The Mathematics Enthusiast

Volume 10
Number 1 *Numbers 1 & 2*

Article 11

1-2013

# Developing problem solving experiences in practical action projects

François Pluvinage

# Developing problem solving experiences in practical action projects

*François Pluvinage*
*Departamento de Matemática Educativa, Cinvestav - I.P.N., México*

**Abstract:** Problem solving doubtless is an essential element of mathematical learning, so that mathematics educators often are satisfied when finding situations that lead their students to such activity. But in many cases, the chosen situations and the ways to guide students' works are not sufficiently analyzed from a didactic point of view. Our goal in the present analysis is to underline the possible ways for managing the situations, and to exhibit the parameters that educators have at their disposition within their role as mediator between students and mathematical knowledge and know-how.

**Keywords:** Problem solving, problem for investigating, project based learning, a priori analysis, mediation.

## 1. Classical problem solving

Problem solving at its broad extent is clearly exposed by Alan Schoenfeld (Schoenfeld, 2006, p.41): "[Problem solving] *includes a child's actions in interacting with its parents, a student working on a mathematics problem in class or in the laboratory, and a teacher's decision-making while teaching a mathematics (or other) lesson."* In this paper, we will only consider problem solving in mathematics instruction. Nevertheless, what we mostly found in mathematics educational literature and in learning material designed for students is a restricted use, with sometimes a reduction to drill for improving skills. In our opinion, this approach is too limited. Thus we first analyze some typical situations used in mathematics education with a problem solving purpose.

Our goal in the following analysis is to underline the possible ways for managing the situations, and to exhibit the parameters that educators have at their disposition within their role as mediator between students and mathematical knowledge and know-how.

### 1.1. Too easy… and how to do it better

The following *Math Problem Solving* example comes from a worksheet of Rhl School. We may consider it as a representative of certain trend in exploiting problem solving in mathematics education.

**“Ryan’s Class**

There are exactly twelve children in Ryan’s class. Only four of the children are boys. The following questions refer to a time when all the children are present in the class. There are no visitors in the class. There might be more than one correct answer to a question.

1. Which of the lettered statements **must** be true?
2. Which of the lettered statements **cannot** be true?
3. Which of the lettered statements could be true **or** not true?

a. There are **twice** as many girls as boys in Ryan’s class.
b. There are eight **more** girls **than** boys in Ryan’s class.
c. There are four **more** girls **than** boys in Ryan’s class.
d. If Ryan is sitting at a table with all the girls, there are exactly nine children at that table.
e. If **only** three of the boys are standing on their heads, one of the boys is not standing on his head.”

**Math Problem Solving, Vol. 8, No. 1, May 5, 2003.** Retrieved from http://www.rhlschool.com/math8n1.htm

Comments about the task: Giving an answer to the lettered statements a, b, c and e only supposes to understand them. The only true *problem* here comes from the fact that we don’t know Ryan’s gender (while mostly used for boys, *Ryan* has been used for girls in the United States since the 1970s). And for the lettered statement d, if Ryan is a boy, the statement is true, and if Ryan is a girl, the statement is false; then the correct answer **for us** is the answer 3, but the correct answer would be different for somebody who knows Ryan's gender. So we note that there is a subtle semantic distinction in that statement; nevertheless, in terms of mathematics knowledge, we will say that the proposed task is *too easy*. That means that the mathematical activity for answering is poor, even though understanding the statements may have a relatively high intellectual cost for the students. Compare with the situation that would result from the following questions, without changing the data (i. e. a class of 12 students, four of them being boys):

> We try to form groups, all having the same number of students (for instance two groups of six students each), or all having the same numbers of girls and boys (for instance two groups, each one composed of four girls and two boys). What numbers of groups can we form in each case?

For such reasons, researchers later followed in France by official curricular commissions, introduced precisions about problem solving, considering *open problems* and *problem for investigating*. What follows is translated from a French text by Roland Charnay (Charnay, 1993).

> "The team of the IREM of Lyon offers the following definition.
> An open problem is a problem that has the following characteristics:
> - The statement is short.
> - The statement does not induce either method, no solution (no questions intermediate or questions like "show that"). This solution should never be reduced to the use or the immediate application of the latest results presented in class.
> - The problem is in a conceptual domain to which students have enough familiarity. Thus, they can easily take "possession" of the situation and engage in testing, conjecture, draft resolution, counterexamples.
>
> **Example 1 (extracted from "Rencontres Pédagogiques", n°12, INRP)**
> I have 32 coins in my piggy bank.
> I only have coins of 2 F and 5 F.
> The total amount of my 32 coins is 97 F.
> How many coins of each value are in my piggy bank?
> **Example 2 (extracted from "Situations problèmes", APMEP, Elem-math IX).**

What is the biggest product of two natural numbers that we can obtain using each digit from 1 to 9 once for writing the numbers?"

The reader can find more precisions about Problem for Investigating in (Houdement, 2009), in particular a list of 24 representative statements, heuristic elements and perspective for research.

### 1.2. OK but limited except for students at an advanced level…

In this section, we present a situation well appropriate for sequences of investigation by 4 or 5 grade students. Thus it generates a good students' training in numerical treatments. But its institutionalization in Brousseau's meaning will be difficult until an advanced level, for instance undergraduate, because of the difficulty for enunciating and proving the general result. So in this case, the teacher can exploit the situation for younger students with an objective of arithmetical training or acquisition of methods, but not of learning new mathematics knowledge.

> "A statement for students of all levels:
> The number 23 can be written in many ways as a sum of natural numbers.
> For example:
> 23 = 11 + 5 + 7. Among these sums, find the one whose product of the terms is the biggest (in our example, the product was 11×5×7 = 385).
> Other statement is: Among the additive decompositions of a natural number, find the one, whose product of its terms is the biggest."

Retrieved from http://educmath.inrp.fr/applet/exprime/plgrprge.pdf and from http://gilles.aldon.free.fr/ensemble/Ensemble/syplgrprtel.pdf

In this situation, we can first discover the answer, either in particular cases (such as 23, that produces 4374 = 3×3×3×3×3×3×3×2 as highest product) or in the general case in a descriptive way. At more advanced levels, a new possibility is to deepen the problem, and express and prove the general solution in an algebraic way. So does Gilles Aldon when he writes what follows.

> The Sloane Encyclopedia (the OEIS Foundation) gives some properties of this biggest product at http://oeis.org/A000792. It quotes a dozen of different definitions of the number a(n) equal to the biggest product obtained with partitions of n in sum. If we take $a(0) = 1$, then we obtain the induction formula: $a(n) = \max\{(n-i)a(i), i<n\}$. This is the definition given in the Encyclopedia.

In such a situation, a teacher does not have possibility to adapt the situation with some changes in the statement. He/she has only to know that the situation is convenient for arithmetic training of young students, and for improving the use of induction by older students.

### 1.3. Too difficult because insufficiently explored by the interested teachers…

**Analyzing a problem of a mathematical contest: the "Math Rally of Alsace"**

We translate here the statement of one of the problems of a mathematical contest, the "Math Rally", followed by the solution and comments given by the organizing team, retrieved from http://irem.u-strasbg.fr/php/index.php?frame=.%2Fcompet%2Fcompet.php&m0=ral&categ=rallye.

> **Exercise 3 (grade 11)**
> In the Valley of Bruche River, a clearing has a circular shape. A treasure is buried near the clearing. An old parchment shows the location of the treasure: "From the great fir located on the circle, go to the poplar in the clearing. Then turn right at a right angle and walk to the edge of the clearing. Still turn right at a right angle and walk as many steps as from fir to poplar. There is buried treasure."
> The clearing has a radius of 20 meters, and the only tree on the clearing is a poplar located at 4 meters from the centre. Unfortunately, on the edge of the clearing, firs disappeared long time ago. Can you find the distance between the centre of the clearing and the treasure?

**Solution**

As is often the case in Math Rally, the statement tells a story, placed here in Alsace. The story is about a treasure, a fir tree, a poplar. Why not denote them T, S, P and label O the centre of the clearing, i.e. the circle.

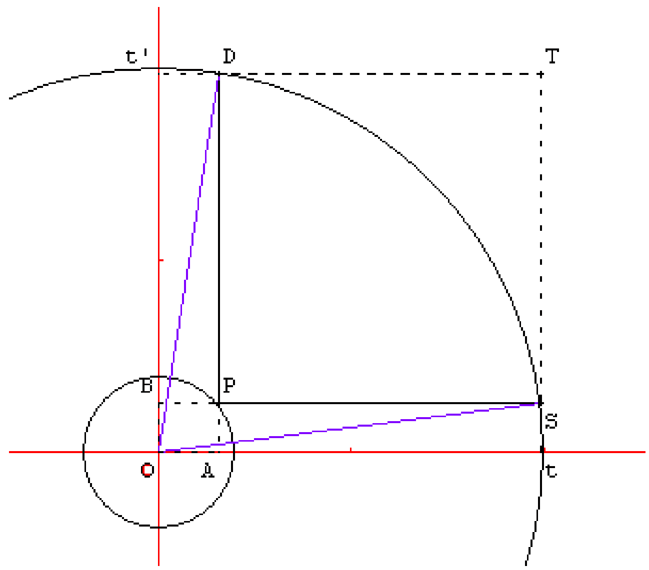


Figure 1: Geometric representation of the Math Rally problem

In the Cartesian coordinate system (O, I, J) with x- axis parallel to the line (SP), let A and B the points obtained by orthogonal projection of P on the coordinate axes. We know that OP = 4, OS = OD = 20. As SPDT is a rectangle, SP = DT and PD = ST. Let $a$ = OA, $b$ = OB, $c$ = PS and $d$ = PD. Applying the Pythagorean Theorem to the triangles OBS, OAD and OAP, we obtain the relationships:

$OS^2 = OB^2 + BS^2 = b^2 + (a + c)^2 = 20^2$ (1)

$OD^2 = OA^2 + AD^2 = a^2 + (b + d)^2 = 20^2$ (2)

$OP^2 = OA^2 + OB^2 = a^2 + b^2 = 4^2 = 1$ (3)

Computing (1) + (2) – (3), we obtain:

$(a + c)^2 + (b + d)^2 = 784.$

Since $(a + c)^2 + (b + d)^2 = OT^2$, then OT = 28.

Thus the distance that we sought is 28 meters.

**Comments by the organizing team**

This is a geometrical exercise and it was the less successful for students. Yet its resolution only involves the Pythagorean Theorem, applied to several right triangles of course.

Candidates who worked on this exercise drew a figure that includes letters often undefined. Didn't they have the habit of labeling a figure in order to make it understandable? The resolution led them to the correct answer (28 meters), but most of them relied on a particular case – for example points S, O and P collinear – and they do not consider the general case. Some say that they were only considering a particular case, but many students seem not to realize it. (…)

**Our analysis**

Thus in its report, the organizing staff itself recognizes that this problem was unsuccessful. But we think that this poor result originates in a lack of mathematical analysis by the organizing team. The members of this team are very experienced teachers and competent mathematicians, but they were more preoccupied in this case by the design of an amazing story than by didactical considerations. As a matter of evidence, we refer to the solution given in the report, which is for us not satisfactory, because it is not convincing. It is like the rabbit that comes out of the hat of a magician: we measured lengths, and it appears that the final result does not depend on the variable elements. A "good" solution of a problem produces a change in our mind with respect to the situation: from mysterious the result becomes evident for the reader. As an example, let us consider a classical result: The three altitudes of a triangle ABC intersect in a single point, called the orthocenter of the triangle. There are several proofs of this theorem, for instance by considering geocenter and circumcenter of the given triangle ABC, or studying angles and finding that the point of barycentric coordinates $\left(\tan\hat{A}, \tan\hat{B}, \tan\hat{C}\right)$ is the searched orthocenter of the triangle. These proofs are of interest, but their cognitive routes are relatively complex. In contrast, the proof illustrated by Figure 2 easily could produce the change of mind that we want to emphasize: We trace respectively by A, B and C the lines parallel to the opposite side of the triangle, that create a new triangle A'B'C'; the altitudes of ABC are the perpendicular bisectors of the sides of A'B'C'. Thus it is obvious that they intersect at the point O equidistant of the three vertices A', B' and C'.

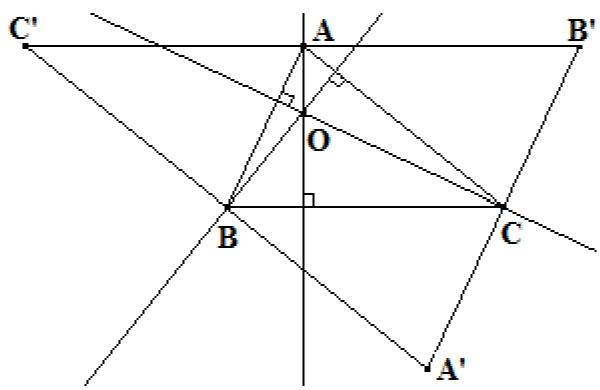


Figure 2: Altitudes of a triangle as perpendicular bisectors of a bigger triangle

For the Math Rally problem, we present below a solution approach that intends to promote the reader's mathematical reflection and analysis. Then we will consider what changes could produce a better result in implementing this problem in the classroom.

To conclude that the distance between the center O of the circle and the point T does not depend on the location of S on the circle is the same as to affirm that the locus of

T when S moves along the circle is a new circle with O as its center. This assertion does not seem easier than the given statement of the problem (Figure 3 left), but…

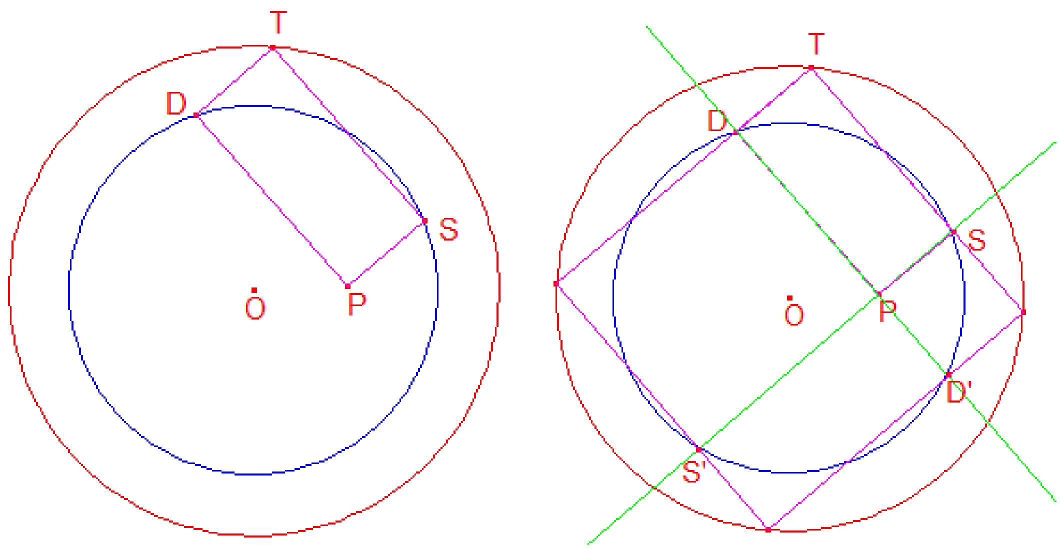


Figure 3. Left: incomplete figure, right: completed figure

If we observe that the situation is mathematically incomplete, our vision of the problem completely changes. And completing the figure is a natural idea when working with software such as Cabri or Geogebra. This involves drawing a complete line as a perpendicular to a given line or segment. After completing (figure above, right), a new statement is: "*Given a circle and a point P inside the disk, we consider two perpendicular lines passing through P and respectively cutting the circle at D and D', and at S and S'. Then the rectangle obtained by tracing by these points parallels to DD' and SS' has its center at O.*" And this is a quite evident result, because the perpendicular bisectors of SS' and DD' are the axes of symmetry of the rectangle and pass through the center O of the circle.

The given statement of the problem did not lead the candidates, of the mathematical contest, a way to focus on a geometric construction. But we see that a teacher in his (her) class can present the same mathematical situation through another statement and in a different environment that changes the didactical situation and facilitates the students’ access to it. In this case, we pretend that a modified presentation could change the vision that students have of that geometric situation.

As a conclusion at this point, we will assert that the kind of students’ work generated by dealing with a problem strongly depends on the way showed by the statement of the situation. But the form of working, particularly when using the technology, has an influence too. This enforces the role of mediator devoted to the teacher, not only when preparing the lesson but also during the class.

## 2. Project based learning (PBL or PjBL)

In the expression “Problem Solving” appears the word “Problem”, which is also present in the pedagogical method called “Problem Based Learning” and often designated by the initials PBL. This method is in use in various disciplines such as biology or medicine or engineering, but problems for the associated topics do not refer to the same kind of mathematical problems that we studied in §1. For our part, in empiric experiments, we studied the potential impact of *projects of practical action* on the teaching-learning process. We were applying a teaching strategy, presented in Cuevas & Pluvinage (2003), close to the method denominated Project Based Learning, which sometimes is also designated by the initials PBL. Here, like other authors, we use the letters PjBL in order to avoid confusions with Problem Based Learning.

Important features of our teaching strategy lie in the systematic use of registers of representation according to Duval’s terminology (Duval, 1995): formation rules, treatments within a register, and conversions from one register to other. For this reason, a first step in the study of a project of practical action is a descriptive phase, the objective of which being to introduce the formation in one register or various registers.

### 2.1. A paradigmatic example: concrete and virtual Tangram

Mexican educative institution presents Tangrams as a medium to organize learning activities at school. The Instituto Latinoamericano de la Comunicación Educativa (ILCE) has a web page on this subject at http://redescolar.ilce.edu.mx/educontinua/mate/imagina/mate3z.htm, with a link to instructions for cutting the puzzle. We retrieved the figure below from this web site. In English, for instance Wolfram's project gives a version at http://demonstrations.wolfram.com/Tangram/.

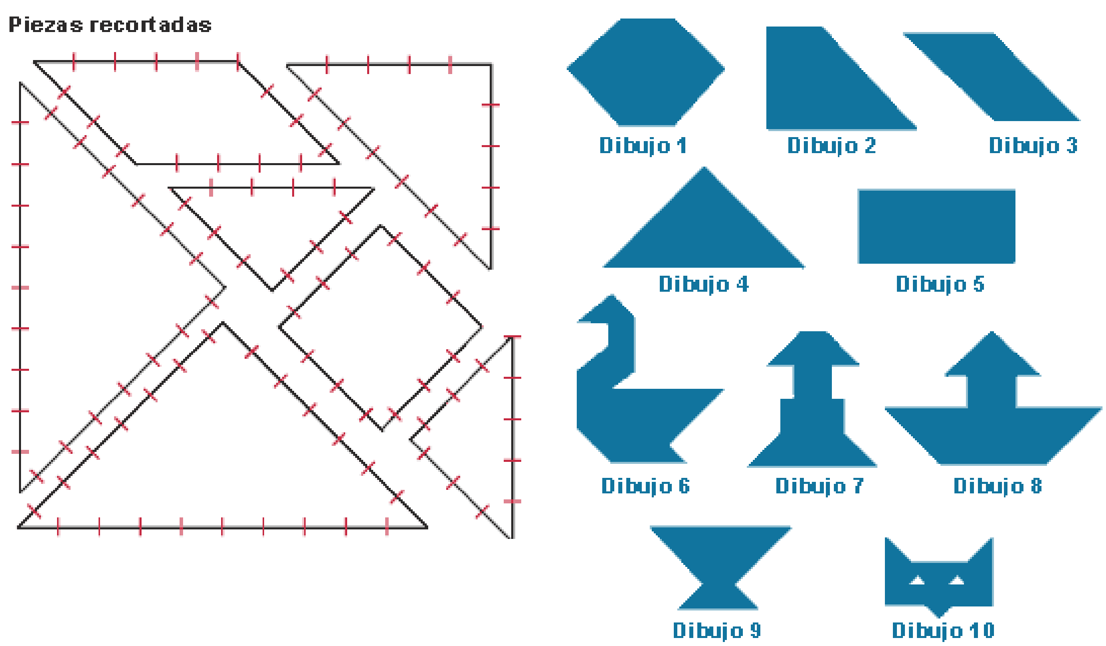


Figure 4: The seven pieces of a Tangram and shapes to be built

The web page of ILCE presents both possibilities for using Tangrams: concrete and virtual (on line). Wolfram's proposal is only a virtual one. We actually observed some use of a concrete Tangram in class and related problems, but it remains to experiment the more extended use that we present in this section.

There are many problems that the students are able to pose by themselves. For example: Can we build a triangle with the seven pieces? With the seven pieces, can we construct a triangle different from the preceding? Can we build with the seven pieces a rhombus different from a square? How many distinct rectangles can we build with the seven pieces? How many distinct rectangles can we build with some of the pieces, but not necessary all pieces? Etc. In that situation, teacher's role can be to encourage the students to invent problems and not only to solve given problems.

An interesting activity in a class might be to classify these distinct problems. Organizing principles for such a classification arise from considerations of shape resulting from values of lengths, angles and areas. For example, the shape of each piece is a polygon with angles 45º, 90º or 135º, and this leads us to a strength restriction for the possible shape of a triangle made with the seven pieces: a triangle necessarily is a right isosceles triangle. With exact lengths, we enter in the symbolic world, because the lengths of the sides of a piece in two distinct directions are incommensurable magnitudes. In the table below, we present the exact lengths of the sides of pieces with the choice of the length of the shortest side as unit.

**Table 1. Names of the Tangram pieces and exact lengths of their sides**

| Name | Side 1 | Side 2 | Side 3 | Side 4 |
|---|---|---|---|---|
| Square | 1 | 1 | 1 | 1 |
| Parallelogram | 1 | $\sqrt{2}$ | 1 | $\sqrt{2}$ |
| Triangle 1 | 1 | 1 | $\sqrt{2}$ | |
| Triangle 2 | 1 | 1 | $\sqrt{2}$ | |
| Triangle 3 | $\sqrt{2}$ | $\sqrt{2}$ | 2 | |
| Triangle 4 | 2 | 2 | $2\sqrt{2}$ | |
| Triangle 5 | 2 | 2 | $2\sqrt{2}$ | |

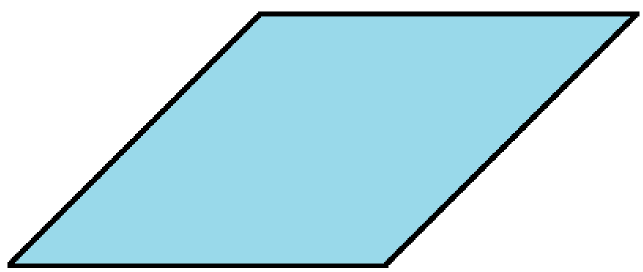

Figure 5: A shape impossible to construct with Tangram

Then, for instance when studying the situation of a rhombus different from a square (Figure 5), we can first observe that its angles might be 45º and 135º and then use the algebraic register and employ inferences like $a+b\sqrt{2}=c+d\sqrt{2} \Rightarrow \begin{cases} a=c \\ b=d \end{cases}$.

**Problem to be solved by the reader:** Prove the impossibility of constructing a rhombus different from a square with the seven pieces of Tangram.

We suggest that all the problems we have seen can be included into a *project of practical action*, and in order to do this, we will replace the concrete material by a virtual one. Then the first step is to construct the material. We used Cabri geometer for this purpose, but the design would be approximately the same if we were using Geogebra. Thus it is necessary to solve a first problem in our project: *How to move a piece in the plane?*

There is a natural relationship between movements and geometric transformations. Nowadays, it is controversial if geometric transformations have a place in the curricula at elementary level. In our opinion, moreover with the reference to a genetic point of view, the geometric transformations are a topic of high interest in mathematics education. We suggest using the introduction of transformations in the learning of geometry at secondary level in order to facilitate the transition from Geometry I (natural geometry) to Geometry II (natural axiomatic geometry) (Houdement & Kuzniak, 1999).

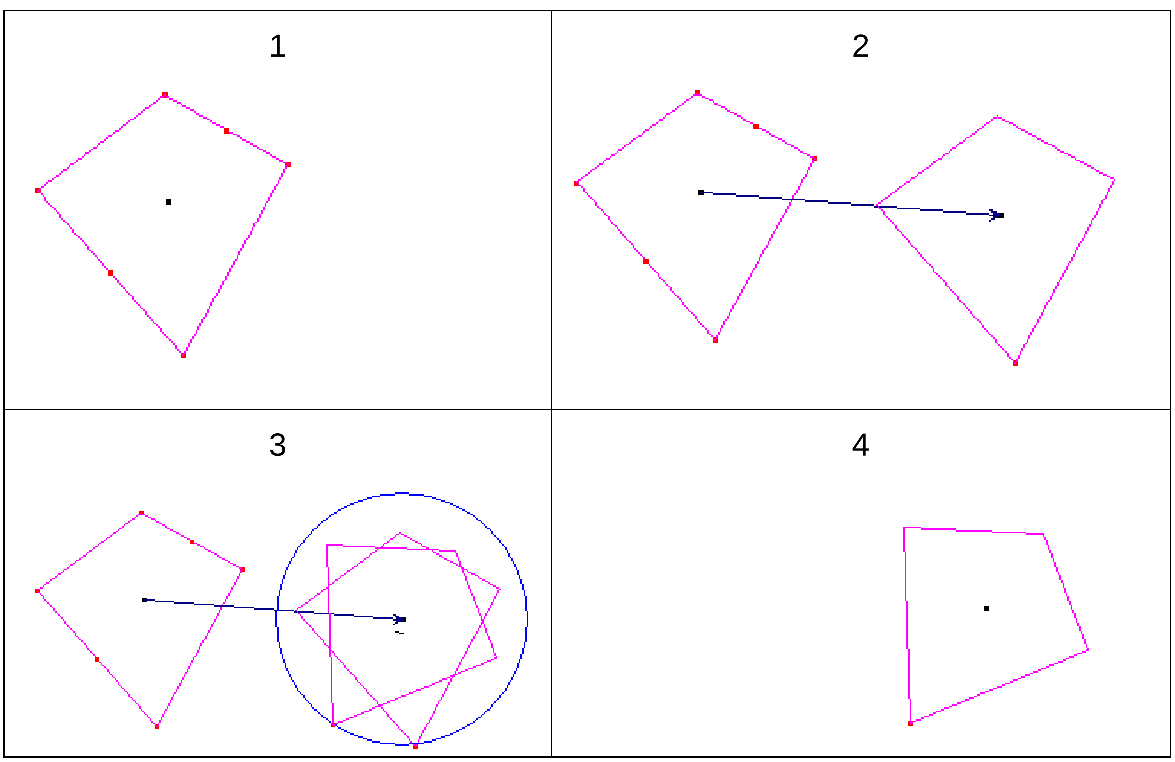


Figure 6. Modeling the displacement of a quadrilateral in the plane: 1- Choosing a center (here the midpoint of two midpoints), 2- Translating the piece, 3- Rotating the translated piece around its center, 4- Showing only the final figure with the two director points (black and red)

Figure 6 illustrates a solution obtained by a Cabri program. For each piece, we choose a center point that is represented by a black dot, and a vertex that is represented by a red dot. When moving the black dot, we translate the piece, and when moving the red dot, we rotate the piece around the black dot. In order to do that, first we build each piece and mark the black and red dots. Then we define the translation and the rotation and apply them to the piece. Finally we hide the initial piece and only show its last image.

The reader can imagine alternative possibilities, and also complete the universe of allowed transformations by adding the reflection of the parallelogram that we do not introduce in our program (with a concrete jigsaw puzzle, we can turn over the parallelogram). And one can find many interactive programs on line in Spanish and in English, but almost all such web pages are only game oriented. For example, we don't see there any place for verbal descriptions or for problems whose solutions are "impossible" to achieve. So, in this case, it seems to us preferable to promote students' work in classroom or at home. The teacher can construct a virtual Tangram and give his/her students a web page like that illustrated in the figure below, or ask them for constructing the virtual Tangram.

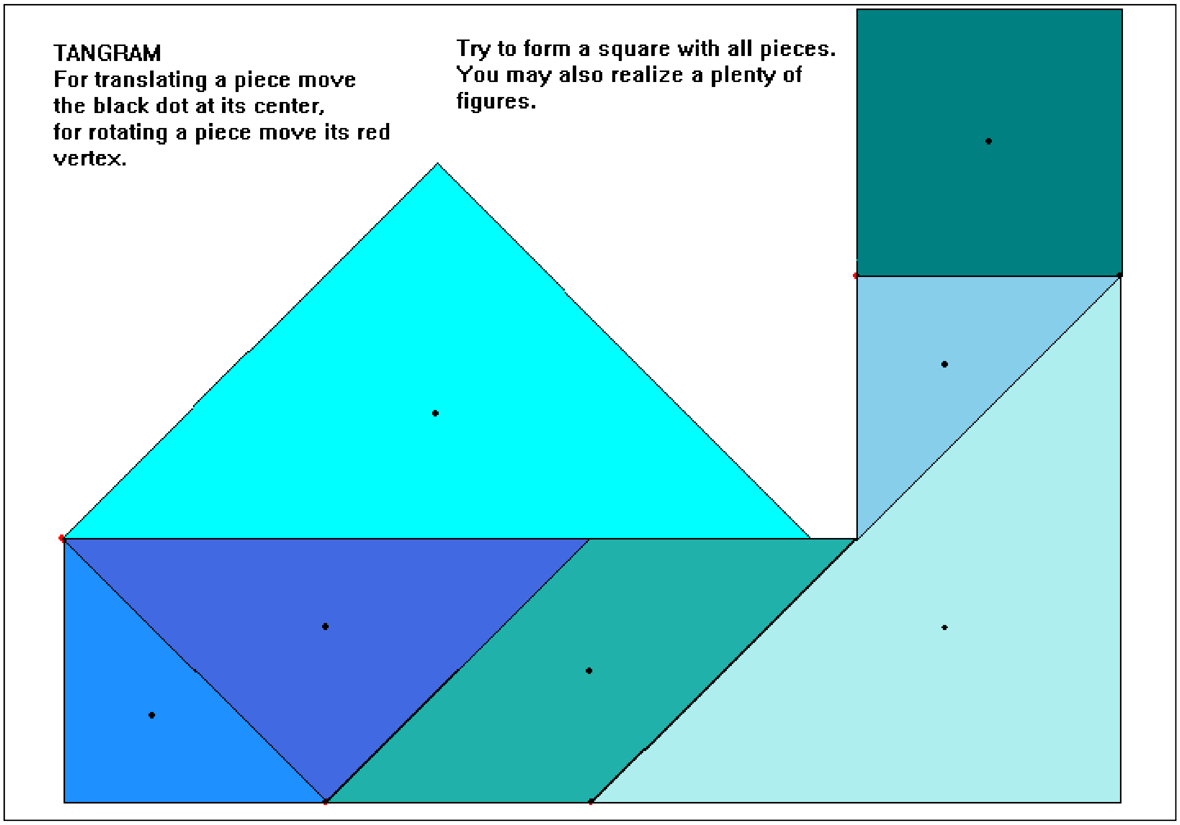


Figure 7: Virtual Tangram designed for students

Assigning the task of constructing a virtual Tangram to students by using geometric software is a good example of what Martin Wagenschein (1977) presents as *Exemplarisches Lehren*. The translation "exemplary teaching" does not exactly reflect the meaning of this expression, which refers to a specific kind of *case analysis*. The specificity of this method is that the studied cases are chosen as representative of the situations to be mastered by the student at the end of the learning process. Here, we pretend that the achievement of the construction of Tangram is significant for acquisitions of certain level of competency in mastering geometric transformations.

### 2.2. Some didactic observations resulting from empirical studies using projects

Most projects of practical action include spatial situations. So their modeling supposes to connect spatial geometry (3D-geometry) and 2D-geometry. This can be made by perspective representations or plane sections and also by nets, a net being *an arrangement of edge-joined polygons in the plane which can be folded (along edges) to become the faces of the polyhedron* (see http://en.wikipedia.org/wiki/Net_%28polyhedron%29). A great family of projects of practical action is the construction of solids subject to certain constraints.

The study of nets associated to polyhedra is a particular case of these spatial projects, which can be applied at high school or more advanced level. When we were experimenting in undergraduate classes, we realized that the majority of students do not master the easiest spatial situations, because of the lack of both knowledge and experience. For instance it was a great surprise for them to know that all convex polyhedron has a net, and that the "tower" made by the superposition of two cubes, the edges of the higher being a third of the edges of the lower, is a polyhedron whose construction with thin cardboard requires two disconnect nets.

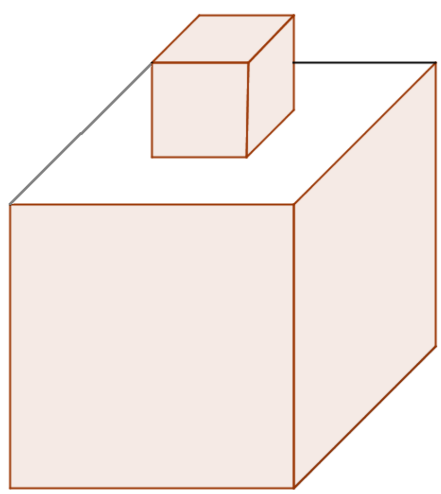

Figure 8. "Tower": Non convex polyhedron that does not have a net

The preceding example illustrates one of the natural problems that arise from studying the nets associated to polyhedra. Other problems are those of uniqueness: Can we obtain two different polyhedra with the same plane net?

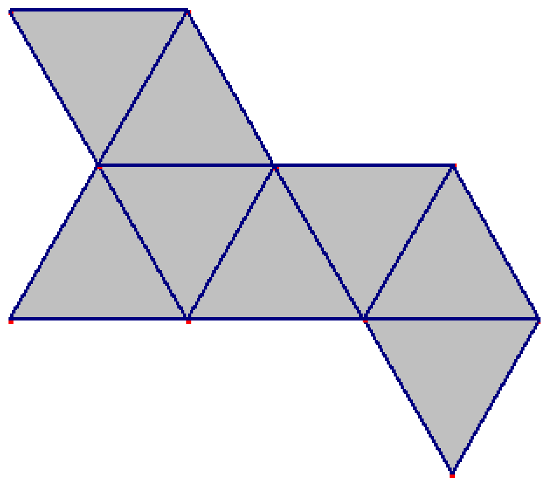

Figure 9: Same net for two different polyhedra?

**Problem to be solved by the reader:** The net above allows constructing a regular octahedron by folding paper. Try to find another (non convex) octahedron that the same net allows constructing.

Modeling spatial situations goes further than 3D-geometry. There is a lot of possible projects about containers (glasses, bottles, cans, recipients, etc.). A task for our students of the Master degree in mathematics education is to design a didactical project, and some of them choose this kind of subject. We present below the statement of a problem, which is a part of such a project that a student presented in a Web site, followed by its English translation and the representation of the solution that we made with Geogebra.

### *Optimización*

*Escrito por Paulo Angel Garcia Regalado*

*Domingo, 05 de Diciembre de 2010 01:32*

1. *Se pretende fabricar una lata de refresco de 335 mililitros de capacidad. ¿Cuáles deben ser sus dimensiones para que se utilice el mínimo posible de metal?*

Retrieved from http://grupolycaon.com/matedu/

Translation of the statement to English:

We want to make a soda can of 335 ml capacity. What should be its size in order to use the minimum amount of metal?

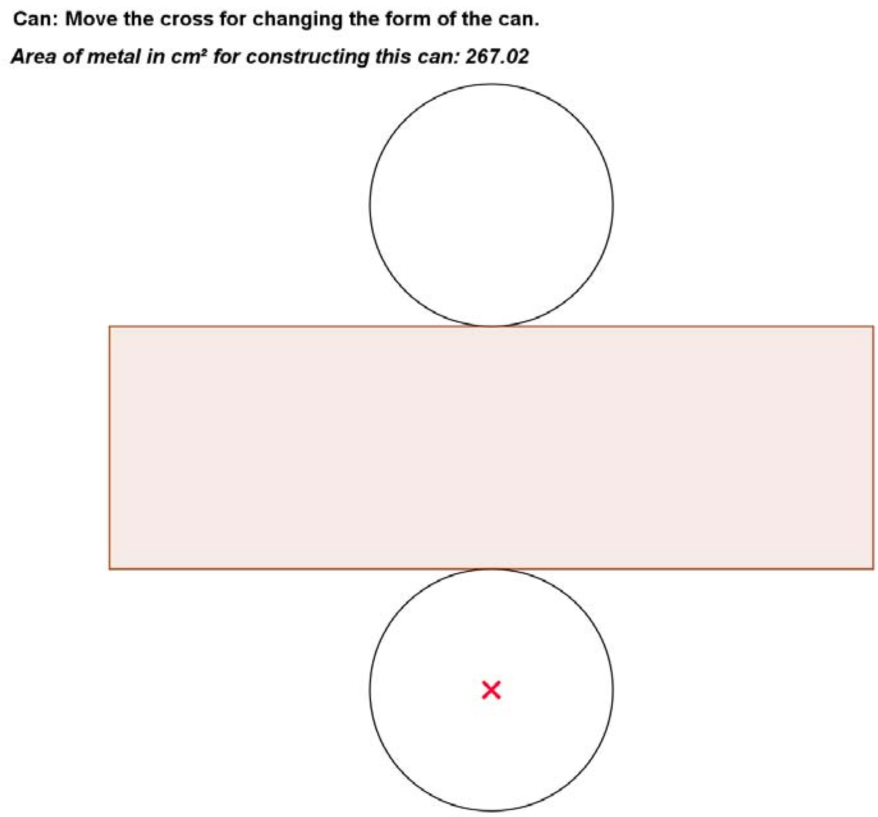


**Figure 10. Problem to be solved by the reader:** In the solution represented above for the problem of constructing a can with minimal quantity of metal, it seems that the height of the can is equal to the diameter of both disks at the bottom and the top. Is that rigorously exact?

In this last example, we saw calculus beside geometry. That is only a sample of the variety of mathematical domains and theories that the study of projects led us to encounter. Particularly, the project of modeling how recipients are filled by a regular flow of water has been a wealth field.

For instance 15-year students in Germany and France were asking by Stölting (2008) to represent in this situation the height of water in the recipient as a function of the time. We reproduce below the answers given to Stölting by a student. The last figure on the right is very interesting, because it shows a trend to the discretization of the phenomenon. Previously we emphasized the importance of verbal descriptions of a phenomenon to be studied. Here we observe another important step in an investigation process for a complex phenomenon: a qualitative approach with the help of a representation. Both verbal description and figural representation were actually present in Stölting's research. Hence, students were interviewed to explain ways they drew the figures.

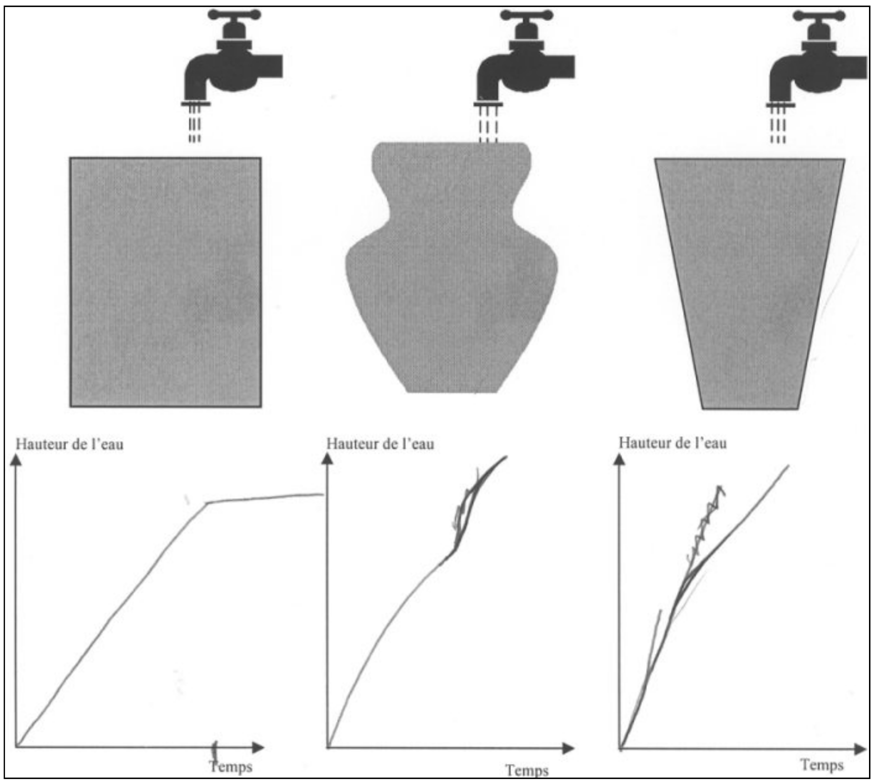


Figure 11: Answers of a (high achiever) French student

Interviews with high school students in order to deepen these observations with learning objectives are reported by Pluvinage & Marmolejo (2012) (see an illustration on next page), who describe the application in a class of the collaborative method ACODESA (Hitt, 2007), characterized by distinct phases of work: individual, group and collective, for exploiting the complete situation of filling recipients. We observed that a first step is important, namely to consider various cylindrical recipients. Indeed, many students think that, in the case of a cylindrical recipient, diameter and height play the same role for the volume. So it is important to pay attention to this case studying the volumes of various cylindrical recipients, for instance as pouring water from one recipient to another that has double diameter and half height. Many students are surprised by the fact that the second recipient is not full. Moreover the same behavior as that related before was observed with our students. With access to the convenient mathematical tools, this would lead us to use an ordinary differential equation for solving the problem.

Nevertheless there is another easier way for modeling the phenomenon, changing the theoretical frame of reference. It is the consideration of reciprocal function. Indeed, the volume of a truncated cone of height *h*, radio of lower disk *a*, and radio of upper disk *b*, is given by $\mathrm{V}_{total} = \frac{\pi h}{3}\left(a^2 + ab + b^2\right)$. Thus we can represent the volume as a function of the height, and then by symmetry we can conversely obtain the height as a function of the height. We used Geogebra in the illustration. But, and we do not know the precise reason, students do not spontaneously go on that way. This needs a help to the students from the teacher.

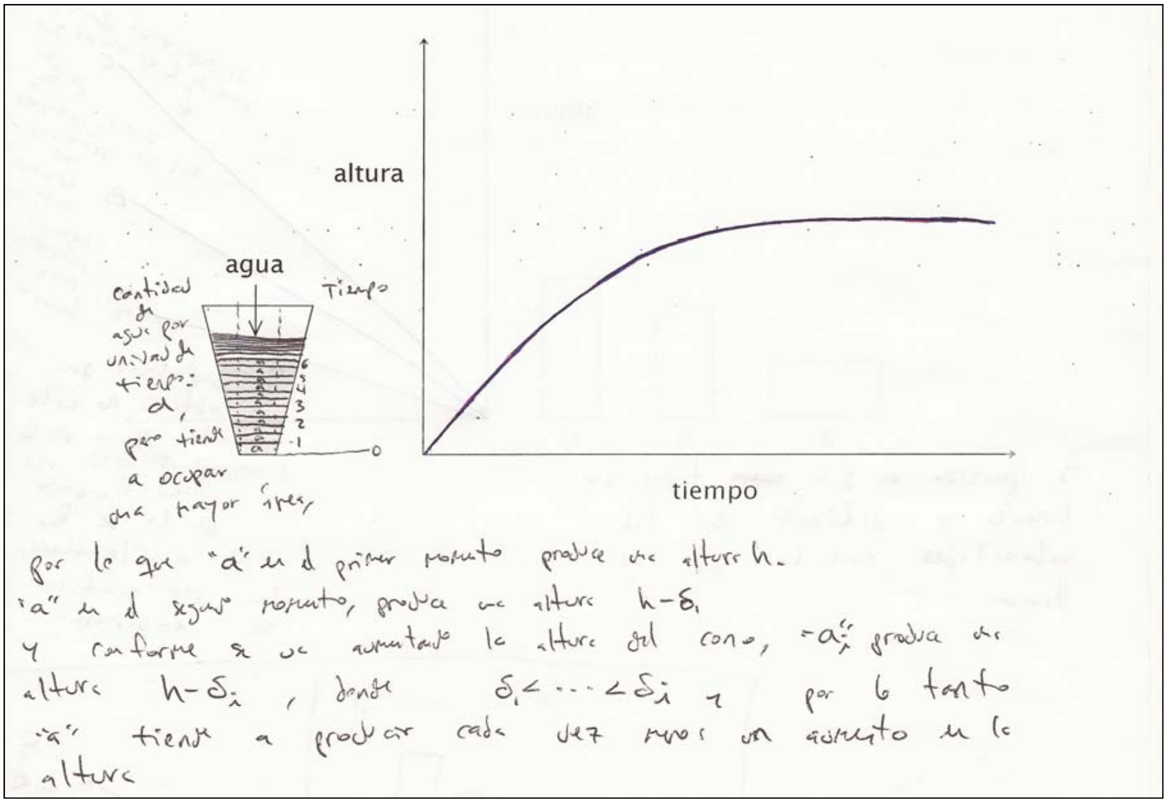


Figure 12. Answer given by an interviewed high school Mexican student: “The same amount a of water produces a rise that always diminishes when the height increases”

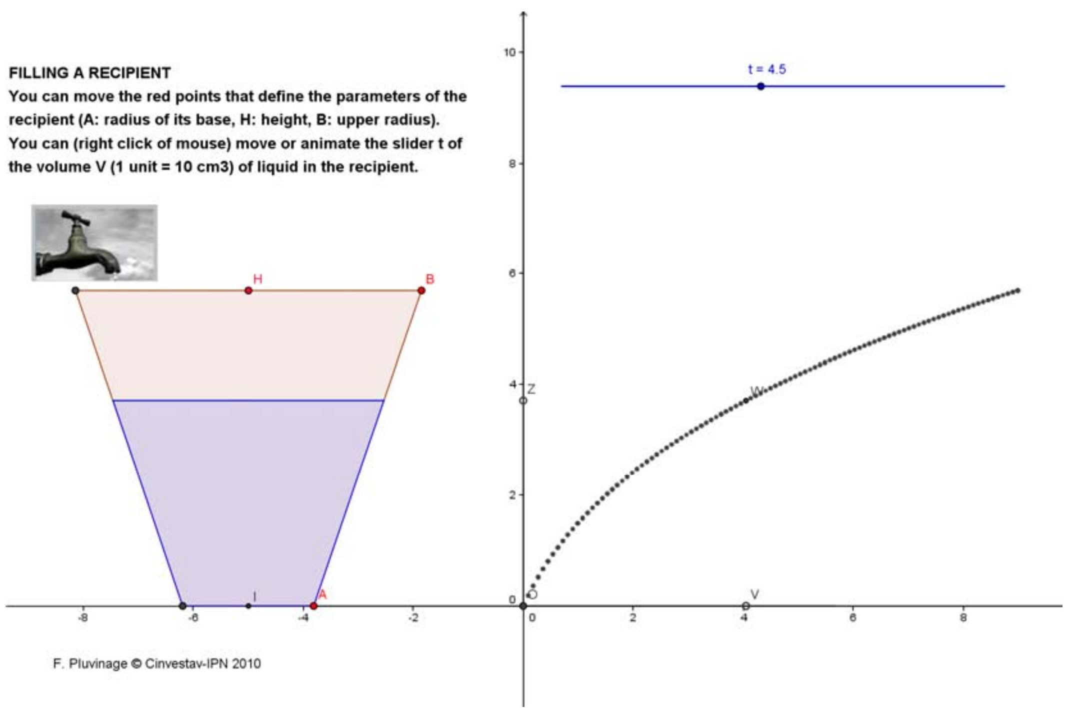


Figure 13: Solution resulting from consideration of the reciprocal function of volume as function of height

How could a teacher exploit these reported observations and considerations, for designing a possible instructional route? The first stages to do this are common to many practical action projects:

(a) A diagnostic, here with a sheet of questions as we have seen about comparing volume of different cylindrical recipients and representing how various recipients are filled by a regular flow;

(b) Some concrete experiment, for instance here to pour water or sand from a recipient to another and observe if the second is fulfilled or not. In our environment it is easy to get recipients that have the required characteristics: shape approximately cylindrical and respective diameters and heights approximately having simple ratios (for instance a box of coarse salt and a glass, or tin cans of varying size).

At the end of each stage, a discussion and a synthesis (e.g. volume of cylinder) will be useful. For the following stages, with the use of computation, the knowledge of a truncated cone and its volume formula are necessary. The teacher has the choice between giving the students these elements, and asking the students to obtain them by searching on Internet. Then the stages are:

a) Figural representation of the transversal section of a truncated cone of varying height and diameters of upper disc and lower disc (see figure on the preceding page), and graphical representation of its volume $V$ as a function of the height $h$ of liquid in the recipient;

b) In order to find now the inverse of the obtained function, i.e. the height $h$ of liquid as a function of the volume $V$, first observe the difficulty to work only with a formula and, as a consequence of this difficulty, introduce the students into the work with geometric register, particularly with the use of reflection. Construction depends on software in use, for instance Cabri could reflect the graph of the function $V(h)$ and Geogebra not, so with Geogebra we had to locate a point on the graph and then to reflect this point and eventually obtain its locus.

The time devoted to this project is a good investment with the objectives of improvement of the functional thinking and acquisition of the difficult *procept* (amalgam of process, mathematical object and symbol introduced by Eddie Gray and David Tall) of an inverse function.

### 3. Concluding

Teaching ways that we found rich of learning perspectives for the students consist in managing practical action projects at school, particularly in collaborative environment, for instance with the so called ACODESA method (Hitt, 2007). An important feature of these teaching ways is that the teacher is not the only one who is posing problems or ideas of problems. Nevertheless, his/her experience is essential for redacting in a correct mathematical language the statements suggested by students. And this step of writing mathematics also is extremely important for the students in terms of improvement of their mathematical experience.

The author is grateful to the reviewers, whose pertinent suggestions allowed to improve the first version of this paper.

### References

Charnay, R. (1993) Problème ouvert, problème pour chercher, *Grand N*, nº51, retrieved from http://www.crdp.ac-grenoble.fr/imel/nx/n51_6.htm

Cuevas, C. A. & Pluvinage, F. (2003). Les projets d'action pratique, éléments d'une ingénierie d'enseignement des mathématiques, *Annales de didactique et sciences cognitives*, Vol. 8, 273–292.

Duval, R. (1995), *Sémiosis et pensée humaine : registre sémiotique et apprentissages intellectuels*, Peter Lang, Berne.

Hitt, F. (2007). "Utilisation de calculatrices symboliques dans le cadre d'une méthode d'apprentissage collaboratif", de débat scientifique et d'autoréflexion. *In M. Baron,* *D. Guin et L. Trouche (Éds.), Environnements informatisés pour l'éducation et la formation scientifique et technique : modèles, dispositifs et pratiques*. Paris: Hermes.

Houdement, C. (2009), Une place pour les problèmes pour chercher, *Annales de Didactique et de Sciences Cognitives*, Volume 14, 31-59

Houdement, C. & Kuzniak, A. (1999). Un exemple de cadre conceptuel pour l'étude de l'enseignement de la géométrie en formation des maîtres, *Educational Studies in Mathematics*, Volume 40, Number 3, 283-312

Pluvinage, F. & Marmolejo Rivas, E. (2012) Une recherche didactique recourant à la modélisation et au travail collaboratif : un cas d'étude de paramètres, in *Formation à la Recherche en Didactique des Mathématiques*, Loze-Dion éditeurs, Québec, Canada, pp. 11-24.

Schoenfeld, A. (2006). Problem Solving from Cradle to Grave. *Annales de Didactique et de Sciences Cognitives*, Volume 11, 41- 73

Stölting, Pascal, (2008), *La pensée fonctionnelle des élèves de 10 á 16 ans. Une étude comparative de son enseignement en France et en Allemagne*, Thèse de doctorat, Université de Paris Diderot

Wagenschein, M. (1977), *Verstehen lernen, genetisch-sokratisch-exemplarisch*, Beltz Verlag: Basel; Weinheim.