Developing the art of seeing the easy when solving problems

Alfinio Flores

Jaclyn Braker

Developing the art of seeing the easy when solving problems

Alfinio Flores & Jaclyn Braker

University of Delaware

Introduction

For Leonardo da Vinci "saper vedere", that is, knowing how to see, or having the art to see, was the key to unlocking the secrets of the visible world. *Saper vedere* included a precise sensory intuitive faculty as well as artistic imagination (Heydenreich, 1954) which were at the root of Leonardo's inventiveness and creativity. According to Leonardo, to understand, you only have *to see things properly* (Bramly 1994, p. 264). Knowing how to see is also important in mathematics. The Italian mathematician Bruno de Finetti (1967) stresses this importance in his book on "Saper vedere" in mathematics. He highlights several aspects of knowing how to see in mathematics, such as knowing how to see the easy, how to see the concrete things, and how to see the economical aspects. He also discusses in what ways knowing how to see also helps us to better recognize the meaning of general and systematic methods of mathematics represented in formulas. His book starts by highlighting the importance of reflection for learning the art of seeing.

Reflection also plays a central role in Polya's *Looking back* stage in problem solving. Polya's heuristics also provide a language to help problem solvers think back about their problem solving experiences. As Lesh and Zawojewski (2007) point out, "by *describing their own processes*, students can use their reflections to develop flexible prototypes of experiences that can be drawn on in future problem solving" (p. 770).

Reading Polya's heuristics and looking at the examples he gives, we can concur with Lesh and Zawojewski that Polya's heuristics are intended to help students go *beyond* current ways of thinking about a problem, rather than being intended only as strategies to help students function better *within* their current was of thinking.

Lesh and Zawojewski (2007, p. 769) point out that when solving problems in complex problematic situations the abilities related to "seeing" are as important as abilities related to "doing". Schoenfeld (1985) found that individuals select solution methods to problems based on what they "see" in problem statements. Schoenfeld's data indicate that mathematical experts decide what problems are related to each other based upon the *deep structure* of the problems, whereas novices tend to classify problems by their *surface structure* (p. 243). Krutetskii (1976) found in his research that one trait of mathematically able students was to strive for a clear, simple, short, and thus "elegant" solution to a problem (p. 283). He also mentions that "a striving for simplicity and elegance of methods characterizes the mathematical thought of all prominent mathematicians" (p. 283-284). Krutetskii also describes how all the capable students, after finding the solution to a problem, continued to search for a better variant, even though they were not required to do so (p. 285). In contrast, average students paid no particular attention in his experiments to the quality of their solutions if there were no special instructions from the experimenter in that respect. Krutetskii observed that capable students "were usually not satisfied with the first solution they found. They did not stop working on a problem, but ascertained whether it was possible to improve the solution or to do the problem more simply" (p. 285-286).

In this article we will focus on learning the art of seeing the easy, by using an example of a problem posed to future secondary mathematics teachers. De Finetti indicates that it is often difficult to see the easy things, that is, to be able to distinguish, in the complexity of circumstances present in a problem, those that are enough to formulate the problem or that allow one to do the formulation as several successive steps that can be carried out easily.

The problem presented below was posed as part of a modeling course. Lesh and Doerr (2003) point out that from a modeling perspective, traditional problem solving is viewed as a special case of model-eliciting activities. Lesh and Doerr emphasize that "for model-eliciting activities that involve a series of modeling cycles, the heuristics and strategies that are most useful tend to be aimed at helping students find productive ways to adapt, modify, and refine ideas that they do have." (p. 22). According to Lesh and Doerr, we need to put "students in situations where they are able to reveal, test, and revise/refine/reject alternative ways of thinking" (p. 26).

We will first present the strategy used by a group of future teachers, and then an approach gained by looking back at the problem and trying to see it at a glance. We finish with a brief discussion of why it would be worthwhile for prospective teachers to look back at the this and other problems..

The problem

During a course for prospective high school teachers, one of the assignments was to present a problem to their fellow students that could be modeled or solved with high school mathematics. The second author posed the following problem to her classmates.

You are attempting to bathe a cat in your kitchen. Unfortunately, the cat is not as open to the bath as you were hoping, and as a result you spill 3 gallons of water in your kitchen. Which brand of paper towel should you use to clean up the spill?

Brand A	Brand B
Paper towel is 1/32 inches thick	Paper towel is 1/64 inches thick
Total diameter of roll is 5 inches	Total diameter of roll is 6 inches
Diameter of hollow inside is 2 inches	Diameter of hollow inside is 2 inches
One sheet absorbs 1.5 fluid ounce	One sheet absorbs 1 fluid ounce
Each sheet is 10 inches long	Each sheet is 10 inches long

The assumption is that the price for the roll is the same for both brands. Notice that it is not necessary to know the width of the sheets, because we know how much each sheet absorbs for each brand. Remember that 1 gallon = 128 fluid ounces.
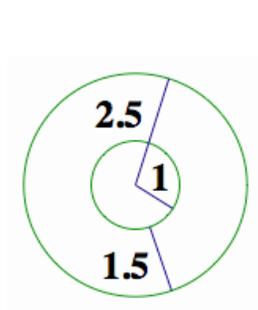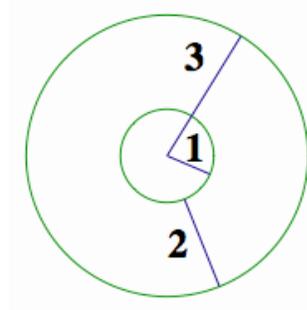

Figure 1a. Cross section of Brand A roll   Figure 1b. Cross section of Brand B roll

The future teachers used an approximation by modeling the spiral cross section of the role of paper as a series of concentric circles. Each successive layer was a little longer because the thickness of each sheet increased the diameter. The approach used by all the future secondary teachers to solve the problem was to find how many rolls of each brand

were needed to clean up the spill. To find this number they decided to compute how much water can be absorbed by one roll of each brand, finding first how many sheets are in each roll.
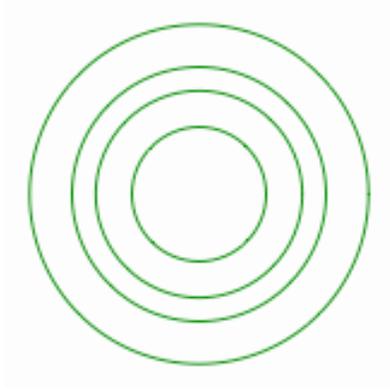

Figure 2. Concentric layers

Thus for Brand A the first layer has a length of $C_1 = 2\pi \times 1$ inches.

For the second and third layers the length in inches is

$$C_2 = 2\pi \times \left(1 + \frac{1}{32}\right)$$

$$C_3 = 2\pi \times \left(1 + \frac{2}{32}\right)$$

and in general, the length in inches of the k-th layer is

$$C_k = 2\pi \times \left(1 + \frac{k-1}{32}\right).$$

The number of layers, n, is given by dividing the thickness of the roll, 1.5 inches,

$$n = \frac{1.5}{\frac{1}{32}} = 48$$

by the thickness of each sheet, 1/32 inches, so .

The total length is thus $C_1 + C_2 + \cdots + C_{48} = 2\pi + 2\pi \times \left(1 + \frac{1}{32}\right) + \cdots +$

$2\pi \times \left(1 + \frac{47}{32}\right)$

$= 2\pi \left(48 + \frac{1}{32}(1 + 2 + \cdots + 47)\right)$

$= 2\pi \left(48 + \frac{1}{32} \times \frac{47 \times 48}{2}\right)$

$= 96\pi + \frac{141}{2}\pi \approx 523$ inches.

Thus the total length of a roll of brand A is 523 inches. The length of each sheet is 10 inches, so there are about 52 sheets per roll. These sheets together can absorb $52 \times 1.5 = 78$ ounces of water. Thus each roll absorbs 78 fl. oz. of water. To clean 3 gallons $= 3 \times 128$ fl. oz. $= 384$ fl. oz. we need $\frac{384 \text{ fl. oz}}{78\frac{\text{fl. oz}}{\text{roll}}} = 4.9$ rolls. That is, we need almost 5 rolls of Brand A to clean the spilled water.

For Brand B the length of each layer is $C_k = 2\pi \times \left(1 + \frac{k-1}{64}\right)$ inches and the

number of layers is $n = \frac{2}{\frac{1}{64}} = 128$. The total length is $C_1 + C_2 + \cdots + C_{128}$

$= 2\pi + 2\pi \times \left(1 + \frac{1}{64}\right) + \cdots + 2\pi \times \left(1 + \frac{127}{64}\right)$

$= 2\pi \left(128 + \frac{1}{64} \times \frac{127 \times 128}{2}\right) \approx 1602$ inches. The total length is thus about 1602 inches. Because each sheet is 10 inches long, that is about 160 sheets. Each sheet absorbs one

fluid ounce, so one roll absorbs 160 fl. oz. To clean 3 gallons we need $\dfrac{384 \text{ fl. oz}}{160\dfrac{\text{fl. oz}}{\text{roll}}} = 2.4$

rolls. Brand B is clearly the better choice for this problem.

Looking back

  Polya points out that when we have obtained a long and involved solution we naturally want to see whether there is a more direct and clear way to solve the problem. He advises one to ask the questions: *Can you derive the result differently? Can you see it at a glance?* (Polya 1973, p. 61). He also points out that even when we have found a satisfactory solution we may still benefit from finding a different solution, which may give us further understanding or allow us to look at the problem from a different perspective. Polya encourages us to study a result and try to understand it better, to see a new aspect of it (p. 64). In the same way that we might get a better perception of an object by using two senses, we might get a better understanding of a problem by finding a solution in two ways. Future teachers need to learn to guide their students on how to find in a result itself indications of a simpler solution.

  The approach used by the future teachers described above has several advantages. One is that it highlights the use of an arithmetic sequence and how the average of its terms can be used to obtain their sum. Because $1 + 2 + \ldots + n$ is an arithmetic series, the average of all the terms will be the average of the first and last terms, $\dfrac{n+1}{2}$. One way to read the formula for $1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$ is that to obtain it we multiply the average

of the terms, $\frac{n+1}{2}$, by the number of terms n. Another advantage is that we can actually find how many rolls of paper we need.

In terms of the original problem posed we may want to look back and ask ourselves what are the essential differences between the two types of paper rolls in this problem. In the situation described, we really want to compare the efficiency to absorb water of the rolls relative to each other in order to determine which brand to use. Once we determine what brand to use then we can compute how many rolls of that brand we need.

Use of proportional reasoning to compare paper towels

A key insight for solving this problem in a different way is to realize that when comparing the rolls, we need to compare their ratios with respect to the different factors that affect their number of sheets and absorption capacity.

In the solution above, the average of the lengths of the layers played an important role. Here we will see how we can use the average in a different way. The number of sheets in a roll will be proportional to the area of the circular ring cross section . The area of this ring can be obtained by multiplying the circumference of the average circle by the width of the ring (Figure 3). If r_1 is the radius of the hollow circular center, r_2 the radius of the paper roll, and d_1 and d_2 their respective diameters, then the area of the cross section is given by

$$A = \frac{1}{2}(d_1 + d_2)\pi(r_2 - r_1) \qquad\qquad (1)$$

Exercise 1. Derive formula (1) for the area of the ring using the difference of areas of concentric circles.

Exercise 2. Discuss in what ways formula (1) is analogous to the formula for the area of a trapezoid.
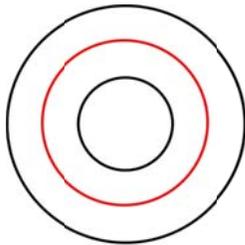

Figure 3. The average circle

Thus, a good way to compute how many sheets are around the roll is by using the circumference of the average circle, in this case the mid circle between the hollow core and the outer layer. For Brand A this average circle has a diameter of $\frac{2+5}{2} = 3.5$ inches, for Brand B a diameter of $\frac{2+6}{2} = 4$ inches. The number of sheets in each roll is proportional to the diameters of its mid circles, and proportional to the useful cross-sectional width of the rolls. The corresponding ratios comparing Brand B to Brand A will

be thus $\dfrac{4}{3.5}$ for the diameters of the average circles, and $\dfrac{2}{1.5}$ for the useful cross-sectional widths of the rolls. The number of sheets will be inversely proportional to the thickness of each sheet, so the ratio between Brand B and Brand A is $\dfrac{1/32}{1/64} = \dfrac{64}{32} = \dfrac{2}{1}$. The two brands have the same lengths of sheets, so to get the ratio of the number of sheets of Brand B to the number of sheets of Brand A, we just need to multiply these three ratios, which yields $\dfrac{4}{3.5} \times \dfrac{2}{1} \times \dfrac{2}{1.5}$. Because the ratio of the absorption efficacy per sheet of Brand B to Brand A is $\dfrac{1}{1.5}$, the ratio of number of ounces of water absorbed by a roll of Brand B to the number of ounces of water absorbed by Brand A is $\dfrac{4}{3.5} \times \dfrac{2}{1} \times \dfrac{2}{1.5} \times \dfrac{1}{1.5} = \dfrac{128}{63}$. So Brand B is about twice as good as Brand A for this task. This agrees with our previous result that the ratio of rolls needed is $\dfrac{4.9}{2.4}$. With this alternative approach of multiplying ratios it would be easy to make adjustments in case the length of the sheets or the price was not the same for both brands. All we would have to do is to multiply the previous product of ratios by the ratios of the prices, and by the ratio of the length of the sheets. In these cases, as with the thickness of the sheets, we would be dealing with inverse ratios.

  To find how many rolls of Brand B we would actually need, we can find the number of sheets in a roll, using the average circumference (4π), multiplying it by the number of layers that fit in the usable cross-sectional width ($2 \div \dfrac{1}{64}$), and dividing by the

length of the sheets (10). So the number of sheets is $4\pi \times 128 \div 10 \approx 161$. (Notice that this result is very close to the result obtained with the other method.) Because each sheet of Brand B absorbs one ounce of water, this is also the number of fluid ounces that each roll can absorb. The total number of rolls required is $\frac{384}{161} \approx 2.4$.

Exercise 3. Derive formula (1) as the limit of polygonal rings formed by trapezoids (see Figure 4).

Exercise 4. Discuss in what ways is formula (1) analogous to the formula for the volume of a torus obtained by rotating a circle around an axis outside the circle. The volume of the torus is equal to the product of the area of the circle times the circumference traced by its center.
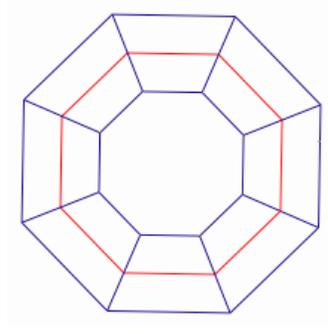

Figure 4. A ring formed by trapezoids

Concluding remarks

When teachers pose a mathematical problem to their students, they often do so because the problem can be solved with a mathematical approach that the teachers want to illustrate. Problem solving can be used as a powerful means to learn mathematical concepts and procedures (Lester & Charles, 2003; Schoen & Charles, 2003). In the above problem, the intent of the preservice teacher who posed it was that students have an

opportunity to use an arithmetic progression and the formula for its sum. Problems can be excellent ways to foster the development and understanding of particular mathematical concepts and procedures. However, students might use an alternative solution process that does not require the concept or process that the teacher wanted to emphasize. Teachers thus need to be aware that students might find alternative solutions that do not involve those concepts or procedures. In that case, the teachers needs to decide at what point, and to what extent, they should discuss those alternative approaches. It is important that teachers look at problems they pose from multiple perspectives, and try to foresee alternative solutions. That way teachers can better plan how and when to use those alternatives so that it becomes an enriching experience for all the students, rather than becoming a situation where some students have the opportunity to develop their thinking with respect to specific mathematical concepts and methods and others do not. Of course, sometimes students may surprise us and find an approach we did not foresee.

Learning to see the easy is one of the possible benefits of looking back at a problem and reflecting on its solution. Finding a simpler solution does not mean that our original approach was less valuable. The first method that occurred to us very likely gave us some insight into mathematical relations of a certain kind in the given situation, and perhaps used mathematical ideas that were freshest in our minds. Furthermore, often we find a simpler path only after we are able to solve the problem in another way. By taking time to consider alternatives once they have found a solution, students may find an easier solution. Students may realize it is not always necessary to apply the most complicated mathematical concepts that they know in order to solve even what appear to be difficult problems.

However, even when we find a simple solution first, it is worthwhile to take a second look at a problem and look for a different solution. The second solution may give us a different kind of insight. As Polya points out, there are also other benefits of looking back, such as establishing connections. A few connections were hinted at above, but a full treatment would go beyond the main focus of this paper.

There are other authors who emphasize the importance of reflection when solving problems. Shulman states that "the more complex and higher-order the learning, the more it depends on reflection—looking back—and collaboration—working with others." (Shulman 2004, p. 319). The importance of reflection is not restricted to mathematics learning. Shulman also describes how studies of expertise in the solving of physics problems indicate that the most able problem solvers do not learn by just doing, that they do not learn from simply practicing the solving of physics problems. Rather they learn from looking back at the problems they have solved and learn by reflecting on what they have done to solve them. Able problem solvers learn, not just by doing, but also by thinking about what they have done. (Shulman, 2004, p. 319).

Good teachers understand and convey to their students the benefits of looking back at a problem. Learning to see the easy is one of the benefits.

References

Bramly, S. (1994). *Leonardo: The artist and the man*. New York: Penguin Books.

De Finetti, B. (1967). *Il "saper vedere" in matematica*. Torino: Loescher Editore.

Heydenreich, L. H. (1954). *Leonardo da Vinci*. New York: Macmillan.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren*. Chicago, University of Chicago Press.

Lesh, R. & Doerr, H. M. (2003). Foundations of a models and modeling perspective on mathematics teaching, learning, and problem solving. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 3 – 33). Mahwah, NJ: Lawrence Erlbaum and Associates.

Lesh, E. & Zawojewski, J. S. (2007). Problem solving and modeling. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 763-804). Charlotte, NC: Information Age Publishing.

Lester, F. K., Jr., & Charles, R. I. (Eds.). (2003). *Teaching mathematics through problem solving: Pre-K–grade 6.* Reston, VA: National Council of Teachers of Mathematics.

Polya, G. (1973). *How to solve it: A new aspect of mathematical method*. (2nd edition). Princeton, NJ: Princeton University Press.

Schoen, H. L., & Charles, R. I. (Eds.). (2003). *Teaching mathematics through problem solving: Grades 6–12.* Reston, VA: National Council of Teachers of Mathematics.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando FL: Academic Press.

Shulman, L. S. (2004). Teaching alone, learning together: Needed agendas for the new reform. In L. S. Shulman, *The wisdom of practice: Essays on teaching, learning, and learning to teach* (pp. 309-333). San Francisco, CA: Jossey-Bass.