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Abstract: In this article, the author, who has written extensively about mathematical problem solving over the past 40 years, discusses some of his current thinking about the nature of problem-solving and its relation to other forms of mathematical activity. He also suggests several proficiencies teachers should acquire in order for them to be successful in helping students become better problem solvers and presents a framework for research on problem-solving instruction. He closes the article with a list of principles about problem-solving instruction that have emerged since the early 1970s.

Keywords: mathematical activity, problem solving, problem-solving instruction, proficiencies for teaching, craft knowledge, research design, teaching as a craft, teacher planning, metacognition..

Introduction

My interest in problem solving as an area of study within mathematics education began more than 40 years ago as I was beginning to think seriously about a topic for my doctoral dissertation. Since that time, my interest in and enthusiasm for problem solving, in particular problem-solving instruction, has not waned but some of my thinking about it has changed considerably. In this article I share some of my current thinking about a variety of ideas associated with this complex and elusive area of study, giving special attention to problem-solving instruction. To be sure, in this article I will not provide much elaboration on these ideas and careful readers may be put off by such a cursory discussion. My hope is that some readers will be stimulated by my ideas to think a bit differently about how mathematical problem solving, and in particular problem-solving instruction, might be studied.

Setting the stage

Most mathematics educators agree that the development of students' problem-solving abilities is a primary objective of instruction and how this goal is to be reached involves consideration by the teacher of a wide range of factors and decisions. For example, teachers must decide on the problems and problem-solving experiences to use, when to give problem solving particular attention, how much guidance to give students, and how to assess students' progress. Furthermore, there is the issue of whether problem solving is intended as the *end* result of instruction or the *means* through which mathematical concepts, processes, and procedures are learned. Or, to put it another way, should teachers adopt "teaching *for* problem solving, "—an *ends* approach—or "teaching *via* problem solving" —a *means* approach?[1] (I say more about means and ends later in this article.) In my view, the answer to this question is that both approaches have merit; problem solving should be both an end result of learning mathematics and the means through which mathematics is learned (DiMatteo & Lester, 2010; Stein, Boaler, & Silver, 2003). Whichever approach is adopted, or if some combination of approaches is used, research is needed that focuses on the factors that influence student learning. Unfortunately, as far as I know, no prolonged, in-depth, programmatic research of this sort has been undertaken and, as a result, the accumulation of knowledge has been very slow. Moreover, the present intense interest in research on teachers' knowledge and

[1] It has become more common to refer to the "means" approach to teaching as teaching *through* problem solving. In Schroeder and Lester (1989) we discuss three approaches to problem-solving instruction: *teaching about, for,* and *via* problem solving. Teaching "via" problem solving is essentially the same as teaching *through* problem solving. Today, teaching *about* problem solving is not generally regarded as a legitimate instructional method, although I suspect that some (many?) teachers and curriculum writers subscribe to this approach.

proficiencies demands that future problem-solving research pay close attention to the mathematical and pedagogical knowledge and proficiencies a teacher should possess (cf., Ball, Thames, & Phelps, 2008; Hill, Sleep, Lewis, & Ball, 2007; Moreira & David, 2008; Zazkis & Leikin, 2010).

But before discussing problem-solving instruction, let me first say a few things about mathematical problem solving. This short discussion will highlight how my thinking has changed about the nature of problem solving and other forms of mathematical activity.

Some claims about Problem Solving

Among the many issues and questions associated with problem-solving instruction I have worried about during my career, several have endured over time. In this section I make five claims related to these enduring issues and offer brief discussions of my current thinking about them.

Claim 1. We need to rethink what we mean by "Problem" and "Problem Solving"

Although there have been at least four distinct problem-solving research traditions within (namely, Gestalt/Cognitive, Learning/S-R, Computer/Information Processing, and Psychometric/Component Analysis), they all agree that a problem is a task for which an individual does not know (immediately) what to do to get an answer (cf., Frensch & Funke, 1995; Holth, 2008). Some representative definitions illustrate this fundamental agreement:

> *A problem arises when a living creature has a goal but does not know how this goal is to be reached. (Duncker, 1945, p. 1)*

A question for which there is at the moment no answer is a problem. (Skinner, 1966, p. 225)

A person is confronted with a problem when he wants something and does not know immediately what series of actions he can perform to get it. (Newell & Simon, 1972, p. 72)

Whenever you have a goal which is blocked for whatever reason . . . you have a problem. (Kahney, 1993, p. 15)

These definitions have two common ingredients: there is a goal and the individual (i.e., the problem solver) is not immediately able to attain the goal. Moreover, researchers irrespective of tradition, view problem solving simply as what one does to achieve the goal. Unfortunately, these definitions and descriptions, like most of those that have been given of *mathematical* problem solving – including those I and other mathematics educators have proposed – are unhelpful for thinking about how to teach students to solve problems or to identify the proficiencies needed to teach *for* or *via* problem solving. A useful description should acknowledge that problem solving is an activity requiring an individual (or group) to engage in a variety of cognitive actions, each of which requires some knowledge and skill, and some of which are not routine. Furthermore, these cognitive actions are influenced by a number of non-cognitive factors. And, although it is difficult to define problem solving, the following statement – which Paul Kehle and I devised a few years ago– comes much closer to capturing what it involves than most of those that have appeared in the literature.

Successful problem solving involves coordinating previous experiences, knowledge, familiar representations and patterns of inference, and intuition in an

> *effort to generate new representations and related patterns of inference that resolve some tension or ambiguity (i.e., lack of meaningful representations and supporting inferential moves) that prompted the original problem-solving activity. (Lester & Kehle, 2003, p. 510)*

The advantage of this description of problem solving over the others lies in its identification of several key ingredients of success: *coordination of experience, knowledge, familiar representations, patterns of inference,* and *intuition*. So, to be a successful problem solver, an individual must have ample relevant experience in learning how to solve problems, strong content knowledge, proficiency in using a variety of representations[2] and a solid grasp of how to recognize and construct patterns of inference. Moreover, it recognizes the importance of intuition in successful problem solving[3]. With the possible exception of intuition, each of these ingredients should be attended to any program aimed at equipping prospective teachers with the proficiencies needed to teach mathematics either *for* or *via* problem solving. I say more about the implications of this description for the education of mathematics teachers later in this article. But first, let me continue with a few more observations about the nature of problem solving.

Claim 2. We know very little about how to improve students' metacognitive abilities.

So much has been written about metacognition and its place both in the teaching and learning of mathematics that a few comments about this elusive construct seem warranted. I remain convinced that metacognition is one of the driving forces behind

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[2] The research perspective on the role of representation in doing mathematics provided by Goldin (2003) is particularly relevant to this discussion.

[3] A reviewer pointed out to me that intuition is itself a very subtle notion and as such the definition we propose is unhelpful. To be sure, intuition is a subtle idea, but I think it essential to include it in any description of what problem solving entails because it serves to point out just how subtle the act of problem solving can be and, consequently, how difficult it has been to make progress in learning how to teach students to be better problem solvers.

successful problem solving (Garofalo & Lester, 1984), but we really know almost nothing about what teachers should do to develop students' metacognitive abilities. To be sure, it is essential that successful problem solvers be able to monitor and regulate their cognitive behaviors. But, almost no research has been done that demonstrates that students' can be taught these behaviors. Within the mathematics education community both Schoenfeld (1992) and I (Lester, Garofalo, & Kroll, 1989), among others, have conducted research aimed at enhancing students' metacognitive abilities, but neither of us has identified the proficiencies teachers need to do this. Instead, we have offered suggestions, with too little evidence to support them. So, any program designed to enhance mathematics teachers' proficiencies that pays heed to metacognition should do so only after acknowledging that there is no conclusive research evidence to support any particular method of metacognition instruction over another.

Claim 3. Mathematics teachers needn't be expert problem solvers; they must be serious students of problem solving.

It is natural to suggest that teachers must themselves be expert problem solvers before they are to be considered proficient mathematics teachers. But, I think this is asking too much of them! George Polya was an expert (and, hence, proficient) problem solver as well as an expert teacher of mathematics, but to expect all teachers to be experts is both unreasonable and unnecessary. After all, expert basketball coaches needn't have been expert basketball players and expert violin teachers needn't have been concertmasters. Of course, teachers should be experienced problem solvers and should have a firm grasp of what successful problem solving involves, but care should be taken not to confuse *proficiency* in teaching students to solve problems with *expertise* as problem solvers.

Claim 4. Problem solving isn't always a high-level cognitive activity.

A fourth observation is that the description of problem solving I have given above blurs the distinction between problem solving and other types of mathematical activity—I have more to say about this blurring later. The distinctions that led historically to the isolation of mathematical problem solving as a research focus from other areas of study and the subsequent distinctions that resulted from this isolation are due in part to strong traditions of disciplinary boundaries (Lester & Kehle, 2003).[4] This isolation led to subsuming mathematical understanding under problem solving. But, the inverse makes more sense; that is, to subsume problem solving (and problem posing) under mathematical understanding and, hence, under mathematical activity. By so doing, emphasis is placed on several other constructs that are important in being able to do mathematics — e.g., model building, generation of representations, constructing patterns of inference — that too often are not considered when problem solving is isolated from other forms of mathematical activity.

Claim 5. Research tells us something about problem-solving instruction, but not nearly enough.

Although research on mathematical problem solving has provided some valuable information about problem-solving instruction, we haven't learned nearly enough (but see also the very last section of this article). In a paper I co-authored about 20 years ago,

[4] Indeed, some years ago, one mathematics education researcher asked me why I (and most other problem-solving researchers) studied problem solving in isolation from learning specific mathematics concepts and processes. I had no good answer; she was correct and I couldn't provide a compelling reason why we did so. Today, I think the reason stems from our reliance on cognitive science for guidance in developing our research agendas and methods.

my co-author and I identified four reasons for this unfortunate state of affairs: (1) relatively little attention to the role of the teacher in instruction; (2) too little concern for what happens in real classrooms; (3) a focus on individuals rather than small groups or whole classes; and (4) the largely atheoretical in nature of problem-solving research (Lester & Charles, 1992). I have discussed the fourth reason elsewhere (Lester, 2005), so will not discuss it here. Instead, let me comment on the other three reasons. (Interested readers may wish to read the provocative analysis of the state of mathematical problem-solving research written by Lesh and Zawojewski (2007). In their analysis they take issue with the nature and direction of nearly all the research over the past 50 years.)

The role of the teacher. More than twenty-five years ago, Silver (1985) pointed out that the typical research report might have described in a general way the instructional method employed, but rarely was any mention made of the teacher's specific role. Some progress has been made since then (see, e.g., the edited volumes by Lester and Charles (2003) and Schoen and Charles (2003) and the review by Schoenfeld (1992)). But, as useful as these efforts have been, they fall short of what is needed. Instead of simply considering teachers as agents to effect certain student outcomes, their role should be viewed as one dimension of a dynamic interaction among several dimensions of a system involving: the role of the teacher, the nature of classroom tasks, the social culture of the classroom, the use of mathematical tools as learning supports, and issues of equity and accessibility. Changing any of the dimensions of this system requires parallel changes in each of the other dimensions (Hiebert et al., 1997).

Observations of real classrooms. Several years ago, my colleague, Randy Charles, and I conducted a large-scale study of the effectiveness of an approach to problem-solving instruction based on ten specific teaching actions (Charles & Lester, 1984). The research involved several hundred fifth and seventh grade students in more than 40 classrooms over the period of one full school year. The results were encouraging: students receiving the instruction benefited tremendously with respect to several key components of the problem-solving process. However, despite the promise of our instructional approach, the conditions under which the study was conducted did not allow us to make extensive, systematic observations of classrooms. Ours is not an isolated instance. In particular, there has been a lack of descriptions of teachers' behaviors, teacher-student and student-student interactions, and the type of classroom atmosphere that exists. It is vital that such descriptions be compiled if there is to be any hope of deriving sound prescriptions for teaching problem solving. In the final section of this article I present a framework for research that, if used, might provide the sorts of rich, detailed descriptions I think we need.

Focus on individuals rather than groups or whole classes. Throughout most of the history research in mathematical problem solving (dating back about 50 years) the focus has been on the thinking processes used by individuals as they solve problems or as they reflect back on their work solving problems. When the goal of research is to characterize the thinking involved in a process like problem solving, a microanalysis of individual performance seems appropriate. However, when our concerns are with classroom instruction, we should give attention to groups and whole classes. To be sure, small groups can serve as an appropriate environment for research on teaching problem solving, but the research on problem-solving instruction cannot be limited to the study of small groups. In order for the field to move forward, research on teaching problem solving needs to examine teaching and learning processes for individuals, small groups, and whole classes.

A Model of Complex Mathematical Activity[5]

In addition to the lack of attention to the role of the teacher in real classrooms and the focus on individuals rather than whole classes, the relative ineffectiveness of instruction to improve students' ability to solve problems can be attributed to the fact that problem solving has often been conceptualized in a simplistic way. This naive perspective has two levels or "worlds": the everyday world of things, problems, and applications of mathematics and the idealized, abstract world of mathematical symbols, concepts, and operations. In this naive perspective, the problem-solving process typically has three steps. Beginning with a problem posed in terms of physical reality, the problem solver first translates the problem into abstract mathematical terms, and then operates on this mathematical representation in order to come to a mathematical solution of the problem. This solution is then translated into the terms of the original problem. According to this view, mathematics may be, and often is, learned separately from its applications and (too often) with no attempt to connect new mathematics concepts to old ones. Teachers who adhere to this perspective are very concerned about developing skillfulness in translating (so-called) real-world problems into mathematical representations and vice versa. However, these teachers tend to deal with problems and applications of mathematics only after the mathematical concepts and skills have been introduced, developed, and practiced. Many of the “problems” found in textbooks often

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[5] The discussion in this section is excerpted with only minor revision from Lester and Kehle (2003).

can be solved exactly as this naive perspective indicates. But for more challenging, substantive problems, the problem solver cannot simply apply a previously learned procedure to solve the problem. In addition to translation and interpretation, these problems also demand more complex processes such as planning, selecting strategies, identifying sub-goals, choosing or creating appropriate representations, conjecturing, and verifying that a solution has been found. For non-routine tasks, a different type of perspective is required, one that emphasizes the making of new meanings through construction of new representations and inferential moves (refer back to the description of problem solving Kehle and I (2003) have proposed).

The new perspective, like the previous one, also contains two levels representing the everyday world of problems and the abstract world of mathematical concepts, symbols and operations. In this perspective, however, the mathematical processes in the upper level are "under construction" (i.e., being learned, as opposed to already learned; coming to be understood, as opposed to being understood) and the most important features are the relationships between steps in the mathematical process (in the mathematics world) and actions on particular elements in the problem (in the everyday world). It is in the forging of these relationships that results in the meaning making that is central to mathematical activity of all kinds. At times the problem solver is learning to make abstract written records of the actions that are understood in a concrete setting. This involves the processes of abstraction and generalization. And, at other times the problem solver attempts to connect a mathematical process to the real-world actions that the mathematical process represents. Also, a problem solver who had forgotten the details of a mathematical procedure would attempt to reconstruct that procedure by imagining the corresponding concrete steps in the world in which the problem was posed. As a result, typically the problem solver moves back and forth between the two worlds—the everyday problem world and the mathematical world—as the need arises.

But, although this perspective is an improvement over the original, it too falls short of what is needed because it does not account for many of the most important actions (both cognitive and non-cognitive) involved during real problem solving. Even the modified perspective regards problem solving as somehow being different from other sorts of mathematical activity. In my view, what is needed is to subsume problem solving within a much broader category, "mathematical activity," and to give a prominent role to the metacognitive activity engaged in by the individual or group.[6]

Figure 1 below depicts mathematical behavior as a complex, involved, multiphase process that begins when an individual, working in a complex context (Box A), poses (or is given) a specific task to solve (the solid arrow between A and B). To start solving the task, the individual simplifies the complex setting by identifying those concepts and processes that seem to bear most directly on the problem. This simplifying and problem posing phase involves making decisions about what should be attended to and what can be ignored, developing a sense about how the essential concepts are connected, and results in a realistic representation of the original situation. This realistic representation is a model of the original context from which the problem was drawn because it is easier to examine, manipulate, and understand than the original situation.

Next comes the abstraction phase (solid arrow from B to C), which introduces mathematical concepts and notations (albeit perhaps idiosyncratic). This abstraction

[6] I should point out that this depiction is a representation of ideal, rather than typical, performance during an individual's work on some mathematical task. It is ideal in the sense that it denotes key actions in which the individual *should* engage in order to obtain acceptable results.

phase involves the selection of mathematical concepts to represent the essential features of the realistic model. Often the abstraction phase is guided by a sense of what a given representation makes possible in the subsequent computation phase. The explicit representation of the original setting and problem in mathematical symbolism constitutes a mathematical representation of both the setting and the task/problem.

Once a problem solver has generated a mathematical representation of the original situation, the realistic problem now becomes a specific mathematical problem related to the representation. This mathematical problem acquires a meaning all its own, becoming an isolated, well-defined mathematical problem (Box C).

The third phase of the process (from C to D) involves manipulating the mathematical representation and deducing some mathematical conclusions—depicted in the figure by the "computing" arrow. During this phase, the person draws upon her or his store of mathematical facts, skills, mathematical reasoning abilities, and so forth. For example, the problem might call for a solution of a system of equations and solving this system of equations does not depend on the original context of the initial problem. The final phase (from D to A, D to B, and D to C), then, should involve the individual in comparing the conclusions/results obtained with the original context and problem, as well as with the mathematical representation (refer to the dashed arrows between boxes). But, the act of comparing does not occur only after conclusions are drawn and a solution is obtained. Rather, it might take place at any time and at any point during the entire process. Indeed, this regular and continual monitoring—metacognitive activity—of one's work is a key feature of success on complex mathematical tasks. In general, the act of comparing the current state of one's work, thinking, and decisions denotes how complex mathematical activity can be. The degree to which the individual chooses to compare her or his current state with earlier states can be considered a determinant of task complexity and, in fact, is the primary way to distinguish “routine” from “non-routine” tasks (i.e., problems). For example, performing routine calculations using whole numbers typically requires little comparing, whereas work on more complex tasks might necessitate quite a lot of comparing throughout ones work on it. In brief, then, the degree to which a task can be considered problematic can be determined by the amount of “comparing” involved.

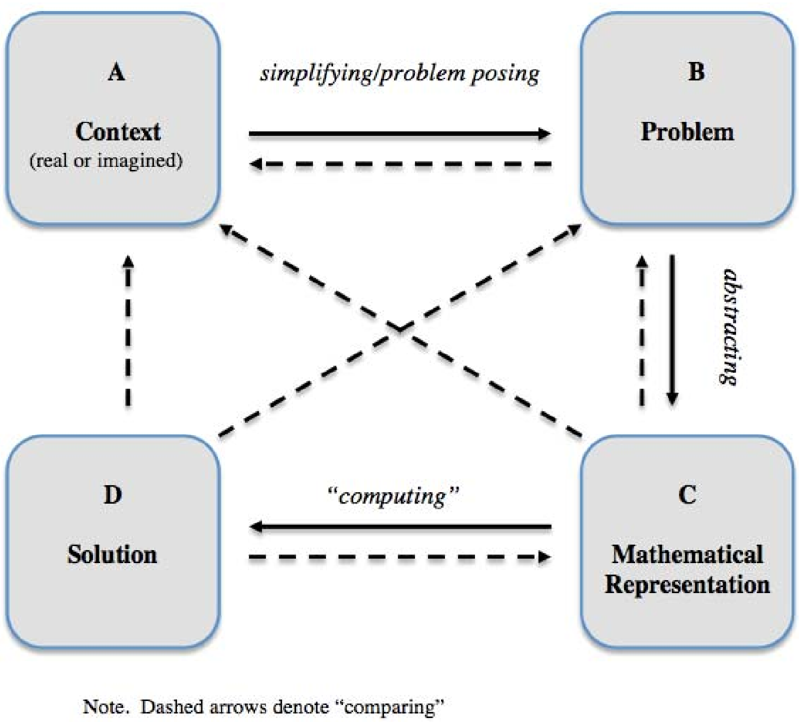

Figure 1. A model of complex mathematical activity

To sum up, what Kehle and I have proposed is a blurring of the distinction between problem solving and other mathematical activity emerging from research on mathematical problem solving and constructivist thinking about learning. Furthermore, we have proposed a blurring of task, person, mathematical activity, non-mathematical activity, learning, applying what has been learned, and other features of mathematical problem solving. A consequence of this blurring is that it necessitates some rethinking about the proficiencies mathematics teachers need. In the next section I discuss these proficiencies in light of the preceding discussions.

Proficiencies for Teaching Mathematics

The debate over the merits of direct (explicit) instruction versus constructivist instruction has been raging for at least 50 years and any consideration of the mathematical proficiencies needed for teaching mathematics must be made in view of this debate. More specifically, the identification of such proficiencies must take into account the assumptions that are being made about the nature of mathematics learning and instruction, as well as about instructional goals. For example, a proponent of direct instruction (e.g., Kirschner, Sweller & Clark, 2006) might view learning as simply a matter of making a change in students' long-term memories. But for a constructivist teacher, in addition to making a change in students' long-term memories, learning involves much more. Constructivist teachers are concerned with (among other things) how to help students select and use good procedures for solving problems (Gresalfi & Lester, 2009). Clearly, these quite different perspectives on what mathematics learning involves will have a tremendous influence on what teachers must be able to do in their classrooms (i.e., what proficiencies they need). Furthermore, there is the matter of the teacher's goals. If problem solving is intended as the *end* result of instruction, one set of proficiencies for teaching is needed, but if problem solving is the *means* through which mathematical concepts, processes, and procedures are learned, then a different set of proficiencies may be called for. For example, the teacher for whom problem solving is a *means*, would likely need to be very proficient at listening to and observing students as they work on mathematical tasks (Davis, 1997; Yackel, 2003). And, quite naturally, listening to students would play a much less important role for a teacher who mostly lectures. Put more directly, consideration of how to include problem solving in a mathematical-proficiencies-for-teaching framework should be done in view of the assumptions the teacher makes about the nature of mathematics learning and the goals of instruction.

But, what of the proficiencies needed to help students learn how to solve problems? One consequence of subsuming problem solving under the broader heading "mathematical activity," is that it becomes more difficult to specify a precise set of proficiencies teachers need. To illustrate, consider the task "Which is more 2/3 or 2/5?" Does this task involve any problem solving on the part of the student? Maybe, maybe not! Of course, one can "cross multiply" to determine that 2/3 is more (or use some other previously learned procedure), or one may have had sufficient experience with fractions to simply "know" that 2/3 is more. In these instances, one could argue that no problem solving is going on. But suppose you are a 3rd grader[7] who does not know of any procedures to decide which is more. Without a prescribed method of attack, this task might be used to help you better understand the meanings of numerator and denominator and also help you see how useful one half can be as a fraction benchmark (Van de Walle, 2003). This is at the heart of what it means to teach *via* problem solving. But, what

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[7] One needn't be a 3rd grader to find this task problematic. Over the past 4 years I have been tutoring unemployed adults who hope one day to pass the US high school equivalency exam (GED). Almost to a person, they do not know how to solve this task when they begin to study with me.

proficiencies must teachers have who subscribe to a teaching *via* problem solving approach? Of course, they must be adept at selecting good problems, at listening and observing, at asking the right questions, at knowing when to prod and when to withhold comment, as well as a host of other actions[8]. These actions make up what Moore (1995) has called the "craft of teaching." Moore's "image of a [proficient] teacher is that of a skilled craftworker, a master machinist say, who knows exactly what she must do, brings the tools she needs, does the work with straightforward competence, and takes pleasure in a job well done. She does her work right every day, and every day's work fits the larger plan of her project" (p. 5). For Moore a craft is a "collection of learned skills accompanied by experienced judgment" (p. 5). So, the question is "How does one become a craftsman?"

Thirty years ago, Randy Charles and I wrote a book in which we laid out an instructional plan for teachers to follow in order to be effective in teaching students how to solve mathematics problems (Charles & Lester, 1982). The plan focused on three phases of instruction—Before, During, and After—and was organized around 10 "teaching actions." Since then, the three phases have appeared in different guises in various American elementary and middle school textbook series (e.g., the middle grades *Connected Mathematics* series organizes activities around Launch, Explore, and Summarize (Pearson Education Inc. 2011)). The features of our plan that most clearly distinguish it from more "traditional" instructional plans have to do with the teacher's role and the nature of the classroom environment. However, this is far from sufficient; knowing about the teaching actions is simply not enough! In addition to knowing *what* to

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[8] I, and various collaborators over the years, have used the word (teaching) "actions" to refer to what the teacher does during the act of teaching.

do, the teacher must also know *when* to do it and what the implications might be of the action taken. In particular, teachers must be adept at: (1) designing and selecting tasks and activities, (2) listening to and observing students as they engage with problem-solving activities, (3) making sure that instructional activities remain problematic for students, (4) focusing on the methods students use to solve problems and being familiar with problem-solving methods (e.g., heuristics, strategies) that are accessible to students, and (5) being able to tell the right thing at the right time (cf., Cai, 2010; DiMatteo & Lester, 2010; Hiebert, 2003). Moreover, teachers and students share responsibility for creating and maintaining a classroom atmosphere that is conducive to exploring and sharing ideas, cooperating with each other, and risk taking (Stephan & Whitenack, 2003). Thus, for me, in addition to myriad other knowledge and skills, a proficient mathematics teacher must be skillful at —

- Designing and selecting appropriate tasks for instruction
- Making sense of and taking appropriate actions after listening to and observing students as they work on a task
- Keeping tasks appropriately problematic for students
- Paying attention to and being familiar with the methods students use to solve problems
- Being able to take the appropriate action (or say the right thing) at the right time
- Creating a classroom atmosphere that is conducive to exploring and sharing.

To be sure, teachers who have command of these and related teaching actions and who also have considerable mathematics content knowledge appropriate for the level at which they are teaching might be considered craftsmen. But, I think what separates a craftsman from others has to do with the amount of planning and reflection that he or she has done prior to and after instruction. Unfortunately, even though it seems clear to me that the type and amount of planning a teacher does have tremendous impact on what happens during instruction, teacher planning has been largely ignored as a factor of importance in research on problem-solving instruction. Indeed, in most studies teacher planning has not even been considered because the teachers in these studies have simply implemented a plan that had been predetermined by the researchers, not the teachers. Furthermore, it is no longer warranted to assume that the planning decisions teachers make are driven totally by the content and organization of the textbooks used and, therefore, need not be considered as an object of research. The challenge, then, is to determine ways to provide these teachers with opportunities to acquire the proficiencies needed to become craftsmen; opportunities that in my view are best provided through apprenticeship experiences in their real-world context and situation (Collins, Brown, & Newmann, 1990). To date, too little attention has been paid to studying the design and implementation of apprenticeship programs for teacher education. This lack of attention is unfortunate because I think apprenticeship training is the approach most likely to result in highly proficient teachers—that is, teacher craftsmen skilled at teaching mathematics *via* problem solving.

A Framework for Research on Problem-solving Instruction

Twenty years ago, Randy Charles and I developed a framework for research on problem –solving instruction that was a synthesis of previous conceptualizations of teaching in general and mathematics teaching in particular (Lester & Charles, 1992). Unfortunately, to my knowledge, other researchers have not adopted this framework. I still think it could serve us well in designing research on problem-solving instruction and I bring this article to a close by offering a slightly-modified sketch of what it consists of.

The framework is comprised of four broad categories of factors that we consider essential in the conceptualization and design of research studies: (1) Non-classroom factors, (2) Teacher planning, (3) Classroom processes, and (3) Instructional outcomes. Of course, the categories overlap and the factors within each interact both within and across categories.

Category 1: Non-classroom Factors

What goes on in a classroom is influenced by many things that exist or take place apart from actual classroom instruction. For example, the teacher's and students' knowledge, beliefs, attitudes, emotions, and dispositions all play a part in determining what happens during instruction. Furthermore, the nature of the tasks used as well as the contextual conditions present outside the classroom also affect instruction (e.g., course schedules, school structures). There are six types of factors: teacher presage characteristics, student presage characteristics, teacher knowledge and affects, student knowledge and affects, tasks features, and contextual (situational) conditions.

Teacher and student presage characteristics. These are characteristics of the teacher and students that are not amenable to change but which may be examined for their effects on classroom instruction. In addition, presage characteristics serve to describe the individuals involved. Typically, in experimental research these characteristics have potential for control by the researcher. But, awareness of these characteristics can useful in non-experimental research as well by helping researchers make sense of what they are observing. Among the more prominent presage characteristics are age, sex, and previous experience (e.g., teaching experience, previous experience with the topic of instruction). Factors such as previous experience may indeed be of great importance as we learn more about the ways knowledge teachers glean from experience influences practice.

Teacher and student knowledge and affects. The teacher's and students' knowledge (both cognitive and metacognitive) and affects (including beliefs) can strongly influence both the nature and effectiveness of instruction. As a category, these teacher and student traits are similar to, but quite different from, presage characteristics. The similarity lies in the potential for providing clear descriptions of the teacher and students. The difference between the two is that affects and knowledge may change, in particular as a result of instruction, whereas presage characteristics cannot.

Task features. Task features are the characteristics of the tasks used for instructional or assessment purposes. Historically, at least five types of features serve to describe tasks: syntax, content, context, structure, and process (see Goldin & McClintock, 1984). *Syntax features* refer to the arrangement of and relationships among words and symbols in a task. *Content features* deal with the mathematical meanings in the problem. Two important categories of content features are the mathematical content area (e.g., geometry, probability) and linguistic content features (e.g., terms having special mathematical meanings such as "less than," "function," "squared"). *Context features* are the non-mathematical meanings in the task statement. Furthermore, context features describe the problem embodiment (representation), verbal setting, and the format of the information given in the problem statement. *Structure features* can be described as the logical-mathematical properties of a task. Structure features are determined by the particular representation that is chosen for a problem. For example, one student may choose to represent a task in terms of a system of equations, while another student may represent the same problem in terms of some sort of guessing process. Finally, *process features* represent something of an interaction between task and student. That is, although problem-solving processes (e.g., heuristic reasoning) typically are considered characteristics of the student, it is reasonable to suggest that a problem may lend itself to solution via particular processes. A consideration of task process features can be very informative to the researcher in selecting tasks for both instruction and assessment.

Contextual conditions. These factors concern the conditions external to the teacher and students that may affect the nature of instruction. For example, class size is a condition that may directly influence the instructional process and with which both teacher and students must contend. Other obvious contextual conditions include textbooks used, community ethnicity, type of administrative support, economic and political forces, and assessment programs. Also, since instructional method provides a context within which teacher and student behaviors and interactions take place, it too can at times be considered a factor within this category. I should add that that these six areas of consideration do not necessarily cover all possible influences; it is likely that there are other influences that may be at least as important as the ones I have mentioned. Rather, my intent is to point out the importance of paying attention to the wide range of factors that can have an impact on what takes place during instruction.

Category 2: Teacher Planning

Teacher planning is not clearly distinct from the other categories, in fact, it overlaps each of them in various ways. Of particular interest for research are the various decisions made before, during and as a result of instruction about student presage characteristics, instructional materials, teaching methods, classroom management procedures, evaluation of student performance, and amount of time to devote to particular activities and topics. Unfortunately, teacher planning has been given too little attention as a factor of importance in problem-solving instruction research. Indeed, in most studies teacher planning has not even been considered because the teachers in these studies have simply implemented a plan that had been predetermined by the researchers, not the teachers. Furthermore, it is no longer warranted to assume that the planning decisions teachers make are driven totally by the content and organization of the textbooks used and, therefore, need not be considered as an object of research. A teacher's behavior while teaching either *for* or *via* problem solving is certainly influenced by the teacher's knowledge and affects. However, some of this behavior is likely to be determined by the kinds of decisions the teacher makes prior to entering the classroom. For example, a teacher may have planned to follow a specific sequence of teaching actions for delivering a particular problem-solving lesson knowing that the exact ways in which these teaching actions are implemented evolve situationally during the lesson. Or, if the knowledge teachers use to plan instruction is knowledge gleaned from previous instructional episodes, then we would search for those cases that significantly shape the craft knowledge teachers use as a basis for planning and action. Future research should consider how teachers go about planning for problem-solving instruction and how the decisions made during planning influence actions during instruction.

Category 3: Classroom Processes

Classroom processes include the host of teacher and student actions and interactions that take place during instruction. Four dimensions of classroom processes are apparent: teacher knowledge and affects; teacher behaviors; student knowledge and affects; and student behaviors.

Both the teacher's and the students' thinking processes and behaviors during instruction are almost always directed toward achieving a number of different goals, sometimes simultaneously. For example, during a lesson the teacher may be assessing the appropriateness of the small-group arrangement that was established prior to the lesson, while at the same time trying to guide the students' thinking toward the solution to a problem. Similarly, a student may be thinking about what her classmates will think if she never contributes to discussions and at the same time be trying to understand what the task confronting her is all about. In our framework, we have restricted consideration to what the teacher thinks about and does to facilitate the students' thinking and what the student thinks about and does to solve a problem. We have not attempted to include a complete menu of objects or goals a teacher might think about during instruction.

Teacher knowledge and affects. These processes include those attitudes, beliefs, emotions, cognitions and metacognitions that influence, and are influenced by, the multitude of teacher and student behaviors that occur in the classroom during instruction. In particular, this dimension is concerned with the teacher's thinking and affects while facilitating students' attempts to understand a task, develop a plan for solving it, carry out the plan to obtain an answer, and look back over the solution effort.

Teacher behaviors. A teacher's knowledge and affects that operate during instruction give rise to the teacher's behaviors, the overt actions taken by the teacher during problem-solving instruction. Specific teacher behaviors can be studied with regard to use (or non-use) as well as quality. The quality of a teacher behavior can include, among other things, the correctness of the behavior (e.g., correct mathematically or correct given the conditions of the problem), the clarity of the action (e.g., a clear question or hint), and the manner in which the behavior was delivered (e.g., the verbal and nonverbal communication style of the teacher).

Student knowledge and affects. Similar to the teacher, this subcategory refers to the knowledge and affects that interact with teacher and student behaviors. The concern here is with how students interpret the behavior of the teacher and how the students' thinking about a problem, their affects, and their work on the problem affects their own behavior. Also of concern here is how instructional influences such as task features or contextual conditions directly affect a student's knowledge, affects, and behaviors.

Student behaviors. These behaviors include the overt actions of the student during a classroom problem-solving episode. By restricting our attention to the problem-solving phases mentioned earlier, we can identify several behaviors students might exhibit as they work on a task.

Category 4: Instructional Outcomes

The fourth category of factors consists of three types of outcomes of instruction: *student outcomes, teacher outcomes,* and *incidental outcomes*. Most instruction-related research has been concerned with short-term effects only. Furthermore, transfer effects,

effects on attitudes, beliefs, and emotions, and changes in teacher behavior have been considered only rarely.

Student outcomes. Both immediate and long-term effects on student learning are included in this category, as are transfer effects (both near and far transfer). Illustrative of a student outcome, either immediate or long-term, is a change in a student's skill in implementing a particular problem-solving strategy (e.g., guess and check, working backwards). An example of a transfer effect is a change in students' performance in solving non-mathematics problems as a result of solving only mathematics problems. Also, of special importance is the consideration of changes in students' beliefs and attitudes about problem solving or about themselves as problem solvers and the effect of problem-solving instruction on mathematical skill and concept learning; for example, how is computational skill affected by increased emphasis on the thinking processes involved in solving problems?

Teacher outcomes. Teachers, of course, also change as a result of their instructional efforts. In particular, their attitudes and beliefs, the nature and extent of their planning, as well as their classroom behavior during subsequent instruction are all subject to change. Each problem-solving episode a teacher participates in changes her or his craft knowledge. Thus, it is reasonable to expect that experience affects the teacher's planning, thinking, affects, and actions in future situations.

Incidental outcomes. Increased performance in science (or some other subject area) and heightened parental interest in their children's school work are two examples of possible incidental outcomes. Although it is not possible to predetermine the relevant incidental effects of instruction, it is important to be mindful of the potential for unexpected "side effects."

Research on teaching in general points to the important role a teacher's knowledge and affects play in instruction. Questions such as the following need to be investigated : What knowledge (in particular, content, pedagogical, and curriculum knowledge) do teachers need to be effective as teachers of problem solving? How is that knowledge best structured to be useful to teachers? How do teachers' beliefs about themselves, their students, teaching mathematics, and problem solving influence the decisions they make prior to and during instruction?

The forgoing analysis of factors to be considered for research on problem-solving instruction is intended as a general framework for designing investigations of what actually happens in the classroom during instruction. As I mentioned earlier, there may be other important factors to be included in this framework and that certain of the factors may prove to be relatively unimportant. Notwithstanding these possible shortcomings, this framework could serve as a step in the direction of making research in the area more fruitful and relevant.

A Final, More Positive Note

I do not intend for my remarks to give the impression that I think mathematical problem solving research has not amounted to much during the past 40 years or that current research efforts are misguided. Indeed, quite the opposite is the case! Several important principles have slowly emerged from the research since the early 1970s. I end this article by listing these principles without comment: each principle could serve as the basis for an article or monograph. My hope is that this list, like much of the rest of my article, will stimulate discussion among those who are interested in pursuing a research agenda that includes problem solving at its core.

1. *The prolonged engagement principle.* In order for students to improve their ability to solve mathematics problems, they must engage in work on problematic tasks on a regular basis, over a prolonged period of time.
2. *The task variety principle.* Students will improve as problem solvers only if they are given opportunities to solve a variety of types of problematic tasks (in my view, principles 1 and 2 are the most important of the seven).
3. *The complexity principle.* There is a dynamic interaction between mathematical concepts and the processes (including metacognitive ones) used to solve problems involving those concepts. That is, heuristics, skills, control processes, and awareness of one's own thinking develop concurrently with the development of an understanding of mathematical concepts. (This principle tells us that problem- solving ability is best developed when it takes place in the context of learning important mathematics concepts.)
4. *The systematic organization principle.* Problem-solving instruction, metacognition instruction in particular, is likely to be most effective when it is provided in a systematically organized manner under the direction of the teacher.
5. *The multiple roles for the teacher principle.* Problem-solving instruction that emphasizes the development of metacognitive skills should involve the teacher in three different, but related, roles: (a) as an external monitor, (b) as a

facilitator of students' metacognitive awareness, and (c) as a model of a metacognitively-adept problem solver.

6. *The group interaction principle.* The standard arrangement for classroom instructional activities is for students to work in small groups (usually groups of three or four). Small group work is especially appropriate for activities involving new content (e.g., new mathematics topics, new problem-solving strategies) or when the focus of the activity is on the process of solving problems (e.g., planning, decision making, assessing progress) or exploring mathematical ideas.
7. *The assessment principle.* The teacher's instructional plan should include attention to how students' performance is to be assessed. In order for students to become convinced of the importance of the sort of behaviors that a good problem-solving program promotes, it is necessary to use assessment techniques that reward such behaviors.

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