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Becoming Aware Of Mathematical Gaps In New Curricular Materials: A Resource-Based Analysis Of Teaching Practice

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Abstract: The study featured in this article, with its central focus on resources-in-use, draws upon salient aspects of the documentational approach of didactics. It includes an a priori analysis of the curricular resources being used by a teacher for the first time, followed by detailed in situ observations of the unfolding of her teaching practice involving these resources. The central mathematical problem of the lesson being analyzed deals with families of polynomial functions. The analysis highlights the teacher's growing awareness of the mathematical gaps in the resources she is using, which we conjecture to be a first step for her in the evolutionary transformation of resource to document, as well as an essential constituent of her ongoing professional development.

Keywords: documentational approach of didactics, documentational genesis, curricular resources in mathematics, families of polynomial functions, mathematical gaps in resources, ongoing professional development, resources-in-use, research on teaching practice with new curricular resources.

Introduction

Mathematical problems suitable for use in high school classrooms can be obtained from a variety of resources, including the internet, newspapers and books, colleagues, and of course textbooks. There is general consensus that most mathematics teachers rely on textbooks for their day-to-day fare of problem-solving items for students (Schmidt, 2011). Over time, these problems and the ways in which they are presented to students get tinkered with and gradually become refined (Gueudet & Trouche, 2010, 2011). However, we are only now beginning to learn a little about the ways in which teachers

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interact with the mathematical resources available to them (Gueudet, Pepin, & Trouche, 2011). Chevallard and Cirade (2010) have raised an additional issue, that of the lack of adequate mathematical resources for teachers when the school program is changed and new problems and problem-solving approaches are introduced. Moreover, as pointed out by Artigue and Houdemont (2007), many teachers who teach mathematics – even at the level of secondary school – are not mathematics specialists and "are quite often not proficient in mathematics, and that the mathematics and didactic formation they receive during their training does not compensate for these limitations" (p. 376). Although a focus on the mathematical resources available to teachers, their supportive role, and their adaptation and adoption is not one that, up to now, has been central to the research agenda of the problem-solving research community, its importance can be argued for, at the very least, on pragmatic grounds: The ways in which resources support (or do not support) teachers in their problem-solving efforts in class clearly impact upon the problem-solving experience of students.

According to Remillard (2005) who conducted a seminal review of teachers' use of curricular materials, the process by which mathematics teachers appropriate and transform such resources, as well as the support that these resources offer, is rather unexplored terrain. In 2000, Adler similarly proposed that "mathematics teacher education needs to focus more attention on resources, on what they are and how they work as an extension of the teacher in school mathematics practice" (Adler, 2000, p. 205). In one such study of teachers using reform-based curricular materials, Manouchehri and Goodman (1998) reported what they viewed as shortcomings in the guidance for teachers provided by the curricula, saying that the curricula "did not provide the teachers

with detailed methods of how to address the content development" (p. 36). Teaching with new resources can thus lead to situations where teachers are not suitably prepared, but which can provide the impetus for new awarenesses of both a mathematical and didactical nature. In this regard, Gilbert (1994) has said: "reflection-in-action occurs when new situations arise in which a practitioner's existing stock of knowledge – their knowledge-in-action – is not appropriate for the situation" (p. 516). This reflection-in-action, which involves critical examination and reformulation of one's existing knowings, is intimately connected to, and synergistic with, one's evolving appropriation and transformation of resources, according to the documentational approach of didactics (Gueudet & Trouche, 2009, 2011).

The Documentational Approach of Didactics

Gueudet and Trouche (2009, 2011) have developed a theoretical research framework based on the premise that documentation work is at the core of teachers' professional activity and professional change. Documentation work includes selecting resources, combining them, using them, and revising them. Even outside a particular reform or professional development program context, such work is deemed central to teaching activity. Gueudet and Trouche employ the term "resource" to describe the variety of artifacts that they consider – such as a textbook, a piece of software, a student's work sheet, a discussion with a colleague. Like Adler (2000), a key aspect of Gueudet and Trouche's (2011) approach is resource-in-use (in-class and out-of-class).

One of the pivotal constructs of their theory is that 'resources' become transformed into 'documents' via a process of documentational genesis – a construct

inspired by and adapted from the parallel process in the instrumental approach whereby artifacts become transformed into instruments via instrumental genesis (Rabardel, 1995). The instrumental approach distinguishes between an artifact, available for a given user, and an instrument, which is developed by the user – starting from this artifact – in the course of his/her situated action. Similarly, a document is developed by a teacher, starting from a resource, in the course of his/her situated action. Gueudet and Trouche represent this process of documentational genesis with the following simplified equation, where the 'scheme of utilisation' refers to the various personal adaptations that are made with respect to using the resource in accordance with a teacher's evolving knowledge and beliefs (Gueudet & Trouche, 2009, p. 209): "*Document = Resources + Scheme of utilization*". Documentational genesis is therefore considered to be a dialectical process involving both the teacher's shaping of the resource and her practice being shaped by it.

In their description of this theoretical approach and its accompanying methodological principles, Gueudet and Trouche (2011) emphasize the professional growth that is intertwined with documentational genesis. They argue that:

> Teachers "learn" when choosing, transforming resources, implementing them, revising them etc. The documentational approach proposes a specific conceptualisation of this learning, in terms of *genesis*. A documentational genesis induces evolutions of the teacher's schemes, which means both evolutions of the rules of action (belonging to her practice) and of her operational invariants (belonging to knowledge and beliefs). Documentation being present in all aspects of the teacher's work, it yields a perspective on teachers' professional growth as a complex set of documentational geneses. (Gueudet & Trouche, 2011, p. 26)

The study featured in this article, with its central focus on resources-in-use within actual teaching practice, draws upon salient aspects of the documentational approach of didactics. More specifically, our research question centered on uncovering key moments

of teacher awareness, particularly those of a mathematical nature, in the process of using new curricular resources in class. We begin with an *a priori* analysis of the curricular resources being used by a teacher for the first time, followed by detailed in situ observations of the unfolding of her teaching practice involving these resources. The analysis highlights the teacher's growing awareness of the mathematical gaps in the resources she is using – conjectured to be a first step for her in the evolutionary transformation of resource to document, as well as an essential constituent of her ongoing professional development.

Methodological Aspects of the Study

The present study is situated within a multi-phase program of research whose current phase is the study of teaching practice in mathematics classes involving the use of digital technology in the teaching of algebra, in particular, the use of Computer Algebra System (CAS) technology. Previous phases of the research integrated tasks that had been designed by members of the research team (see, e.g., Kieran, Tanguay, & Solares, 2011). This phase examines teaching practice in technology-supported classroom environments where commercially-developed curricular resources, such as textbooks, are in use.

Participants in this phase of the study included three teachers from three different public high schools. They responded positively to our request for volunteers who were using technology in their regular teaching of high school algebra and who would be willing to be observed and interviewed for our research study. We observed and videotaped each teacher's practice for five consecutive days in all of their regular mathematics classes. We intended to capture, as much as would be possible under the videotaping circumstances, their natural teaching practice involving whatever resources

they had chosen to make use of. We also interviewed each teacher twice – once at the beginning of the week and once at the end. The analysis presented in this article focuses on the practice of one of the three teachers, Mae (a pseudonym), during one of her lessons of the week.

Mae taught all three of the senior year (17-year-old students) mathematics classes in her school. She was one of the pioneers of her school on the use of technology in the teaching of mathematics. In her own classes, she regularly used a whiteboard hooked up to her computer and all students had CAS calculators available to them. She was technically very savvy and could respond easily to all students' questions regarding the use of technology. Her academic background included a doctorate in education with a thesis on the use of graphing calculator technology. Her mathematical knowledge seemed, however, less developed than her technological skill. She made a regular practice of asking students to read ahead in the text because – as she mentioned during an interview – they would soon be graduating and had to learn to be autonomous adults who were responsible for their own learning. However, this practice also led students to pose questions of a mathematical nature that went beyond what they had been able to extract from their textbook. Such questions were not, in general, handled with the same expertise and knowledge base with which Mae handled their technological questions.

The analysis of Mae's teaching practice that we present in this article does not focus on her integration of technology into the teaching of mathematics, but rather on the mathematical content at stake in her lesson within the framework of the documentational approach of didactics (Gueudet & Trouche, 2009), a key construct of which is the evolutionary nature of documentational genesis whereby resources gradually become

transformed into documents. The resources that Mae was using during the period in which our classroom observations occurred were new to her that year. The provincial curriculum guidelines had changed the year before and new textbooks were developed that would adhere more closely to the new guidelines. Mae tended to rely on both the student textbook and accompanying teacher guide to plan the mathematical content of her lessons. We were interested in following the process of her integration of these resources into her teaching practice, the way in which she was adapting and transforming them, and the way in which they might be co-transforming her practice and her knowledge – that is, in capturing the reciprocal nature of the documentational genesis that was occurring.

Although the analysis we present in this article is focused on a very small part of Mae's teaching practice, on one lesson in fact, the approach to our analysis is broader in scope. We begin with an analysis of the two text-based resources she used for her lesson on families of polynomial functions, tracing back in these resources to some of the earlier notions that served as foundation for the development of the lesson's mathematical content. Then we analyze the videotape of the unfolding of the classroom lesson. This latter analysis attempts to draw out the dynamics and forces that came into play as the prepared mathematical content was elaborated in the classroom setting, examining in particular those moments that seemed critical to the further development of her teaching practice and to the evolutionary process whereby a resource becomes a document. The videotapes of the interviews with the teacher also serve to illuminate some of the underlying aspects of her teaching practice.

Analysis of the Resources Used by the Teacher in Preparing her Lesson

Herein we present pertinent extracts from the resources used by the teacher, Mae, in preparing her lesson, as well as some of our own mathematical and didactical commentary related to these extracts. The two resources she used were the student textbook *Advanced Functions 12* (Erdman, Lenjosek, Meisel, & Speijer, 2008a) and the accompanying teacher guide *Advanced Functions 12, Teacher's Resource* (Erdman et al., 2008b). The lesson was on Families of Polynomial Functions (Section 2.4 of Erdman et al., 2008a). Our analysis of the resources used by the teacher focuses primarily on the issue of the mathematical links between factors written in the form ($x - b/a$) versus the form ($ax - b$) for given families of polynomial functions.

The Student Textbook

Background mathematical material from Section 2.2. In Section 2.2 of the textbook (Erdman et al., 2008a), students are presented with the Factor Theorem (see Figure 1). This section of the textbook provides some of the support for the theoretical affirmations regarding the desired form for factors of a polynomial that are made in the later section on Families of Polynomial Functions with respect to rational roots.

Factor Theorem

$x - b$ is a factor of a polynomial $P(x)$ if and only if $P(b) = 0$.

Similarly, $ax - b$ is a factor of $P(x)$ if and only if $P\left(\frac{b}{a}\right) = 0$.

Figure 1. The Factor Theorem (drawn from p. 95 of Erdman et al., 2008a)

This theorem allows for determining whether a certain binomial is or is not a factor of a given polynomial on the basis of numerical evaluation. The textbook does not prove this theorem; it merely provides the following affirmation, which allows for some misinterpretation: "With the factor theorem, you can determine the factors of a

polynomial without having to divide" (p. 95). No explanation is provided as to why the numerical evaluation $P(b/a)$, when it yields zero, should in fact be sufficient for determining a factor of the polynomial. However, the central issue for our analysis is the following: if $P(b/a) = 0$, *why write the factor in the form (ax – b) and not in the form of* (x - b/a)? It clearly makes for an easier long-division calculation when written in the form of ($ax - b$). But what happens, mathematically speaking, when one expresses ($x - b/a$) as ($ax - b$)? Are the two forms equivalent? What mathematics is hidden in expressing the former form as the latter? How does one convert one form to the other and maintain equivalence?

Subsequent pages of the student textbook expand on the Factor Theorem by means of two additional theorems, the Integral Zero Theorem (p. 97) and the Rational Zero Theorem (p. 100), illustrated in Figure 2. However, once again, no further explanation is provided for the case of the polynomial *P(x)* having a rational zero *a/b*, either as to why *a* should be a factor of the leading coefficient of *P(x)* or the issue regarding the form to be used for the factor of *P(x)* corresponding to the rational zero.

Integral Zero Theorem	**Rational Zero Theorem**
If $x - b$ is a factor of a polynomial function $P(x)$ with leading coefficient 1 and remaining coefficients that are integers, then b is a factor of the constant term of $P(x)$.	Suppose $P(x)$ is a polynomial function with integer coefficients and $x = \frac{b}{a}$ is a zero of $P(x)$, where a and b are integers and $a \neq 0$. Then, • b is a factor of the constant term of $P(x)$ • a is a factor of the leading coefficient of $P(x)$ • $ax - b$ is a factor of $P(x)$

Figure 2. The Integral Zero and Rational Zero Theorems
(drawn from Erdman et al., 2008a, pp. 97 & 100)

The textbook provides several examples that show the advantages of using these two latter theorems when the task is to find the factors of a polynomial. However, the relevance of writing the factor in the form ($ax - b$) when $x = b/a$ is a root of the

polynomial *P(x)* is never discussed. This can have repercussions, didactically speaking, at the moment when the teacher introduces the theory underlying families of polynomial functions, coming up in Section 2.4. The intervening section 2.3, on the solving of polynomial equations, adds no further theory related to the Factor Theorem.

Families of Polynomial Functions. Before giving a general definition of families of polynomial functions, the textbook offers several examples that illustrate that one obtains different members of the same family of polynomial functions for different values of the parameter *k* (see Figure 3 for one such example).

Example 1 Represent a Family of Functions Algebraically

The zeros of a family of quadratic functions are 2 and −3.

a) Determine an equation for this family of functions.

b) Write equations for two functions that belong to this family.

c) Determine an equation for the member of the family that passes through the point (1, 4).

Solution

a) The factor associated with 2 is $x - 2$ and the factor associated with −3 is $x + 3$.

An equation for this family is $y = k(x - 2)(x + 3)$, where $k \in \mathbf{R}$.

b) Use any two values for k to write two members of the family.

For $k = 8$, $y = 8(x - 2)(x + 3)$.
For $k = -3$, $y = -3(x - 2)(x + 3)$.

c) To find the member whose graph passes through (1, 4), substitute $x = 1$ and $y = 4$ into the equation and solve for k.

$4 = k(1 - 2)(1 + 3)$
$4 = k(-1)(4)$
$4 = -4k$
$k = -1$

The equation is $y = -(x - 2)(x + 3)$.

Figure 3. Algebraic representation of a family of polynomial functions (drawn from Erdman et al., 2008a, p. 115)

As is illustrated in Figure 3, the family of polynomial functions that has as zeros 2 and –3 can be represented algebraically as $y = k(x-2)(x+3)$. But, if we look at part (b) of the solution of this example, the information that is given suggests that different values of *k* yield different members of the same family of polynomial functions. This can lead those who are using this textbook as a resource to a false mathematical conception if they

do not distinguish the crucial role being played by the root of the polynomial in terms of whether it is a whole number or a rational. In other words, if the zeros of the polynomial are not whole numbers, but rather rational numbers, then the value of *k* can vary according to the form of the factor, without changing the member of the polynomial family. For example, if the zeros of a family of polynomial functions are 3 and –1/2, then the family has as its function $P(x) = k(x-3)(x+1/2)$. And so, a member of this family is: $P(x) = 2(x-3)(x+1/2)$, if *k* = 2. But if we write the factor $(x+1/2)$ as (2*x* + 1), the value of *k* changes from 2 to 1 for the same member of the polynomial family, that is, $P(x) = 1(x-3)(2x+1)$. The algebraic transformation involved in changing the form of the factor $(x+1/2)$ to (2*x* + 1) is as follows: $(x+1/2)$ = 2/2 $(x+1/2)$ = $1/2$ (2*x* + 1). Thus, the conversion of $(x+1/2)$ to (2*x* + 1) involves also multiplying the rest of the expression by $1/2$, thereby yielding the new *k*-value of 1 (from multiplying the previous *k*-value of 2 by $1/2$). This example shows that, if we have a family of polynomial functions expressed algebraically as $P(x) = k(x-a_1)(x-a_2)...(x-a_n),\ k \in \Re,\ k \neq 0,$ we cannot say that different values of the parameter *k* necessarily imply different members of a given family of polynomial functions, unless of course all the zeros are whole numbers.

The examples provided in the textbook are then followed by the general definition of families of polynomial functions (see Figure 4).

KEY CONCEPTS

- A family of functions is a set of functions with the same characteristics.
- Polynomial functions with graphs that have the same x-intercepts belong to the same family.
- A family of polynomial functions with zeros $a_1, a_2, a_3, \ldots, a_n$ can be represented by an equation of the form $y = k(x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n)$, where $k \in \mathbb{R}, k \neq 0$.
- An equation for a particular member of a family of polynomial functions can be determined if a point on the graph is known.

Figure 4. Definition of families of polynomial functions (drawn from Erdman et al., 2008a, p. 118)

With this general definition, where there are no conditions on the zeros of a polynomial function, the earlier suggestion to write the factor as $(ax - b)$ when the zero is rational receives no consistent theoretical support. In fact, the general definition would seem to suggest that the factored form for a given zero *a*, be it a rational or whole number, is of the form $(x - a)$.

The Teacher Guide

We now examine the nature of the support offered in the teacher guide with respect to preparing lessons on families of polynomial functions (Erdman et al., 2008b). This resource presents only a few teaching suggestions, most of which could be considered, at the very least, quite incomplete from a didactical point of view (see Figure 5).

- In **Examples 1** and **2**, point out that the equation for the family of functions must include the constant k to represent the leading coefficient. Encourage students to use fractions and not decimals where appropriate. Apply Chapter 1 skills to graph the polynomials, as shown in **Example 2** part d).

Figure 5. A typical suggestion found in the teacher guide from the section dealing with families of polynomial functions (drawn from Erdman et al., 2008b, p. 52)

In fact, the guidance noted in Figure 5 where students are to be encouraged to use fractions and not decimals is contradicted in another suggestion within the same resource a few lines later (see Figure 6).

- The factors corresponding to the rational zeros in **questions 10** and **11** should be expressed in the form $ax - b$. All equations should be expanded and simplified.

Figure 6. An explicit suggestion in the teacher guide (Erdman et al., 2008b, p. 52) to write all factors involving rational zeros in the form $(ax - b)$

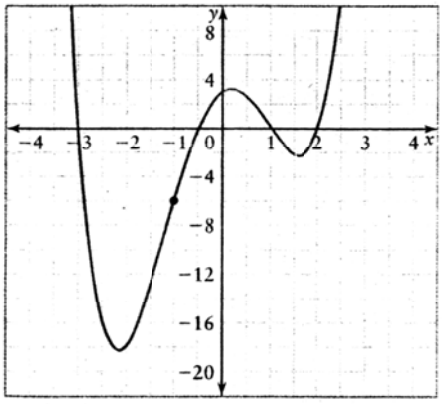
The advice displayed in Figure 6 is not accompanied by any justification for the use of the form ($ax - b$), nor is there any discussion as to how a teacher might respond to potential students' questions regarding the issue as to why they are to use the form $(ax - b)$ and not ($x - b/a$). In fact, the teacher is not even alerted to the possibility that such a question might arise. Additionally, no explanation is provided as to why "all equations should be expanded and simplified." Question 10, to which the suggestion given in Figure 6 refers, reads as follows: *Determine an equation for the family of quartic functions with zeros –5/2, –1, 7/2, and 3.* In accordance with the directive given in Figure 6, the equation for the given family of quartic functions ought to be written as $P(x) = k(2x+5)(x+1)(2x-7)(x-3)$. But an obvious question is why one might not instead write the function in the following form: $P(x) = k(x+5/2)(x+1)(x-7/2)(x-3)$.

Analysis of the Unfolding of Mae's Lesson on Families of Polynomial Functions

The mathematical problem on which Mae had decided to focus in her lesson on families of polynomial functions was one that involved a rational root. It was a variation of an example that was worked out in the student textbook (see Figure 7).

Example 4 **Determine an Equation for a Quartic Function From a Graph**

Determine an equation for the quartic function represented by this graph.



Solution

From the graph, the x-intercepts are -3, $-\frac{1}{2}$, 1, and 2.

The corresponding factors are $x + 3$, $2x + 1$, $x - 1$, and $x - 2$.

An equation for the family of polynomial functions with these zeros is

$y = k(x + 3)(2x + 1)(x - 1)(x - 2)$.

Select a point that the graph passes through, such as $(-1, -6)$.

Substitute $x = -1$ and $y = -6$ into the equation to solve for k.

$$-6 = k[(-1) + 3][2(-1) + 1][(-1) - 1][(-1) - 2]$$
$$-6 = k(2)(-1)(-2)(-3)$$
$$-6 = -12k$$
$$k = 0.5$$

The equation is $y = 0.5(x + 3)(2x + 1)(x - 1)(x - 2)$.

Figure 7. The mathematical problem that was used as a basis for Mae's lesson on families of polynomial functions (drawn from Erdman et al., 2008a, pp. 117-118)

Mae had taught the same lesson in two other Grade 12 classes earlier the same day. We note that during each class, including the third, she had emphasized that all members of a family of polynomial functions share the same *x*-intercepts. The only features that differed for each member of a given family, she said, were the value of the leading coefficient *k* and the accompanying stretching or compressing of the graph. She had also drawn students' attention to the fact that there would be only one correct value of the parameter *k* for any given member of a family of polynomial functions. We now

recount the unfolding of the lesson as it occurred in the third class of the day. The reader will notice how closely Mae follows the textbook presentation of the problem she chose to use in her mathematics lesson.

The lesson on Families of Polynomial Functions began with Mae presenting the following three functions – all of them being members of one family of polynomial functions:

$$f_1(x) = 2x^3 - 4x^2 - 10x + 12$$
$$f_2(x) = -x^3 + 2x^2 + 5x - 6$$
$$f_3(x) = -2.5x^3 + 5x^2 + 12.5x - 15\,.$$

Mae displayed the definitions of the functions on the whiteboard at the back of the room [see Figure 8] and asked students to copy them down and then to graph them on their CAS calculators: "Open up a graphs page on your calculator. Given f_1, f_2, and f_3, what do you notice about all three functions?"

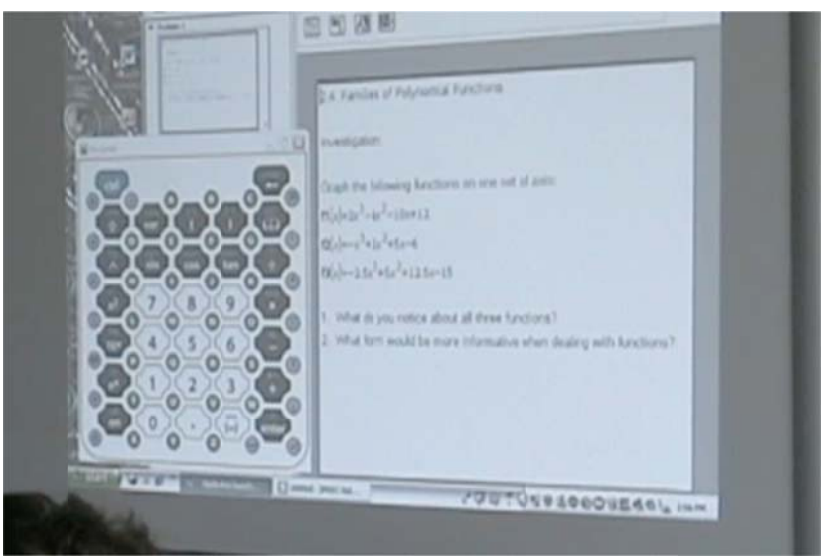


Figure 8. The opening of the lesson, with its accompanying CAS and whiteboard technology

After students had spent some time trying to find appropriate graphing windows, Mae asked them what common characteristics the functions shared. One student mentioned that they were all of degree three and another that they had the same *x*-intercepts. Following up on the latter idea, Mae asked if they were able to tell from looking at the given expanded forms that the three functions had the same *x*-intercepts. "So how could you make it more obvious?", she asked. When one student suggested "factoring them", Mae responded: "Yes, when you factor them, you have your function in a form where you can easily see that the *x*-intercepts are similar." She then asked students to split their graphs page in two so that they could insert a calculator page for the factoring of the three functions. It is noted that a certain amount of time was devoted to taking care of technical aspects of the CAS, such as splitting a page in two and then copying the three functions to that page.

The factored form of the three functions was as follows:

$$f_1(x) = 2(x+2)(x-1)(x-3)$$
$$f_2(x) = -(x+2)(x-1)(x-3)$$
$$f_3(x) = -2.5(x+2)(x-1)(x-3)$$

Mae then continued with her lesson, as illustrated by the following extract of classroom dialogue. It was soon to lead to the problem associated with a factor that corresponds to a given rational zero.

Teacher: So, in factored form, right away you can see that they all share –2, 1, and 3 as *x*-intercepts. So, if you are looking at all of these three graphs and they all share the *x*-intercepts, why do they look so different on your graphs page?

Student1: Coefficients and translations.

Student2: Leading coefficients.

Teacher: So you can express it in different ways: leading coefficients, stretches, compressions. OK, so if you look at the leading coefficients, there's a two in one of them, negative one in the other one, and negative two point 5. Alright.

So, this section (2.4) is titled, Families of Polynomial Functions. And by definition if you have polynomial functions, all with the same *x*-intercepts, they're within the same family. Is everyone OK with that?

So another way I can ask you questions would be something like this. So here [referring to the whiteboard where the general form for families of polynomial functions was displayed: $f(x) = k(x - a_1)(x - a_2)(x - a_3)...(x - a_n)$, where $k \in \Re$, $k \neq 0$] is the basic definition of the functions you were dealing with before, where if you have all the zeros, all the *x*-intercepts being the same, and the only thing that differs is your value – and here they label it *k* – in front, basically you can say that this family of polynomials, they share the same characteristics, they're in the same family.

Then I can ask you something like question #3 [which was then displayed on the whiteboard]:

A function has x-intercepts –3, –(1/2), 1, and 2, with point (–1, –6) on the function. Determine the equation of the polynomial function.

What #3 is asking you to do, you're given specific *x*-intercepts, they want you to find the equation of the polynomial function. But along with the four *x*-intercepts, they also give you a point. What do you think the point is going to help you determine?

Student1: the *k*.

Teacher: Right, the *k*. Thank you very much. So try to give me the equation of the polynomial function. And remember there are two ways to present the equation of a

polynomial function, or two forms. It's up to you which form you want to present. But obviously in factored form, you can get the *k* easily.

After working on the problem, various students stated the values that they had obtained for *k*, not all of them arriving at the same value. So, Mae asked a few students with different answers to go to the board to show their work, but first insisted that they all use the basic form, which she wrote at the board as follows:

$f(x) = k(x - a_1)(x - a_2)(x - a_3)(x - a_4)$. She also asked that everyone show the factors they were using and how they were substituting-in the coordinates of the given point.

One student began writing at the board the following equation (see Figure 9):

$f(x) = k(x + 3)(2x + 1)(x - 1)(x - 2)$, clearly using the factored form $(ax - b)$ to represent the rational zero –1/2.

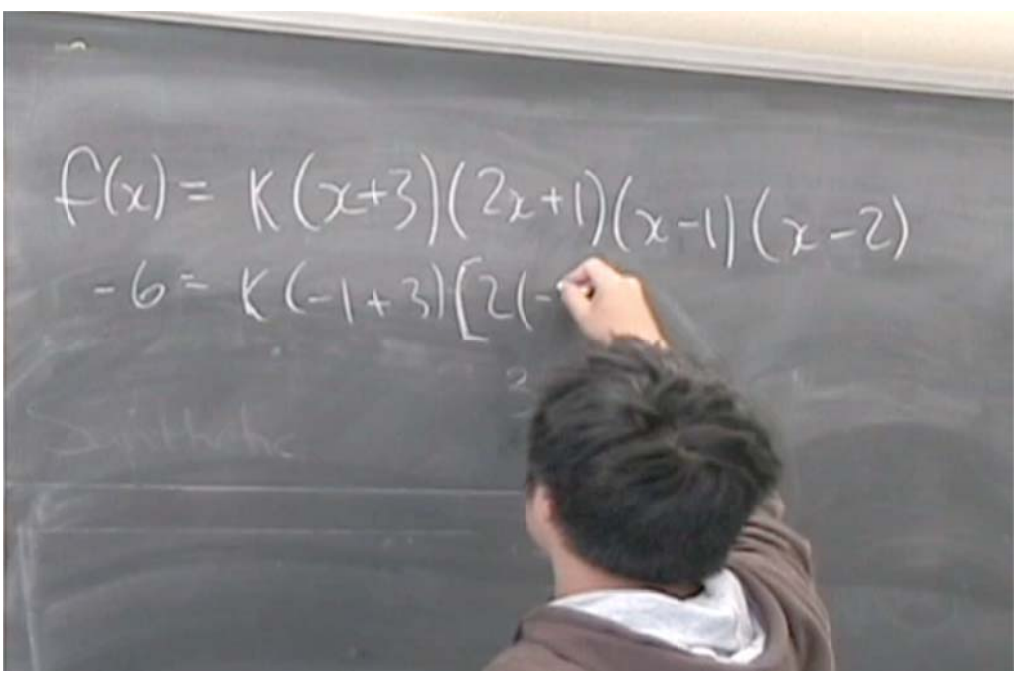


Figure 9. At the blackboard, one student writing his version of the requested equation

The other students who were working at the board used a similar form for the second factor. This was clearly a reflection of the work they had done earlier in the week on the Factor Theorem. Despite the fact that Mae had just a few minutes earlier mentioned that they all should employ the general form, whose factors were of the form, $x - a$, she did not remark on the students' use of the form $ax - b$. It conformed, after all, to the form suggested in the teacher's guide. The student, after substituting-in the coordinates of the point for the *x*'s and *f(x)*, obtained the result of $1/2$ for *k*. So too did all of the others who were showing their work at the board. The various erroneous values that they had earlier obtained for *k* were self-corrected.

Teacher: Well, so, we all got a half. So you all determined your polynomial function equation all in the same way. Did anyone happen to write their function differently?

Student1: Well, you could expand your function first and then plug it in.

Teacher: Actually, that's correct. So, it actually turns out to be the same thing. But did anyone write this part differently [pointing to the four factors of the expression]? [No one said anything]. So, everyone was able to write their factors as either *x* plus or minus *b*, or *ax* plus or minus *b*. Is everyone OK with that?

Student3: Why can't you use $(x + 0.5)$ for the *x*-intercept of $-1/2$? Like for $(2x + 1)$.

The teacher seemed unsure as to what Student3 was proposing. So, she asked him to come forward to write it at the board, which he did: [he wrote $x + 0.5$].

Teacher: Ooh! Very good question. So. Let's all try this. Instead of using $(2x+1)$, use $(x+1/2)$. Tell me what happens when you use $(x+1/2)$ instead of $(2x+1)$.

Student 4: You get 1.

Teacher: OK, you get 1. So you get something completely different. Right. So why do you get something completely different?

Student 5: Divide that part by 2 and then write in the rest of it [clearly referring to the $2x+1$, but his technique was not clearly and completely stated].

Teacher: OK, good [without expanding on the student's partial suggestion], so your entire expression is actually completely different.

Here in lies the crux of the mathematical difficulty. The teacher appears to see the function with its different value of *k* as another member of the family of polynomial functions, and not as the same member: that is, that $f(x)=1/2(x+3)(2x+1)(x-1)(x-2)$ and $f(x)=1(x+3)(x+1/2)(x-1)(x-2)$ are two different members of the same family. We reiterate that neither of the resources she was using had led her to think otherwise. She attempted to explain this phenomenon to the class in the following manner, focusing on the fact that the zeros were the same, but the *k*'s were different:

Teacher: So your *x*-value here [in 2*x* + 1] is –1, so when you go 2 times –1 plus 1, you get –1. But when you put –1 in here [in $x+1/2$] plus $1/2$, you get $-1/2$. Right, so you get two totally different values, so your *k* will be different.

Student1: Isn't that also right though?

Teacher: Is this [pointing to $2x + 1$] a different intercept from this [pointing to $x+1/2$]? We have $2x + 1$ and $x+1/2$ [she writes on the board $2x+1=0$ and $x+1/2=0$]. So, what does *x* equal in the two cases? So, they're the same answer, right [i.e., the same zero or *x*-intercept]. But we're getting different values [for each] because, in $2x+1=0$, you double something and then you add, and in this [$x+1/2$] you just add something. So, according to the order of operations, you get different values of *k* here. Right.

Student6: So how do you know it's not $(4x+2)$, because the *x*-intercept is still $-1/2$?

Teacher: That's very good, but you actually don't know that. You don't know if that would be $(4x+2)$. Although again what you're trying to do is figure out what kind of leading coefficient you have there. OK.

Mae's 'explanation' of the phenomenon at hand showed her to be oblivious at that moment to any consideration that the two algebraic forms might be equivalent. If she had realized that the factoring of $(2x+1)$ as $2(x+1/2)$, followed by the multiplication of the 2 with the *k*-value of $1/2$, would yield an equivalent second form of the given expression, the problem might have been resolved. Furthermore, Student6's question regarding the possibility of using $(4x+2)$ for the $(2x+1)$ factor (or any of an infinite number of other possibilities for the factor representing the *x*-intercept of $-1/2$) might have been discussed in terms of there being no difference whether one uses one form of the factor or another, because the resulting different value of *k* would maintain the equivalence. The following are all equivalent: $1(x+3)(x+1/2)(x-1)(x-2)$;

$1/2(x+3)(2x+1)(x-1)(x-2)$; $1/4(x+3)(4x+2)(x-1)(x-2)$; and so on. They are all the same member of a certain family of polynomial functions, despite their having different *k*'s. Mae's distinction between different members of the same family, based on the criterion of having different *k*'s, had failed to take into account the role played by different possible forms of a factor that represent the same *x*-intercept, or zero, when it is a rational number. The textbook resources she had just begun to use had not alerted her to this phenomenon.

As if to prove her point about the two functions being distinct members of the same family, Mae then suggested to the class that they expand the two – but was somewhat taken aback by the result. When the expanded results came out to be the same, the teacher wondered aloud if she had not mistakenly entered the same expression twice into her computer, which was hooked up to the whiteboard. The following classroom discussion ensued.

Student1: Even though the *k* is different, it is still the same thing. Whatever you do to the factor, you are also doing to *k* [not quite correct, but on the right track]

Teacher: I am not sure that you are all following this. For the second one, we got a different value of *k*. And what do you find when you do it [that is, expand the expression: $1(x+3)(x+1/2)(x-1)(x-2)$]?

Several students at once: The same thing!

Student1: Witchcraft!

Teacher: [recovering somewhat from her surprise, but still at a loss for words] Does it make a difference? [Looking around the class] Do you understand why that, even

though, because of how you are phrasing the question, or your factors, you are going to get your different values of *k*. Remember some people were saying that when you expand it, you should still get the same thing anyway [what had actually been suggested earlier by one of the students was related to expanding just one expression that was in factored form and not expanding two seemingly different expressions]. Well, when you expand it [the two seemingly different factored forms], you can see that the functions are still the same. Generally, we do use the $ax \pm b$ form, but obviously you can see that we are dealing with the same function. Right. So thank you very much for your question, Student3.

At this moment, the teacher quickly brought her lesson on families of polynomial functions to an end. The mathematical issues that had arisen clearly required further reflection on her part.

Discussion

The issues we wish to discuss here are threefold: the mathematical gaps of textbook resources, the process of becoming aware of and overcoming these gaps which constitutes a form of ongoing professional development for a teacher, and the evolutionary nature of documentational genesis whereby resources gradually become transformed into documents.

The new textbook and teacher guide that Mae had used as resources for her lesson had not provided the level of mathematical support that she needed. They had not alerted her to the issues surrounding the two forms of a factor representing a given rational zero of a function, and the accompanying impact on the value of the parameter *k*. The

resources had been silent about both the technique for converting from one factored form to another and the equivalent nature of the two. Chevallard and Cirade (2010) have discussed the question of missing mathematical resources and have identified this as a major praxeological problem for the profession.

It was in the act of teaching her three classes on a given day that Mae became aware of the mathematical deficiencies of the textual resources with which she had prepared her lesson on families of polynomial functions. She had not been equipped to handle the questions put to her by her students and had to react on the fly in an ad hoc and inadequate fashion. Nevertheless, she seemed to learn from the experience. Zaslavsky and Leikin (2004) have pointed out that, by listening to students and observing their work, and by reflecting on this work, teachers learn through their teaching. Mason (1998) has emphasized that it is one's developing awareness in actual teaching practice that constitutes change in one's knowledge of mathematics and mathematics teaching and learning.

By taking seriously her students' questions regarding the relationship between two seemingly different factored forms of a function, Mae became sensitized to mathematical aspects of the given area of study that she had not heretofore considered. Her knowledge of families of polynomial functions was in the process of being transformed by what transpired in her class, especially by the thought-provoking queries of her students. According to Zaslavsky and Leikin (2004), such in-practice activity can be an effective vehicle for teachers' own professional growth. Although Mae's primary preoccupation was the teaching of the material on families of functions, she was at the same time engaging in the problem that she was putting to the students. She, with the

collaboration of her students, was developing her knowledge of the mathematics of this area.

In their theoretical paper on documentation systems for mathematics teachers, Gueudet and Trouche (2009) introduce a general perspective for the study of teachers' professional evolution, where the researcher's attention is focused on the resources, their *appropriation* and *transformation* by the teacher or by a group of teachers working together. Their approach aligns with Adler's (2000), who claims that, "in mathematics teacher education, resources in practice need to become a focus of attention" (p. 221) and with Remillard's (2005) whereby the evolution of the curriculum material actually used and a teacher's professional development are viewed as two intertwining processes.

With respect to this intertwining process, Gueudet and Trouche (2009) point out that:

> A teacher draws on resource sets for her documentation work. A process of genesis takes place, producing what we call *a document*. … A given teacher gathers resources: textbooks, her own course, a previously given sheet of exercises... She chooses among these resources to constitute a list of exercises, which is given to a class. It can then be modified, according to what happens with the students, before using it with another class during the same year, or the next year, or even later. The document develops throughout this variety of contexts. (p. 205)

We suggest that the awarenesses acquired by Mae in her teaching of families of polynomial functions with new resources will be instrumental in enabling her to modify these resources, thereby leading to the gradual transformation of a resource into a document for her. However, Gueudet and Trouche (2009) emphasize that "documentational genesis must not be considered as a transformation with a set of resources as input, and a document as output. It is an ongoing process … that *continues in usage*. We consider here accordingly that a document developed from a set of resources

provides new resources, which can be involved in a new set of resources, which will lead to a new document etc. Because of this process, we speak of the *dialectical* relationship between resources and documents" (p. 206).

We close our discussion by turning to a relevant comment made by Adler (2000) that puts the focus not on producing more (or better) resources, but rather on better understanding how teachers use the resources they have, change them, and in the process engage in a form of ongoing, personal, professional development: "Our attention shifts away from unproblematised calls for more [resources] and onto the inter-relationship between teacher and resources and how, in diverse, complex contexts and practices, mathematics teachers use the resources they have, how this changes over time, and how and with what consequences new resources are integrated into school mathematics practice" (p. 221). In this article, we have attempted to illustrate the complex interrelationship within actual teaching practice between a teacher and a new set of resources, by describing the nature of the classroom experiences whereby a teacher becomes aware of the mathematical gaps of new resources and thus better positioned to make changes to them over time. Such an approach both situates resources and their adaptive use within a documentational framework and re-centers professional development within the actual practice of teaching.

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