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## Proof and Problem Solving at University Level

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**Abstract:** This paper will be concerned with undergraduate and graduate students' problem solving as they encounter it in attempting to prove theorems, mainly to satisfy their professors in their courses, but also as they conduct original research for theses and dissertations. We take Schoenfeld's (1985) view of problem, namely, a mathematical task is a problem for an individual if that person does not already know a method of solution for that task. Thus, a given task may be a problem for one individual, who does not already know a solution method for that task, or it may be an exercise for an individual who already knows a procedure or an algorithm for solving that task.

**Keywords:** Hungary, mathematics education, mathematics competition, Olympiads, international comparative mathematics education, problem solving, creativity, mathematically talented students.

### A Continuum of Tasks from Very Routine to Very Non-routine

While what is a problem depends on what a solver knows, it is possible for most mathematics teachers to judge what is difficult for most students in a given class. Thus, we see mathematical tasks for a given class, such as a calculus class, on a continuum from those that are very routine to those that are genuinely difficult problems (Selden, Selden, Hauk, & Mason, 2000). At one end, there are *very routine* problems which mimic sample worked problems found in textbooks or lectures, except for minor changes in wording, notation, coefficients, constants, or functions that are incidental to the way the problems are solved. Such problems are often referred to as exercises (and might not be considered to be problems at all in the problem-solving literature).

The vast majority of exercises in calculus textbooks are of this nature. Lithner (2004) distinguished three possible solution strategies for typical calculus textbook exercises: identification of similarities (IS), local plausible reasoning (LPR), and global plausible reasoning (GPR). In IS, one identifies surface features of the exercise and looks for a similar textbook situation -- an example, a rule, a definition, a theorem. Without consideration of intrinsic mathematical properties, one simply copies the procedure of that situation. In LPR, one identifies a slightly similar textbook situation, but one in which a few local parts may differ. The solution strategy is to copy as much as possible from that similar situation, modifying local steps as needed. In GPR, the strategy is mainly based on analyzing and considering intrinsic mathematical properties of the exercise, and using these, a solution is constructed and supported by plausible reasoning. Lithner selected a textbook used in Sweden [Adams' *Calculus: A Complete Course* (5th ed.), Addison-Wesley], and worked through and classified solution strategies for 598 single-variable calculus exercises. He found 85% IS, 8% LPR, and 7% GPR. Furthermore, he concluded that "it is possible in about 70% of the exercises to base the solution not only on searching for similar situations, but on searching only the solved examples."

Moving toward the middle of the continuum, there are *moderately routine* problems which, although not exactly like sample worked problems, can be solved by well-practiced methods, for example, ordinary related rates or change of variable integration problems in a calculus course.<sup>1</sup> Moving further along the continuum, there are *moderately non-routine* problems, which are not very similar to problems that students

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<sup>1</sup> Sandra Marshall (1995) has studied how students can develop schema (well-practiced routines) to reliably guide the solution of arithmetic word problems.

have seen before and require known facts or skills to be combined in a slightly novel way, but are "straightforward" in not requiring, for example, the consideration of multiple sub-problems or novel insights. This is the type of problem we used on the non-routine test in our three studies of undergraduate students' calculus problem solving. One of those problems was: *Find values of  $a$  and  $b$  so that the line  $2x+3y=a$  is tangent to the graph of  $f(x) = bx^2$  at the point where  $x=3$ .* (Selden, Selden, Mason, & Hauk, 2000, p. 133).

Finally, at the opposite end of the continuum from routine problems, there are *very non-routine* problems which, while dependent on resources in one's knowledge base, may involve considerable insight, the consideration of several sub-problems or constructions, and use of Schoenfeld's (1985) behavioral problem-solving characteristics (heuristics, control, beliefs). For such problems a large supply of *tentative solution starts* (Selden, Selden, Mason, & Hauk, 2000, p. 145), built up from experience, might not be adequate to bring to mind the resources needed for a solution, while for moderately novel problems it probably would. Often the Putnam Examinations include such very non-routine problems.<sup>2</sup>

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<sup>2</sup>The following problem was on the 59th Annual William Lowell Putnam Mathematical Competition given on December 5, 1998: Given a point  $(a, b)$  with  $0 < b < a$ , determine the minimum perimeter of a triangle with one vertex at  $(a, b)$ , one on the  $x$ -axis, and one on the line  $y = x$ . You may assume that a triangle of minimum perimeter exists.

This appears to be a calculus problem, but it only requires clever use of geometry. An elegant solution (posted by Iliya Bluskov to the sci.math newsgroup) involves extending the construction "outward" by reflecting across both the lines  $y = x$  and the  $x$ -axis and noticing that the perimeter of the triangle equals the distance along the path from  $(b, a)$  to  $(a, -b)$ . Probably only very experienced geometry problem solvers could have previously constructed images of problem situations containing a tentative solution start that would easily bring this method to mind.

Most U.S. university mathematics teachers would probably like undergraduate students who pass their lower-level courses, such as calculus, to be able to work a wide selection of routine, or even moderately routine, problems. In addition, we believe that many such teachers would also expect their better students to be able to work moderately non-routine problems, and would think of the ability to do so as functionally equivalent to having a good conceptual grasp of the course. In other words, we conjecture that the ability to work moderately non-routine problems based on the material in a university mathematics course, such as calculus, is often considered part of the implicit curriculum and taken as equivalent to good conceptual grasp. However, no research has yet been done to substantiate this conjecture.

### **Tentative Solution Starts**

An individual who has reflected on a number of problems is likely to have seen (perhaps tacitly) similarities between some of them. He or she might recognize (not necessarily explicitly or consciously) several overlapping problem situations, each arising from problems with similar features. For example, after much exposure, many lower-level university students would probably recognize a problem as one involving, for example, factoring, several linear equations, or integration by parts.<sup>3</sup> Such problem situations can act much like concepts (perhaps without signs or labels). While they may lack names, for a given individual they are likely to be associated with mental images,

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<sup>3</sup>Although the kinds of features noticed by students in mathematical problem situations do not seem to have been well studied, the features focused on in physics problem situations have been observed to correspond to an individual's degree of expertise. Novices tend to favor surface characteristics (e.g., pulleys), whereas experts tended to focus on underlying principles of physics (e.g., conservation of energy) (Chi, Feltovich, & Glaser, 1981).

that is, strategies, examples, non-examples, theorems, judgments of difficulty, and the like. Following Tall and Vinner's (1981) idea of concept image, we have called this kind of mental structure a *problem situation image* and have suggested that some such images may, and others may not, contain what we have called *tentative solution starts* (Selden, Selden, Hauk, & Mason, 2000, p. 145). These are tentative general ideas for beginning the process of finding a solution. The linking of problem situations with one or more tentative solution starts is a kind of (perhaps tacit) knowledge. For instance, the image of a problem situation asking for the solution to an equation might include "try getting a zero on one side and then factoring the other." It might also include "try writing the equation as  $f(x) = 0$  and looking for where the graph of  $f(x)$  crosses the  $x$ -axis," or even "perhaps the maximum of  $f$  is negative so  $f(x) = 0$  has no solution." We suggest that an individual's problem-solving processes are likely to include the recognition of a problem as belonging to one or more problem situations, and hence, bring to mind one or more tentative solution starts contained in that individual's problem situation image. This, in turn, may mentally prime the recall of resources from that individual's knowledge base. Thus, a tentative solution start may link recognition of a problem situation with the recall of appropriate resources. We have suggested that problem situations, their images, and the associated tentative solution starts all vary from individual to individual and that the process of mentally linking recognition (of a problem situation) to recall (of requisite resources) through problem situation images might occur several times in solving a single problem, especially when an impasse occurs (Selden, Selden, Mason, & Hauk, 2000, pp. 145-147).

## **The Genre of Proof**

We consider proofs, those that occur in advanced university mathematics textbooks and research journals, as being written in a special genre. It is clear that not every mathematical argument can be considered a proof, and much has been written in the mathematics education research literature about the distinction between argumentation and proof. (See, for example, Duval, as reported in Dreyfus, 1999, and Douek, 1999.) In this paper, we are considering proofs of the sort that advanced undergraduate students and beginning graduate mathematics students are expected to produce for their professors. We are aware that many upper-level U.S. mathematics majors just beginning their study of proof-based courses such as abstract algebra and real analysis often have great difficulty producing such proofs, despite the fact that many of them have previously taken a transition-to-proof course (Moore, 1994), usually in their second year of university. Students in such transition-to-proof courses often have trouble knowing what to write, especially when asked to prove simple set theory theorems, perhaps because the results are “too obvious” or are verifiable using examples or Venn diagrams. Thus, learning the genre of proof is important. Indeed, to help students learn the genre of proof, we have considered two aspects (or parts) of a final written proof: the formal-rhetorical part and the problem-centered part. The *formal-rhetorical* part of a proof (also sometimes referred to as a *proof framework*) is the part of a proof that depends only on unpacking and using the logical structure of the statement of the theorem, associated definitions, and earlier results. In general, this part does not depend on a deep understanding of, or intuition about, the concepts involved or on genuine problem solving in the sense of Schoenfeld (1985, p. 74). We call the remaining part of a

proof the *problem-centered* part. It is the part that *does* depend on genuine mathematical problem solving, intuition, and a deeper understanding of the concepts involved. (See Selden & Selden, 2009).

A sample proof framework is given below for a proof of the following theorem: If  $f$  and  $g$  are real valued functions of a real variable continuous at  $a$ , then  $f + g$  is continuous at  $a$ .

Proof. Let  $f$  and  $g$  be functions and suppose they are continuous at  $a$ . Suppose  $\varepsilon$  is a number  $> 0$ . Because  $f$  is continuous, there is a  $\delta_f > 0$  so that for all  $x$ , if  $|x - a| < \delta_f$ , then  $|f(x) - f(a)| < \underline{\hspace{1cm}}$ . Also because  $g$  is continuous, there is a  $\delta_g > 0$  so that for all  $x$ , if  $|x - a| < \delta_g$ , then

$$|g(x) - g(a)| < \underline{\hspace{1cm}}. \text{ Let } \delta = \underline{\hspace{1cm}}. \text{ Note that } \delta > 0. \text{ Let } x \text{ be a number.}$$

Suppose that  $|x - a| < \delta$ . Then  $|f(x) + g(x) - (f(a) + g(a))| = \dots < \varepsilon$ . Thus,  $|f(x) + g(x) - (f(a) + g(a))| < \varepsilon$ . Therefore  $f + g$  is continuous at  $a$ .

The problem-centered part of the proof consists of cleverly filling in the blanks using, for example  $\frac{\varepsilon}{2}$ , minimum, and the triangle inequality. This is not to say that filling in the blanks is easy. Indeed it can be very difficult for an individual with little or no experience with such real analysis proofs.

Being able to write a proof framework can be very helpful for students because it not only improves their proof writing, bringing it in line with accepted community norms, but also because it can reveal the nature of the problem(s) to be solved. Having once learned to write a number of proof frameworks, students can then concentrate their creative energies on solving the actual mathematical problems involved. In addition, for students, writing the formal-rhetorical part of a proof, and whatever else they can



regarding the actual problem(s) to be solved, can enable their university mathematics teachers to give more helpful and targeted criticism and advice.

### **The Close Relationship Between Problem Solving and Proving**

A number of authors have remarked on the close relationship between problem solving and proving (e.g., Furinghetti & Morselli, 2009; Mamona-Downs & Downs, 2009; Moore, 1994), and our division of proofs into their formal-rhetorical and problem-centered parts (described above) can make this explicit for students. However, having good ideas for how to solve the problem-centered part of a proof is not equivalent to having a proof. Mamona-Downs and Downs (2009) have given university students informal arguments suggesting a way to solve tasks and asked them to convert those arguments into acceptable mathematical form. They concluded that “proof production [from an intuitively developed argument] can involve significant problem solving aspects. ... A particularly frustrating circumstance for a student is when he/she can ‘see’ a reason why a mathematical proposition is true, but lacks the means to express it as an explicit [mathematical] argument.” Thus, there are actually two distinct kinds of problem solving that can occur during proof construction, namely, solving the actual mathematical problem(s) that enable one to get from the given hypotheses to the given conclusion, and converting one’s (informal) solution into acceptable mathematical form. Neither of these problem-solving tasks is easy and students may require instruction and practice with each. How informal arguments are converted into acceptable mathematical form has been very little researched.

Sometimes one's informal argumentation (developed during one's initial problem solving) can be converted rather seamlessly into a written proof that is acceptable to the mathematical community. Boero, Douek, Morselli, and Pedemonte (2010, p. 183) suggested that "in some cases this [problem-solving] argumentation can be exploited by the student in the construction of a proof by organizing some of the previously produced arguments into a logical chain," and in such situations, writing proofs is easier for students. They refer to this situation as one of structural *cognitive unity*.

We have experienced such structural cognitive unity with beginning graduate mathematics students. They find the following theorem comparatively easy to prove: *If  $g$  is a function continuous at  $a$  and  $f$  is a function continuous at  $g(a)$ , then  $f \circ g$  is continuous at  $a$ .* We conjecture that, if they think of an "appropriate representation," (see Figure 1), they can sketch an epsilon neighborhood of  $f(g(a))$ , and invoking the continuity of  $f$ , "pull it back" to a delta neighborhood of  $g(a)$ . Then they can use that delta,  $\delta_g$ , as a new epsilon in applying the continuity of  $g$  to arrive at the needed final delta,  $\delta_f$ .

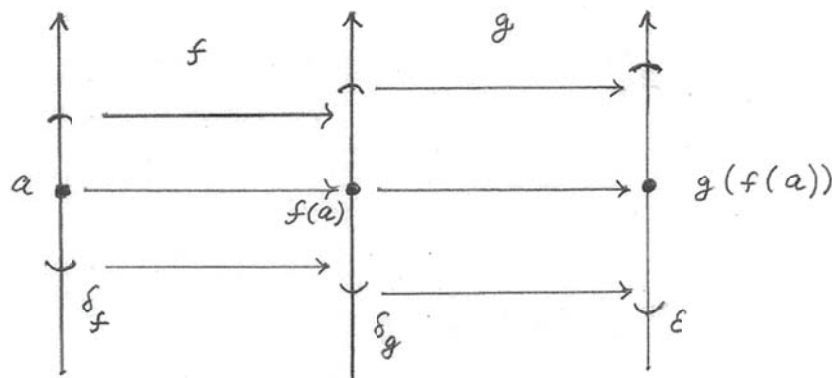


Figure 1. A “picture” of  $g \circ f$  with the epsilon and delta neighborhoods indicated.

However, the theorem whose proof framework was illustrated above, namely, *If  $f$  and  $g$  are real valued functions of a real variable continuous at  $a$ , then  $f + g$  is continuous at  $a$* , and whose proof involves using minimum and the triangle inequality cannot be easily obtained from informal intuitive argumentation about adding together the ordinates of the Cartesian graphs of  $f$  and  $g$ .

### **The Importance of Problem Reformulation and Selection of Appropriate Representations**

A number of researchers (e.g., Boero, 2001; Gholamazad, Liljedahl & Zazkis, 2003; Zazkis & Liljedahl, 2004) have noted that reformulating a problem by making an appropriate choice of representation is often useful, sometimes even necessary, to make progress. Furinghetti and Morselli (2009) reported the unsuccessful problem-solving behavior of two fourth-year Italian university mathematics education students during attempts to prove that *The sum of two numbers that are prime to one another is prime to each of the addends*. One student, with the pseudonym Flower, after some initial panic and working with examples, succeeded in producing a potentially helpful representation (using the Prime Factorization Theorem), but could not exploit it. The other student, with the pseudonym Booh, first chose the representation of Least Common Multiple that “synthesizes [captures the essence of] the property, but doesn’t allow algebraic manipulation ... being non-transparent” and realized that it was “without future.” So he considered another representation (also using the Prime Factorization Theorem), but also could not exploit it. In a separate earlier paper, Furinghetti & Moriselli (2007) reported

the unconventional, metaphorical thinking of another student who chose to think of, and draw, a frog jumping from stop to stop (i.e., from integer to integer on the number line) and successfully proved the same theorem. They noted that “The choice of the representation ... may foster or hinder transformational and anticipatory thinking, which are two key issues in the proving process” (Furinghetti & Morselli, 2009, p. 74).

Concepts can have several (easily manipulated) symbolic representations or none at all. For example, prime numbers have no such representation; they are sometimes defined as those positive integers having exactly two factors or being divisible only by 1 and themselves. It has been argued that the lack of an (easily manipulated) symbolic representation makes understanding prime numbers especially difficult, in particular, for preservice teachers (Zazkis & Liljedahl, 2004).

Symbolic representations can make certain features transparent and others opaque.<sup>4</sup> For example, if one wants to prove a multiplicative property of complex numbers, it is often better to use the representation  $re^{i\theta}$ , rather than  $x+iy$ , and if one wants to prove certain results in linear algebra, it may be better to use linear transformations,  $T$ , rather than matrices. Students often lack the experience to know when a given representation is likely to be useful. More research is needed on the effect of one’s choice of representation(s) on successful problem-solving behavior.

### **How Mathematicians Solve Problems**

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<sup>4</sup> For example, representing 784 as  $28^2$  makes the property of being a perfect square transparent and the property of being divisible by 98 opaque. For more details, see Zazkis and Liljedahl (2004, pp. 165-166).

It would be very informative to have research on how advanced university mathematics students or mathematicians actually construct proofs in real time, but such a study has not yet been conducted. However, there is research on how mathematicians solve mathematical problems of various kinds: Carlson and Bloom (2005) investigated how mathematicians manage their well-connected conceptual knowledge and make decisions during problem solving; DeFranco (1996) replicated Schoenfeld's work on the use of resources, heuristics, control, and beliefs with mathematicians; and Stylianou (2002) investigated how mathematicians use diagrams in problem solving.

However, the problems given to the mathematicians in these studies were not the sort encountered by advanced undergraduate or graduate mathematics students when constructing proofs for their courses or by professors when conducting research. For example, one problem whose solution Carlson and Bloom (2005, p. 55) discussed at length was: *A square piece of paper ABDC is white on the front side and black on the back side and has an area of 3 square inches. Corner A is folded over to point A' which lies on the diagonal AC such that the total visible area is  $\frac{1}{2}$  white and  $\frac{1}{2}$  black. How far is A' from the fold line?* One problem used by DeFranco (p. 212) was: *In how many ways can you change one-half dollar?*

Still some of the results are interesting, so we briefly recall them here. Based on interviews with 12 mathematicians, Carlson and Bloom (2005) developed a "problem solving framework" that has four phases (orientation, planning, executing, and checking). As part of the planning phase, there was a *conjecture-imagine-evaluate* sub-cycle, in which the mathematicians typically imagined a hypothetical solution approach, followed by "playing out" and evaluating whether that approach was viable. If it was not viable,

the *conjecture-imagine-evaluate* sub-cycle was repeated until a viable solution path was identified. Carlson and Bloom (2005, p. 45) stated, “The effectiveness of the mathematicians in making intelligent decisions that led down productive paths appeared to stem from their ability to draw on a large reservoir of well-connected knowledge, heuristics, and facts, as well as their ability to manage their emotional responses.”

DeFranco (1996) studied the problem-solving behaviors of eight research mathematicians who had achieved national or international recognition in the mathematics community (e.g., had altogether 12 honorary degrees and had been awarded prizes such as the National Medal of Science) and eight who had not achieved such recognition, but had published from three to 52 articles. He concluded that the former were problem-solving experts, as well as content experts, and had superior metacognitive skills, whereas the latter were content experts with only modest problem-solving skills.

Stylianou (2002) was interested in the interplay between visualization and analytical thinking and asked mathematicians the following problem: *Given a right circular cylinder cut at an angle* (shown in her accompanying diagram), *describe the resulting truncated cylinder’s net, that is, the “unrolled” truncated cylinder.* She observed that the “mathematicians consistently attempted to infer additional consequences from their visual action. Each time a mathematician either constructed a new diagram or modified a previously constructed one, he took a few seconds to ‘extract’ any additional information . . . and to understand any possible implications.”

In addition, there have been studies (e.g., Burton, 1999) that have included the reflections of mathematicians upon their own ways of working; however, these are often too general to be useful for an in-depth understanding of problem solving or for obtaining

suggestions for teaching. For example, Burton (1999) found some of the mathematicians likened problem solving and research to working on jigsaw puzzles or to climbing mountains.

### **The Role of Affect in Proving and Problem Solving**

While strong affect can play both a positive and a negative role during proving and problem solving, more research is needed on the role of various kinds of affect from beliefs and attitudes to emotions and feelings. Furinghetti and Morselli (2009, p. 82) considered how negative affective factors influenced the problem-solving behavior of their two unsuccessful students. They noted that Flower panicked immediately after reading the statement of the theorem writing, “Help! I’m not familiar with prime numbers!” Later, after constructing some examples, Flower wrote, “Help! I cannot do it, I do not see anything. Deepest darkness.” Then when she came up with the prime factorizations, Flower apparently expected to “conclude the proving process in an almost automatic way ... [without] the possibility of dead ends and failures,” illustrating that beliefs and expectations are also important factors influencing problem-solving outcomes.

In their study of mathematicians’ problem solving, Carlson and Bloom (2005) concluded, “The effectiveness of the mathematicians ... appeared to stem from their ability to draw on a large reservoir of well-connected knowledge, heuristics, and facts, as well as their *ability to manage their emotional responses* [italics ours].” In a study of non-routine problem solving, McLeod, Metzger, and Craviotto (1989) found that both experts (research mathematicians) and novices (undergraduates enrolled in college-level

mathematics courses), when given different experience appropriate problems, reported having similar intense emotional reactions such as frustration, aggravation, and disappointment, but the experts were better able to control them.

DeBellis and Goldin (1997, 2006) have considered affect (i.e., values, beliefs, attitudes, and emotions) as an internal representational system that is not merely auxiliary to cognition, but as “a highly structured system that encodes information, interacting fundamentally – and reciprocally – with cognition.” They have introduced the construct of *meta-affect*, by which they mean not only affect about affect, but also cognition that acts to monitor and direct one’s emotional feelings. They also suggested that one might characterize individuals’ *affective competencies*, such as the ability to act on curiosity or to see frustrations as a signal to modify strategy, but did not suggest how to do so.

In addition, we see nonemotional cognitive feelings of appropriateness and of rightness or wrongness as giving direction to one’s problem-solving efforts. As Mangan (2001, Section 6, Paragraph 3) said, “In trying to solve, say, a demanding math problem, [a feeling of] rightness/wrongness gives us a sense of more or less promising directions long before we have the actual solution in hand.” Below we give the example of Mary, a returning graduate student, who did not get a feeling of appropriateness with regard to using fixed, but arbitrary elements in her real analysis proofs for at least half a semester.

Also there have been working groups on affect and mathematical thinking at several recent CERME conferences,<sup>5</sup> but the discussions there seem to have been mainly concerned with methodological issues and such topics as changing teachers’ and students’ motivation and attitudes towards mathematics. Still we feel that the interplay

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<sup>5</sup> CERME is the Congress of the European Society for Research in Mathematics Education, many of whose Proceedings are available online.



between cognition and affect during problem solving and proving needs further investigation.

### **How University Mathematics Students Prove Theorems**

Much of the research on university students' proving has been concerned with difficulties they encounter or competencies needed. These include the use of logic, especially quantifiers (Dubinsky & Yiparaki, 2000; Epp, 2009; Selden & Selden, 1995); the necessity to employ formal definitions (Edwards & Ward, 2004); the need for a repertoire of examples, counterexample, and nonexamples (Dahlberg & Housman, 1997); the requirement for a deep understanding of the concepts and theorems involved (Weber, 2001); the need for strategic knowledge of which theorems are important (Weber, 2001), the selection of appropriate representations (Kaput, 1991); and the importance of being able to validate (i.e., read and check) one's own and others' proofs for correctness (Selden & Selden, 2003).

### **Teaching Proving to University Mathematics Students**

For several years, we have been developing methods for teaching proof construction to advanced undergraduate and beginning graduate mathematics students. We have developed an inquiry-based Modified Moore Method course (Mahavier, 1999; Coppin, Mahavier, May, & Parker, 2009) for advanced undergraduate and beginning graduate mathematics students who need help with proving (hereafter referred to as the "proofs course" and described in Selden, McKee, & Selden, 2010) and a voluntary

proving supplement for undergraduate real analysis (hereafter referred to as the “supplement” and described in McKee, Savic, Selden, & Selden, 2010).

In the proofs course, the students are given self-contained notes consisting of statements of theorems, definitions, and requests for examples, but no proofs. The students construct their proofs at home and present them in class. The proofs are then critiqued, sometimes extensively, and additionally suggestions for improvements in the notation used and the style of writing are given. There are no formal lectures, and all comments and conversations are based solely on students’ work. The specific topics covered are of less importance than giving students opportunities to experience as many different kinds of proofs as possible so we select theorems from sets, functions, real analysis, semigroups, and topology.

We have developed some theoretical underpinnings for the two courses. One such theoretical underpinning involves having students develop a proof framework first in order to reveal the mathematical problem(s) to be solved. (See the above description in “The Genre of Proof” section.) While students with little experience in proof writing, at first can find constructing a proof framework to be a problem of moderate difficulty, eventually through practice, writing a proof framework can become routine or very routine.

In addition, our proofs course notes are constructed to give students opportunities to prove theorems that are successively more non-routine. But non-routineness is not unidimensional; it is not simply a matter of whether the students have seen the concepts before or have the necessary factual knowledge but cannot bring it to mind (as was the case for the students in our calculus studies). In our proofs course notes we have “built

in” non-routine theorems, which we refer to as theorems of Types 1, 2, and 3. *Type 1 theorems* have proofs that can depend on a previous result in the notes. These theorems are included to encourage students to look for helpful previous results, as we have found that students often attempt to prove theorems directly from the definitions without recourse to previous results. *Type 2 theorems* require formulating and proving a lemma not in the notes, but one that is relatively easy to notice, formulate, and prove, whereas *Type 3 theorems* require formulating and proving a lemma not in the notes, but one that is hard to notice, formulate, and prove. An example of a Type 3 theorem is: *A commutative semigroup  $S$  with no proper ideals is a group*, given after a brief introduction to the ideas of semigroup and ideals thereof. What is needed for a proof of this theorem is the observation that  $aS$  is an ideal and hence  $aS=S$ . (This is the first lemma needed.) This is followed by the nontrivial observation that  $aS=S$  implies that equations of the form  $ax=b$  are solvable for any  $b$  in  $S$ . Using some clever instantiations of this equation, one can obtain an identity and inverses, and hence, conclude  $S$  is a group. To date only two students have been able to produce a proof without help or hints, and several mathematics faculty (whose speciality is not semigroups) have found that proving this theorem takes time and a certain amount of reformulation. This convinces us that this theorem can be considered at least moderately non-routine. More research is needed on what makes a problem non-routine (for an individual or a class), that is, what are the various dimensions or characteristics contributing to non-routineness.

#### *The Co-construction of Proofs in the Supplement*

We have implemented this method three times to date in small (at most 10 students) supplementary voluntary proving classes for real analysis. The supplement is

intended for students who feel unsure of how to proceed in constructing real analysis proofs. At the beginning of a supplement class period, the statement of a theorem entirely new to them, but similar to a theorem assigned for homework, but not a template theorem, is written on the board. The students themselves, or one of us if need be, offer suggestions about what to do, beginning with the construction of a proof framework. For each suggested action, such as writing up the hypotheses or an appropriate definition, drawing a sketch, or introducing cases, one student is asked to carry out the action at the blackboard. The intention is that all students reflect on the actions and later perform similar actions autonomously on their assigned homework (McKee, Savic, Selden, & Selden, 2010).

For example, if the students are to prove a sequence  $\{a_n\}_{n=1}^{\infty}$  converges to  $A$ , they would typically begin by writing the hypotheses, leave a space, and write the conclusion. After unpacking the conclusion, they would write “Let  $\varepsilon > 0$ ” immediately after the hypotheses, leave a space for the determination of  $N$ , write “Let  $n \geq N$ ”, leave another space, and finally write “Then  $|a_n - A| < \varepsilon$ ” prior to the conclusion at the bottom of their nascent proof. This would conclude the construction of a proof framework and bring them to the problem-centered part of the proof (Selden & Selden, 2009), where some “exploration” or “brainstorming” on the side board would ensue. The entire co-construction process, and accompanying discussions, is a slow one – so slow that only one theorem can be proved and discussed in detail in a 75-minute class period. More research is needed on how to foster such mathematical “exploration” and “brainstorming.”

## **Theoretical Underpinnings: Actions and Behavioral Schemas**

### ***Actions in the Proving Process***

We see proving as an activity, that is, as a sequence of actions, that are either physical (such as writing or drawing) or mental (such as attempting to recall a definition or theorem). Each action is paired with, and is a response to, a situation in a partly completed proof. By a situation we mean a reasoner's inner, or interpreted, situation as opposed to an outer situation that may be visible to an observer. Although we are referring to a person's inner situation, we have found in teaching that we can often gauge approximately what the inner situation is from the outer, observable, situation and the ensuing action. For example, below we will interpret Sofia's staring blankly at the blackboard during tutoring sessions as the situation of not knowing what to do next.

If a person engages in proving several theorems, then he or she is likely to experience a number of similar situations yielding similar actions. The first such situation-action pair is likely to have a conscious warrant based on, say, heuristics, logic, strategy, or known mathematics. However with time and (sometimes considerable) repetition, the need for a conscious warrant may disappear. The situation may then become linked, in an automated way, to a tendency to carry out the corresponding action; and the individual will not be conscious of anything happening between the situation and the action. We see such automated situation-action pairs as persistent mental structures and have called the smallest of them *behavioral schemas* (Selden, McKee, & Selden, 2010; Selden & Selden, 2011). By a small situation-action pair, we mean one that is not equivalent to any sequence of smaller such pairs. While the word "schema" has been used in several ways in the literature, we only mean such a persistent mental structure.

### ***Behavioral schemas***

The formation of behavioral schemas, whether beneficial or detrimental, requires the development of a way of recognizing particular kinds of situations, and in response, enacting particular kinds of actions. It is possible that neither the kind of situation nor the kind of action for a potential behavioral schema exists as a concept in the surrounding culture. In that case, constructing a behavioral schema entails noticing, either explicitly or implicitly, similarities among situations and among the corresponding actions, and eventually reifying these into what amounts to conceptions (usually without any need for formal designations).

### ***Properties of Behavioral Schemas***

- (1) Within very broad contextual considerations, behavioral schemas are immediately available. They do *not* normally have to be recalled, that is, searched for and brought to mind.
- (2) Behavioral schemas operate outside of consciousness. A person is not aware of doing anything immediately prior to the resulting action – he/she just does it. Furthermore, the enactment of a behavioral schema that leads to an error is not under conscious control, and one should not expect that merely understanding the origin of the error would prevent its future occurrences.
- (3) Behavioral schemas tend to produce immediate action, which may lead to subsequent action. One becomes conscious of the action resulting from a behavioral schema as it occurs or immediately after it occurs.
- (4) A behavioral schema that would produce a particular action cannot pass that information, outside of consciousness, to be acted on by another behavioral

schema. The first action must actually take place and become conscious in order to become information acted on by the second behavioral schema. That is, one cannot “chain together” behavioral schemas in a way that functions entirely outside of consciousness and produces consciousness of only the final action. For example, if the solution to a linear equation would normally require several steps, one cannot give the final answer without being conscious of some of the intermediate steps.

(5) An action due to a behavioral schema depends on conscious input, at least in large part. In general, a stimulus need not become conscious to influence a person’s actions, but such influence is normally not precise enough for doing mathematics. Also, non-conscious stimuli that lead to action usually originate outside of the mind, not within it (as often happens in proof construction).

(6) Behavioral schemas are acquired (learned) through (possibly tacit) practice. That is, to acquire a beneficial schema a person should actually carry out the appropriate action correctly a number of times – not just understand its appropriateness. Changing detrimental behavioral schemas, many of which have been tacitly acquired, requires similar, perhaps longer, practice (Selden, McKee, & Selden, 2010; Selden & Selden, 2011).

Implicitly acquired detrimental behavioral schemas can be enacted automatically in problem-solving situations. For example, some experienced teachers may have noticed that giving a counterexample to a student who consistently makes an errorful calculation, such as  $(3a+b)/3c = (a+b)/c$  or  $\sqrt{a^2+b^2} = a+b$ , is often not very effective. This can be so even when the student seems to understand the counterexample. Our view of

behavioral schemas suggests an explanation. If an incorrect algebraic simplification is caused by the enactment of a behavioral schema, then the resulting action (the incorrect simplification) would follow directly from the situation, that is, would not be under conscious control. To change the student's behavior, one might try to change the detrimental behavioral schema not only by providing a counterexample, but also by suggesting a number of relevant problems and some monitoring.

### **Using our Theoretical Underpinnings to Teach Proving**

Having students write a proof framework first, enables them to “get started” on writing a proof and reveals the mathematical problem(s) to be solved. What happens next depends on a student's ability to solve various mathematical problems. Informally, one of our graduate students has reported that writing a proof framework helped her organize her thoughts on her high stakes mathematics PhD comprehensive examinations. Also, looking for students' detrimental behavioral schemas and trying to help them replace them with beneficial schemas has enabled us to help students with proof construction. Sometimes acquiring a beneficial schema can take a long time.

### ***Mary's Reaction to Considering Fixed, but Arbitrary Elements***

There are theorems, particularly in real analysis, that involve several quantifiers. For example, proving a function  $f$  is *continuous* at  $a$  involves proving that for all  $\varepsilon > 0$  there is a  $\delta > 0$  so that for all  $x$ , if  $|x-a| < \delta$  then  $|f(x)-f(a)| < \varepsilon$ . For such proofs, one needs to consider a fixed, but arbitrary  $\varepsilon$ . Students are often reluctant to do this. We conjecture this is because they do not feel it right or appropriate to do so.



Mary, an advanced mathematics graduate student, was interviewed about events that took place two years earlier when she was taking both a pilot version of our proofs course and Dr. K's graduate real analysis course. In the homework for Dr. K's course, Mary needed to prove many statements that included phrases like 'For all real numbers  $\varepsilon > 0$ ,' where  $\varepsilon$  represented a variable (the situation). In her proofs, Mary needed to write something like 'Let  $\varepsilon > 0$ ,' where  $\varepsilon$  represented an arbitrary, but fixed number (the action).

When Mary was interviewed about this situation-action pair she said the following:

Mary: At that point [early in Dr. K's real analysis course] my biggest idea was, well he said to "do it", so I'm going to do it because I want to get full credit. And so I didn't have a sense of why it worked.

Interviewer: Did you have any feeling ... if it was positive or negative, or extra ...

Mary: Well, I guess I had a feeling of discomfort ...

Interviewer: Did this particular feature [having to fix  $\varepsilon$ ] keep coming up in proofs?

Mary: ... it comes up a lot and what happened, and I don't remember [exactly] when, is that instead of being rote and kind of uncomfortable, it started to just make sense ... By the end of the semester this was very comfortable for me.

Mary told us that, after completing each such proof, she attempted to convince herself that considering a fixed, but arbitrary element resulted in a correct proof.

However, only after repeatedly executing this situation-action pair, and convincing

herself that her individual proofs were correct, did she develop a feeling of appropriateness.

***Willy's Focusing Too Soon on the Hypotheses***

We have observed that after writing little more than the hypotheses, some students turn immediately to focusing on using the hypotheses, rather than unpacking the conclusion to see what is to be proved, after which they often cannot complete a proof. For example, late in our proofs course, Willy was asked to prove the theorem: *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  be a homeomorphism of  $X$  onto  $Y$ . If  $X$  is a Hausdorff space, then so is  $Y$ .* Because only ten minutes of class time remained and Willy had indicated that he had not yet proved the theorem, we asked him to “do the set-up”, that is, construct a proof framework (Selden, McKee, & Selden, 2010; Selden & Selden, 2011).

On the left side of the blackboard, Willy wrote:

*Proof.* Let  $X$  and  $Y$  be topological spaces.

Let  $f : X \rightarrow Y$  be a homeomorphism of  $X$  onto  $Y$ .

Suppose  $X$  is a Hausdorff space.

...

Then  $Y$  is a Hausdorff space.

Then, on the right side of the board which was for scratch work, he listed one after the other: “homeomorphism, one-to-one, onto, continuous ( $f$  is open mapping)”. He then looked perplexedly back at the left side of the board. Even after two hints to look at the final line of his proof, Willy said, “And, I was just trying to just think,

homeomorphism means one-to-one, onto, ...” After some discussion about the meaning of homeomorphism, the first author said, “There is no harm in analysing what stuff you might want to use, but there is more to do before you can use any of that stuff”, meaning that the conclusion should be examined and unpacked first.

We inferred that Willy was enacting a behavioral schema in which the situation was having written little more than the hypotheses, and the action was focusing on the meaning and potential uses of those hypotheses before examining the conclusion. We conjectured that Willy and other students, who are reluctant to look at, and unpack, the conclusion feel uncomfortable about this, or perhaps feel it more appropriate to begin with the hypotheses and work forward.

### ***Sofia’s Reaction to Not Having an Idea***

Sofia was a diligent first-year graduate student; however, as our proofs course progressed, an unfortunate pattern in her proving attempts emerged. When she did not have an idea for how to proceed, she often produced what one might call an “unreflective guess” only loosely related to the context at hand, after which she could not make further progress. Although we could sometimes speculate on the origins of Sofia’s guesses, we could not see how they could reasonably have been helpful in making a proof, nor did she seem to reflect on, or evaluate, them herself. We inferred that Sofia was enacting a behavioral schema: she was recognizing a situation, that is, that she had written as much of a proof as she could, and had a feeling of not knowing what to do next. This situation was linked in an automated way to the action of just guessing any approach that usually was only loosely related to the problem at hand without much reflection on its usefulness.

Using our idea of behavioral schemas, we devised an intervention that was used in tutoring sessions with Sofia. We attempted to deflect implementation of her “unreflective guess” schema, by suggesting that she write the first and last lines of a proof, unpack the conclusion, and then do something else, such as draw a diagram, review her class notes, or reflect on everything done so far. These suggestions and guidance helped Sofia construct a beneficial behavioral schema. As the course ended, this intervention of directing Sofia to do something else was beginning to show promise. For example, on the in-class final examination Sofia proved that if  $f$ ,  $g$ , and  $h$  are functions from a set to itself,  $f$  is one-to-one, and  $f \circ g = f \circ h$ , then  $g = h$ . Also on the take-home final, except for a small omission, she proved that the set of points on which two continuous functions between Hausdorff spaces agree is closed. This shows Sofia was able to complete the problem-centered parts of at least a few proofs by the end of the course, and suggests her “unreflective guess” behavioral schema was weakened (Selden, McKee, & Selden, 2010; Selden & Selden, 2011).

### **Future Research on Proof and Problem Solving**

The above discussion has not only synthesized some of the literature on proof and problem solving, it has highlighted several areas that could use more research. These are: how informal arguments are converted into acceptable mathematical form; how representation choice influences an individual’s problem-solving and proving behaviour and success; how students’ and mathematicians’ prove theorems in real time (especially when working alone); how various kinds of affect, including beliefs, attitudes, emotions, and feelings, are interwoven with cognition during problem solving; which characteristics

make a problem non-routine (for an individual or a class), that is, what are the various dimensions contributing to non-routineness; and how one might foster mathematical “exploration” and “brainstorming” as an aid to problem solving.

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