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Abstract: Mathematical tasks are key ingredient to foster teachers and students' development and construction of mathematical thinking. The use of distinct computational tools offers teachers a variety of ways to represent and explore mathematical tasks which often extends problem solving approaches based on the use of paper and pencil. We sketch a framework to characterize ways of reasoning that emerge as result of using computational technology to solve a task that involves dealing with variation phenomena.

Keywords: problem solving, framework, the use of computational tools.

Introduction

It is widely recognized that the use of computational technology offers teachers and students different ways to represent and explore mathematical problems or concepts. There is also evidence that different tools might offer learners different opportunities to think of problems in order to represent, explore, and solve those problems. What tools and how should teachers integrate them in their teaching environments? What instructional goals should teachers aim with the use of technology? In accordance to Hegedus & Moreno-Armella (2009) "technology is here to transform thinking, and not to serve as some prosthetic device to prop up old styles of pedagogy or curriculum standards" (p. 398). Thus, it becomes important for teachers to discuss approaches to use technology in order to guide their students to develop ways of thinking that favour their comprehension of mathematical concepts and problem solving experiences. In particular, teachers should discuss the extent to which the use of the tools helps them represent and

explore mathematical tasks in ways that enhance and complement problem solving processes that rely on the use of paper and pencil environment. The use of computational tools in learning scenarios implies that teachers need to pay attention to and reflect upon aspects that involve:

(a) The process shown by the subject to transform the artefact (material object) into an instrument to represent, to comprehend mathematical ideas, and to solve problems;

(b) The type of tasks used to foster students' mathematical thinking;

(c) The ways of reasoning exhibited by the subjects during problem solving activities;

(d) The role of teachers during problem solving sessions; and in general,

(e) The structure and dynamics of scenarios that promote the use of different tools to learn mathematics and solve problems.

We introduce a pragmatic framework for teachers to organize learning activities that promote the systematic use of technology. The framework provides teachers with the opportunity to discuss aspects related to the presentation and exploration of mathematical tasks through the use of a dynamic software in problem solving environments. The aim is to identify and reflect on possible routes that teachers or researchers can follow to structure and organize problem-solving activities that enhance the use of technology with the purpose of furthering mathematics learning. We highlight a set of questions that teachers can think of as a way to delve into the problem through the use of technology.

To this end, we chose a generic[1] task that involves a variation phenomenon to illustrate how the use of the tool fosters an inquiring approach to make sense of the posed statement and to promote different ways of reasoning to explore and solve the task (NCTM, 2009). Thus, focusing on ways to represent a variation phenomenon through the tool demands that teachers identify, express, and explore mathematical relationships in terms of visual, numeric, graphic, and algebraic approaches. "Conceptualization of invariant structures amidst changing phenomena is often regarded as a key sign of knowledge acquisition" (Leung, 2008, p. 137). Thus, teachers need to work on tasks where the use of the tools provides them a set of affordances to identify and perceive what parameters vary and what are maintained invariant within the problem structure.

Background and Rationale

Lester (2010) quotes the online *Encarta World English Dictionary* to define a framework: "a set of ideas, principles, agreements, or rules that provides the basis or the outline for something that is more fully developed at a later stage" (p. 60). Our notion of framework includes initial arguments that describe patterns associated with the use of a dynamic software in mathematical problem solving. " A framework tells you what to look at and what its impact might be" (Schoenfeld, 2011, p. 4). It is a pragmatic framework that consists of episodes that could help practitioners re-examine and contrast those frameworks that explain learners competences exhibited in paper and pencil environments. It becomes a scaffolding tool to reflect on issues related to the use of tools in learning scenarios.

[1] Generic in the sense that the task represents a family of tasks where it is possible to explore or examine optimization behaviours of the parameters involved in the task.

Schoenfeld (1985) proposed a framework to explain students' problem solving behaviours in terms of what he calls basic resources, cognitive and metacognitve strategies, and students' beliefs. Schoenfeld's framework came from analyzing and categorizing experts and students' problem solving approaches that involve mainly the use of paper and pencil tools. What happens when subjects use systematically computational tools to make sense of problem statement, represent, explore and solve problems? We argue that the use of technology introduces new information to characterize the problem solver's proficiency. For instance, one of the tasks used by Schoenfeld involves asking the students to draw with straightedge and compass a circle that is tangent to two intersecting lines where one point of tangency is a given P on one line. Schoenfeld reports that students formulated several conjectures about the position of the centre of such a tangent circle: (a) The centre of the tangent circle C is the midpoint of the line segment between P and the point Q, where P and Q are equidistant from the point of intersection V (Figure 1a); (b) The centre of the circle is the midpoint of segment of the circular arc from P to Q that has centre V and radius |PV| (Figure 1b), etc. (Schoenfeld, 2011, p. 31).

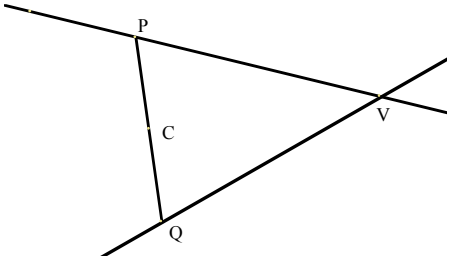

Figure 1a: A student conjecture

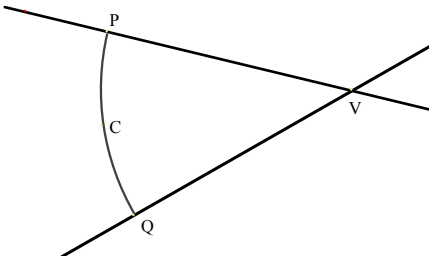

Figure 1b: Another student conjecture

Schoenfeld stated that the students picked up the straightedge and compass, tried out their conjecture, and either accepted or rejected it on the basis of how good their

drawing looked. With the use of a dynamic software "good drawing" doesn't depend on subject's skills to manage the straightedge and compass; rather, the tool provides the affordances (precision of drawings, parameter movement, quantification of parameters, loci, etc.) to deal or explore conjectures. That is, the use of a dynamic software provides teachers ways to initially visualize and test empirically conjectures and, they often access or develop relevant knowledge needed to verify and prove those conjectures (Moreno-Armella & Sriraman, 2005; Santos-Trigo, 2010). For example, in Figures 2a and 2b, the dotted circle drawn with the software provides elements to reject the corresponding conjectures. Thus, the use of the tool offers relevant information to characterize and foster the students' problem solving competences. For example, students can explore visually that the centre of the tangent circle lies on the perpendicular line to line PV at P (Figure 2c) and use that information to construct a formal approach based on properties embedded in that visual approach.

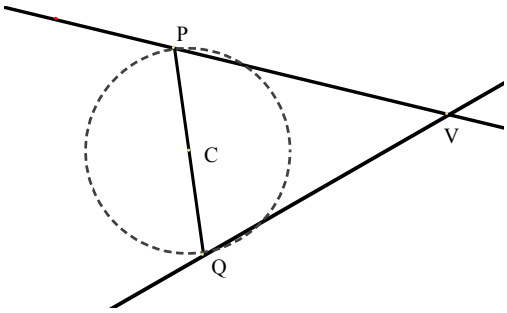

Figure 2a: A student conjecture

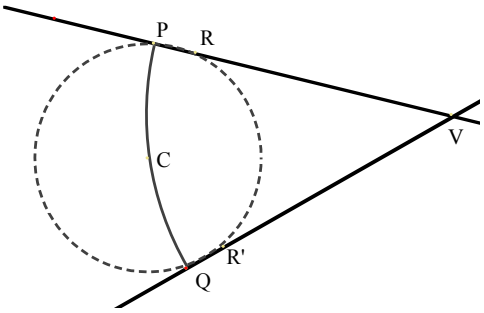

Figure 2b: Another student conjecture

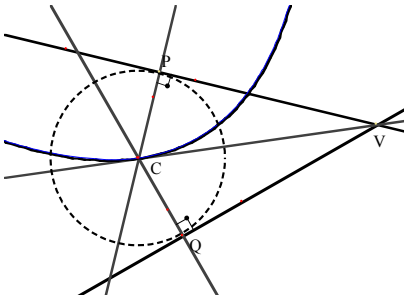

Figure 2c: The centre of the tangent circle lies on the perpendicular to PV that passes through P.

We argue that practitioners interested in using computational tools in their learning activities can find in the problem solving episodes described in the next section a quick reference to the type of mathematical discussions that might emerge during the problem solving sessions. In addition, the episodes might provide directions to structure a lesson plan where empirical, visual, graphic, and formal approaches can be considered to organize a didactic route. We contend that the episodes can provide relevant information that relates to what Jackiw and Sinclair (2009) call first and second order effects of the use of the software (referring to *The Geometer's Sketchpad*) in learning. "First-order effects are a direct consequences of the affordance of the environment; second-order effects are then a consequence of these consequences, and usually relate to changes in the way learners think, instead of changes in what learners *do*" (p. 414). That is, teachers could use the affordances associated with the software to encourage their students to think of novel ways to represent dynamically problem situations. Software' affordances (dragging, finding loci, quantifying parameters, etc.) provide ways to observe changes or invariance of involved parameters. As a consequence, the use of the tool allows the problem solver to develop ways of reasoning to examine parameters behaviours that emerge as a result of moving mathematical objects within the task representation or configuration. Heid & Blume (2008) stated "[t]he nature of a mathematical activity depends not only on the mathematical demands of the task but also on the process of the task as constructed by the doer" (p. 425). Thus, teachers with the

use of the tool might guide their students to think about the problem in different ways and to discuss concepts and processes that appear during the exploration of the task.

A problem-solving episodes to deal with phenomena of variation

An example is used to illustrate, in terms of episodes, a route to think of the use of technology to represent and explore the area variation of an inscribed parallelogram. The first episode emphasizes the relevance for the problem solvers to comprehend the statement in order to construct a dynamic representation that can help them visualize parameter behaviours.

The task

Given any triangle ABC, inscribe a parallelogram by selecting a point P on one of the sides of the given triangle. Then from point P draw a parallel line to one of the sides of the triangle. This line intersects one side of the given triangle at point Q. From Q draw a parallel line to side AB of the triangle. This line intersects side AC at R. Draw the parallelogram PQRA (Figure 3). How does the area of inscribed parallelogram APQR behave when point P is moved along side AB? Is there a position for point P where the area of APQR reaches a maximum value? (Justify).

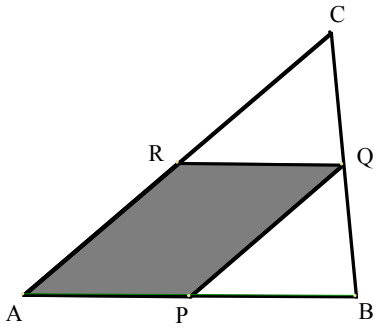

Figure 3: Drawing a parallelogram inscribed into a given triangle.

Comprehension Episode

Polya (1945) identifies the process of understanding the statement of a problem as a crucial step to think of possible ways for solving it. Understanding means being able to make sense of the given information, to identify relevant concepts, and to think of possible representations to explore the problem mathematically. The use of technology could help teachers focus on the construction of a dynamic model as a means to pose and explore questions that lead them to comprehend and make sense of tasks.

The comprehension stage involves questioning the statement and thinking of the use of the tool to make sense and represent the task. For instance, what does "for any given triangle" mean and how this can be expressed through the software?, what information does one need to draw *any* triangle?, are there different ways to inscribe a parallelogram into a given triangle?, and how can one draw a dynamic model of the problem? are examples of questions where the problem solver could rely on the tool to explore and discuss the problem. Thus, a route to answer these questions might involve using *Cabri-Geometry* or *The Geometer's Sketchpad* to draw triangle ABC (Figure 4) and from P on AB draw a parallel line to CB (instead of AC). This line intersects side AC and from that point of intersection, one can draw a parallel line to AB that intersects BC, thus, the two intersection points and point P and B form an inscribed parallelogram, the problem solver can ask: how is the former parallelogram related to the one that appears in Figure 3? Do they have the same area for the same position of P? How can we recognize that for different positions of point P the area of the parallelogram changes? This problem

comprehension phase is important not only to think of the task in terms of using the software commands, but also to identify and later examine possible variations of the task. For example, how does the area of a family of inscribed parallelograms, generated when P is moved along AB, change (Figure 4)?

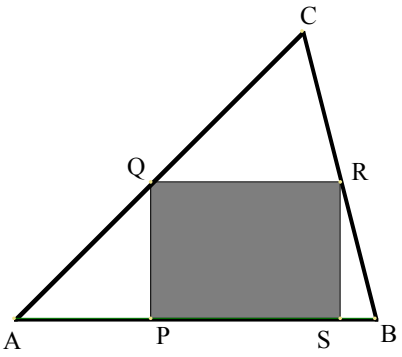

Figure 4: Another way to inscribe a parallelogram in a given triangle.

Comment

Making sense of the problem statement is a crucial step in any problem solving approach. The use of a dynamic software plays an important role in initially conceptualizing the statement as an opportunity to pose and explore a set of questions. That is, the use of the tool demands that the problem solver thinks of the statement in terms of mathematical properties to use the proper software commands to represent and explore the problem (Santos-Trigo & Espinosa-Pérez, 2010). In this case, teachers can work on the task in order to identify task's sketches that can help their students focus their attention to particular concepts or explorations. Of course, the posed questions don't include all possible routes to examine the statement; rather they illustrate an inquiry method to guide the problem solver's reflection.

A Problem Exploration Episode

Teachers can use the software to draw a triangle by selecting three non- collinear points and discuss conditions needed to draw it when for example three segments (instead of three points) are given (the triangle inequality). The use of the software allows moving any vertex to generate a family of triangles. This process broadens the cases for which the problem can be analyzed. Then, they can select a point P on side AB to draw the corresponding parallels to inscribe the parallelogram. With the help of the software it is possible to calculate the area of the parallelogram and observe area values change when point P is moved along side AB. Thus, it makes sense to ask whether there is a position of P in which the area of the inscribed parallelogram reaches either its maximum or minimum value. By setting a Cartesian system (an important heuristic) as a reference and without using algebra, it is possible to construct a function that associates the length of segment AP with the area value of the corresponding parallelogram. Figure 5 shows the graphic representation of that function. The domain of the function is the set of values that represents the lengths of AP when point P is moved along side AB. The range of that function is the corresponding area values of the parallelogram associated with the length AP. This graphic representation can be obtained through the software by asking: What is the locus of point S (the coordinates of point S are length AP and area of APQR) when point P moves along the side AB? It is important to observe that the graphic representation is obtained without defining explicitly the algebraic model of the area change of the parallelogram.

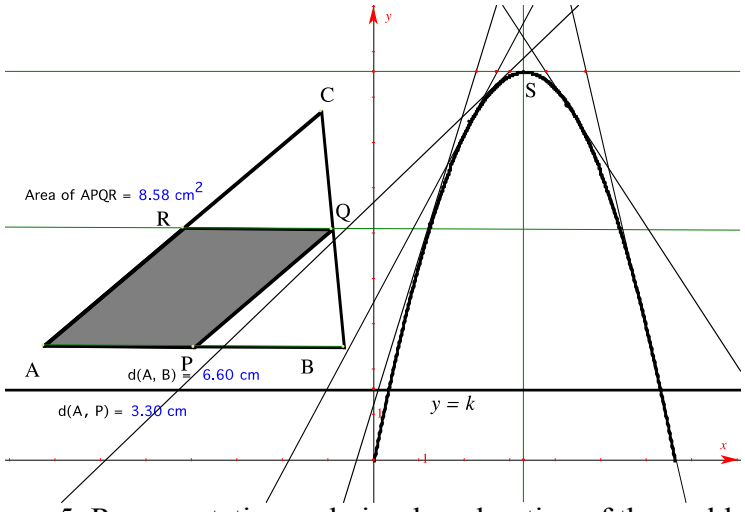

Figure 5: Representation and visual exploration of the problem.

This graphic approach to solve the problem provides an empirical solution. Both visually and numerically it is possible to observe that in the given triangle the maximum area of the inscribed parallelogram is obtained when P is situated at 2.30 cm from point A. At this point, the area value of the parallelogram is 8.56 cm^2. Based on this information a conjecture emerges: *When P is the midpoint of segment AB, then the corresponding inscribed parallelogram will reach the maximum area value*. Graphically the behaviour of tangent line to the curve behaves at different points can be observed (Figure 5). It can be seen that when the slope of the tangent line to the area graph is positive the function increases, but when the slope is negative the function area decreases.

Are there other ways to inscribe a parallelogram in triangle ABC? Figure 6 shows three ways to draw an inscribed parallelogram and all of them have the same area for different positions of point P. Also, Figure 7 shows that when point P is the midpoint of side AB then triangle ABC can be divided into four triangles with the same areas.

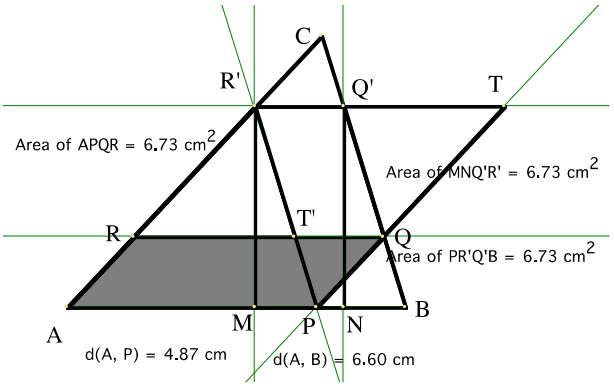

Figure 6: Inscribing three parallelograms in triangle ABC.

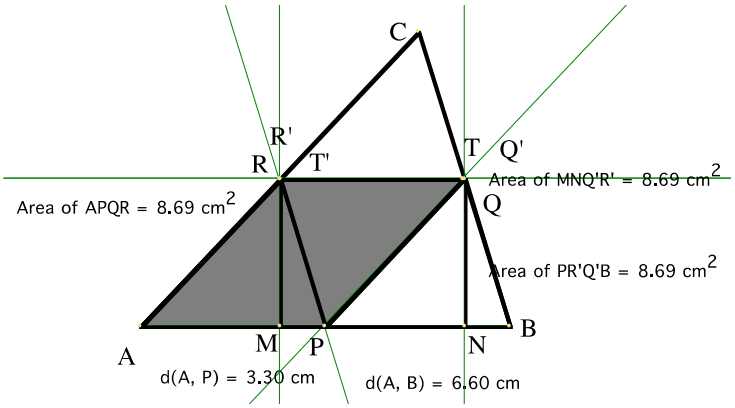

Figure 7: When point P is situated at the midpoint of segment AB, then triangles APR, PQR, PBQ, and RQC have the same area.

From Figures 6 and 7 two conjectures emerge: (i) the three inscribed parallelograms always have the same area for different positions of point P, and (ii) when point P is the midpoint of segment AB, the four triangles always have the same area and the maximum area of the inscribed parallelogram is half the area of the original or given triangle. Thus, the use of the tool provides an opportunity for the problem solver to simultaneously examine properties of figures that within the configuration. These conjectures are proved further down.

Comment

The dynamic representation becomes a source that generates mathematical conjectures as a result of moving objects within the configuration. Exploring different ways to inscribe the parallelogram leads to formulate two related conjectures. In addition, the use of the tool allows graphing the area's variation without defining explicitly an algebraic model. Thus, it is possible to think of a functional approach, without defining the function algebraically, that associates the position of point P (for example, the distance between AB, BP or AC) with the corresponding area value. Figure 5 provides a visual and numerical approach to describe the parallelogram's area behaviour.

The Searching for Multiple Approaches Episode

We argue that if students are to develop a conceptual understanding of mathematical ideas and problem solving proficiency, they need to think of different ways to solve a problem or to examine a mathematical concept. In this context, the visual and empirical approaches used previously to explore the problem provide a basis to introduce other approaches. We argue that each approach to the problem demands that the problem solver not only think of the problem in different ways; but also to use different concepts and resources to solve it.

Analytical approach

In this approach, the use of the Cartesian system becomes important to represent the objects algebraically. The problem can be thought in general terms as shown below.

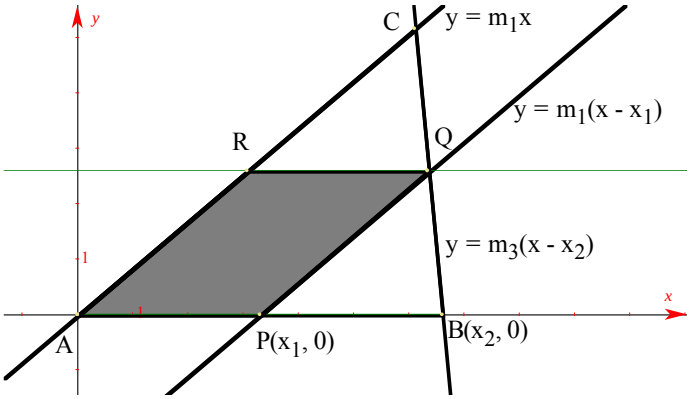

Figure 8: Using a Cartesian system to construct an algebraic model of the problem

General case

Without loosing generality, we can always situate the Cartesian system in such a way that one side of the given triangle can be on the x-axis and the other side on line $y = m_1 x$ (Figure 8). Point P will be located on side AB and its coordinates will be $P(x_1, 0)$. Point $B(x_2, 0)$ is vertex B of the given triangle (Figure 8). The general goal is to represent the area of parallelogram APQR in terms of known parameters. This process leads to represent the area in terms of one variable ($AP = x_1$) as:

$A(x_1) = \dfrac{m_1 m_3 \left(x_1^2 - x_2 x_1\right)}{m_1 - m_3}$. The roots of $A(x_1)$ (a quadratic function) are 0 and x_2. Also, this function has a maximum value if and only if $\dfrac{m_1 m_3}{m_1 - m_3} < 0$. We are assuming that $m_1 > 0$. The assumption on the triangle location guarantees that m_3 and $m_1 - m_3$ have opposite signs. By a symmetric argument, $A(x_1)$ reaches its maximum at the midpoint of the interval $[0, x_2]$, that is, at $x_1 = \dfrac{x_2}{2}$.

Another way to determine the maximum value of this expression is by using calculus concepts: $A'(x_1)=\dfrac{m_1 m_3(2x_1-x_2)}{m_1-m_3}$, the critical points are obtained when $A'(x_1)=0$, we have that $x_1=\dfrac{x_2}{2}$ which is the solution of the equation, then the function $A(x_1)$ will reach its maximum value at $x_1=\dfrac{x_2}{2}$. This is because $A''(x_1)=\dfrac{m_1 m_3}{m_1-m_3}<0$. Thus, this result supports the conjecture formulated previously in the graphic approach.

General case

It is possible to use a hand-held calculator to find the maximum area for the case shown in Figure 9. In this case, we have that $m_1=72/85;\ m_3=-10.33$; and $x_2=6.6cm$.

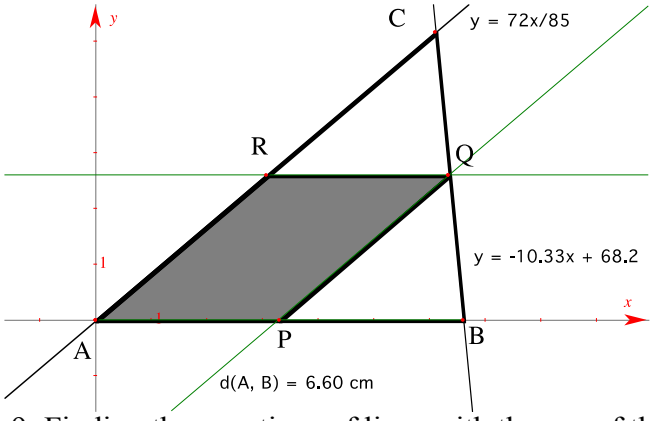

Figure 9: Finding the equations of lines with the use of the tool.

Figure 10 shows the algebraic operation carried out to get the point where the function reaches its maximum value and Figure 11 shows its graphic representation.

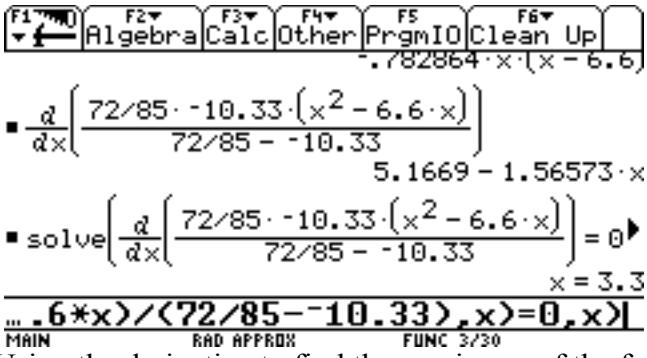

Figure 10: Using the derivative to find the maximum of the function area.

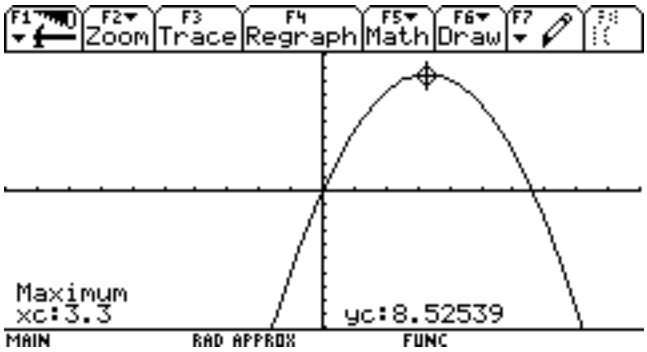

Figure 11: Graphic representation of the function area.

A Geometric approach

The goal is to use geometric properties embedded in the problem's representation to construct an algebraic model. In Figure 12, it can be seen that:

Triangle ΔABC is similar to triangle ΔPBQ, this is because angle PQB is congruent to angle ACB (they are corresponding angles) and angle ABC is the same as angle PBQ. Based on this information, $h_1 = \frac{h(a-x)}{a}$ and the area of APQR can then be expressed as $A = xh_1$, that is, $A(x) = x\left(\frac{h(a-x)}{a}\right)$. This latter expression can be written as $A(x) = xh - \frac{hx^2}{a}$. This expression represents a parabola. $A'(x) = h - \frac{2hx}{a}$, now if

$A'(x) = h - \frac{2hx}{a} = 0$, then $x = a/2$. Now, we observe that $A'' < 0$ for any point on the domain defined for $A(x)$, therefore, there is a maximum relative for that value.

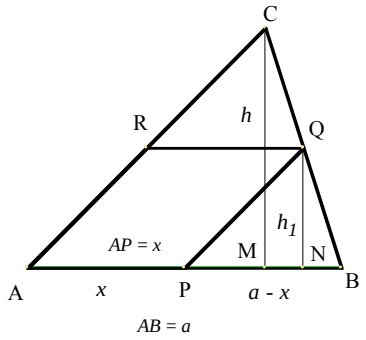

Figure 12: Relying on geometric properties to construct an algebraic model.

During the comprehension and exploration episodes two conjectures emerged, the first one (area of parallelogram APQR is the same as area of parallelogram PBQ'R') can be proved by considering parallelogram APTR' (Figure 7). It is observed that triangles APR' and TR'P are congruent and triangles RR'T' and TQQ' are also congruent (SSS). Then, we have that quadrilaterals APT'R and T'QQ'R' have equal areas, also, the area of triangle PQT' is the same as the area of triangle PQB. Based on this information, we have that the area of APQR is the same as area of PBQ'R'.

The second conjecture that involves showing that the four triangles have the same area can be proved by observing that the triangles are part of three parallelograms (APQR, PBQR and PQCR) that overlap each other (Figure 7). Then the overlapping triangle PQR has the same area as the others because they share a diagonal as a side of each corresponding parallelogram. Therefore, the maximum value of the inscribed parallelogram is half the area of the given triangle (ΔABC).

Comment

An important feature of the frame is that teachers should always look for different ways to solve and examine the tasks. The common goal in the task is to represent and explore the area model, however the approaches used to achieve this goal offer tearcher the opportunity to focus on diverse concepts and resources as a way to construct the model. For example, the algebraic model relies on representing and operating mathematical objects analytically while the geometric approach is based on using triangles' properties to define the area model. It is also observed that the general model can be tested by assigning particular coordinates to the original triangle vertices. Thus, problem solvers have the opportunity to test their initial conjectures obtained visually and empirically by using now the general result (Figure 10 and 11). The use of a hand-held calculator, in general, makes easy to operate the algebraic expressions and as a consequence learners could focus their attention to discuss the meaning of the results. Each approach relies on using different concepts and ways to deal with the involved relations. As a consequence, the problem solver can contrast strengths and limitations associated to each approach.

An extension

In figure 13, we draw a line passing through points PR (vertices of parallelogram APQR). With the use of the software, we ask for the locus of line PR (envelope) when point P is moved along side AB. Visually, the locus (tangent points) seems to be a conic section, the goal is to show that it holds properties that define that figure.

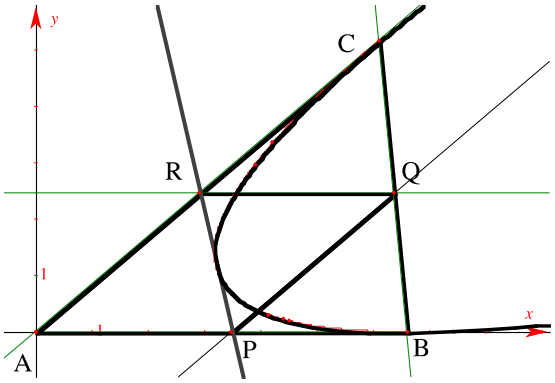

Figure 13: What is the locus of line PR when point P is moved along side AB?

Again, with the use of the tool it is shown that the locus is a parabola whose focus and directrix are identified in Figure 14. It is also shown that when point M is moved along the locus the distance from that point to the directrix (L) and to point F (focus of the parabola) is the same (this property defines a parabola).

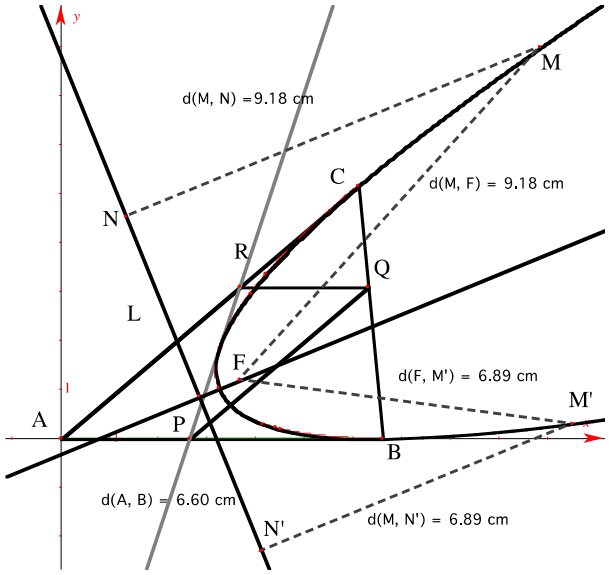

Figure 14: The locus of line PR when point P is moved along segment AB is a parabola.

Comment

Some serendipitous results or relations might appear as a result of introducing other objets within the configuration. In this case, adding a line PR to the configuration led to identify a conic section. Thus, the use of the tool offers a means to think of

mathematical connections that are not easy to identify with the only use of paper and pencil approaches.

The Integration Episode and Reflections

It is important and convenient to reflect on the processes involved in the distinct phases that characterize an approach to solve mathematical problems that fosters the use of computational technology. Initially, the comprehension of the problem's statements or concepts involves the use of an inquiry approach to make sense of relevant information embedded in those concepts or statements. This enquiry process provides the basis to relate the use of the tools and ways to represent dynamically the problem or situation. A dynamic model becomes a source from which to explore visually and numerically the behaviour of parameters, as a result of displacing some elements within the problem representation. In particular, it might be possible to construct a functional relationship between a variable, for example the variation of the side AP of the parallelogram and its corresponding area.

Two distinct ways to construct an algebraic model of the area variation were pursued; one involves the use of the Cartesian system to identify the equations associated with some elements of the model. The second way relies on identifying similar triangles in the inscribed parallelogram whose properties led to the construction of the area model. Both approaches, the analytic and geometric, converge in the search for the algebraic model. The algebraic model represents the general case and it can be "validated" by considering the information of the triangle used to generate the visual model. In addition, it can be used to explore some of the relations that were detected

during the visual approach. For example, to identify the intersection points of line $y = k$ and the area model $A(x_1) = \frac{m_1 m_3 \left(x_1^2 - x_2 x_1\right)}{m_1 - m_3}$ (Figure 5) we solve the equation $k = \frac{m_1 m_3 \left(x_1^2 - x_2 x_1\right)}{m_1 - m_3}$ for x_1. Thus, the discriminant of this quadratic equation $\Delta = (m_1 m_3 x_2)^2 + 4 m_1 m_3 (k m_3 - k m_1)$ provides useful information to interpret the relationship between line $y = k$ and the graph of the area model $A(x_1) = \frac{m_1 m_3 \left(x_1^2 - x_2 x_1\right)}{m_1 - m_3}$. When the discriminant is zero the line intersects the graph at the maximum point, when it is greater than zero, there are two intersection points and when the discriminant is less than zero, then the line does not intersect the area's graph.

Concluding, the systematic use of computational tools in problem solving approaches led us to identify a pragmatic framework to structure and guide learning activities in such a way that teachers can help the students develop mathematical thinking. A distinguishing feature of the problem solving episodes is that constructing a dynamic model of the phenomena provides interesting ways to deal with them from visual and empirical approaches. Later, analytical and formal methods are used to support conjectures and particular cases that appear in those initial approaches. The NCTM (2009) recognizes that reasoning and sense making activities require for students to gradually develop levels of understanding to progress from less formal reasoning to more formal approaches.

The use of computational tools provides a basis not only to introduce and connect empirical and formal approaches, but also to use powerful heuristics as dragging objects

and finding loci of particular objects within the dynamic problem representation. As Jackiw & Sinclair (2009) pointed out "Dynamic Geometry is revealed as a technological capability to produce seemingly limitless series of continuously-related examples, and in so doing, to represent visually the entire phase-space or configuration potential of an underlying mathematical construction" (p. 414). Throughout the problem solving episodes we show that it is important for teachers to conceive of a task or problem as an opportunity for their students to represent, explore and examine the task from diverse perspectives in order to formulate conjectures and to look for ways to support them. The diversity of approaches allows them to contrast and relate different concepts and ways to reason about their meaning and applications. In this context, the use of the tools opens up new windows to frame and encourage teachers and students' mathematical discussions

Remarks

Is there any way to characterize forms or ways of mathematical reasoning that emerge as a result of using computational tools in problem solving approaches? In which ways does this reasoning complement problem solving approaches that rely on the use of paper and pencil? Thinking of the task in terms of the affordances provided by the tools demands that problem solvers focus their attention on ways to take advantage of the opportunities offered by the tool to represent and explore the problem. For example, the use of the tool to construct a dynamic model of a task not only becomes relevant to identify and formulate series of conjectures or mathematical relations but also to reason about the task in terms of graphic and visual approaches without relying, at this stage, on an analytic model. In addition, with the use of the software becomes natural and easy to

extend the analysis of a case to a family of cases. For example, by moving any vertex of triangle ABC, it is possible to verify that all the relations found during the analysis of the task are also true for a family of triangles that result when moving one the vertices. With the use of the tool it is often possible to generate loci of points or lines within the model or to identify parameter behaviours without defining the corresponding algebraic model. In addition, the empirical and visual approaches often provide important information to present formal arguments to support conjectures. In this context, it is clear that the software approach could play an important role to complement and construct formal or analytic approaches.

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