladder as the radius of a circle
	Puzzlement	Search for mental image Dynamic and interactive image with GeoGebra
	Stimulus, motivation	Technological manipulation with the computer Pictorial representation with GeoGebra
	Satisfaction	Pictorial representation with “trace on” GeoGebra Full construction from scratch Come up with a final solution
	GLOBAL AFFECT	Positive self-concept
Student 6 Pathway-2	Curiosity	Problem reading
	Puzzlement	Global visualization of problem Pictorial image
	Bewildermen	Search doe an instrumental image with GeoGebra
	Frustration	Computer handling skills
	Anxiety	Inability to visualize the ladder as the radius of a circle and using “trace on”
	Fear/despair	Needing help to find the solution
	GLOBAL AFFECT	Negative self-concept

Conclusion, limitations and further research

The results of this study suggest that various factors are present in conjunction with visual thinking. The first appears to be the study group's belief that visual thinking and their goal to become teachers would be furthered by working with technology (Cobb, 1986). The data shows that all the student teachers believed that visual thinking is essential to solving mathematical problems. That finding runs counter to other studies on visualization and mathematical ability, which reported a reluctance to visualize (e.g., Eisenberg, 1994). However, different emotions were associated with this belief. The belief about using computers and that software is a tool that contributes to overcoming negative feelings has an impact on motivated behavior and enhances a positive self-concept as a mathematical learner. Despite this advantage, however, student teachers may still misunderstand or misinterpret and therefore misuse computer information, unknowingly in some cases, and surrender all authority to the computer.

While prospective teachers resort to GeoGebra software to help maintain a productive affective pathway and foster visual thinking, student 20's experience with problem 5 is significant, for it shows that the tool by itself is not enough. If the software is unable to deliver the dynamic geometric capability that students want to use for the concepts at hand, it is useless and may even have an adverse impact on their affective pathway, possibly resulting in feelings of defeat such as reported by student 20. Her experience provides further evidence of the importance and complexity of mathematics teacher training, as documented by researchers studying the issue from an instrumental approach (e.g., Artigue, 2002). The mere provision of tools cannot be expected to necessarily raise the frequency of productive affective pathways. Rather, thought needs to be given to how those tools are integrated into classrooms to support the development of visualization skills. Some students (as in item 4.2) think of graphs as a photographic image of a situation due to a primarily static understanding of functional dependence. That might be attributed to the fact that the pointwise view of mathematical objects tends to prevail in the classroom, where the dynamic view is underrepresented (institutional dimension of visualization).

The results of this study bring to mind the *progressive modelling in visual thinking* notion introduced by Rivera (Rivera, 2011: 270). Furthering visualization processes in teaching involves more than just drawing "pretty pictures": it requires sequenced progression of the thought process. This in turn calls for awareness of the transition in dimensional modelling phases from the iconic to the symbolic and the change of mindset. For the problem proposed, "geometric locus", each transition can be associated with mathematical explanations and symbol notation and the proficient use of the visual tool to reify the mathematical concept. Therefore, one question that would be open for research is the definition of the components of an overarching theory of visualization for problem-solving in technological environments where this progression is explicit. While this study was conducted in a classroom context, it focuses on the individual only, not on interaction among individuals. Future studies might profitably explore the role of external affect and others' (i.e., teachers', community's, institution's) external affective representations. Such interaction impacts meta-affect and may potentially either help maintain or interrupt productive affective pathways.

Finally, as explained in the introduction, the teacher training model pursues the development of students' awareness and ability to apply their knowledge in complex contexts, integrating knowledge with their own attitudes and values and therefore developing their personal and professional behavior. From this standpoint, teacher training programs should adopt a more holistic approach (cognitive, didactic, technical and affective). The present paper aims to provide a preliminary framework to help teacher educators or mathematical cognitive tool designers select and analyze interaction techniques. A secondary aim is to encourage the design of more innovative interactive mathematical tools.

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