Cognitive processes developed by students when solving mathematical problems within technological environments

Fernando Barrera-Mora

Aarón Reyes-Rodríguez

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Fernando Barrera-Mora & Aarón Reyes-Rodríguez
Universidad Autónoma del Estado de Hidalgo

Abstract: In this paper we document and discuss how the use of digital technologies in problem solving activities can help students to develop mathematical competences; particularly, we analyze the characteristics of reasoning that students develop as a result of using Cabri Geometry software in problem solving. We argue that the dynamical nature of representations constructed with Cabri, and the availability of measure tools integrated to it are important elements that enhance students' ability to think mathematically and foster the implementation of several heuristic strategies in problem solving processes.

Keywords: Problem solving, digital technologies, mathematical thinking.

Introduction

Mathematical problem solving has been widely recognized as a framework to analyze learning mathematical processes in which it plays dual relevant roles. On one side, it guides performing research in mathematics education (Schoenfeld, 1985) and on the other hand, it supports the development of curricular proposals (NCTM, 2000). In learning approaches, based on problem solving, it is considered that students construct mathematical knowledge by solving problems (Harel, 1994) in a community that fosters development of an inquisitive attitude. Students' participation in a community of practice has been recognized as a fundamental element of what constitutes mathematical thinking (Schoenfeld, 1992; Santos-Trigo, 2010), since in this community they have opportunities to reflect on their own thought processes through listening and reflecting upon ideas of other members of it.

> [In a community of inquiry] participants grow into and contribute to continual reconstitution of the community through critical reflection;

inquiry is developed as one of the forms of practice within the community
and individual identity develops through reflective inquiry. (Jaworski,
2006, p. 202)

Problem solving is an activity involving conceptualization of the discipline "as a set of dilemmas or problems that need to be explored and solved in terms of mathematical resources and strategies" (Santos-Trigo, 2007, p. 523) and that promotes students' engagement in a variety of cognitive actions that can allow them to relate diverse mathematical concepts, facts, procedures and forms of reasoning to construct learning with understanding (Hiebert et al., 1997) through posing and pursuing relevant questions.

In problem-solving learning approaches, students need to conceptualize the construction of mathematical knowledge as an activity in which they have to actively participate in order to identify and communicate ideas that emerge when they are approaching mathematical situations (Moreno-Armella & Sriraman, 2005), as well as to pose questions around problematic tasks that lead them to recognize relevant information needed to give meaning to mathematical concepts. In this line of thinking, Santos-Trigo (2010, p. 301) has stated that: "An overarching principle that permeates the entire problem-solving process is that teachers and students should transform the problem statement into a set of meaningful questions to be examined".

Some classical approaches to problem solving have identified necessary steps for solving problems (Polya, 1945), and central variables that influence students' behaviors and ways of reasoning. For instance, Schoenfeld (1985) considers four categories of variables that are useful to characterize students' mathematical performance: (i) resources, (ii) heuristics, (iii) control and (iv) belief systems. However, since these

theoretical categories were developed based on experiences carried out in paper and pencil environments, when using technological tools, those categories necessary have to be reviewed since the use of technological tools offers students new opportunities to discuss mathematical tasks from perspectives where visual and empirical approaches are widely enhanced and by doing this, students can gain a deeper understanding of mathematical concepts.

> Technology based tools are now used on a daily basis in fields ranging from the sciences to the arts and the humanities, as well as in professions from agriculture to business and engineering […] And, these new conceptual tools are more than simply new ways to carry out old procedures; they are radically expanding the kind of problem solving and decision-making situations that should be emphasized in instruction and assessment. (Lesh & Doerr, 2003, p. 15)

Technological tools allow students experiment, observe mathematical relations, formulate conjectures, construct proofs, and communicate results in ways that can enhance and complement paper and pencil approaches, supporting mathematical learning by offering opportunities to expand students' capabilities to visualize, experiment, obtain feedback, and consider the need to prove mathematical results (Arcavi & Hadas, 2000).

In order to examine the potential of using particular computational tools, in terms of characteristics of reasoning developed by students when solving problems, and the type of cognitive processes performed by learners as a result of the use of these tools, in the international research agenda in mathematical problem solving it has been identified some important questions that can shed light on our understanding about the effect of using these tools in learning mathematics through problem solving, such as: To what extent does the systematic use of technological tools help students to think mathematically? Which aspects of mathematical thinking can be enhanced through the

use of digital technologies in mathematical problem solving? What type of reasoning do students develop as a result of using diverse computational tools in problem solving? (Santos-Trigo, 2007). In this line of thinking, the aim of this paper is to identify and analyze how the use of computational tools could help teachers, enrolled in a master program in mathematics education, to propose problem solving strategies and give arguments to justify and validate conjectures that emerge in the course of solving optimization problems.

Digital technologies as cognitive reorganizers

According to Pea (1987), cognitive technologies are media that help us transcend some limitations of mind such as capacity for storing and processing information based only on biological memory. These cognitive technologies are characterized by externalizing the intermediate thinking products, allowing us to operate, analyze and reflect upon them. Furthermore, the representations that can be constructed with technological artifacts are dynamical and manipulable. This dynamical character of computational representations enable students to construct, for instance, families of configurations, and to establish links among diverse representations, so that when a representation is modified, the change is reflected immediately on the other representations, allowing students to interact, operate or modify the representation and its relations more directly than in a paper and pencil environment.

How does the systematic use of digital technologies impact cognitive structures? Digital technologies can be considered as amplifiers or reorganizers of human cognition. The term "amplify" means doing the same things that one could do without technology,

but performing it in a faster or a better way, without transforming qualitatively our actions; for example, a calculator is an amplifier if it is used only to perform arithmetic computations. On the other hand, "reorganize" means doing new things that one cannot do without technology, or those that were not practical to do. A technological tool can be considered as a reorganizer if it modifies cognitive processes and allows us to establish a dialectical relationship among our actions, forms of thinking and tool's functionalities, which affect our modes of approaching the acquisition of knowledge.

The use of digital tools in learning activities, promotes that students pay attention on the structural aspects of problem solving, by facilitating the performance of routine procedures, opening the possibility of approaching problems which were difficult to discuss within paper and pencil settings, and modifying the cognitive processes that they develop to construct or to operate representations of mathematical objects. For instance, to sketch the graph of a function within a paper and pencil environment, students could proceed to explore and evaluate an algebraic expression, defining the function, for some values of the variable, then those values need to be plotted in a coordinate system, and finally students sketch the graph. However, graphing a function with a calculator or a computational tool only requires introducing in the system the algebraic expression that defines the function, and the software performs intermediate steps required to sketch the graph. That is, computational tools simulate cognitive processes that formerly were exclusive of human beings, attribute that Moreno-Armella and Sriraman (2005) have called *executability*.

Although the use of computational tools offers students advantages to learn mathematics, technological tools by themselves are not enough for constructing learning

with understanding. Mathematical learning with understanding also requires developing an appreciation to practice genuine mathematical inquiry, and disposition to construct connections among diverse mathematical concepts, ideas and procedures. The ability to construct connections is supported by the conceptual structure of the problem solver, a term that we use to indicate how the problem solver's resources are used to approach the examination and solution of a problem; that is, the extent in what these resources can be coordinated in order to articulate different concepts and results when students develop a mathematical activity.

The use of technology to approach learning activities involves considering its impact on the principles and concepts associated with the frame that guides research or instructional processes. As Santos-Trigo and Barrera-Mora (2007, p. 84) have stated "…any conceptual framework or perspective constantly needs to be examined, refined or adjusted in terms of the development of the use of tools (particularly computational tools) that influences directly the ways students learn the discipline". Thus, it is important to consider to what extent the systematic use of technology allows us to examine, test, refine and expand some elements of mathematical thinking considered in problem solving frameworks such as (i) students' access to basic resources or knowledge, (ii) implementation of problem solving strategies that involves ways to represent and analyze the problems, (iii) the use of metacognitive strategies and, (iv) the construction of justifications to validate conjectures and mathematical results.

> […] mathematical problem solving as a research and practice domain has
> evolved along the development and availability of computational tools
> and, as a result, research questions and instructional practices need to be
> examined deeply in order to characterize principles and tenets that support
> this domain. (Santos-Trigo, 2007, p. 524)

We argue that computational tools "incorporates a mathematical knowledge accessible to the learner by its use" (Mariotti, 2000, p. 37), and by doing this, several consequences arise. Among them, the use of technology allows that some resources inherent to the tool could be incorporated to students' resources when they solve problems. For example, when students solve problems using computational tools, they need a lesser amount of explicit mathematical resources to approach a task since students can develop forms of reasoning based on visual and empirical approaches, enhanced by the tools, and therefore their mathematical conceptual structure can be extended incorporating to it some inherent tool's characteristics.

Methodology

Six high school teachers (Jacob, Sophia, Daniel, Emily, Peter and Paul) participated in three hours-weekly problem-solving and problem-posing sessions during one semester. These teachers were enrolled in a master program in mathematics education. They had some experience in using computational tools such as Cabri-Geometry and a hand-held calculator (Voyage 200). All teachers had completed a Bachelor Science degree, majoring in mathematics or engineering, and they had teaching experience ranging from one to five years.

During the semester there were twenty work sessions. The first two sessions were employed to show teachers basic functionalities of Cabri-Geometry through the construction of some common geometrical figures, and to illustrate the form to implement several heuristic strategies such as: to consider that the problem has been solved, relaxing problem conditions, add auxiliary elements to geometric configurations

or to solve a simpler problem. The aim of these sessions was that teachers should

comprehend that a valid construction in Cabri geometry must be based in the properties

and relationships defining the geometrical figures, and that dynamic behavior of figures

is based on the hierarchy of construction procedure.

> The core of the dynamics of a DGE figure, as it is realized by the dragging function, consists of preserving its intrinsic logic, that is, the logic of its construction. The elements of a figure are situated in a hierarchy of properties; this hierarchy is defined by the construction procedure and corresponds to a relationship of logical conditionality. (Drijvers, Kieran & Mariotti, 2009, p. 119)

In the following three sessions, teachers discussed The Church View Task: A car

is driven on a straight roadway. Aside, there is an old church and the driver wants to stop

so that his friend (the passenger) can appreciate the facade of the church. At what

position of the roadway should the driver stop the car, so that his friend can have the best

view? In the process to solve this task, teachers used Cabri to construct a dynamic model

of the situation, and developed numerical and graphical strategies to quantify and

understand the relationship between the car's position on the roadway, and the view of

the church's facade. Besides, through exploration of relationships among elements of the

dynamic configuration, teachers transformed the original problem in an equivalent

geometrical problem: draw a tangent circle to line l (roadway) that passes through points

A and B (representing the church's façade). They conjectured that tangency point of the

circle and line l is the place where the observer gets the best view of the church (Santos-

Trigo & Reyes-Rodríguez, 2011).

The analyzed tasks in this paper were developed within the sixth to eighth

sessions. During the sessions, the teachers were encouraged to use Cabri Geometry and a

hand held calculator to solve problems involving construction of dynamic configurations. The teachers worked on solving problems that come from different contexts: mathematical, hypothetical and real world (Barrera-Mora & Santos-Trigo, 2002). The researchers documented how the tools helped teachers to propose strategies to solve the problems and give arguments to justify and validate conjectures that emerged in the course of the solution process.

The didactical approach employed during the sessions involved teachers working in pairs and plenary discussions in which each pair of teachers communicated and discussed their approaches and strategies employed to solve the problems. Two researchers coordinated the sessions and participated as members of a community, encouraging the development of an inquisitive approach to perform the tasks, and promoting a collaborative work not only to solve the problems, but also to review and reflect on mathematical content and ideas that emerged during problem-solving processes.

The sessions were video recorded and recordings were transcribed. Each pair of teachers handed in a report that included the software files. The transcripts and teachers' reports constituted basic research data. The unit of analysis was the work shown during the sessions by pairs of teachers, however sometimes attention was focused on the work the entire community. The reduction of data was performed by identifying and selecting some chunks of the transcripts or reports, which offered information about strategies employed by teachers to solve the problems or forms of reasoning used to justify their conjectures.

The main tasks analyzed in this paper are three: (a) find the rectangle of maximum area among all rectangles of given perimeter, (b) find the rectangle of maximum perimeter among all rectangles of given area, (c) given a wire, split it into two parts; and with one of the parts construct a square and with the other, construct a circle. Where should you cut the wire so that the sum of the areas of the square and the circle will be minimal?

First task

Peter and Paul constructed a dynamic configuration in Cabri to solve the first problem. They drew a segment *AB* representing the given perimeter of the rectangle that they wanted to construct. Then, they obtained midpoint *M* of segment *AB*, traced segment *MB* and put a point *C* on segment *MB*. The teachers transferred measures *MC* and *CB* to the horizontal and vertical axes, respectively, to construct a rectangle. After that, they verified that the dynamic construction fulfilled the conditions of the problem (the perimeter of the rectangle should be equal to the length of the segment *AB*) measuring the length of the segments *MC* and *CB*, and comparing these lengths with the length of rectangle's sides (Figure 1). The aim of these actions was to verify that there were no mistakes during the construction process, and to provide evidences that the dynamic construction works properly.

Peter computed the rectangle's area using Cabri tools, and dragged point *C* until he obtained a numerical approximation of the maximum area, conjecturing that this one is not attained when the rectangle is a square. His conjecture was based on numerical results, since apparently the maximum area is reached when the measures of the

rectangle's sides are 3.42 cm and 3.29 cm. In this phase of teachers' activity, the tool

acted as a cognitive reorganizer since it enabled them to formulate conjectures based on

the relationship between visual and numerical representations mediated by dragging, as

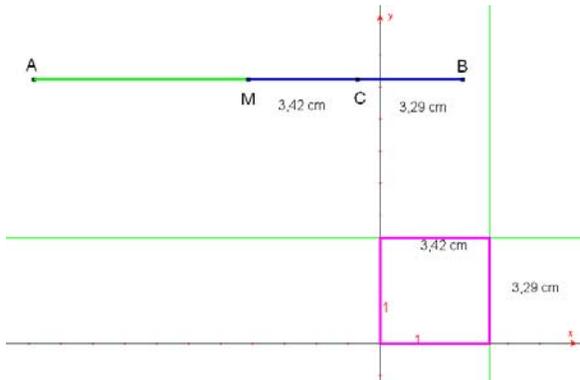well as to construct justifications supported and expressed via the software's resources.

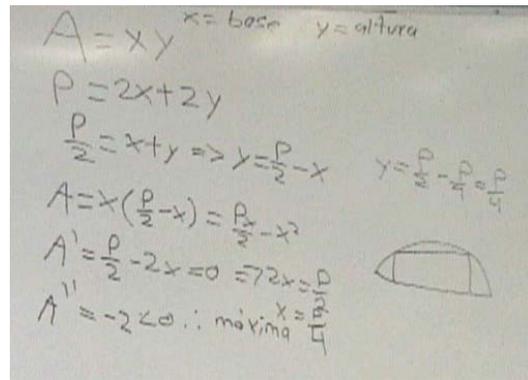Figure 1. Dynamical model constructed in
Cabri Geometry.

Figure 2. Algebraic procedure developed
by Peter to obtain the problem solution.

Peter and Paul considered necessary to take an algebraic approach in order to

obtain the "exact" solution of the problem using calculus techniques. Peter and Paul

denoted by x and y the base and height of the rectangle, respectively. Then, they

represented algebraically the area A as a function of x, and differentiated this function to

obtain the critical points and the value that maximizes the area of the rectangle (Figure 2).

Based on this algebraic procedure, Peter and Paul were convinced that the maximum area

is attained when the rectangle is a square, and obtained evidence that their initial

conjectures was wrong. This conjecture was based on both, visual perception obtained by

manipulating the dynamic configuration, and prior problem solving experiences of Peter

with other optimization problems whose solution do not correspond to a square. For

instance, the following problem: find the rectangle of maximum area inscribed in a

semicircle (see bottom right corner from figure 2).

Peter and Paul obtained additional certainty about the correctness of the solution associating the geometric problem with a similar algebraic problem: maximize the product of two numbers whose sum is given.

> At first, I thought that the rectangle of maximum area would not be square, since there is a classical problem of finding the maximum area of a rectangle inscribed in a semicircle, and the square is not the figure with maximum area. After we obtained the solution $x = y$ algebraically, the result of the problem became logical to me, because if the area is equal to xy, you can prove that the product of two numbers, whose sum is given, is maximum if both numbers are equal (Extracted from Peter and Paul's report corresponding to the sixth session).

Daniel and Emily drew the segment AB and its midpoint (C). After this, they placed point E between points B and C, without considering that C should move on segment BC. For this reason, point E can be dragged over the entire segment AB and not only over BC. Teachers also drew point D, symmetric to point E respect to point C, but this point was not used. Teachers transferred lengths EB and CE to the horizontal and vertical axes, respectively, to draw a rectangle (Figure 3).

Daniel and Emily computed the rectangle's area and transferred this value to the vertical axis; then employed the "Locus" tool to construct a graphic representation of the area function (Figure 3). The teachers were astonished to observe the graph (Figure 3), since they expected that it was only a portion of a parabola. The graph behavior was due to the way that Daniel and Emily developed the geometrical construction, since this rectangle does not always meet the problem's conditions. For some positions of point E, rectangle's perimeter is greater than the length of segment AB (Figure 3, right). It is important to notice that Daniel and Emily, unlike Peter and Paul, did not verify that their

geometrical model satisfied the problem conditions, although the graph they visualized

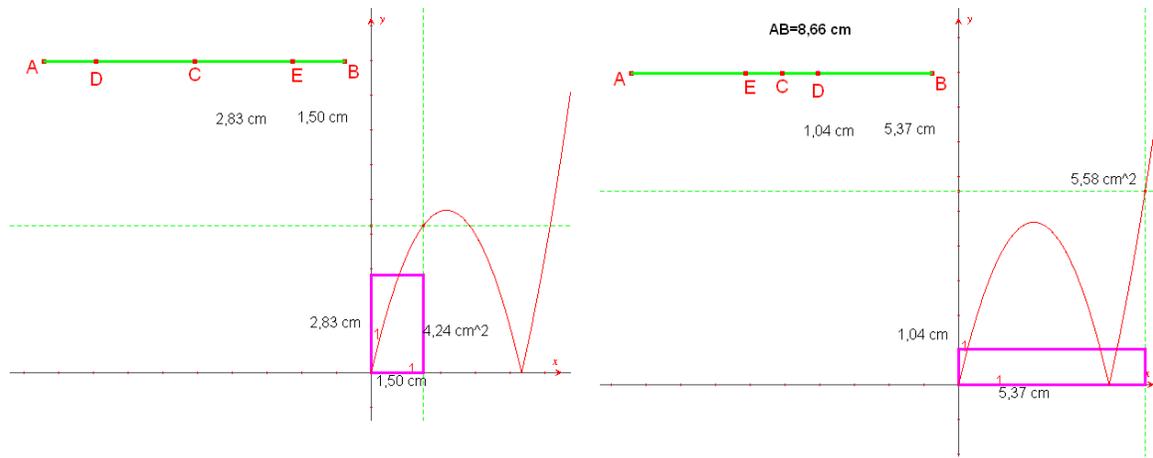on the screen did not correspond to what the teachers had anticipated.

Figure 3. Dynamical model elaborated by Daniel and Emily.

The problem solving behavior shown for this pair of teachers to approach the first

task is representative of the activity performed by them to approach all tasks. Daniel and

Emily had some difficulties to construct dynamic configurations that met the conditions

of problem statement. Additionally, in this task, they did not consider relevant to use the

resources offered by the software, such as measure tools to check the accuracy of their

dynamical construction. However, these teachers employed the graph of the function area

to conjecture that the maximum area is attained when the rectangle is a square, so the use

of the tool allowed teachers to formulate conjectures, which is an important element of

what constitutes mathematical thinking.

Other important feature of Emily and Daniel´s problem solving behavior was that

they showed difficulties to implement algebraic procedures to exploring solution routes,

although, plenary presentations allowed them consider the importance to develop this

type of strategies to prove or refute conjectures posed using the resources offered by

Cabri. In this context, the use of a dynamic software offered teachers opportunities to

approach tasks with less amount of algebraic resources in relation to the requirements of a paper and pencil setting.

Jacob and Sophia traced a segment *AB*, a point *C* on *AB*, and midpoint (*M*) of segment *AC*. They transferred lengths of segments *MC* and *CB* to the vertical and horizontal axes, respectively, and traced a rectangle based on these measures. Teachers verified that rectangle's sides had the same measures as segments *CB* and *MC*, however they did not realize that rectangle's perimeter is not the length of segment *AB*. The mistakes made in the constructions process led them to formulate a wrong conjecture: the base of the rectangle of maximum area must be twice its height (Figure 4).
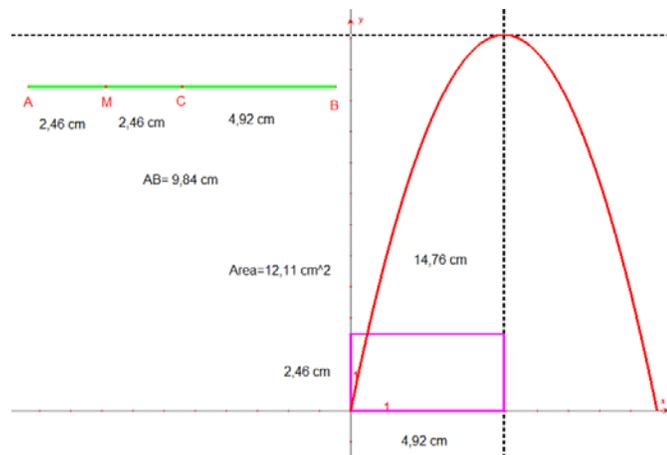
Figure 4. Dynamic model elaborated by Sophia and Jacob.

Comments

The results of this task show that, in general, Cabri acted as a reorganizer, since it allowed teachers to develop procedures to approach the task that could not be done in paper and pencil environments, such as formulating conjectures based in the observation of variation of numerical attributes of figures, as was the case of Peter and Paul's approach; or the visualization of a relationship between two quantities obtained without the previous formulation of an algebraic expression as in the approaches developed by

Daniel, Emily, Jacob and Sophia. That is, the teachers were able to access the resources incorporated in the tool, specifically numerical and graphical resources available in Cabri, to develop a particular form of thinking to approach the problem.

Concerning the justification process, Peter and Paul considered important to verify empirically that the construction satisfied the conditions stated in the problem and elaborated an algebraic proof of their conjecture. In this case, the use of measure tools was a mean to establish the validity of their construction; and the algebraic proof was employed to obtain an "exact" and not only an "approximated" solution. However, it can be observed that not all teachers verified that geometric configurations were constructed properly, neither all of them were aware of the importance to provide justifications using the means offered by the tool or external to it. These results differ from other research works that analyze the same problem. In those, it is concluded that the transition from a geometric conjecture to an algebraic proof, emerges from a discrepancy between a conjecture and the approximate results obtained with the tool, which suggested a different result (Olive, 2000).

The plenary discussion supported Daniel, Emily, Jacob and Sophia to identify pitfalls in their work and reflect about some important mathematical ideas such as the domain of a function and the importance to provide justifications. Besides this, the interaction among member of the learning community allowed Peter and Paul to incorporate a visual approach to their repertoire of problem solving strategies.

Second task

Peter and Paul selected a point *A* on the horizontal axis. The distance between point *A* and the origin *O* of the coordinate system represents the length of a side of the rectangle that teachers wanted to construct. Teachers used the "Numerical Edit" tool to define a quantity that represents the area of the rectangle, and to calculate the length of other side of it, they divided the length of segment *AO* by the area, and obtained the value *c*. Then, they transferred *c* to the vertical axis and obtained point *B*. Then, teachers drew rectangle *OABC*, calculated its area and dragged point *A* to verify that the area remain constant. Next, teachers obtained the perimeter of rectangle *OABC* to construct a graph relating a side of the rectangle and the corresponding rectangle´s area (Figure 5). In this problem, Peter and Paul incorporated to their repertoire of strategies the graphical approach discussed in the plenary session corresponding to the first task.
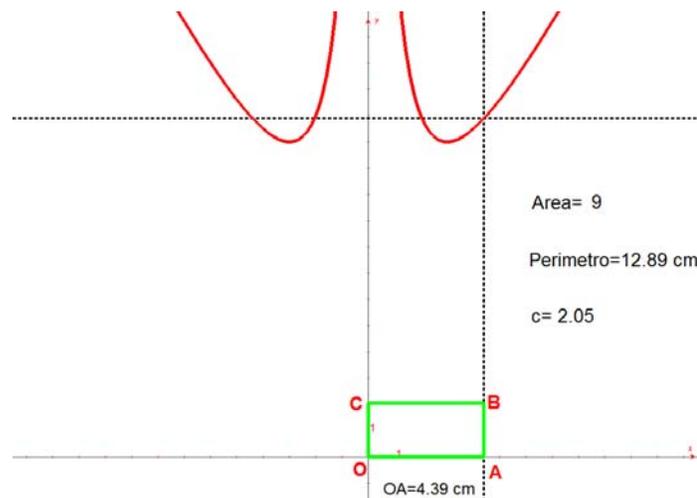
Figure 5. Perimeter of rectangle *OABC*, as a function of a length of side *OA*. (Graph, elaborated by Peter and Paul)

Teachers conjectured that the graph of the perimeter, as a function of side *OA*, consists of a branch of a hyperbola. Peter and Paul determined that although the locus was split in two branches, it was enough to consider one of them. Teachers tried to test their conjecture, first, by using the "Equation or Coordinates" tool, but the software did

not display the equation corresponding to the locus. Secondly, teachers selected five points on one of the branches for tracing a conic that, visually overlapped the graph, and by this mean they were convinced that the locus corresponded to a hyperbola. In the same way as in the first task, the software acted as a reorganizer, since it allowed Peter and Paul to develop graphical approaches to obtain evidence support their conjectures that are difficult to implement in paper and pencil settings.

Peter and Paul also conjectured that minimum perimeter is reached when the rectangle is a square. Teachers did not construct an algebraic proof of their conjectures; they were convinced of their results based on the visual and numerical evidence provided by the software. The problem solving behavior of these teachers differs from that shown by them to solve the previous problem, in which they considered important to formulate and solve the problem algebraically.

Sophia and Jacob approached this problem drawing a segment *AB* whose length represents the rectangle´s area. Then, they put a point *C* on *AB*, and stated that the length *AC* would represent one of rectangle's sides. Sophia and Jacob transferred the measure of *AC* to the horizontal axis to obtain a point *X*. Then, they obtained the length of the other rectangle side computing the quotient AB/*AC*, and transferred this measure to the vertical axis to obtain point Y, finally they drew the rectangle OXZY (Figure 6). Later, teachers measured rectangle's area to verify that this measure coincided with the length of segment *AB*. In this action, it can be observed the effect of interaction in a learning community, since in the previous problem; this pair of teachers does not considered the use of measure tools to verify their construction was correct.
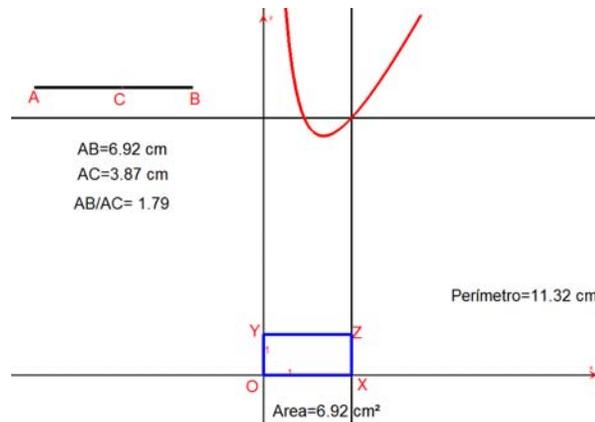
Figure 6. Graph of the perimeter of rectangle *OXZY*, as a function of side OX, elaborated by Sophia and Jacob.

Then, Sophia and Jacob constructed the graph that relates the perimeter of rectangle OXZY and the length OX, and conjectured that the minimum perimeter is reached when the rectangle is a square, based on dragging point C and the visualization of perimeter function. They did not elaborate an algebraic justification of their conjecture.

Emily and Daniel had difficulties to build a rectangle of constant area in Cabri, and they tried to solve a simpler problem with the aim of using this to solve the original problem. They proposed constructing a triangle of constant area, and carried out the construction fixing the triangle's base, and putting the third triangle's vertex on a parallel line to the base of a triangle. Teachers verified, with the "Area" tool, that the triangle they constructed satisfied the condition of having constant area, and conjectured that the triangle of minimum perimeter is an isosceles triangle (Figure 7). In the process to approaching this task it can be observed that Daniel and Emily incorporated the use of measure tools to their repertoire of resources to verify accuracy of a dynamical construction, strategy which was discussed during plenary discussion of the first task.
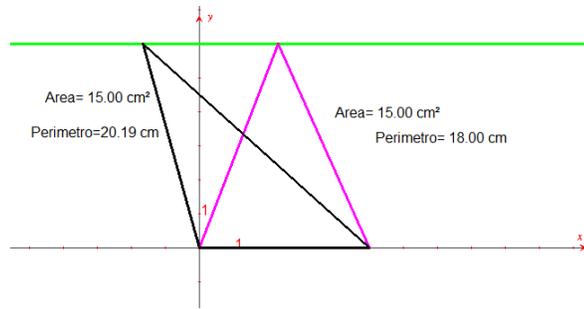
Figure 7. A family of triangles of constant area.

In the same line of thinking, Daniel and Emily considered relevant to provide justifications. For instance, the teachers were able to justify that triangles they constructed have constant area since the base is fixed and all triangles of the family have the same height. Daniel and Emily also tried to use algebraic procedures to verify that the triangle of minimum perimeter is an isosceles one, but they were unable to algebraically formulate the problem, as can be observed in the figure 8. The analysis of the activity developed by Daniel and Emily, allows us to obtain evidence that the use of Cabri increases the number of problems that students, with a low ability to manage algebraic procedures, can tackle.
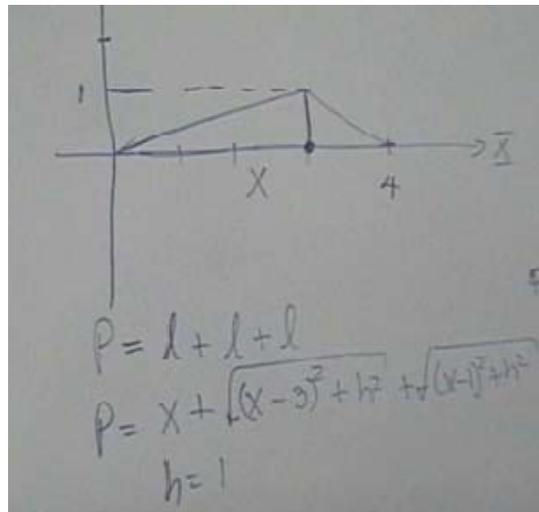
Figure 8. Algebraic formulation for the minimum perimeter triangle problem. (Elaborated by Daniel and Emily)

Sophia and Jacob were interested in solving the previous problem, and they tried
to find, by algebraic means, the triangle of minimum perimeter given the conditions
stated by Emily and Daniel. The teachers formulated algebraically the problem (Figure 9,
left) and using calculus tools and a hand held calculator to perform algebraic operations,
they obtained the point that maximizes the perimeter of the triangle, and concluded that
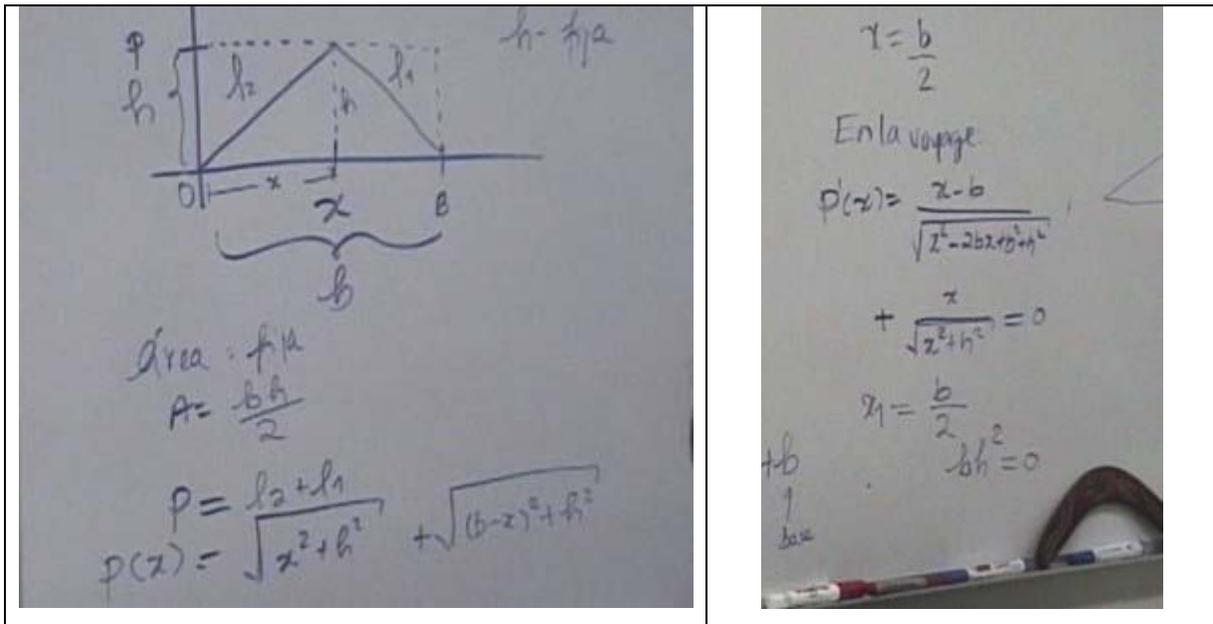the triangle of maximum perimeter is an isosceles triangle. (Figure 9, right).

Figure 9. Algebraic solution proposed by Sophia and Jacob.

It was observed that discussion developed into the community, influenced the
problem solving behavior of Sophia and Jacob, since these teachers incorporated the use
of algebraic procedures to their repertory of justifications. On the other hand, when
teachers solved this task, they used calculator Voyage 200 as an amplifier, since the tool
was only employed to perform computations such as the derivative of the perimeter
function ☐ and to solve the equation ☐ . However, the use of the calculator
allowed teachers to reflect about the results not encountered in paper and pencil settings.
When Sophia and Jacob solved the equation ☐ , they obtained as a result

[   ]. Sophia commented that the expression [   ] means that the

minimum perimeter is also attained if the base or height of the triangle is equal to zero,

but in this case the triangle dissapears.

Comments

This task allowed us to observe that Cabri transformed teachers' forms of thinking

and reasoning. For instance, approaching tasks within a paper and pencil environment

leads to consider the meaning of a variable with restricted properties, basically based on

representing it with a symbol, say x. Meanwhile, using a dynamic software to approach

the task, allowed teachers to construct the idea of a variable, not only as a symbol, rather

as an amount that changes, as it can be observed when teachers dragged the point

representing the independent variable to approximate the value that produces the

minimum perimeter. That is, the use of Cabri, particularly the executability of

representations, gives rise to a different meaning of the concept of variable, since the tool

helps to perceive the idea of variation as the work of Peter, Paul, Jacob and Sophia has

shown. We argue that the exploration of ideas such as variation and co-variation, through

the use of a dynamic software, favors a reorganization of students' cognitive processes,

since it helps them to give meaning to ideas and concepts involved in the solution of

optimization problems, such as the function concept. It is attained by means of

visualization and perceiving how one quantity changes when the other does.

Third task

$\boxed{}$. Moreover, the area of the circle can be computed as $\boxed{}$ (figure 11, left).

Then, Peter expressed the sum of areas as a function of x, and used the calculator to obtain the derivative of the function and its critical points (Figure 11, right).
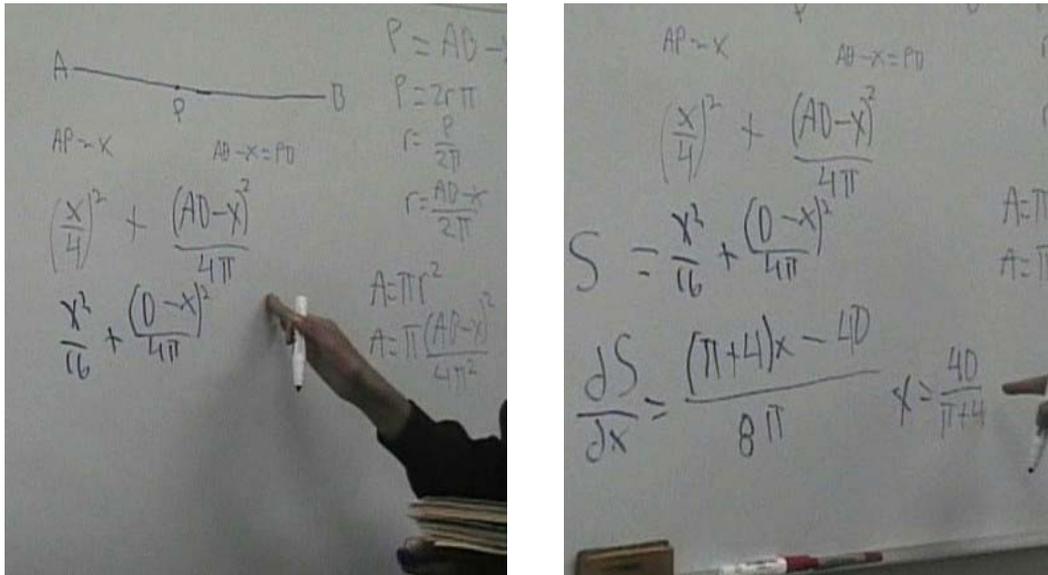

Figure 11. Algebraic formulation of the wire problem.

Peter expressed that in the dynamic configuration he approached the point which minimizes the sum of the areas and compared it with the result obtained by substituting the particular values into the algebraic solution.

The process employed by Daniel and Emily to solve the problem consisted in drawing a segment *AB* to represent the wire, and put a point *C* on *AB* which is the point where it is cut. Then, teachers constructed a square by considering as one of its sides the segment *AC,* they traced midpoint (*D*) of segment *CB*, and drew a circle with center *D* and radius *DB*. Daniel and Emily also computed the areas of the square and circle, and computed their sum *S*. Finally, teachers constructed a graph relating length of segment *AC* and the area *S* which is the sum of areas (Figure 12). The activity developed by the

teachers showed that they did not understand the problem statement, since the perimeter of the circle and square are not the lengths of segments *AC* and *BC*, respectively. Daniel and Emily had difficulties to understand the problem, even after Peter and Paul showed how they solved the problem; Daniel and Emily did not understand why the cable should be divided into four equal parts to construct the square.
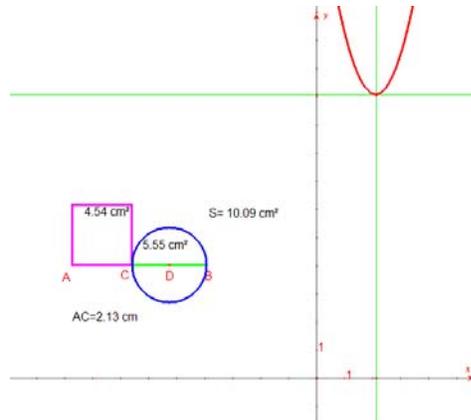

Figure 12. Dynamical configuration representing the wire problem, elaborated by Daniel and Emily.

Comments

Approaching this task, using technology, required by the problem solvers to think about the geometric objects in terms of actions, for instance, the actions to be consider to construct a square given a segment, are different from those when paper and pencil environment is used. The difference has to do with a "new quality" that the representation of the objects have when using Cabri, the executability property.

In analogy with the previous tasks, the use of Cabri software allowed the problem solvers to relate geometric and algebraic aspects of the problem as well as to coordinate them into a wider conceptual network. For instance, it allowed them to assign the variable, which represented the side of a square or the radius of a circle, a more concrete

meaning in terms of variation and not only its representations as a symbol. Besides, as in the previous tasks, the idea of dependence between variables acquired a more robust meaning in terms of the concept of a function. The plenary discussion allowed the participants to reflect about the properties of a function concerning points where it reaches maxima or minima as well as its domain.

Closing remarks

The use of digital tools allowed teachers not solely to remember facts or apply algorithms, but most importantly, it helped them to formulate conjectures, and develop visual schemas to provide justifications. Mainly, measuring attributes and dragging elements in geometric constructions allowed teachers to formulate conjectures (Arzarello, Olivero, Paola & Robutti, 2002) and observe relations among mathematical objects that can be a departure point to develop a deeper mathematical understanding.

It was observed that the use of technology helped teachers to develop ways of reasoning and forms of reflecting about the meaning and connections among mathematical objects. For example, the dynamic software enabled teacher to search for various forms of justifying a conjecture, in which the use and integration of visual, empirical and deductive arguments were useful.

Based on the activities developed, we noticed that teachers founded their forms of reasoning strongly on the visual representations, a result previously reported by Arcavi (2003). The dynamism of representations helped teachers to think about variation of particular instances and provided them with empirical basis to formulate conjectures. The software provided feedback to the teachers (Arcavi & Hadas, 2000), but not all teachers

were able to give meaning to this feedback, which was observed in the form that Sophia and Jacob, and Emily and Daniel have solved the first task.

The analysis of the tasks has shown a way in which the conceptual structure of the problem solver can be extended by incorporating the resources of the tool through the use of it in the process of solving problems. This was explicit when teachers used the tool to provide a visual representation of the information and by doing so, to approach a solution of the problem, which used algebraic setting as well as visual ones. That is, the capabilities of the tool as a cognitive reorganizer were based on the different possibilities that the tool offers to establish connections and to act as an extension of the cognitive structure of the teachers.

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