The Mathematics Enthusiast

Volume 10
Number 3 *Number 3*

Article 9

7-2013

# Mathematical Habits of Mind for Teaching: Using Language in Algebra Classrooms

Ryota Matsuura

Sarah Sword

Mary Beth Piecham

Glenn Stevens

Al Cuoco

Follow this and additional works at: https://scholarworks.umt.edu/tme

Part of the Mathematics Commons

Let us know how access to this document benefits you.

## Recommended Citation

Matsuura, Ryota; Sword, Sarah; Piecham, Mary Beth; Stevens, Glenn; and Cuoco, Al (2013) "Mathematical Habits of Mind for Teaching: Using Language in Algebra Classrooms," *The Mathematics Enthusiast*: Vol. 10 : No. 3 , Article 9.
Available at: https://scholarworks.umt.edu/tme/vol10/iss3/9

This Article is brought to you for free and open access by ScholarWorks at University of Montana. It has been accepted for inclusion in The Mathematics Enthusiast by an authorized editor of ScholarWorks at University of Montana. For more information, please contact scholarworks@mso.umt.edu.

# Mathematical Habits of Mind for Teaching: Using Language in Algebra Classrooms[1]

Ryota Matsuura[2]
St. Olaf College

Sarah Sword
Education Development Center, Inc.

Mary Beth Piecham
Education Development Center, Inc.

Glenn Stevens
Boston University

Al Cuoco
Education Development Center, Inc.

**ABSTRACT:** The notion of mathematical knowledge for teaching has been studied by many researchers, especially at the elementary grades. Our understandings of this notion parallel much of what we have read in the literature, but are based on our particular experiences over the past 20 years, as mathematicians engaged in doing mathematics with secondary teachers. As part of the work of Focus on Mathematics, Phase II MSP, we are developing, in collaboration with others in the field, a research program with the ultimate goal of understanding the connections between secondary teachers' mathematical knowledge for teaching and secondary students' mathematical understanding and achievement. We are in the early stages of a focused research study investigating the research question: *What are the mathematical habits of mind that high school teachers use in their professional lives and how can we measure them?* The main focus of this paper is the discussion of the habit of using mathematical language, and particularly how this habit plays out in a classroom setting.

**Keywords:** Mathematical habits of mind, mathematical language, algebra

---

[1]This material is based upon work supported by the National Science Foundation under Grant No. 0928735. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.
[2] matsuura@stolaf.edu

***The Mathematics Enthusiast*, ISSN 1551-3440, Vol. 10, no.3**, pp.735-776
2013©The Author(s) & Dept. of Mathematical Sciences-The University of Montana

## Our Philosophy and Approach

Building on two decades of prior work, the *Focus on Mathematics* (FoM) Math and Science Partnership program (MSP) has, over the last decade, developed and refined a distinctive framework for a mathematics-centered approach to developing teacher leaders, and it has built a mathematical community based on that framework. The FoM approach involves teachers, mathematicians, and educators working together in professional development activities. The common thread running through this tightly connected set of activities is an explicit focus on mathematical habits of mind.

We define *mathematical habits of mind* (MHoM) to be the web of specialized ways of approaching mathematical problems and thinking about mathematical concepts that resemble the ways employed by mathematicians (Cuoco, Goldenberg, & Mark, 1997, 2010; Goldenberg, Mark, & Cuoco, 2010; Mark, Cuoco, Goldenberg, & Sword, 2010). These habits are not about particular definitions, theorems, or algorithms that one might find in a textbook; instead, they are about the thinking, mental habits, and research techniques that mathematicians employ to develop such definitions, theorems, or algorithms. Some examples of MHoM follow:

- Discovering the structure that is not apparent at first by experimenting and seeking regularity and/or coherence.
- Choosing a useful representation—or purposefully toggling among various representations—of a mathematical concept or object.
- Purposefully transforming and/or interpreting algebraic expressions (e.g., rewriting $x^2-6x+10$ as $(x-3)^2+1$ to reveal its minimum value).

- Using mathematical language to express ideas, assumptions, observations, definitions, or conjectures.

Our work over the past decade has convinced us of the importance of MHoM for students and for *teachers* of mathematics, particularly at the secondary level. These habits foster the development and use of general purpose tools that make connections among various topics and techniques of secondary school mathematics content; they can bring parsimony, focus, and coherence to teachers' mathematical thinking and, in turn, to their work with students. In this sense, we envision MHoM as a critical component of mathematical knowledge for teaching (Hill, Rowan & Ball, 2005) at the secondary level (i.e., the knowledge necessary to carry out the work of teaching mathematics).

We begin this paper by describing the mathematical community that we have built and the impact that it has had on our teachers, in particular, the impact on teachers' mathematical understanding and instructional practices. Then we discuss the research that grew out of our desire to study scientifically how MHoM might be an indicator of teacher effectiveness. Lastly, we shed light on one habit that emerged prominently in our research—using mathematical language. We examine how a teacher might use this habit in a classroom, possible implications for student learning, and how use of the habit relates to teachers' use of other mathematical habits in the classroom.

We end this section with a few remarks. Although we describe our research on MHoM, the emphasis of this paper is not on our study, on its particular outcomes, or on the measurement instruments in development. Instead, we intend to illustrate, using examples, our motivation for why we think these mathematical habits are important. Hence, the main focus of the paper is the discussion of the habit of *using mathematical language*, and particularly how this habit plays out in a classroom setting. We include a detailed discussion of the FoM MSP, partly to situate our work within the MSP context in this special issue of *The Mathematics Enthusiast*. We also want to provide background for the research that emerged from and is motivated by our ongoing MSP work with secondary teachers. Indeed, our study of teachers' MHoM and corresponding instrument development arose from our desire to measure progress in and continue to improve our work with our own FoM teachers.

## Focus on Mathematics

*Focus on Mathematics* (NSF DUE 0314692) is a targeted MSP funded by the National Science Foundation since 2003. Our partnership is devoted to improving student achievement in mathematics through programs that provide teachers with solid content-based professional development sustained by *mathematical learning communities* in which mathematicians, educators, administrators, and teachers work together to put mathematics at the core of secondary mathematics education.

The original FoM district partners include the Massachusetts school systems of Arlington, Chelsea, Lawrence, Waltham, and Watertown. These systems range from suburban to urban, with middle and high school student populations from 1,300 to 6,000. Over the years, FoM has offered a variety of professional opportunities for teachers, including: (a) a public colloquium series devoted to mathematics and education; (b) partnership-wide mathematics seminars; (c) week-long summer institutes for teachers; (d) online problem-solving courses; and (e) a new *Mathematics for Teaching* Masters Program at Boston University. Two activities deserve special mention.

- PROMYS for Teachers summer institute, a six-week intensive immersion in mathematics, engages participants in experiencing mathematics as mathematicians do, solving problems and pursuing research projects appropriate for them. Each summer, the institute combines teachers from multiple districts, Grades 5–12.
- Academic-year study groups are district-based—often building-based—groups that meet biweekly for two to three hours over the course of a year. Though focused on *doing* mathematics (rather than being taught its results or how to teach it)—again, experiencing mathematics as a mathematician would—these trade the intensity and immersion of the summer institute for long-term, ongoing study.

These mathematical learning communities with core involvement of mathematicians are designed to help teachers develop the mathematical habits of mind that are central to the discipline of mathematics. Our teachers have responded enthusiastically, with comments such as:

- "[The study group] is the best 'professional development' that I have been involved in throughout my 35-year teaching career. I guess the best testament for the success of Focus on Mathematics comes from the continued attendance of so many teachers. We continue to talk about the topics discussed at our study groups long after the weekly session is over" (Cuoco, Harvey, Kerins, Matsuura, & Stevens, 2011).
- "The [Masters] program has expanded my knowledge of mathematics and deepened my understanding of how children learn mathematics, but—more importantly—I am now connected to people who are as passionate about children learning and doing mathematics as I am" (Cuoco, Harvey, Kerins, Matsuura, & Stevens, 2011).

To study the impact of FoM's professional development programs on teachers' professional lives, the Program Evaluation Research Group at Lesley University (FoM's evaluators) collected and analyzed teacher and student data over five years (Lee, Baldassari, Leblang, & Osche, 2009) and conducted case studies of teachers (Baldassari, Lee, & Torres, 2009). Below are those findings most strongly informing our current work:

- *Teacher beliefs and attitudes about the nature of mathematics*: In interviews, teachers reported understanding the structure of mathematics in greater depth—how topics and ideas are connected and how they are developed through the grade levels. Teachers referred to developing a more complete picture or understanding of mathematics as a system and understanding the connections between different threads within it (Lee, Baldassari, & Leblang, 2006; Lee, Baldassari, Leblang, Osche, & Hoyer-Winfield, 2007).
- *Teacher changes in instructional practice*: Many of the instructional changes teachers reported stem from the ways in which they experienced learning through FoM (Lee et al., 2006). When teachers developed a deeper understanding of mathematics, their confidence often increased and they developed more flexibility in their teaching and the ability to adjust lessons based on student responses.

Through our work in FoM, we have seen that MHoM is indeed a collection of *habits* teachers can acquire, rather than some static you-have-it-or-you-don't way of thinking. And teachers report to us that developing these habits has had a tremendous effect on their teaching. We have collected ample anecdotal evidence, but recognize the need for scientifically-based evidence to establish that these teachers have indeed learned MHoM and that these habits have had a positive impact on their teaching practices. We also recognize the need to study student outcomes affected by teachers' uses of MHoM.

## Mathematical Habits of Mind for Teaching Research Study

*Focus on Mathematics, Phase II: Learning Cultures for High Student Achievement* (NSF DUE 0928735) is an MSP project that began in 2009. In FoM-II, we continued to refine our mathematical learning communities and began an exploratory research study focused on teachers' mathematical habits of mind.

As a basis for beginning the research study, we used the theoretical frameworks developed by Clarke and Hollingsworth (2002) for their "Interconnected Model of Teacher Professional Growth," which is characterized by networks of "growth pathways" among four "change domains" in teachers' professional lives—the *external* domain (E), the personal domain (K) (of *knowledge*, beliefs and attitudes), and the domains of *practice* (P) and *salient outcomes* (S). Significant, from our point of view, is the Clarke-Hollingsworth theory of *professional growth* (as distinct from simple *change*), which they represent as "an inevitable and continuing process of learning" (p. 947). They aptly distinguish their framework from others: "The key shift is one of agency: from programs that change teachers to teachers as active learners shaping their professional growth through reflective participation in professional development programs and in practice" (Clarke & Hollingsworth, 2002, p. 948). The agency of teachers in their own professional growth characterizes virtually all FoM programs, so we see the Clarke-Hollingsworth model of professional growth as well suited for our purposes.

We illustrate our use of the Clarke-Hollingsworth framework with an example. Shown in Figure 1 is a change environment diagram for "Ms. Crew," a middle school teacher and active member of the FoM learning community. The diagram represents the change domains as four boxes, labeled E, K, P, and S, as explained above. The solid arrows refer to growths due to *enactment*, while the dashed arrows depict those due to *reflection*. The loop on the box E refers to interaction between study groups and the immersion.

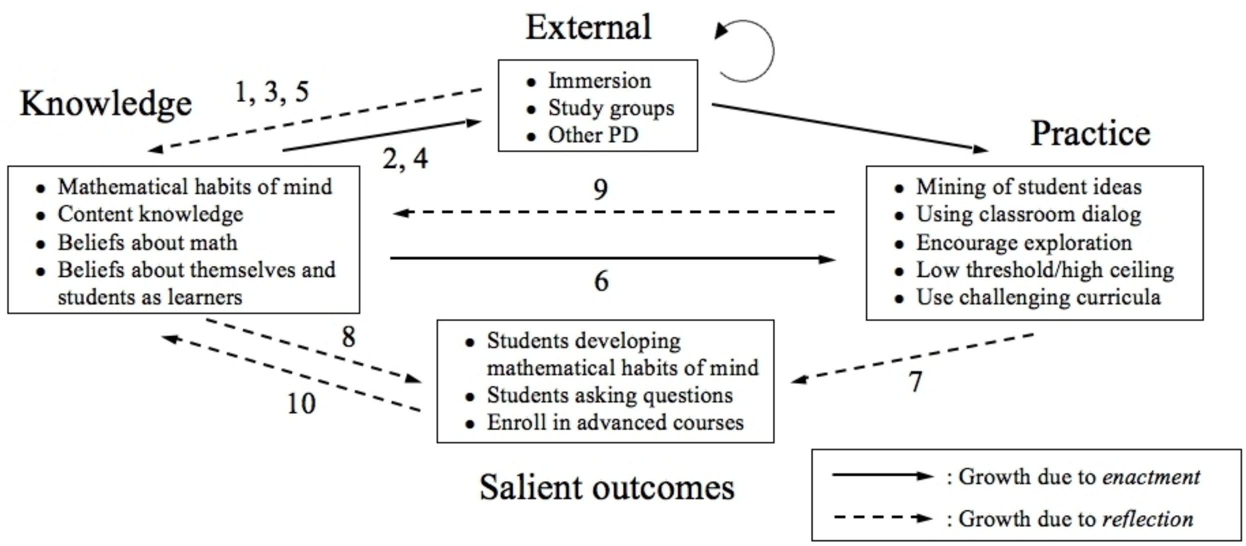


Figure 1. Schematic diagram of Ms. Crew's change environment

This particular diagram depicts activity related to Ms. Crew's research on Pythagorean Triples and shows how this activity led to her growth, both mathematically and as a teacher. Each arrow represents a growth in Ms. Crew that occurred as a result of a change in her professional life. For example, arrow 6 depicts how her increased belief about herself (a change in box K, the personal domain) leads to Ms. Crew encouraging her students to perform more explorations (a change in box P, the domains of practice). Moreover, arrow 6 is solid, because the change in her classroom is due an enactment, i.e., a particular course of action that she took as a teacher. The arrows are numbered in chronological order, so arrow 1 denotes a growth in Ms. Crew that occurred before that depicted by arrow 2, and so on. The dashed arrow from box E to K has multiple numbers (as does the solid arrow from K to E). Here, the dashed arrow may be interpreted as three separate arrows (arrow 1, arrow 3, and arrow 5)—we simply condensed them into one arrow to save space in the diagram.

Ms. Crew first encountered the concept of Pythagorean Triples while studying Gaussian integers during her summer immersion experience. The topic left such an impression on her (reflective arrow 1) that she pursued it (enactive arrow 2) as a research project under the guidance of an FoM mathematician. Through months of hard work—familiarizing herself with Pythagorean Triples through dozens of examples, making careful data recording and analysis, discovering beautiful patterns, coming up with interesting conjectures (some were true, some were false), and finally writing down clear and concise propositions and proving them—she came to understand (reflective arrow 3) features of Pythagorean triples that would have been beyond her conception before this experience. Ms. Crew produced an independent research paper and a one-hour mathematics talk for her peers (enactive arrow 4).

Neither the summer immersion experience nor the independent research project was easy for Ms. Crew, who came into our program with a rather weak mathematics background. But completing this project had a significant effect on her mathematical self-confidence (reflective arrow 5). The loops of this upward spiral between domains K and E repeated many times. Amongst her peers, Ms. Crew became one of the leaders in her study group (4). In her curriculum planning, she now has more belief in her decision-making abilities (5). And in her classroom, she engages her students in performing mathematical exploration (6). This new classroom atmosphere, as well as her new attitude towards mathematics, led to more curiosity and questions from her students (7, 8). And while she may not be able to answer all of them on the spot, she now welcomes mathematical dialogs and uncertainty in her classroom (9, 10). All of this represents significant professional growth and Ms. Crew's change diagram enables us to see the elements of that growth at a glance.

Looking at Ms. Crew's change diagram, one cannot fail to notice the intense activity taking place around the node K, which includes growth in Ms. Crew's knowledge of mathematics. But it seems to us that more is involved than simply knowing mathematics as a body of knowledge. Ms. Crew is learning mathematics in a certain way. Her beliefs about the nature of mathematics are changing. She is acquiring certain mathematical habits of mind and she is finding these habits useful for her work in the classroom and also for leadership roles in the school.

Applying this framework of teacher change, we began to build for ourselves a theoretical understanding of how MHoM plays a role in the work of teaching. Recognizing the need for a scientific approach to test the theory, and indeed investigate the ways in which MHoM is an indicator of teacher effectiveness, we conducted an exploratory study titled *Mathematical Habits of Mind for Teaching* that centers on the following question:

> *What are the mathematical habits of mind that secondary teachers use in their profession and how can we measure them?*

To investigate this question, we developed a detailed definition of MHoM and have been building the following two instruments:

- A *paper and pencil (P&P) assessment* that measures how teachers engage MHoM when doing mathematics for themselves.

- An *observation protocol* measuring the nature and degree of teachers' uses of MHoM in their teaching practice.

We emphasize that both instruments are needed, because in our work with teachers, we have seen those who have very strong MHoM for themselves but do not necessarily employ the same mathematical habits in their teaching practices.

Our current work fits into a larger research agenda that we are developing in collaboration with leaders in the field, with the ultimate goal of understanding the connections between secondary teachers' mathematical knowledge for teaching and secondary students' mathematical understanding and achievement.

**Operationalizing MHoM**

To operationalize the MHoM concept, we relied on our own experiences as mathematicians doing mathematics with secondary teachers (Stevens, 2001). We also studied existing literature—in particular, Dewey's (1916) and Dewey and Small's (1897) earlier treatments of habits and habits of mind, the Study of Instructional Improvement (SII) and the Learning Mathematics for Teaching (LMT) projects to develop measures of mathematical knowledge for teaching (MKT) for elementary teachers (Ball & Bass, 2000; Ball, Hill, & Bass, 2005; Hill, Schilling, & Ball, 2004; Hill, Ball, & Schilling, 2008), and the description by Cuoco et al. of mathematical habits of mind (1997, 2010). And we consulted the national standards, i.e., the NCTM Principles and Standards for School Mathematics (National Council of Teachers of Mathematics [NCTM], 2000) and the Common Core Standards for Mathematical Practice (National Governors Association Center for Best Practices and the Council of Chief State School Officers [NGA Center & CCSSO], 2010). But above all, we went into the classrooms of FoM teachers, where we observed a broad sampling of MHoM strengths. Some teachers exhibited precise use of language and careful reasoning skills; others had strong exploration skills, were good at designing mathematical experiments, or showed special strength at generalizing from concrete examples.

From these various sources, we began to compile a list of habits that constitute MHoM. As the list grew, we identified four broad and overlapping categories into which our mathematical habits naturally fell:

- Seeking, using, and describing mathematical structure
- Using mathematical language
- Performing purposeful experiments
- Applying mathematical reasoning

Indeed, these are categories of mathematical practices that are ubiquitous in the discipline. And in order to conduct a fine-grained study of these categories, we teased apart multiple habits within each category that we wanted to measure, some of which were identified earlier. That being said, we primarily envision MHoM as being comprised of the four categories, with the list of habits within each category providing more detail and texture to these four. By no means is our list final. In fact, we consider it an evolving document that we will continue to revise as we obtain more data using our instruments. From our data, we will learn which habits are more prominently used by secondary teachers, both when *doing* and *teaching* mathematics.

**Paper and Pencil (P&P) Assessment**

We developed a pilot P&P assessment that measures how secondary teachers use MHoM while doing mathematics. This assessment contains seven open-ended problems and is designed to be completed in one hour. In particular, we developed problems that most teachers have the requisite knowledge to solve, or at least begin to solve. And what we are assessing is *how* they go about solving it. It is the *choice* of their approach that we are interested in, as opposed to whether or not they have the necessary knowledge/skills to solve it. Each item is designed to reveal what habits and tools teachers choose to use in familiar contexts. To date, we have gone through several rounds of design, pilot-test, data analysis, and revision of this instrument. For our latest pilot-test in the summer of 2011, we administered the P&P assessment to 43 secondary mathematics teachers participating in the NSF-funded study *Changing Curriculum, Changing Practice* (NSF DRL 1019945). We will carry out another field test with approximately 50 teachers in the summer of 2012.

To gather initial data on the role that teachers' approach to solving mathematics problems plays in their approach to mathematics instruction, we asked a follow-up question to some of our P&P assessment problems: *What strategies would you want your students to develop for a problem like this?* Our 43 respondents almost unanimously reported that they want their students to approach the problems exactly as they did themselves. (Note: A few teachers wanted their students to appreciate a variety of approaches.) This finding provides initial evidence that teachers' own mathematical work may be indicative of how they choose to explain/formulate the subject matter for their students. Recognizing the need for further study of this hypothesis, we began to create an observation protocol.

**Observation Protocol**

We are in the process of designing an observation protocol and coding scheme that measure the nature and degree of teachers' uses of MHoM in their classroom instruction. To develop the instrument, we conducted live and videotaped observations of two to three consecutive mathematics lessons collected from a total of 30 secondary teachers to identify teacher behaviors that reflect the uses of a particular mathematical habit. In addition, we developed a simple protocol for pre- and post- interviews with teachers we videotape. We also collected classroom artifacts (lesson plans, in-class worksheets, homework, and assignments) from each classroom we observed.

An important feature of our observation protocol is that it measures how teachers *use* MHoM in their instruction. Thus teachers are coded not for *possessing* certain mathematical habits in the abstract, but for *choosing* to bring them to bear in a classroom setting. To develop such an instrument, we are currently studying our videos and slicing these lessons into small episodes—i.e., short instructional segments lasting 30 seconds to 4 minutes. In each episode, we determine whether there were behavioral indicators that reflected teachers' uses of MHoM, and we create codes that generalize and characterize these teacher classroom behaviors. We emphasize that our current focus is on *teacher* behaviors and uses of MHoM in the classroom. We are still a step away from connecting teaching practices centered on MHoM to students' development of MHoM and to student achievement—partly because we do not yet have the instruments to assess these habits in students—but impacting students, of course, is our ultimate goal.

Later, we describe three teachers from whom we gathered video data for our observation protocol development. Specifically, we will discuss how they apply the habit of using mathematical language in their classroom instruction. We will also consider how teacher use of this particular habit may affect student understanding.

## Relevant Literature and Related Work

The theory of mathematical habits of mind is philosophically grounded in Dewey's (1916) and Dewey and Small's (1897) earlier treatments of habits and habits of mind. Their seminal work has since encouraged educators (Duckworth, 1996; Meier, 1995) and education researchers (Kuhn, 2005; Resnick, 1987; Tishman, Perkins, & Jay, 1995) to further operationalize the concept of habits of mind—that is, to respond to the general question: *What do habits of mind look like in the context of learning?* Not as evident in the literature are the habits of mind that promote successful learning in specific disciplines. In the case of mathematics, the question that has gained research attention within the last decade is: *What do habits of mind look like in the context of learning and doing mathematics?* While addressing this question is not an unfamiliar task (Hardy, 1940; Polya, 1954a, 1954b, 1962), what is less familiar is the task of gathering evidence of mathematical habits of mind from *teachers* of mathematics. We began this work in our FoM-II study; we are in the long-term process of developing valid and reliable instruments that will allow us to more rigorously investigate the relationship between teachers' own MHoM, their uses of MHoM in their teaching practice, and student achievement.

As mentioned earlier, we envision MHoM as an integral component of MKT at the secondary level. The notion of MKT has been studied by many researchers (Ball, 1991; Ball, Thames, & Phelps, 2008; Heid, 2008; Heid & Zembat, 2008; Heid, Lunt, Portnoy, & Zembat, 2006; Hill et al., 2008; Kilpatrick, Blume, & Allen, 2006; Leinhardt & Smith, 1985; Ma, 1999; Stylianides & Ball, 2008). Our understandings of this notion parallel much of what we have read in the literature, but are based on our particular experiences over the past 20 years, as mathematicians engaged in doing mathematics with secondary teachers.

As mathematicians working in schools and professional development, we have come to understand some of the ways in which teachers know and understand mathematics. These fit into four large and overlapping categories:

(1) *Teachers know mathematics as a scholar*: They have a solid grounding in classical mathematics, including its major results, its history of ideas, and its connections to precollege mathematics.

(2) *Teachers know mathematics as an educator*: They understand the thinking that underlies major branches of mathematics and how this thinking develops in learners.

(3) *Teachers know mathematics as a mathematician*: They have experienced a sustained immersion in mathematics that includes performing experiments and grappling with problems, building abstractions from the experiments, and developing theories that bring coherence to the abstractions.

(4) *Teachers know mathematics as a teacher*: They are expert in uses of mathematics that are specific to the profession, including the ability to "think deeply of simple things" (Jackson, 2001, p. 696), the craft of task design, and the "mining" of student ideas.

The first two of these ways of knowing mathematics are common to most pre-service and in-service professional development programs. FoM has paid particular attention to the last two, which typically receive less emphasis. We have become convinced that (3) greatly enriches and enhances the other ways of knowing mathematics and that many teachers who go through such an experience develop the habits of mind used by many mathematicians. Furthermore, we have seen that participation in a mathematical learning community helps such teachers "bring it home" in the sense that they create strategies for helping their students develop the mathematical habits that they themselves have found so transformative.

Other researchers are developing instruments to assess secondary teachers' content knowledge and use of mathematics in their classrooms (Bush et al., 2005; Ferrini-Mundy, Senk, McCrory, & Schmidt, 2005; Horizon Research, Inc., 2000; Measures of Effective Teaching Project, 2010; Piburn & Sawada, 2000; Reinholz et al., 2011; Shechtman, Roschelle, Haertel, Knudsen, & Vahey, 2006; Thompson, Carlson, Teuscher, & Wilson, n.d.). In developing our own instruments, we have drawn insight from all of these projects. But we have most closely followed the model developed by Ball and Hill—specifically, their MKT assessment and Mathematical Quality of Instruction (MQI) protocol for documenting MKT in elementary teachers (Hill et al., 2005; Learning Mathematics for Teaching, 2006). Their instruments measure "specialized" mathematical knowledge, that is, knowledge that teachers use, as distinct from the mathematical knowledge held by the general public or used in other professions, whose components include representation of mathematical ideas, careful use of reasoning and explanation, and understanding unique solution approaches. These skills resemble the kinds of mathematical habits that we are interested in studying at the secondary level.

The collective efforts of the field will all contribute to what we know about MKT, but there are important differences between our instruments and those of others. The differences are listed below.

- A focus on MHoM—the methods and ways of thinking through which mathematics is created—rather than on specific results (Cuoco et al., 1997). It is impossible, even in three or four years of high school mathematics aligned with the Common Core, to equip students with all of the facts they will need for college and career readiness. But learning to think in characteristically mathematical ways is a ticket to success in fields ranging from business, finance, STEM-related disciplines, and even building trades.

- The core involvement, at every level, of mathematicians who have thought deeply about the implications of their own habits of mind for precollege mathematics curricula, teaching, and learning (Bass, 2011; Schmidt, Huang, & Cogan, 2002).

Our instruments are, therefore, aimed at discerning the extent to which secondary classrooms are centered on the practice of *doing* mathematics rather than on the special-purpose methods that often plague secondary curricula (Cuoco, 2008). In our work with teachers, we have seen how expert teachers use core mathematical habits of mind in their profession—in class, in lesson planning, and in curricular sequencing. And, as the Common Core becomes the nationally accepted definition of school mathematics, teachers will be expected to make the development of mathematical habits an explicit part of their teaching and learning agenda. Our work, therefore, makes a unique contribution to the field's increasing level of attention to secondary mathematics teaching.

## Using Mathematical Language

In this section, we will focus on a specific mathematical habit—*using mathematical language*—and examine how teachers use this core habit in their instructional practice. We will also consider its potential implications for student learning, and how this habit may work in conjunction with other mathematical habits in the classroom.

In particular, we will discuss examples of three teachers whose Algebra 1 classrooms we observed in our research study. We will begin with Mr. Hart, who uses mathematical language to encapsulate the experiences, observations, and discoveries of his students. Second, we will look at Ms. Graham, who uses precise and operationalizable language as a way of promoting conceptual understanding and ease of problem-solving. And third, we will describe an example of a teacher, Mr. Braun, whose choice of language can interfere with students' engagement in activities designed to promote other MHoM.

All three of these teachers have shown evidence of strong MHoM in their own doing of mathematics. Mr. Hart has held formal and informal leadership roles in a number of FoM's mathematical learning communities; and in those roles, he has exhibited strong MHoM. The other two teachers performed well on our P&P assessment. The names of these teachers have been altered to protect their identities.

**Mr. Hart**

We consider Mr. Hart, an Algebra 1 teacher who uses mathematical language to encapsulate the underlying structure that students discovered through experimentation. The mathematical topic of the day is recursive rules. The class begins with students working on the following warm-up problem.

> *A function follows [this rule] for integer valued inputs: The output for a given input is* $\frac{3}{2}$ *greater than the previous output. Make a table that matches the description. Can you make more than one table?*

Note that the rule is incomplete, because it is missing the base case. Students experiment with this rule, creating input/output tables and trying to derive closed-form equations.

Because of their different choices of base cases, they come up with different functions defined by expressions of the form $f(x)=\frac{3}{2}x+b$. Students conclude that the graphs of these functions are parallel lines with different $y$-intercepts. Mr. Hart also asks, “So what’s the part where you get to be creative in making these tables?” He then explains, “So you get to pick one number, and then everything else is decided by the part that I gave you [in the warm-up]. But there’s still an awful lot of different numbers.” Here, he is foreshadowing the *need* to fix the base case.

Then Mr. Hart formally introduces the notions of recursive rule and base case to summarize students’ experiences and to capture the underlying structure they observed when working on the warm-up problem. He says,

> A recursive rule, that’s just the description that tells us how to get from an output—to an output from the previous ones. So basically, what we were doing. Now as you saw, there’s another piece that’s not really enough information. It’s just me telling you how to get from one, to the next, to the next. To have a complete rule, we also need to know where to start. Because otherwise, we won’t know if we have the rule that—the first rule, the second rule, the third rule, or some other rule completely. (Video transcript, February 14, 2011.)

Next, the class studies the function described by the following table:

| $n$ | $f(n)$ |
|---|---|
| 0 | 3 |
| 1 | 8 |
| 2 | 13 |
| 3 | 18 |

| | |
|---|---|
| 4 | 23 |
| 5 | 28 |
| 6 | 33 |

In this table of data, students recognize the +5 pattern, i.e., "You add 5 to the output." Through discussion, Mr. Hart guides them to articulate the relationship more precisely: $f(5) = f(4) + 5$. Using this concrete example, students are able to derive a general equation: $f(n) = f(n-1) + 5$.

To make sense of this recursive rule, Mr. Hart points out that the equation $f(n) = f(n-1) + 5$ "lets us relate any output to a previous one." In essence, it is the symbolic representation of what he told students in the warm-up problem. Then he describes the need for the base case, saying, "But that wasn't quite enough because lots of you wrote down different rules. And [Student 1] had one, [Student 2] had a different one, [Student 3] had a different one probably, and so on. So we need something else to sort of fix it in place."

Here, a student interrupts and proposes a closed-form rule: $f(n) = 5n + 3$. There are now two ways to describe the function at hand, namely the (still incomplete) recursive rule $f(n) = f(n-1) + 5$ and the closed form rule $f(n) = 5n + 3$. He says, "[The recursive rule] tells us how to work our way down the table. If I know one value, I know 23, I can find the next one really easily. Now this one's [points to the closed-form rule] nice too because it lets me work across the table. If I know the input, I can say the output really quickly." In this short episode, Mr. Hart uses the symbolic representation of each rule to discuss its underlying structure.

Mr. Hart returns to the equation written on the board (i.e., $f(n) = f(n-1) + 5$) and says, "But still, this—this rule almost tells me the whole table, but it doesn't quite because

I'm missing one critical piece of information." A student chimes in, "Well, you don't know what you started with." Mr. Hart responds with, "That's a good point. Yeah, so like [Student]'s saying this 3 in the table, that's where we're starting. So we kind of need to know that. So the way (pause) a good way that we can sort of keep track of this and write our rule..." Almost 20 minutes into the lesson, Mr. Hart finally introduces the complete notation

$$f(n)=\begin{cases} 3 & \text{if } n=0, \\ f(n-1)+5 & \text{if } n>0. \end{cases}$$

He explains this new equation by saying, "So this formula captures exactly what we did. The key part is the recursive part that we had written down already. And this just adds that last bit, the base case, so we can summarize it into one compact rule."

Instead of being a starting point, this notation is the *culmination* of the structures that students discovered through their experimentation and the follow-up discussion. Students readily make sense of the new notation and the accompanying ideas that it encapsulates, because the experience gained through their "struggles" allows them to connect the new language to already-established ideas.

Mr. Hart uses the structure that students found through their experiments to motivate the language needed to describe their observed results. For instance, students' experiments with the warm-up problem, in which they propose different functions that all satisfy the given rule, make the *need* for the base case come alive for them. Indeed, his mathematical habits of mind allow Mr. Hart to create a learning environment where students build new knowledge from their experiences (NCTM, 2000).

**Ms. Graham**

Through Ms. Graham, we look at how an Algebra 1 teacher uses precise and operationalizable language as a way of promoting ease of problem-solving. More specifically, she helps students make sense of the objective of the given problem and, subsequently, provides insight into how to proceed.

In this episode, a student asks about the following question:

*Determine if* $r = -2$ *is a solution to* $6r + 2 = 12 + r$.

Ms. Graham asks, “Did we not understand what they were asking?” The student confirms, “Yeah, obviously there’s an easier way to do it, but I just didn’t know how.” Then the following dialogue occurs, in which Ms. Graham presses for the meaning of the word “solution”:

Teacher (T): All right. When we use the word “solution,” all right, we’ve talked a lot about what a solution is. What does “solution” mean?

Student (S): Like, does—it—when it works.

T: When you said “it works,” what do you mean? Because I think you’re on the right track.

S: Like, does it make sense?

T: Be a little more specific.

S: I don’t know how, like…

T: What does “solution” mean, anyone know? All right.

New student (SN): The answer?

T: “The answer.” We talked about this a lot. What’s a solution to an equation?

SN: Something that can go into make an equation work.

T: Something that makes the equation true, OK?

As we will see later in Mr. Braun's example, "works" is often used by students and teachers to describe what it means for a number to be a solution to an equation. Ms. Graham does not settle for this nor other oft-used phrases such as "it makes sense" and "the answer." The language used by students does not help them unravel the problem to understand what they are being asked to do. Only after the operational definition of "solution" has been given can Ms. Graham continue with an explanation of how to proceed.

T: We're stating that $6r+2$ will be equal to $12+r$. And they're asking, "Is $r=-2$ a solution?" So you got to test it out, just as I asked you to test out that one that we just did. So $6r+2=12+r$. Substitute in $r=-2$. So 6 times $-2$ plus 2—does that have the same value as 12 plus $-2$? And we have to test. All right? We're asking ourselves the question of, does this equal that? [Points to each side of the equation.] OK?

Then Ms. Graham leads the class through the process of substituting $r=-2$ into the equation and concluding that it is not a solution, since $r=-2$ yields unequal values of $-10$ and 10 for the two sides of the equation. The student who originally inquired about this question says, "Ok. Now I get it." The definition of "solution" provided by Ms. Graham—namely, "something that makes the equation true" is operational (i.e., students can *use* this definition to understand and accomplish the task posed by the given question). Indeed, once the definition has been given, substituting $r=-2$ and checking if it makes the equation true is a natural next step.

Ms. Graham concludes this episode by foreshadowing what students will be learning next, by providing them with another definition:

T: We're getting to the point where we're going to ask you, "What is the value of *r* that makes the equation true?" And that's called *solving the equation*.

Throughout the lesson, Ms. Graham consistently uses language carefully. She corrects a student who writes $82+8=90\div 3=30-5=25$, calling it a "run-on sentence in math." When a student describes two sides of an equation by saying, "It's equals," Ms. Graham immediately responds, "They're equal to each other." She repeatedly tells students to check their answer after solving an equation, reminding them what "solution" means. She is also precise in her instructions (e.g., asking the students to "write an expression for the right side of the equation, so that you've got an equation that works and is true when $x=3$").

**Mr. Braun**

One of the issues we have encountered in the development of our observation protocol is, "What counts as evidence of *non-use* of MHoM?" In the case of the habit of using mathematical language, we do see moments in which teachers choose less careful language. For example, a teacher might choose to use informal language. Sometimes there is evidence that the teacher is making this choice because the informal language seems more accessible to students. But such choices—if not made carefully—can lead to student confusion.

In the following example, Mr. Braun is setting up an investigation that aims to lay the foundation that the graph of an equation is a representation of the solution set of the equation (Education Development Center, Inc., 2009b). To launch the investigation, Mr. Braun writes the equation $3x+2y=12$ on the overhead projector and asks students, "What's the answer?" He then describes some of the solutions students offer as "that works" or "that doesn't work." The following is an excerpt from the launch of the investigation. There are two things to note. First, Mr. Braun is modeling how students might experiment with numbers as a way of making sense of the relationship between graphs and equations. Second, observe how frequently he uses the word “works.”

T: $3x+2y=12$. What’s the answer?

SN: It’s complicated.

T: Oh, no. What do you think?

SN: 1 and 2?

T: You think I can use 1 and 2?

S: *x* is 1 and *y* is 2.

T: *x* is 1 and *y* is 2. How would I find out if [name] is right? I could put in the numbers that he gave me, so I’m going to put in 1 for *x* and I’m going to put in 2 for *y*, and do I get 12, like I’m supposed to? What’s $3\times1$?

Students (Ss): 3.

T: What’s $2\times2$?

Ss: 4.

T: What’s 3 + 4?

Ss: 7.

T: Did I get 12?

Ss: No.

T: Man, [name], that’s a bummer. OK, so—

SN: Oh, I know it.

T: —that was something that didn’t work. It’s not bad to find out things that don’t work. Sometimes, you’re going to be asked in these investigations to find things that don’t work, so remember how we did that.

At this point, the teacher continues to take student guesses for *x* and *y*. Students make guesses and one student suggests $x = 2$ and $y = 3$. Mr. Braun tries that suggestion, and sees that indeed, $3(2) + 2(3) = 12$.

T: OK, so we found out that 1 and 2 did not work; we found out that 2 and 3 did work. Do you think there are any more things that don't work?

SN: Yes.

T: A lot more things that don't work. OK, do you think there are any more things that do work?

S: Yes.

T: Can you think of another thing that does work? [...]

SN: 3(3)…

T: OK, if I put a three there, OK.

S: And then, the 2*y* is 2, 2(1).

T: $2 \times 1$. OK, this is 9, right? Plus 2, makes 11 instead of 12. So, we found another thing that doesn't work. So, I—[name], you must have been right, there were more things that do not work. Can you find anything else that does work?

SN: 4 and 1.

T: You think 4 and 1 works? Where do I put my 4, for *x* or for *y*?

S: For *x*, yeah.

T: OK, so I put in 3(4) + 2(1), that gives me 12 + 2 = 14. We found another thing that doesn't work.

S: Actually, put 3 for *y*, plus 1.5.

T: […] 2(1.5), what are we going to get?

Ss: It's 3.

T: 3, and we had 9. Is 3 + 9 = 12?

Ss: Yes.

T: Hey, look at that. All right, now, that's the kind of thing I want you to do. You're just going to try some things. Some of them will work; some of them won't work.

Mr. Braun has modeled a detailed investigation of looking for points that satisfy the equation $3x+2y=12$, using the word "works" as a substitute for "satisfies the equation." He uses the phrases "works" and "doesn't work" repeatedly. He then hands out a worksheet for investigation that includes the problems:

Each point in the following table satisfies the equation $x+y=5$.

a) Complete the table.

| $x$ | $y$ | $(x, y)$ |
|---|---|---|
| 1 | 4 | (1, 4) |
| 2 | | |
| –3 | | |
| | 0 | |
| $\frac{1}{2}$ | | |
| | –2 | |
| | $-\frac{11}{3}$ | |

b) Graph the $(x, y)$ coordinates that satisfy the equation $x+y=5$. [Grid supplied.]

c) What shape is the graph?

and

Use the equation $2x+3y=12$.

a) Find five points that satisfy the equation.

b) Find five points that do not satisfy the equation.

Students begin the investigation. Some do not know what it means for a point to "satisfy an equation." Mr. Braun had created the worksheet based on problems in an Algebra 1 textbook—in the book, students are reminded that "If a point's coordinates make an equation true, the point 'satisfies the equation'" (Education Development Center, Inc., 2009a, p. 251). Mr. Braun had left that reminder off of his worksheet, and some of the students get stuck. For example:

S: … Please!

T: You just told me, though. [Laughter] What are we trying to do? What's it asking you to do?

S: Find this point…

T: OK, what does "satisfy" mean? That's the same equation we played with at the beginning of class, right?

S: I don't know.

T: It is, right? We didn't say "satisfy" and "not satisfy"; what were the words that we used?

S: I don't know. I don't know.

T: When [name] gave us 3 and 1.5, what did we say?

S: Decimal?

T: Well, we said they were decimals, we sighed at [name], but beside that, what else did we say? What does this side equal?

S: *x*? *y*? What?

T: What's $3 \times 3$?

S: 9.

T: What's $2 \times 1.5$?

S: 3.

T: What's 9 + 3?

S: 12.

T: So, what did we say? "[Name]'s solution..."

S: Works?

T: Works! "Works" is another word for "satisfies." If you want to sound smart, you say, "It satisfies the equation." OK? All right.

Similarly, another student asks:

S: I don't understand what it's asking us! [Laughter]

T: All right, fair enough. It says, "Sketch a graph of all the $(x, y)$ coordinates that satisfy"—work—"in this equation," and here's my equation.

On one hand, this is not a big deal. The teacher can travel from group to group, reminding them what "satisfies the equation" means, but he usually simply says that "it means 'works.'" However, "works" as a description is not operational. When students are solving problems, they repeatedly ask about the phrase "satisfies the equation." Rather than offer the operationalizable definition: "if a point's coordinates make an equation true, the point satisfies the equation," Mr. Braun returns to the phrase "works."

It is worth noting that the following day, Mr. Braun poses a warm-up question to his class: "What does it mean to be a solution?" Although he does not specifically address the definition of a point satisfying an equation (and the issue continues to persist for students), he does start working on unpacking that language for students.

**Common Themes in the Examples**

Several observations and questions emerge for us in these examples. First, what strikes us again and again is the complexity of teachers' uses of MHoM. These habits cannot be deployed independently in the classroom any more than they can be when teachers (and mathematicians) do mathematics for themselves. In fact, we saw that the habit of using mathematical language can either complement or get in the way of student experimentation and inquiry, depending on how the teacher uses the habit. In Mr. Hart's class, the precise definition of recursive function is motivated by the structure that his students discovered through experimentation. And, in turn, Mr. Hart plans to use this function notation as an investigative tool to explore further topics (e.g., the connection between linear and exponential functions). Mr. Braun also brings experimentation into his classroom. Indeed, his students conduct an investigation to explore the relationship between an equation and its graph. However, some students have difficulty beginning the investigation, because they do not understand the language they encounter in the task. Here, an operational definition of the phrase "satisfies the equation" may have led them to understand the problem statements and given them insight into how to proceed.

Throughout these examples, we also saw how the use of mathematical language can support students' understanding. In Ms. Graham's class, we see how she pushes her students to clearly state the meaning of the word "solution." And its definition becomes a vehicle that facilitates the problem-solving process. In contrast, we see Mr. Braun whose students encounter the phrase, "satisfy the equation." Instead of providing a usable definition, he offers an alternative, namely "works." We believe Mr. Braun is well-intentioned here. Specifically, there is evidence that he is trying to make the language less intimidating for students by offering a more informal phrase. Indeed, he says, "'Works' is another word for 'satisfies.' If you want to sound smart, you say, 'It satisfies the equation.'" But as discussed earlier, "works" is a phrase that is difficult to operationalize. It leads to confusion for his students, because they do not know how to *use* it. One of the mathematical practices advocated by the Common Core is attending to precision. The Common Core states that, "Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning" (NGA Center & CCSSO, 2010, p. 7). That "usability" of language is an important part of communicating precisely, and one that seems especially important for teachers.

In particular, the careful use of mathematical language not only helps clarify ideas for students, as it did in Ms. Graham's class, but it helps them understand the mathematics itself in a deeper way. We see this in Mr. Hart's lesson, where the recursive formula for $f(n)$ captures the properties of the function that students found through their investigations. Indeed, this formula is both a product and a reflection of their experiences. In our work with FoM teachers, we have found that encapsulating various insights into precise language—as we saw in Mr. Hart's class—helps one better understand the ideas themselves.

Mr. Hart also recognizes the power of precise language to drive further investigations. Later in the school year, these students will use function notation to study transformations of functions (e.g., stretches, shrinks, and translations). He adds, "I think that will be a place where students will really appreciate the function notation in representing those transformations more easily."

Mr. Hart concludes the post-interview by describing how today's lesson is part of a bigger unit and how it sets the foundation for later lessons. He plans to use these recursive rules as a vehicle for better understanding their closed-form counterparts. In a future lesson, students will investigate the connection between linear and exponential functions. "I want my students to see that recursively, exponential functions are very, very similar in their representation to linear functions. I think that will provide a nice foundation for studying exponents," he says. Here, Mr. Hart is using the language of recursive functions to shed light on the connections between their corresponding closed-form representations.

Our own goals in watching these videos have been to better understand teachers' uses of MHoM, and to learn about how we might measure that use. Part of our desire to measure the use stems from our desire to understand (eventually) the link between teachers' uses of MHoM and learning outcomes for students, particularly if we can measure students' uses of MHoM or students' facility with Common Core's Mathematical Practices, which include significant overlap with MHoM. Within the context of the examples in this paper, might teachers' use of language have an impact on student achievement? Even to begin to answer such a question, we must have some objective way of deciding whether or not a given teacher is using clear, usable, and precise language. This, too, is complex. Establishing what counts as "clear, usable, and precise" language depends very much on the classroom context. Mr. Braun uses the word "works" so consistently in his classroom discussion, that if it did not cause confusion, surely we would want to "rate" that as totally acceptable language, taken as shared by the whole classroom.

## Impact and Next Steps

We began our research work partly because we wanted to assess the effects of our own MSP professional development programs using tools that were consistent with the goals of our MSP, and partly because we wanted to understand the MHoM of secondary teachers better. We did not find instruments that measured teachers' MHoM—either when doing mathematics for themselves or teaching mathematics in their classrooms—in existence in the field, so we began to create our own. Although we expected to learn from the data gathered using our instruments, we did not anticipate the immediate implications that our research would have on the professional development programs in our MSP. For example, based on what we had learned from our research, we piloted the *Mathematical Habits of Mind Shadow Seminar* in the summer of 2011, geared toward teacher participants returning to PROMYS for Teachers (our summer immersion program) for a second summer. Through discussions, readings, curriculum analyses, and lesson designs, the goal of this seminar was to explore (a) the ways in which secondary teachers know and use MHoM in their profession, and (b) the effects that a learning environment that stresses MHoM might have on secondary students. We will continue to offer and refine this course as part of our summer immersion program for teachers.

We also did not anticipate the potential for impact on the field. While development and validation of truly reliable tools is beyond the scope of the current FoM-II study, we have been laying the groundwork for our MHoM instruments—the P&P assessment and the observation protocol—over the last few years. This exploratory phase of instrument development also coincided with the emergence of the Common Core State Standards and its adoption by 45 states (NGA Center & CCSSO, 2010). Our MHoM construct is closely aligned with the Common Core, especially its Standards for Mathematical Practice, and there is considerable overlap in the two. For example, both place importance on seeking and using mathematical structure, uses of precision, and the act of abstracting regularity from repeated actions. As we presented our preliminary findings at national conferences (Matsuura, Cuoco, Stevens, & Sword, 2011; Matsuura, Sword, Cuoco, Stevens, & Faux, 2011), we received several requests to use our instruments, even though they were in the pilot phase of development. One district leader wanted to diagnose the preparedness of her teachers to teach from a curriculum based on the Common Core. Others wanted to use the instruments as pre- and post- measures for evaluating professional development programs aligned to the Common Core. We have become abundantly aware of the national need for *valid and reliable* instruments to measure teachers' knowledge and use of MHoM/Mathematical Practices, as well as guidelines for acceptable use of such instruments. Thus, in the next phase of our research, we plan to subject our pilot instruments to rigorous scientific testing. The examples in this paper are exemplars of those that provide both the content basis for the P&P assessment and the behavioral indicators for the observation protocol.

## References

Baldassari, C., Lee, S., & Torres, R. T. (2009). *The case of a high school mathematics teacher*. Retrieved from http://focusonmath.org/FOM/PERG

Ball, D. L. (1991). *Research on teaching mathematics: Making subject matter knowledge part of the equation*. In J. Brophy (Ed.), Advances in research on teaching (Vol. 2, pp. 1–47). Greenwich, CT: JAI Press.

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 83-104). Westport, CT: Ablex.

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Mathematical Educator, 29*(3), 14–17, 20–22, 43–46.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407.

Bass, H. (2011). Vignette of doing mathematics: A meta-cognitive tour of the production of some elementary mathematics. *The Montana Mathematics Enthusiast, 8*(1&2), 3–34.

Bush, W., Karp, K., Ronau, B., Thompson, C., McGatha, M. & Brown, T. (2005, April). *The reliability and validity of diagnostic mathematics assessments for middle school teachers*. Paper presented at the annual meeting of the American Education Research Association. Montreal, Canada.

Clarke, D.J. & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education, 18*(8), 947–967.

Cuoco, A. (2008). Introducing Extensible Tools in Middle- and High-School Algebra. In C. Greenes (Ed.), Algebra and Algebraic Thinking in School Mathematics: 2008 Yearbook of the National Council of Teachers of Mathematics (NCTM) (pp. 51–62). Reston, VA: NCTM.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1997). Habits of mind: An organizing principle for mathematics curriculum. *Journal of Mathematical Behavior, 15*(4), 375–402.

Cuoco, A., Goldenberg, E. P., & Mark, J. (2010). Contemporary curriculum issues: Organizing a curriculum around mathematical habits of mind. *Mathematics Teacher, 103*(9), 682–688.

Cuoco, A., Harvey, W., Kerins, B., Matsuura, R., & Stevens, G. (2011). *What is a "Community of Mathematical Practice"?* [PDF document]. Retrieved from http://cbmsweb.org/Forum4/Presentations/C.1_Cuoco_Harvey_Kerins_Matsuura_Stevens.pdf

Dewey, J. (1916). *Democracy and education: An introduction to the philosophy of education*. New York, NY: The Free Press.

Dewey, J., & Small, A. W. (1897). *My pedagogic creed & the demands of sociology upon pedagogy*. New York, NY: E.L. Kellogg.

Duckworth, E. (1996). *The having of wonderful ideas & other essays on teaching & learning* (2nd ed.). New York, NY: Teachers College Press.

Education Development Center, Inc. (2009a). *CME Project Algebra 1*. Boston, MA: Pearson Education, Inc.

Education Development Center, Inc. (2009b). *CME Project Algebra 1, Teacher's Edition*. Boston, MA: Pearson Education, Inc.

Ferrini-Mundy, J., Senk, S., McCrory, R., & Schmidt, W. (2005, May). *Measuring secondary school mathematics teachers' knowledge of mathematics for teaching: Issues of conceptualization and design*. Working session at the International Committee on Mathematics Instruction Study 15 Meeting, Lindoia, Brazil.

Goldenberg, E. P., Mark, J., & Cuoco, A. (2010). Contemporary curriculum issues: An algebraic-habits-of-mind perspective on elementary school. *Teaching Children Mathematics, 16*(9), 548–556.

Hardy, G. H. (1940). *A mathematician's apology*. New York, NY: Cambridge University Press.

Heid, M. K. (2008, July). *Mathematical knowledge for secondary school mathematics teaching*. Paper presented at the 11th International Congress of Mathematics Education, Monterrey, Mexico.

Heid, M. K., & Zembat, I. O. (2008). Understanding mathematical entities as objects or as inscriptions. [Paper in progress.]

Heid, M. K., Lunt, J., Portnoy, N., & Zembat, I. O. (2006). Ways in which prospective secondary mathematics teachers deal with mathematical complexity. In Proceedings of the 27th Annual Meeting, International Group for Psychology of Mathematics Education North American Chapter [CD]. Merida, MEXICO.

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *Elementary School Journal*, *105*, 11–30.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371–406.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372–400.

Hill, H. C., Blunk, M., Charalambous, C., Lewis, J., Phelps, G., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction, 26*(4), 430–511.

Horizon Research, Inc. (2000). *Inside the classroom observation and analytic protocol*. Retrieved from http://www.horizon-research.com/instruments/clas/cop.php

Jackson, Allyn. (2001). Interview with Arnold Ross. *Notices of the AMS, 48*(7), 691-698.

Kilpatrick, J., Blume, G., & Allen, B. (2006). *Theoretical framework for secondary mathematical knowledge for teaching*. Unpublished manuscript. University of Georgia and Pennsylvania State University.

Kuhn, D. (2005). *Education for thinking*. Cambridge, MA: Harvard University Press.

Learning Mathematics for Teaching. (2006). A coding rubric for measuring the mathematical quality of instruction (Technical report LMT1.06). Unpublished technical report. Ann Arbor: University of Michigan, School of Education.

Lee, S., Baldassari, C., & Leblang, J. (2006, May). *Focus on Mathematics: Creating learning cultures for high student achievement year 3 evaluation report*. Retrieved from http://focusonmath.org/FOM/PERG

Lee, S., Baldassari, C., Leblang, J., Osche, E., & Hoyer-Winfield, S. (2007, May). *Focus on Mathematics: Creating learning cultures for high student achievement year 4 evaluation report*. Retrieved from http://focusonmath.org/FOM/PERG

Lee, S., Baldassari, C., Leblang, J., & Osche, E. (2009, January). *Focus on Mathematics: Creating learning cultures for high student achievement summative evaluation report*. Retrieved from http://focusonmath.org/FOM/PERG

Leinhardt, G., & Smith, D. A. (1985). Expertise in mathematics instruction: Subject matter knowledge. *Journal of Educational Psychology, 77*(3), 247–271.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

Mark, J., Cuoco, A., Goldenberg, E. P., & Sword, S. (2010). Contemporary curriculum issues: Developing mathematical habits of mind. *Mathematics teaching in the Middle School, 15*(9), 505–509.

Matsuura, R., Cuoco, A., Stevens, G., & Sword, S. (2011, April). *Mathematical habits of mind for teaching: Assessing mathematical knowledge for teaching at the secondary level.* Paper presented at the National Council of Supervisors of Mathematics Annual Conference, Indianapolis, IN.

Matsuura, R., Sword, S., Cuoco, A., Stevens, G., & Faux, R. (2011, April). *Mathematical habits of mind for teaching.* Paper presented at the National Council of Teachers of Mathematics Research Presession, Indianapolis, IN.

Measures of Effective Teaching Project. (2010, September). *Content knowledge for teaching and the MET project.* Retrieved from http://www.metproject.org/downloads/Teacher_Knowledge_092110.pdf

Meier, D. (1995). *The power of their ideas: Lessons for America from a small school in Harlem*. Boston, MA: Beacon.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

National Governors Association Center for Best Practices and the Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics.* Retrieved from http://www.corestandards.org/the-standards/mathematics

Piburn, M., & Sawada, D. (2000). *Reformed teaching observation protocol* (Technical Report No. IN00-3). Tempe, AZ: Arizona State University, Arizona Collaborative for Excellence in the Preparation of Teachers. (ERIC Document Reproduction Service No. ED447205)

Polya, G. (1954a). *Mathematics and plausible reasoning: Induction and analogy in mathematics* (Vol. 1). Princeton, NJ: Princeton University Press.

Polya, G. (1954b). *Mathematics and plausible reasoning: Patterns of plausible inference* (Vol. 2). Princeton, NJ: Princeton University Press.

Polya, G. (1962). *Mathematical discovery: On understanding, learning, and teaching problem solving* (Vol. 1). New York, NY: John Wiley.

Reinholz, D., Levin, M., Kim, H., Champney, D., Floden, R., Katwibun, D., Lepak, J., Louie, N, Nix, S., Sanchez, J., Schoenfeld, A., Seashore, K., Shah, N., & Wernet, J. (2011, April). *Capturing what counts: Classroom practices that lead to robust understanding of Algebra.* Paper presented at the annual meeting of the American Educational Research Association, New Orleans, LA.

Resnick, L. B. (1987). *Education and learning to think*. Washington, DC: National Academy Press.

Schmidt, W., Huang, R., & Cogan, L. (2002, Summer). A coherent curriculum: The case of mathematics. *American Educator, 26*(2), 10–26, 47. Retrieved from http://www.aft.org/pdfs/americaneducator/summer2002/curriculum.pdf

Shechtman, N., Roschelle, J., Haertel, G., Knudsen, J., & Vahey, P. (2006, April). *Measuring middle school teachers' mathematical knowledge of teaching rate and proportionality*. Paper presented at the annual meeting of the American Educational Research Association, San Francisco, CA.

Stevens, G. (2001). Learning in the Spirit of Exploration: PROMYS for Teachers. *Mathematics Education Reform Forum, 13*(3), 7–10.

Stylianides, A. J. & Ball, D. L. (2008). Understanding and describing mathematical knowledge for teaching: Knowledge about proof for engaging students in the activity of proving. *Journal of Mathematics Teacher Education, 11*(4), 307–332.

Thompson, P., Carlson, M., Teuscher, D. D., & Wilson, M. (n.d.). *Project Aspire: Defining and assessing mathematical knowledge for teaching secondary mathematics* [Abstract]. Retrieved from The Math and Science Partnership Network, National Science Foundation: http://aspire.mspnet.org/index.cfm/profile

Tishman, S., Perkins, D. N., & Jay, E. (1995). *The thinking classroom: Learning and teaching in a culture of thinking*. Boston, MA: Allyn and Bacon.