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Stephen Hegedus

# Young Children Investigating Advanced Mathematical Concepts With Haptic Technologies: Future Design Perspectives

*Stephen Hegedus*[1]
*Kaput Center for Research and Innovation in STEM Education*
*University of Massachusetts Dartmouth*

## 1. Introduction

In this chapter, we focus on how new technologies can be used with young children to investigate mathematical ideas and concepts that would normally be introduced at a later age. In particular, we focus on haptic technologies that allow learners to touch and feel objects through force feedback in addition to visual images on a screen. The main purpose of this paper is to describe how these technologies can be used to enable young learners to construct meaning about geometric shapes and surfaces as well as attributes of particular mathematical constructions in multiple dimensions (particularly 2D and 3D for purposes of this chapter). Such learning environments enable various forms of mediation both through the devices and software used as well as socially, as students work together to develop meaning and create models of complex ideas.

We begin by describing how and why young learners in particular should be working in such learning environments in order to provide a rationale for our work. In Section 2, we provide some background on how these technologies have evolved and their use in other disciplines and how we have built on prior research in the use of dynamic geometry in mathematics education. Section 3 presents how relevant these new learning environments can be with some specific examples from preliminary work at the Kaput Center. The section also contains some theoretical reflections on how we can

[1] shegedus@UMassD.Edu

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begin to analyze and understand how students work and construct meaning in such environments. Section 4 then concludes by offering some design principles for future research and development.

## How Should Young Children be Doing Mathematical Problem-Solving in the Future?

We believe that the answer to this question lies in three areas that focus on the early introduction of mathematical ideas, the use of technology, and engagement.

**Early introduction**. Several researchers have promoted the idea of introducing mathematical ideas earlier in the curriculum and even introducing the foundation of advanced mathematical thinking in the early grades (Kaput, Carraher & Blanton, 2008; Kaput, 1994). If not then, many children will never be exposed to important mathematics and engage in fruitful and relevant investigations. This can have detrimental effects throughout a child's educational career, reducing their desire to want to learn mathematics because of its lack of relevance or inaccessible representations.

**Technology use**. Technology is often not a major part of elementary school classroom teaching due to a lack of resources and perception of its role and use. The predominant form of technology use in most elementary school classrooms in the U.S. is PowerPoint presentations. Some researchers (Carraher & Schliemann, 2000) believe that the introduction of technology is not enough:

> It is important to provide a social analysis in consonance with a cognitive one. Because technology does not act directly on learners, but only exerts an influence on the social activities and contexts in which it is employed, introducing technology into the mathematics classroom ultimately entails questions such as the following: What is the teacher's role; what are the students trying to achieve in the tasks … . (p. 174)

While we agree that these questions are important, new technologies can have a more participatory and collaborative role rather than be a prosthetic device to prop up existing pedagogical practices. New technologies can actually re-structure interaction in the classroom and allow the introduction of advanced mathematical ideas through radically new mathematical representation systems. The interactions of teachers, students and technologies within a learning environment can modify and transform activity structures (Jonassen, 2000).

Technological affordances can also be mathematical affordances providing a symbiotic link between how mathematical activity can occur. Mathematizing technological affordances is an important step and one we discuss in detail later.

**Engagement**. By integrating activity structures with the affordances of new technologies, the learning environment should be simple enough to establish engagement—to motivate curious young minds to explore, question, and be encouraged to want to continue to learn. It should allow them to construct meaning in open-ended tasks, which have been carefully designed to have mathematical purpose. It should allow them to share, collaborate, and feel free to use non-scholastic language as they conduct their mathematical investigation.

We take a very broad view of what is mathematical problem-solving viewing it as an enterprise of collaborative investigation where multiple approaches are valid. It is not just about solving a specific problem, which has a specific answer or application into the real world, but rather it is an investigation that might have multiple approaches and where students can make multiple observations. Also, most of our activities might best be described as “tasks” rather than “problems.”—that is, they are goal directed activities.

Students are seldom at a loss for ideas to pursue. They are not stuck; they are not frustrated; and, their progress often does not fit the metaphor of moving along a single path that is somehow temporarily blocked. Instead, our environments are carefully engineered so that students can make parallel progress along a variety of interacting paths. Our initial tasks involve exploring, categorizing attributes of geometric shapes or objects, making sense of a set of objects and constructing broad and specific meaning. These tasks, in a broad sense, could be described as modeling (Lesh, 2007). We will continue to use the phrase mathematical problem-solving throughout this chapter but in the spirit of the position described above.

We have referred to new technologies, but we focus on a particular type of learning environment that utilizes haptic or multi-modal devices. Multiple modalities are used in real-world applications. We make sense of problem conditions in the world by using sight, touch, and hearing to name a few. Hence, in our research and development, we have focused on new technologies that use multiple modes of input in early mathematics classrooms. First, let us describe the evolution of such technologies in contrast to the predominance of visualization software in mathematics education.

## 2. Background to New Technology

Haptic literally means “ability to touch” or “ability to lay hold of” (Revesz, 1950) and has evolved to be an interface for users to virtually touch, push, or manipulate objects created and/or displayed in a visual environment (McLaughlin, Hespanha, & Sukhatme, 2002). Recently, this has rapidly evolved to include multi-touch environments. In these environments, learners literally lay their hands on objects via a screen interface, mathematical objects can be manipulated and resultant actions be investigated. Let us first examine the background of educational technology involved in dynamic visual mathematics before extending to haptic technologies which is the focus of this chapter.

Traditionally, dynamic, interactive, mathematical, visual environments—including Computer Algebra Systems such as Mathematica and Maple, as well as multi-dimensional Graphing software such as Avitzur's Graphing Calculator—are used to aid students to visualize complex surfaces in various coordinate systems and complete computationally intensive tasks. The Geometer's Sketchpad® is used in classrooms ranging from elementary grades through to undergraduate programs to allow users to construct, interact and explore geometric figures and shapes, and so engage in model-eliciting activities in various mathematical topics. But these environments are not responsive to users' physical interactions apart from mouse pointing.

The experience of visual mathematics, particularly three-dimensional mathematics, is often very brief for U.S. mathematics and science students. Following a school curriculum of Euclidean Geometry rarely expanding to non-Euclidean geometry or solid geometry, there is a rapid progression in most university curriculum from three-dimensional geometry, which is embedded in third or fourth semester Calculus courses, to the abstract intangibles of higher dimensional mathematics. In fact, given the very nature of multi-dimensional mathematics—that it can examine real life objects and phenomena all around us—it is interesting that such a small proportion of a student's formal mathematical life is spent examining the subject. Such mathematics provides a vocabulary for understanding fundamental modeling equations, for example, weather, heat, planetary motion, waves, and later, multi-dimensional mathematics, finance, epidemiology, quantum mechanics, bioinformatics and many more. Yet, there is a growing emergence of technologies in the scientific workplace that apply, manipulate, and model three-dimensional representations

A wide range of technologies are used in the teaching and learning of multi-dimensional mathematics in various contexts, ranging from relatively expensive Computer Algebra Systems (CAS) such as Mathematica™ and Maple™ (Meel, 1998; Park & Travers, 1996), industrial design packages, (e.g., AutoCAD™), through to Java Applets freely downloadable from the WWW. During reform periods, Mathematics and Science departments have been encouraged to integrate CAS technology into their classes as it can help students with visual and conceptual problems (Zorn, 1987; 1992). As technology becomes more sophisticated, the opportunity cost of training time and money spent on learning how to use a particular software and how to successfully integrate it into school curriculum is sufficiently high to dissuade teachers from the investment.

Dynamic geometry environments offer point-and-click tools to construct geometric objects that can be selected and dragged by mouse movements. All user-defined mathematical relationships are preserved, thus providing environments for students to conjecture and generalize by clicking and dragging hotspots on the object. These hotspots dynamically re-draw and update information on the screen as the user drags the mouse, and in doing so, efficiently testing large iterations of the mathematical construction (Moreno & Sriraman, 2005; Moreno & Hegedus, 2009; Moreno, Hegedus & Kaput, 2008).

Such environments aim to develop spatial sense and geometric reasoning by allowing geometric conjectures to be tested, offering “intelligent” constructivist tools that constrain users to select, construct or manipulate objects that obey mathematical rules (Mariotti, 2003)— that are largely used in secondary and not primary schools.

In summary, these dynamic mathematic environments are responsive to users' interaction but are still more structured in their feedback and lack the expressive capabilities of using physical interaction and force-feedback.

We believe that students naturally need more haptic, kinesthetic avenues through such the combination of dynamic visual environments and haptic technologies to explore the mathematics of change and variation in a more sensory environment to connect to the symbolic formalisms of the mathematical ideas (Nemirovsky & Borba, 2003). Change and variation occurs in multiple school subjects, in particular algebra, geometry and data analysis. In allowing students the combined affordances of multi-touch interaction, visual feedback and force feedback where possible, the technological environment can become a semiotic mediator of mathematical thinking and investigation. Young learners can have access to new forms of mathematical problem-solving or investigation through direct manipulation of mathematical objects linked to varying attributes (e.g. area).

To this aim, we have focused on integrating two types of haptic technologies: (1) Sensable's PHANTOM Omni—a force-feedback device and (2) iPad with a dynamic geometry application—a multi-touch/multi-input device.

Sensable's PHANTOM Omni® (http://www.sensable.com/haptic-phantom-omni.htm)—hereon referred to as Omni—is a desktop haptic device with six degrees of freedom for input (x, y, z, pitch, roll, yaw), and three degrees of output (x, y, z). The Omni's most typical operation is via a stylus-like attachment that includes two buttons (see Figure 1a). The Omni has a very robust community and SDK behind it. The SDK

(OpenHaptics-Academic Edition) allows for two levels of programmatic control: precise programmer created feedback—such as vibration—and pre-programmed feedback—such as springs and dynamic/static friction. The Omni provides up to 3 forces of feedback for x, y, and z. It is primarily used in research, with a significant presence in dentistry and medicine but growing in mathematics education (Hegedus & Moreno, 2011).

Figure 1b is an example of how the Omni can be used with a graphical environment. Users can click, push, or drag a "bug" across a surface using the haptic device. Here, the bug is being pushed around a saddle surface.  A saddle surface has a maximum and a minimum at the same place (the place where you would sit on the saddle). A sample activity would be to ask the student to push the bug to that place on the surface and describe how the surface changes during its motion, both in terms of the visualization but more importantly in terms of its feel. Mathematically speaking this potentially enables an interpretation of differentiability of the surface. This approach can offer new access to profoundly important mathematical ideas for younger children.

*Figure 1a.* Sensable's Omni Haptic Device.

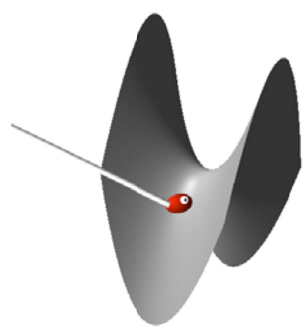

*Figure 1b.* Graphical interface prototype.

In addition to shape, feedback from the Omni can be linked to particular mathematical attributes, particularly varying quantities. Figure 2 highlights a triangle whose shape can be dynamically transformed by moving the vertices (A, B, or C) or by moving the parallel lines apart (via D). Here the feedback can be linked to the area of the triangle and children can investigate what properties impact the area before being introduced to a formal equation.

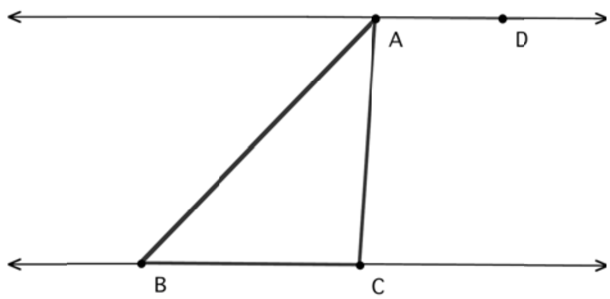


*Figure 2*. Area of a triangle.

It should also be noted that many mass-market whiteboards, tablet PCs and gesture-based PDAs (e.g., iPod Touch/iPhone) and larger forms such as the iPad now feature pressure-sensitive styli of some form; this pressure-input becomes a limited form of force-feedback in software that "responds" differentially to variable pressure. At this stage, feedback is visual, and does not include the additional sensory information of forces or kinesthetics that are directly linked to the properties of the objects being manipulated on a screen. But the growing spread of these devices and their use with educational applications creates an entry point for the use of peripheral computational devices that extend the use of mouse as a physical pointing action, to an environment where users can touch, push or move objects more directly.

In multi-touch/multi-input platforms such as the iPad, the learner can use multiple modes of input and outputs— their natural modes of seeing and feeling, to make sense of the task. The iPad offers a direct (almost zero-interface) mode to touch and directly manipulate mathematical objects, and offer multiple inputs to one mathematical object hitherto impossible on a single-input computer (mouse as pointer and selector).

With both hardware, the technological affordances are tightly coupled with mathematical affordances in such that the technology offers mathematical meaningful tools or avenues to investigate. Hence a new form of mathematical problem-solving originates because of new mathematization routes. We will exemplify these affordances in the next section. In terms of software deployment, key representational features include high-resolution visualization of mathematical objects and constructions which can be made transparent to see their interaction with other objects, and direct manipulation of objects allowing users to rotate and navigate “around” objects and flexible notation systems to allow users to observe outputs (e.g., changing area of a shape) based upon their input.

We now describe how such environments are relevant to mathematics education and offer examples of how they can advance mathematical investigations and inquiry.

## 3. Relevance: Future Mathematical Problem-Solving

Situations inside and outside of formal learning environments involve visualization, multi-modal investigations, using and interpreting multiple representations, connecting mathematical attributions and concepts to real world phenomenon, e.g., form, shape of objects and models (visual surfaces), features and attributes. So what is modeling in a problem-solving context for early learners? And is it relevant or necessary for early learners to be introduced to such ideas? We think it is and it goes deep into what a mathematical problem-solving environment is for a life-long learner. In addition, making sense of the environment in a mathematical way is not just physical or visible (i.e., tangible) but also occurs at the nano-level. Macro images, surfaces, and objects can simulate phenomena, which cannot be seen or felt, e.g., cell structures. Census datasets cannot be understood at a macro level without a deep understanding of the micro. What constitutes a dataset? In a similar vein, what constitutes variation at all across mathematical models? The heart of the research reported here is to establish conditions by which early learners can advance their mathematical inquiry at stages that are hitherto not required by standardized frameworks but are still complimentary. Our primary research question is: Can we establish learning environments by which advanced mathematical ideas can be more readily accessed, understood and used to solve problems? Such understanding is mediated through the affordances of technological devices and at the same time social interaction between peers. Young children can construct meaning through collaboration (one form of mediation) but supported and additionally mediated through the tools afforded to them through mathematically-enhanced technologies.

Holland et al. (2004) outlines the role of a mediating device:

> A typical mediating device is constructed by the assigning of meaning to an object or a behavior. This symbolic object or behavior is then placed in the environment so as to affect mental actions. (p.36)

In our design of a learning environment integrating new technologies in mathematically relevant ways, we adhere to a socio-cultural perspective of learning and analyze the interaction of the students in terms of mathematically-relevant discourse as mediated by the various tools and supports available to them. The affordances of the technological environment are cultural devices. Children can modify the environment to make sense of the attributes of the geometric objects and configurations through investigation and interaction with each other. Vygotsky (1980) explains how activity structures the social environment of interaction and the very behavioral routines of members of that environment. We adhere to that position in our design and observe the technological devices to not be the only mediating device in the learning environment but the interaction between the children as meaning-making becomes a collaborative enterprise. Both are forms of semiotic mediation and result from co-action (Moreno & Hegedus, 2009) between the various participants. The children guide the discussion by interacting with visuals on a screen, receiving visual and haptic feedback loops, which are iteratively discussed and compared within the group and as such the technology reciprocally guides the resulting investigation, decisions in how to further interact, and conjectures or refutations from the resulting actions. Such embodied actions of pointing, clicking, grabbing and dragging parts of the geometric construction also allows a semiotic mediation (Falcade, Laborde & Mariotti, 2007; Kozulin, 1990; Mariotti, 2000; Pea, 1993) between the object and the user who is trying to make sense of, or induce some particular attribute of the diagram or prove some theorem.

Based upon this theoretical perspective, we present two different technologies from our preliminary research and development at the Kaput Center. These have been field tested in informal and formal learning settings. This preliminary work was conducted with 4th graders in a high achieving elementary school in Massachusetts.

**Omni Force-Feedback Device**

In the Omni environment, we developed an exploration activity using solids and a plane to explore how these objects interact—in particular, what different types of planar intersections can be constructed. Our environment includes crisp visuals of these objects, which can be navigated by dragging and moving the stylus on the Omni so that different views of the objects could be explored. Through iterative design, we found that certain colors and use of transparency helped the young learners focus their attention and interpretation on the interaction and their reference to certain attributes. In addition, we combined the haptic affordances of the Omni to add additional feedback to the investigation. We found that magnetism was an important design principle to further aid the learners to focus their attention and aid their discovery. In magnetizing the surfaces, the children could lock onto the intersection of the two shapes and consider what they felt in conjunction with what they saw. Two examples are shown in Figures 3a and 3b. The first shows the planar intersection of a cube, which can result in a set of intersections from a point (plane resting on a vertex), a line (plane on an edge), and 3-gon to 6-gons. The second illustrates the planar intersection of a square based pyramid, which can result in a similar set of intersections up to a 5-gon. Children, in groups of 4 with one device, mainly explored a variety of triangles, quadrilaterals and pentagons. Such an activity is challenging to undergraduates and the children had no prior experience with such an investigation, but we discovered that their engagement in discovering various types of intersection was immediate and endured for almost an hour. They did have prior experience with 2D geometric shapes such as 3-gons to 5-gons but had only a basic knowledge of the attributes of these shapes. For example, they did not classify 4-gons as quadrilaterals but squares and rectangles. They did know how many sides each shape should have which gave rise to interesting discussions as they explored what they saw and how it contrasted with what they felt. In one investigation, the children thought they saw a pentagon, but on tracing around the magnetized shapes they felt a 4-sided shape (by counting edges) and concluded it was trapezoid through group discussion. This illustrates a classic issue of cross-modality where our vision and touch can be in conflict. The pseudo-3D representation on a flat screen is not sufficient, even with dynamic interaction tools such as rotating and navigating the objects—more feedback is necessary for young learners to make sense of certain specific mathematical attributes of the overall geometric configuration. More work is needed in establishing activity structures that help students make mathematical classifications of varying shapes. For example, can force feedback help develop a sense of angle measure (acute, right, obtuse) in classifying all types of triangles?

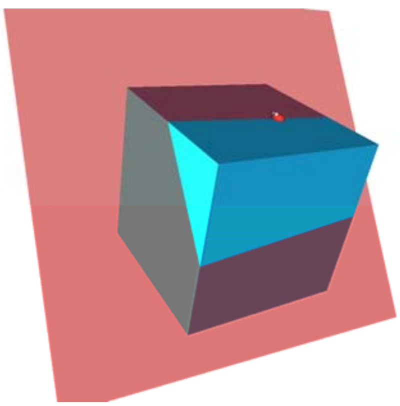

*Figure 3a.* Planar intersection of a cube.

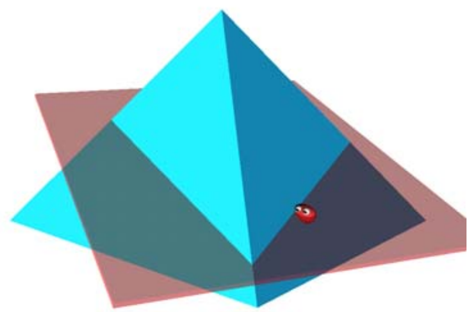

*Figure 3b.* Planar intersection of a square-based pyramid.

In collaboration with KCP Technologies, we developed a set of activities for use with SketchExplorer for the iPad, a viewer application of the widely popular Geometer's Sketchpad® software. This application is available in the Apple Store. Activities were constructed in Sketchpad and then transferred to the iPad through email or other forms of file exchange. All activities are pre-configured for the children to use—as no construction tools are presently available in this version for the iPad. Children directly interacted with objects in the pre-configured activity including geometric objects (e.g., points), iterative counters through flicking, or buttons that had been configured to perform a set of operations (e.g., reflection of an image). Two examples are illustrated below. The first (Figure 4a) allows students to make successive attempts at translating a pre-image onto its pre-destined image (i.e., it has been fixed). They interact by moving the reflection line and pressing the reflect button. This activity calls for two reflections to make one translation. We found that all children in our preliminary field work in 4th grade classrooms eventually discovered how to complete this activity through a variety of methods, and develop an understanding of the relationship between reflections and translations.

The second activity (Figure 4b) maximizes the affordance of multi-touch in a mathematical way. Point 1 can be moved laterally and Point 2 vertically (they are constrained to move along two perpendicular lines that have been hidden). The output of these movements is a blob. This blob will simultaneously move in the directions of the two input Points 1 and 2. The size of the blob can be changed by moving Point H along a slider and the color can be changed by moving a point across the spectrum. In this activity, we asked students to make the blob trace a circle. This was a rich mathematical activity in that two inputs can make one output and many of the children in our preliminary field work discovered this idea. More formally, the construction of a circle is parameterized with two perpendicular actions. Again, this activity was extremely engaging, especially when we added the time to establish a competition of who can make the best circle in the least amount of time. Here, haptics is in the form of multi-touch and can be done by one child (multiple fingers) or single-touch by multiple children. We found the latter to be more fruitful as children had to make sense of the effect of each other's actions and to collaborate to complete the task.

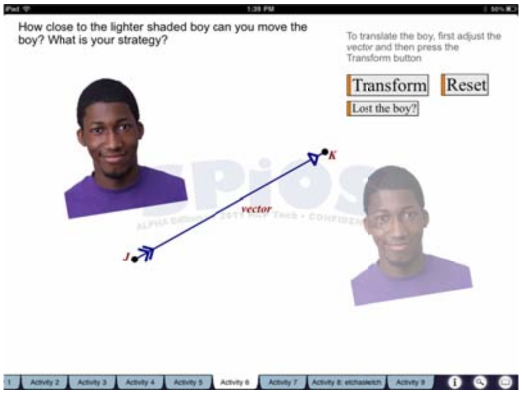


*Figure 4a.* Translation as a composition of two reflections

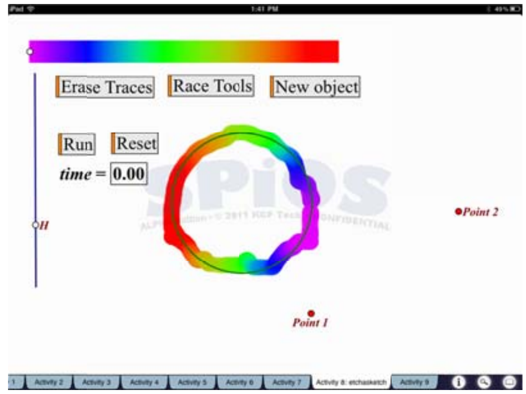


*Figure 4b.* Etchasketch

Our preliminary work with two integrated visual and haptic environments have yielded many positive results in terms of mathematical investigation in which young learners can be engaged. Engagement is of a mathematical kind. Indeed, many of our participating children were highly excited about the possibilities of using new technologies (particularly the iPad which over half of them had at home). But the engagement rapidly became one of a mathematical kind with many forms of scholastic and non-scholastic language being used. Interaction was heightened since not every child had one of either of the devices; collaborative group work was imperative. Such devices enable meaning-making to rise to a small group or whole class level. Discussions resulted from two forms of modality—the visual and the haptic. Sometimes these caused some interference but it was still fruitful in enabling the children to infer, construct and refute ideas in constructing meaning.

Finally, mathematical activities can be designed to create investigations that are difficult or impossible in other technological environments.

## 4. Future Design Principles

Technological affordances should become mathematical affordances and it is in the mathematization of technological affordances that meaningful integration of new multi-modal learning environments can be developed. We conclude with a set of design principles that have evolved from our preliminary work, introduced in this paper, that have the potential to profoundly affect teaching and student learning in the early grades.

### Executable Representations

Mathematical objects and configurations should allow learners to dynamically manipulate and execute operations on the representations in the learning environment. Instead of dealing with static objects or computational outputs, representations that are flexible allow young learners to adapt the configuration and test out their conjectures in an iterative manner.

### Co-action

The learner and learning environment should be collaborative. In dealing with flexible and executable representations, the actions of the learner can guide the environment (re-configure representations) and be guided by the resulting actions of the learning environment.

### Navigation

The integration of dynamic visuals with meaningful haptic feedback forms should allow the learner to navigate the various attributes of the mathematical configuration and construct meaning.

**Manipulation and Interaction**

Objects in such learning environments should be manipulable, and deformed into a wide (if not infinite) set of similar objects, e.g., recall our triangle-area activity earlier, in such a setup all triangles can be configured through direct manipulation.

**Variance/Invariance**

Understanding how quantities vary or not under certain interactions allows a large wealth of mathematics to be explored. In addition to annotations such as measurement, linking variation to force feedback allows meaningful feedback to help guide the learner to make sense of important features, co-varying relationships or invariance.

**Mathematically Meaningful Shape & Attributes**

We naturally use touch to explore the composition of objects in nature as well as varying attributes. In addition to shape, form and texture, haptic feedback can be linked to attributes to aid the learner in their investigation.

**Magnetism**

A natural force is magnetism and this can be used to help learners focus on particular features or relationships between geometric shapes and surfaces. Some objects, or features of objects (where there is a particular mathematically-meaningful interest) can be magnetized and all other attributes de-magnetized.

**Pulse/Vibration**

Pulse in the form of vibro-tactile feedback or oscillating devices (such as the Omni) can similarly aid learners to focus their attention on certain parts of the activity, or offer some form of numerical feedback. For example, the frequency and amplitude of the pulse/vibration can be regulated to vary with some quantity.

**Construction**

Building on the affordances of dynamic geometry, allowing learners to use visual and haptic tools to construct mathematical configurations can help learners to make sense of what objects relate to each other (e.g., co-varying quantities) and communicate with others their understanding or production of a mathematical model.

**Aggregation**

Learning environments often have the affordance of wireless connectivity. Constructions, or evolving discoveries within the learning environment can be easily shared across networks as part of larger models to be aggregated on another computer, or to be contrasted with the work of other students working on the same project. Consider transferring a haptic force with a visual across a network where others can "feel" what you have felt.

We hope that these principles and our preliminary work provide ground-breaking insights into effective generative activity design by future researchers and developers in the future.

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