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Articles appearing in the journal address issues related to mathematical thinking, teaching and learning at all levels. The focus includes specific mathematics content and advances in that area accessible to readers, as well as political, social and cultural issues related to mathematics education. Journal articles cover a wide spectrum of topics such as mathematics content (including advanced mathematics), educational studies related to mathematics, and reports of innovative pedagogical practices with the hope of stimulating dialogue between pre-service and practicing teachers, university educators and mathematicians. The journal is interested in research based articles as well as historical, philosophical, political, cross-cultural and systems perspectives on mathematics content, its teaching and learning. The journal also includes a monograph series on special topics of interest to the community of readers. The journal is accessed from 110+ countries and its readers include students of mathematics, future and practicing teachers, mathematicians, cognitive psychologists, critical theorists, mathematics/science educators, historians and philosophers of mathematics and science as well as those who pursue mathematics recreationally. The editorial board reflects this diversity. The journal exists to create a forum for argumentative and critical positions on mathematics education, and especially welcomes articles which challenge commonly held assumptions about the nature and purpose of mathematics and mathematics education. Reactions or commentaries on previously published articles are welcomed. Manuscripts are to be submitted in electronic format to the editor in APA style. The typical time period from submission to publication is 8-12 months.

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"It is the misanthrope alone who has clarity"

- Manu Joseph

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Editorial: NSF's Math-Science Partnership Projects- Measuring the trickle-down effect of American tax dollars

Bharath Sriraman
The University of Montana

STEM (Science Technology Engineering and Mathematics) is viewed as one of the cornerstones of maintaining competitiveness in an increasingly globalized work force. In the United States, the National Science Foundation (NSF) is one of the many federal agencies that allocate funding of STEM initiatives ranging from school projects onto to the training of graduate students in specialized fields. In 2011, the White House released a report compiled by its *Committee on STEM Education* that revealed its complete STEM portfolio¹. The portfolio consisted of over 250 STEM related investments totaling over 3 billion dollars across all the federal agencies receiving such funding. How does one compare this figure to the total allocation received by the NSF alone which is over 7 billion dollars as of 2012? The NSF budget is subject to the vicissitudes of the Congress and can vary considerably from year to year. In general the trend has been an increasing allocation from Congress. The NSF STEM budget is between one-sixth or one-seventh of the total allocation and further split into the categories of *Education and Training*, and *Research and Development*. The former receives an even smaller allocation from NSF than the latter- approximately one-fourth of the one-sixth (or one-seventh), in other words an apportionment that is between one-twenty fourth and one-twenty eighth of the total congressional pie. Translated into dollars, this amounts to approximately 250 million \$, of which MSP partnerships receive about 13 million \$ annually². In the larger scheme of things, 13 million \$ out of the total budget of over 7 billion dollars is 0.2 % or only 2 *out of every 1000* NSF dollars going towards Math Science Partnership projects! One could say that K-12 education is relegated to the trickle-down effects of the system in place, despite the political rhetoric of advancing the educational needs of our students.

Now that one has read the limited amount of resources allocated to K-12 educational projects, the good news is that much has been accomplished in terms of math-science partnerships as this journal issue will reveal. The guest editors Ruth M. Heaton & Wendy M. Smith have gathered together a collection of papers from the 2012 Learning Network Conference in Washington, D.C that brought together MSP leaders, namely higher education faculty from STEM disciplines, school partners, and project evaluators. The goal of the conference was to provide the various stakeholders in MSP projects to share what they were learning about mathematics and science education through their work. I.e., to articulate progress made through partnerships targeting science and/or mathematics teaching and learning in specific grade bands or disciplinary areas, as well as institute partnerships focusing on developing teacher leadership. The result is reports of seven MSP projects that are constitutive of what has been happening in the United States in terms of how the teaching and learning of mathematics has been improved in K-12 as a result of NSF dollars. The papers speak for themselves and are interesting for anyone wishing to learn more about the practical dimension of K-12 reform initiatives in the U.S. and what has been accomplished given the limited resources that are allocated for such work.

Kırşehir, Turkey
June 30, 2013

¹ http://www.whitehouse.gov/sites/default/files/microsites/ostp/costem_federal_stem_education_portfolio_report.pdf

² Educational Research in the No Child Left behind Environment. Journal for Research in Mathematics Education, 2003 34(3): 185-190

Anatolian Plateau, Turkey [June 2013] ©Bharath Sriraman

**Developing Effective Mathematics Teachers through National Science Foundation
Funded Math and Science Partnership Program Grants¹**

Guest Editors

Ruth M. Heaton² & Wendy M. Smith
University of Nebraska-Lincoln

Every year the National Science Foundation (NSF) gathers together leadership teams of funded Math and Science Partnership programs (MSP) at a Learning Network Conference in Washington, D.C. The purpose of the annual conference is to bring together teams of MSP leaders who represent institution higher education (IHE) faculty from STEM disciplines, IHE education faculty, school partners, and project evaluators, to give them an opportunity to learn across projects, and provide opportunities for individual projects to reflect on their progress. For the last two years, 2011 and 2012, we were part of the conference's organizing committee. During the two-day conference, project teams were invited to articulate their theories of action for preparing teachers to be effective STEM teachers and to describe in broad strokes or in fine grain detail what was happening within their projects' professional development opportunities. Projects also had the opportunity to share within a public forum the preliminary, incomplete, or final results emerging from projects' evaluations or research efforts aiming to determine whether the MSP projects were deepening teachers' content and pedagogical knowledge, changing teachers' practices, and, ultimately, positively impacting students' success.

¹ The development of this special issue is supported by the National Science Foundation (Grant DUE 1143844). The opinions expressed in this document are those of the authors and do not necessarily reflect those of the foundation.

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While the Learning Network Conferences are intended to be for leaders within the MSP community, what MSPs are learning about STEM teaching and learning and professional development are worth sharing to a wider community.. Thus, as follow up to 2012 Learning Network Conference, we proposed to help MSP teams publish articles focused on mathematics teaching and learning accessible to a community broader than other MSP projects. Dr. Bharath Sriraman, editor of *The Mathematics Enthusiast*, generously offered us the opportunity to publish this special issue.

We approach the task of guest editors as empathetic solicitors and reviewers of scholarship associated with MSP projects. We are leaders, ourselves, for multiple MSP projects, and have been since 2004, first for a middle school mathematics project (*Math in the Middle Institute Partnership*, <http://scimath.unl.edu/MIM/>) and now for a K-12 mathematics project (*NebraskaMATH*, <http://scimath.unl.edu/nebraskamath/index.php>); Smith is also a leader on a Research, Evaluation, and Technical Assistance (RETA) project (*Data Connections*, <http://scimath.unl.edu/dataconnections/index.php>). We understand the time-consuming nature and inherent challenges of trying to create meaningful professional development with teams of interdisciplinary IHE faculty, and partner with school districts, to offer professional development and study its impact on teachers and their students in the dynamic life of real districts, schools, and classrooms. We have experienced the learning of teachers and their students to be neither linear nor quick, therefore, we understand that studying STEM teaching and learning is messy, long term, and anything but straightforward. We understand that, for the most part, it is the same MSP leaders who are offering professional development as who are trying to study its effectiveness and that frequently the days are not long enough to do both simultaneously.

Thus, we find MSP projects with their own rhythm and life, waxing and waning their research efforts in concert with their professional development offerings, with one or the other receiving more attention at any given point in time. All MSP project leaders must balance a set of teaching and research priorities in ways that never quite feel satisfactory. These are priorities and tensions that we, indeed, understand from the inside.

We sent out a call for articles to the 2012 Learning Network Conference participants following the conference, and a motivated, hard working group of authors, who double as leaders for mathematics focused MSP projects, responded, some of whom are publishing their scholarship for the first time in this special issue. They have taken their 2012 conference presentation proposals and presentations focused on the theme of effective STEM teaching and created manuscripts. Peers reviewed each manuscript and offered authors constructive feedback. The authors have responded to feedback from those reviewers as well as worked with feedback from us, as the guest editors of this special issue.

What has resulted is a collection of seven thoughtful articles representing MSP projects from across the United States, all with the common goal of aiming to improve mathematics teaching and learning at various points in the K-12 spectrum of schooling. Across all seven articles, the authors see essentially the same challenge and in some sense, the same solution—how best to build mathematics teachers' capacities by increasing and deepening teachers' mathematical and pedagogical knowledge and, in turn, impact student learning. However, each MSP project has its own ideas about how best to leverage change in teacher knowledge and practice, and, ultimately, student learning. Each project is at a different stage in the process, from programs in their infancy to ones that are more mature.

Each project appears to be having success, but how individual programs define success and the degree to which the individual projects have rigorous research designs and data to support their assertions of success varies greatly.

Some of the seven articles have the look and feel of research manuscripts. Others do not. Nevertheless, the authors of each of these seven articles, as leaders of MSP projects, each have a worthwhile story to tell. We have organized them by their longevity as NSF funded projects. The projects include “young” ones that are several years into their project and have had a first cohort of teachers experience their professional development. These projects are positioned to be able to offer a rationale and detailed description of the content of their professional development and anecdotes from their own and their participants’ experiences. Other projects are more “mature” and have been in the MSP business for nearly a decade. These projects have a wealth of wisdom and insight to offer through the results of quantitative analyses of longitudinal data on teachers’ and students’ learning or findings from qualitative data on how teachers and students seem to learn and promising vehicles of teacher change.

We begin with the article by Teixidor-i-Bigas, Schliemann, and Carraher, of the MSP project at Tufts University and TERC, who created *The Poincaré Institute for Mathematics Education* in 2010. The project is an interdisciplinary partnership among faculty in mathematics, physics, education, and nine school districts in three states with the overarching goal of improving the teaching and learning of mathematics in middle schools. Interestingly, this project has chosen to focus their professional development on the topic of functions as a common mathematical topic in the elementary, middle, and high school curricula. Functions also serve as an interdisciplinary connection between mathematics

and physics and provide a “common ground” for three graduate level courses designed to support the mathematical and pedagogical learning of middle school teachers.

The article features a detailed description of the three courses that make up *The Poincaré Institute for Mathematics Education*, designed to help teachers learn the mathematical content they need to know to be able to teach the concept of functions to their students and develop and plan meaningful activities that integrate mathematics and science which they can use with their students. The first of three cohorts of teachers recently completed the program. Teixidor-i-Bigas, Schliemann, and Carraher note within the article how they have continually revised the details of their course offerings based on continual assessment of the learning of the teachers. The authors are just in the beginning stages of assessing the impact of their program based an evaluation of teachers’ performance on course assignments, teachers’ and their students’ level of mastery of mathematical content on project designed assessments, videos of teachers’ classroom practice, and students’ performance on state mandated math assessments.

The next article is co-authored by Kinzer, Bradley, and Morandi, a team of mathematics educators, research mathematicians and public school leaders, who lead a MSP project, the *Mathematically Connected Communities Leadership Institute for Teachers* (LIFT) at New Mexico State University. This K-12 project is similar to *the Poincaré Institute for Mathematics Education* project in that the professional development focuses on strengthening mathematical and pedagogical knowledge. However, the teacher leaders who participate work closely together for two years and have the opportunity to earn a masters degree in teaching mathematics. Teacher leader participants take pairs of courses,

designed and taught by teams of mathematicians and educators to offer parallel learning opportunities in both content and pedagogy.

A unique feature of the LIFT project, as Kinzer, Bradley, and Morandi describe, is the use of descriptive feedback in multiple forms as formative assessment to improve instruction and support learning at every level of teaching and learning involved within both the LIFT project and K-12 classrooms of mathematics teacher leaders. The authors offer specific examples of how instructors, teacher leaders and their peers all give one another feedback in a variety of forms in an effort to support learning from experience in a collaborative and constructive manner. The authors describe how the feedback has influenced changes in the teaching and learning practices of all stakeholders.

The third article in this special issue is by Lewis, Fischman, Riggs, and Wasserman, and features the *Noether Project*, a MSP project that uses an intensive two week summer institute followed by academic year lesson study teams, as the major organizational structure for providing learning opportunities for teachers of grades four, five and six across multiple school sites to develop mathematical and pedagogical content knowledge. The focus of this article is on describing the three lesson study teams' experiences, and analyzing similarities and differences across the experiences. In doing so, Lewis et al. tell a story from the experiences of each team while using each team's experience to address one of the following questions: what teachers are learning from lesson study groups, why it appears that teachers learn from lesson study experiences, and how the learning of teachers within lesson study groups seems to happen.

Lewis et al. tell their stories in the article based on notes taken by the lesson study group facilitators during the group meetings. They also draw on examples of student work

discussed within the lesson study group meetings as well as piece together and analyze conversations within lesson study group team meetings based on notes taken during the meetings and snippets of transcripts made from periodic video recordings of lesson study team meetings. The result is a set of interesting stories of teachers learning together about teaching, children, and mathematics from practice. The authors are hopeful that the district will, over time, assume leadership responsibility for the lesson study teams and that long after NSF funding, the lesson study teams will exist as a sustainable model of teacher professional development.

The fourth article, by Gningue, Peach, and Schroder, is about the *Mathematics Teacher Transformation Institutes* (MTTI) for middle and high school teachers in New York City, led by an interdisciplinary team of mathematicians and education faculty from Lehman College working with school district leaders. Like the other projects in this special issue, the professional development offered to teachers includes challenging mathematical content. However, this project adds an additional component of action research, offered in a two-part course series. Through action research, MTTI teacher leaders study the effectiveness of their own teaching practices by gathering data and systematically examining the learning of their students.

This is the first article in the special issue to describe the project's intentional research efforts to better understand participants' mathematical and pedagogical learning, any resulting impact on classroom practice, and the degree to which the participants' students are showing evidence of increasing their mathematical engagement. Gningue, Peach, and Schroder describe data collection instruments being used to assess impact as well as some of their preliminary findings.

The fifth, sixth, and seventh articles in this special issue represent mature MSP projects which have benefitted from long-term NSF funding and, thus, have been providing professional development to teachers and studying impact on teacher and student learning for a number of years. They are also well-documented projects so all of their stories of teacher learning in their articles are supported by data analyses that offer insights into both how and what teachers are learning about mathematical content and mathematical practices or habits of mind.

The MSP project based at Virginia Commonwealth is featured in the fifth article, by Whitenack and Ellington. The authors work from the premise that the K-8 teachers in their project have acquired content knowledge as part of their participation in a Mathematics Specialist Program. Whitenack and Ellington focus on the description and analysis of a single class discussion to better understand how teachers may have developed new mathematical understanding as participants in their program. In the article, the authors carefully describe tasks given to teachers, the intentions underlying the task, and how teachers responded. This article helps to further understanding about the process of teacher learning.

The sixth article, by Sayler, Apaza, Kapust, Roth, Carroll, Tambe, and St. John, features *Promoting Reflective Inquiry in Mathematics Education* (Project PRIME), a MSP project based at Black Hills State University that has been offering various forms of professional development to strengthen K-12 practicing teachers mathematical and pedagogical content knowledge for the last nine years. This project has extensive longitudinal data that hint at positive impacts on changing classroom practice and provide some evidence of closing the achievement gap for disadvantaged students. What is

particularly interesting about this project, however, is that the professional development offered to teachers over the years has been varied and complex, making connecting changes in practice or student learning to particular forms of professional development quite difficult. This project is the only one in the series with longitudinal data. However, the complexity of the features of Project PRIME, as a whole, while being rich in what has been offered to teachers, limits the causality claims about the changes in practice and improvement in student learning.

The final article in this special issue, by Matsuura, Sword, Piecham, Stevens, and Cuoco, represents the longstanding work of an interdisciplinary team of mathematicians, mathematics educators and classroom teachers, who have been working for nearly two decades on the notion of mathematical habits of mind. Their MSP, *Focus on Mathematics* was funded first as an institute, and later as a phase II grant. The article features an operational definition of habits of mind and a discussion of efforts to develop and use a survey instrument and observation protocol to measure the nature and degree of teachers' uses of mathematical habits of mind in teaching practice. The article describes and then compares and contrasts three teachers' uses of mathematical habits of minds as both learners and teachers of mathematics.

Following the seventh article, Marilyn Strutchens and Gary Martin more information about MSP context as well as a brief commentary on the articles themselves. Strutchens and Martin first talk about their own MSP, TEAM-Math, focusing on the power of the learning communities that have developed over time. Strutchens and Martin relate their work on TEAM-Math to the work of the seven MSPs featured here in this special issue, and highlight commonalities and differences across projects. All of the projects have the

ultimate goal of increasing levels of student success, and all are attempting to do so through teacher professional development. Within that broad vision, each MSP project has taken a unique approach to developing effective mathematics teachers and all are seeing positive results in terms of teachers' learning and students' achievement.

**Integrating Disciplinary Perspectives:
The Poincaré Institute for Mathematics Education¹**

Montserrat Teixidor-i-Bigas² & Analúcia D. Schliemann
Tufts University

David W. Carraher
TERC

Abstract: We describe the development of the Poincaré Institute, an NSF-MSP supported program developed through Tufts University Departments of Mathematics, Education, and Physics and by TERC, in partnership with nine school districts in Massachusetts, New Hampshire, and Maine. We focus on the challenges of developing an inter-disciplinary program aimed at improving the teaching and learning of mathematics from grades 5 to 9, the choice of mathematical and educational content of the program, the course structure, and the progress of the first cohort of participant teachers. We also outline the changes we are implementing for future cohorts.

Keywords: middle school mathematics, algebra, functions, collaboration between mathematicians and educators

Overview of the Institute & Aims of the Article

In 2010 Tufts University, TERC, and several school districts from Massachusetts, New Hampshire, and Maine created the Poincaré Institute for Mathematics Education, a graduate program of studies providing professional development for in-service teachers. The Institute was named in honor of Henri Poincaré, a distinguished mathematician and physicist from the turn of the 20th century who recognized the importance of mathematics

¹ This research is funded by the National Science Foundation, Grant # DUE-0962863 . Findings presented in this paper represent the work of the authors and not necessarily the funding agency.

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education. Naming the Institute after Poincaré reflects our view that teachers need to broaden and deepen their grasp of mathematics, how children think and learn, how teachers teach, and how mathematics can be used to understand scientific and worldly phenomena.

The Institute seeks to transform and improve the teaching and learning of mathematics in middle school and the connections between the elementary, middle, and high school curricula. It highlights the connections by showing how functions implicitly permeate and potentially unify content throughout the K-12 curriculum. In particular, it uses the language of algebra as well as the geometry of functions to bring together otherwise disparate mathematical topics.

The Institute leverages expertise from mathematicians, educational researchers, physicists, teachers, and teacher leaders in school districts to: (a) offer graduate-level online courses on mathematical content, research in mathematics education, and knowledge relevant for teaching mathematics to three cohorts of 60 in-service teachers each (grades 5 to 9) from participant districts and a small group of pre-service teachers; (b) support long-term discussion forums in schools, where teachers plan, review, and improve their lessons; and (c) conduct research on teacher development and student learning.

The idea is to help teachers develop expertise suitable for whatever curriculum their school has adopted rather than provide them with ready-made lessons. Along with course activities aimed at deepening mathematical content, the teachers regularly examine video clips from classroom research on teaching and learning. They interview students on mathematics problems related to the curricula, and they plan, implement, and document

their own learning activities in the classroom.

The attainment of substantial improvements in middle-school mathematics education requires special kinds of interdisciplinary and cross-institutional collaborations that must be carefully nurtured and sustained. In this article we describe the behind-the-scenes evolution of structures, working relations, and decisions that took place in the first two years of the Institute's existence, as we collectively sought to negotiate an interdisciplinary yet reasonably coherent and collaborative approach to a diversity of topics and issues.

The focus of this article will be on how we are merging the different perspectives brought to the project by mathematicians, mathematics education researchers, scientists, and the administrators and teachers in partner districts. In our analysis, we highlight key decisions we faced while attempting to set the scope and sequence of topics, as well as the roles of various contributors to the Institute. As the Institute moves forward and on the basis of what we are learning, we are revising the courses and improving the way we are working and collaborating. We hope the following discussion, although based on our limited experience with an ongoing program of professional development, may prove useful for other groups who are attempting to develop interdisciplinary approaches to middle-school teacher education.

We begin by describing and examining previous interdisciplinary collaboration by the Institute partners at Tufts and TERC that contributed to its content and pedagogical approach, taking into account perspectives from mathematics, mathematics education, and science. Next we provide an outline of the courses offered to teachers. We then introduce

some issues that called for special adjustments in the roles, expectations, and interactions of the partners. At the end of the article, we outline how we plan to evaluate the impact of the project on the teaching and learning of teachers and their students, as well as some preliminary findings on changes we have observed among teachers in the first cohort.

Groundwork

Creating a truly interdisciplinary professional development program requires special sorts of collaboration. The Poincaré Institute needed mathematicians to do justice to the mathematics content, specialists in mathematics education to give proper due to issues of student learning and teacher development, and scientists to contribute expertise related to mathematical reasoning about physical quantities and modeling phenomena in the world beyond mathematics. We were fortunate to be able to draw on a decade-long program of early algebra research conducted by members from Tufts and TERC who would participate in the Institute. The algebra research furnished vivid video classroom examples related to the mathematics content of the courses. These video clips complemented future videotaped presentations by the mathematicians and software applets later designed by Poincaré teams. In-house teams carried out the Institute's own research and provided support for teachers as they designed and implemented their course-related projects for their students in the districts.

Any hopes that the Institute might exert a lasting contribution to classrooms require the input of teachers and other professionals from the participating schools and districts. However, teachers and district leaders' work primarily for schools and districts. They understand that their role as graduate students at a university is a temporary one, and the success of their graduate studies is valued according to its perceived benefits for their work

in schools. This simple fact underlies many decisions we undertook during the first two years of the Institute, including:

1. The creation of, or support of existing, long-term teacher discussion groups in the schools;
2. The inclusion, in the online courses, of weekly challenge questions in which teachers were encouraged to explicitly respond by taking into account their work in classrooms.
3. The designation of every third week of each unit as revolving around the theme, “Engaging Students”. During this week participants partner with colleagues from their schools in planning lessons or interviewing students about the topics of the prior two weeks.

As we will describe next, the Institute, in its current form, has its roots in years of previous work and discussions among the partners in the project. By working closely with the districts from early on, we realized that it would be better to offer courses throughout the school term instead of during the summer or over a few weekends. The teacher leaders helped us identify and handle issues such as defining clear expectations for participants, compensating cohort and non-cohort teachers for attending after-hours meetings, and managing the technical resources provided to each participant.

Despite excellent reviews, the first proposal we submitted was not funded. We were instead encouraged to expand the work beyond Greater Boston and beyond Massachusetts. This delay in initiating the work ultimately proved beneficial. It allowed us to expand the program to target districts in rural Massachusetts, as well as districts in New Hampshire and Maine. It also gave us an additional year to establish the identity of the Institute and the roles of the various contributors. Buoyed by the enthusiastic commitment of the nine school districts, we submitted an improved proposal for the “Poincaré Institute, An MSP

Partnership for Mathematics Education”, in August of 2009. The Institute officially began to function in June of 2010.

Initial Interdisciplinary Collaboration

The Poincaré Institute’s interdisciplinary partnership was built on over a decade of prior collaboration rooted in research on algebra in the early grades, in the education of teachers and researchers, and on the efforts of Education, Mathematics, and Science faculty at Tufts University to improve mathematics teaching and learning at all levels.

The collaboration began through NSF-funded research projects such as the TERC-Tufts Early Algebra, Early Arithmetic project (<http://earlyalgebra.org>). This series of classroom investigations led to key publications about young students’ learning of algebra. The research contributed in a fundamental way to the directions of the Poincaré Institute.

While Tufts University’s Education Department became increasingly engaged in mathematics education research, it also created structures that fostered interaction with faculty from the Mathematics, Physics and Engineering Departments of the same university. For example, candidates in Tufts (Masters of Arts in Teaching) program for the preparation of middle or high school teachers take a minimum of two courses in the discipline they would specialize, in consultation with faculty from the corresponding departments. Each math teacher has two advisors, one from the Department of Education, another from the Department of Mathematics. This led to initial collaborations among the mathematicians and mathematics educators at Tufts.

In 2003, Tufts University created a masters and doctoral program in Mathematics, Science, Technology, and Engineering Education (MSTE). The program prepares researchers and future leaders in Math, Science or Technology Education and demands a

greater knowledge of math, science, and technology. This led to increased collaboration among Mathematics, Science, and Education faculty. For example, faculty members from the different departments commonly serve together on doctoral dissertation committees. The graduate students often take part in Math Club activities and interact regularly with their peers from graduate programs in Mathematics.

In 2005 Tufts University created the Fulcrum Institute for Leadership in Science Education, an NSF funded MSP project with contributions from faculty from Tufts University's Departments of Education and Physics and from TERC. This program has prepared science educators to implement and lead research-centered science learning and teaching in their schools and districts. Participants advance their professional knowledge and status through the Institute's three online graduate course sequence. These courses, created during the NSF support period, are now part of Tufts' regular course offers and form the basis for a new program, the Tufts University Certificate Program for Science Education teachers. At the end of 2007, we began planning the Poincaré Institute for Mathematics Education, an interdisciplinary project focusing on the needs of school districts in the Greater Boston area. Our first challenge was to find a unifying topic for the math curriculum in grades 5-9 and engage mathematicians, scientists and education specialists around the topic.

Function as an Unifying Concept in K-12 Mathematics Education

We soon realized that functions could provide such a common ground. The concept of function is exceedingly important in modern mathematics. It traditionally enters the curriculum only in high school and beyond. Yet there were compelling arguments, which

the mathematics educators themselves had championed (Carraher, Schliemann & Schwartz, 2007; Schliemann, Carraher, & Brizuela, 2007), that functions underlie much of early mathematics, including the operations of arithmetic. The scientists also viewed functions as critical tools for fitting data to models. In short, there was a strong consensus that functions would offer a basis for substantial contributions from all three fields, even though each field had slightly different takes on what functions were about, how they were used, and why they were important. Provided we defined functions in a coherent way, we decided it would be useful to allow approaches from mathematics, education, and science to highlight different facets of functions. In a sense, this reflected our view that the teaching of mathematics requires respect for mathematical concepts and definitions while considering its applications, as well as sensitivity about how students and teachers make sense of it. Maintaining an eclectic perspective has been a constant concern throughout the development of the Institute.

The school districts were deeply concerned about the discontinuities in mathematics education across the K-12 curriculum, especially concerned about the transition from Elementary to Middle School and Middle School to High School. They also identified algebra as the topic that created or brought down barriers in these transition processes. They favorably viewed the prospect of teachers from early grades working alongside colleagues from later grades. One district suggested that the Institute range from grades 5 through 9, rather than 4 through 8 (as we had originally proposed), in order to address the transitions between elementary, middle, and high school mathematics.

Most districts were already committed to the idea that algebra needed to be made accessible to all of their students. Although most districts had not focused on the concept

of functions as one means of helping them achieve this, they were invariably receptive to the idea.

Multiple Representations

During the proposal development phase that led to a second proposal, and after the project was approved, the core members of the Institute met regularly to map out the content and rationale of the three graduate courses to be offered. This allowed the members from different disciplines to identify key topics and ideas for framing the course content.

Early on we recognized that the notion of “multiple representations” would be very useful to the teachers, allowing them to recognize the connections among a number of topics that they normally teach in isolation. It was also of great importance to the mathematicians and the specialists in mathematics education. To illustrate what is meant by “multiple representations” it is useful to recall that functions are conventionally represented mathematically through tables of values, algebraic expressions, arrow diagrams, displacements on number lines, graphs in a coordinate space, input-output “machines,” and various kinds of descriptions in natural language. In the field of mathematical learning, one also includes personal representations of functions that may or may not be consistent with standard mathematical conventions. The team scientists commonly referred to representations as *models* of extra-mathematical phenomena (data, processes, mechanisms). Meanwhile, teachers normally consider the teaching of algebra as manipulation of symbols and the geometric representation of graphs of functions as separate lessons. We decided to leave the definition of representations somewhat open to

interpretation so that it could serve well in mathematical, learning, and scientific contexts and to present multiple representations to the teachers as often as possible throughout the courses.

Interdisciplinary Perspectives

The individual members of the Poincaré Institute often have experience in more than one of the Institute's three foundational disciplines (Mathematics, Mathematics Education, and Science). For example, all of the research mathematicians serve as mathematics educators at Tufts University, and at least some of the Institute's researchers in mathematics education and science have familiarity with mathematics beyond the high school level.

Different disciplines tend to emphasize different aspects regarding *what* teachers should learn to become better teachers of mathematics, *why* they should learn it, and *how* they might best engage students in learning. Such assumptions are not set in stone nor necessarily fully consistent within any discipline. Nonetheless they are important to mention, insofar as they underlie recurring discussions about how the graduate courses should be structured and how the work in the school districts should proceed.

Here we will outline some of the thinking behind various perspectives in the Institute.

Perspectives from Mathematics Education

Our pedagogical approach has its roots in Piaget's constructivist theory of cognitive development and in socio-cultural approaches to learning and development inspired by Vygotsky's work. Their insights into the long-term development of children's understanding of basic logical and mathematical principles provide a rich starting point for

mathematics education work. However, their contribution does not directly consider how learning and understanding is reorganized through appropriation of specific mathematical symbol systems and tools such as the conventions of the decimal system, fractional and graphical notation, transformations across conventional measuring units, etc. (Carraher & Schliemann, 2002; Schliemann & Carraher, 2002). While teaching and learning of mathematics as a discipline should unfold from children's basic logical and mathematical understandings, they must lead to more general, complex, and explicit knowledge. To acknowledge this, however, is not enough. We need to analyze how children's logical and mathematical intuitive understandings can be further expanded as children learn mathematics (Vergnaud, 1996). Ultimately, as Piaget stressed, we need to find "the most adequate methods for bridging the transition between (...) natural but nonreflective structures to conscious reflection upon such structures and to a theoretical formulation of them" (Piaget, 1970, p. 47).

Mathematics educators have been arguing for many years that algebra should pervade the curriculum instead of appearing in isolated courses in middle or high school (Schoenfeld, 1995). The weaving of algebra throughout the K-12 curriculum could lend coherence, depth, and power to school mathematics, and replace late, abrupt, isolated, and superficial high school algebra courses (Kaput, 1998). To this goal, in our approach (Brizuela & Earnest, 2007; Carraher, Schliemann, & Brizuela, 2000; Carraher, Schliemann, & Schwartz, 2007; Schliemann, Carraher, & Brizuela, 2007), functions and their multiple representations (e.g., natural language, line segments, function tables, Cartesian graphs, and algebra notation) play a critical role as an integrative concept, as proposed by Seldon

and Seldon, (1992), Dubinsky and Harel (1992), and Schwartz and Yerushalmy (1992, 1995).

Our approach rests on the premise that a deep understanding of arithmetic requires mathematical generalizations and understanding of basic algebraic principles. We view algebra in elementary and middle school as a generalized arithmetic of numbers and quantities and the introduction of algebraic activities as a move from computations on particular numbers and measures toward thinking about relations among sets of numbers and variables. A key idea behind this view is that an algebraic, functional approach to arithmetic topics will lead to better teaching and learning of arithmetic operations, fractions, ratios, proportion, and geometry, main topics in the middle school curriculum. It also leads to considering isolated examples and topics as instances of more abstract ideas and concepts. Multiplication by two, for example, is a table of number facts ($1 \times 2 = 2$; $2 \times 2 = 4$; $3 \times 2 = 6$; $4 \times 2 = 8$) but it also can be understood as a subset of a function over the integers, $f(n)=2n$, that maps each element from the domain to the co-domain. As such it lays the groundwork for the real-valued, continuous function, $f(x)= 2x$, which can be represented as a line in the Cartesian plane. In this approach, topics of ordinary arithmetic foreshadow increasingly abstract and symbolic topics.

In addition, in elementary and middle school, the contexts and situations in which mathematics problems are embedded play important roles in learning. Research from diverse perspectives (e.g., Moschkovich & Brenner, 2002; T. N. Carraher, Carraher, & Schliemann, 1985, 1987; Nunes, Schliemann, & Carraher, 1993; Schwartz 1996; Smith & Thompson, 2007; Verschaffel, Greer, & De Corte 2002) has shown that the young learner uses a mix of intuition, beliefs and presumed facts coupled with principled reasoning and

argument, instead of relying solely on logic and syntax. However, although rich problem situations provide important points of departure for identifying and working with more abstract structures and syntax, students will eventually need to derive conclusions directly from written system of equations or x-y graphs drawn in the plane.

Likewise, we have often found it useful to begin focusing on students' current ideas, including those that may have arisen outside the classroom. The challenge for teachers in their classrooms, as well as for us in the planning of Poincaré courses, has been to design problems and situations that would trigger the learners' motivation for understanding, their own representations, and their initial intuitive approaches towards solutions. The role of the teacher should then be to further promote reasoning about specific situations, to provide access to new concepts and conventional representation tools, and to allow for abstract knowledge about mathematical objects and structures to emerge. Thus, when working on a given problem, we hope to provide conditions that engage learners in using their own perspectives, ideas, and ways of representing the problem as they come into contact with more advanced mathematical content. Consequently, teachers need be aware of students' typical ways of approaching specific mathematical content, as documented by mathematics education research or by his or her own explorations about actual students in the classroom, together with a view of how students' ideas may relate to the mathematical content to be learned.

Our three longitudinal classroom research investigations revealed the positive impact of this approach (Schliemann et al., 2003; Schliemann, Carraher, & Brizuela, 2012). For example, in a classroom intervention study we implemented from third to fifth grades,

teaching weekly early algebra lessons based on the above described views, we found that, at the end of fifth grade treatment students fared better than controls on algebra problems included in the project's written assessments, as well as in problems included in State mandated tests. And the benefits of the intervention appear to have persisted two to three years later, when the treatment students were more successful than their peers in learning to solve more advanced algebra problems (see Schliemann, Carraher, & Brizuela, 2012).

The following is an example of classroom activities we developed in the early algebra project that proved relevant to the work of Poincaré teachers. We presented the following problem to fourth grade students (see Carraher, Schliemann, & Schwartz, 2007):

Mike and Robin each have some money. Mike has \$8 in his hand and the rest of his money is in his wallet. Robin has altogether exactly three times as much money as Mike has in his wallet. How much money could there be in Mike's wallet? Who has more money?

Fourth graders in our intervention study easily accepted the suggestion that w can stand for "whatever money there is in Mike's wallet." The instructor then listed, in a table drawn on the blackboard, the various amounts in the wallet in the first column, followed by Mike's total amounts in the middle column, and Robin's amount in the third column. For the first several rows in the table, students determine Mike's and Robin's amounts by recalling the story. For each possible amount in the wallet, they compute the values in each column. They discuss whether Robin has three times as much money as Mike, or three times as much money the amount in Mike's wallet. At a certain point a student notes that Mike's amount is always 8 greater than w . Someone suggests writing w and $w+8$ as headers for the left and middle columns. Later someone suggests that, because Robin's amount is

three times the amount in the wallet, Robin's column be labeled $w \times 3$. From this moment on, students are able to immediately determine the values of columns two ($w+8$) and three ($w \times 3$) from those in column one (w). Inferences can be made solely on the basis of the written forms without having to refer back to the story that generated the forms.

Eventually the students conceptualize $w + 8$ and $3 \times w$ as functions free to vary across all values of w . When they plot these functions in the Cartesian space with w along the x axis they recognize that at one and only one value of w do the graphs intersect, namely, when $w = 4$. They come to realize that this is the only value of w for which the equation, $w + 8 = 3 \times w$ happens to be true. When Mike has less than \$4 in his wallet, then Robin will have more than Mike. The situation is reversed when Mike has more than \$4 in his wallet. The only time they have the same amount is when $w = 4$.

In the activities of the first cohort Poincaré Institute teachers, we have seen children taking this big step towards more abstract thinking and the use of variables. In particular, in a fourth grade classroom, while a teacher was introducing the idea of displacement of a graph in the plane using both tables and graphs, children spontaneously started to use letters instead of numbers and wrote relationships among these symbolic representations in the form of equations (with two variables).

Perspectives from Mathematics and Science

Building upon the pedagogical and research expertise described above, the interdisciplinary work undertaken since the first planning steps of the Institute has greatly expanded, transformed, and deepened by the joint contribution of mathematicians, mathematics education researchers, and physicists. The following ideas are perhaps the

most salient, for they constituted some of the original key topics on which the mathematicians, educators, and scientists first focused their attention upon. And quite a few of the ideas ultimately assumed prominent roles in the courses for teachers. They are:

1. Elementary and middle school children are far more capable of algebraic reasoning than they were thought capable of just a couple of decades earlier.
2. The mathematical concept of function, normally introduced at the onset of high school, has considerable potential in uniting diverse topics in early mathematics and bringing out the algebraic character of arithmetic.
3. Mathematical concepts are intricately associated with representations that are used for making sense of diverse situations, inside and outside of mathematics.
4. Much of young students' burgeoning knowledge about algebra and functions is bound up in trying to explain extra-mathematical situations, hence modeling.

The focus on functions was one of the critical decisions we faced early in finding a common ground on which the three basic disciplines could work together with the middle school teachers from the partner districts. This meant having a clear sense of the objects of study as well as some sense as to how these objects could contribute to teaching and learning in the districts. "Algebraic reasoning" and "early algebra," although generally consistent with our planned focus, are not well defined mathematically and thus do not offer the needed traction for an interdisciplinary partnership. Algebra itself is a vast domain of mathematics as well as a language for expressing mathematical ideas in many sub-domains of mathematics.

It should be recognized, however, that functions are rarely prominent in middle-school curricula. On the contrary, they are mainly associated with high school grade levels

in the Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2010). Although NCTM's (2000) standards are generally compatible with function-based approaches to middle school mathematics, implementation of the NCTM's standards are often framed in terms of pattern extension, a relatively ill-defined notion, as opposed to assignment rules of functions.

In mathematics, functions have proven to be a high-level construct of special importance in the history of modern mathematics. Functions are well defined and susceptible to rigorous mathematical examination. For scientists, functions are perhaps *the* key mathematical tool for modeling properties and processes of the physical world through relations among measured variables. Scientists regard functions as lying at the heart of modeling. Their focus on physical quantities and on functions to describe and explain physical and real world phenomena is at the core of our pedagogical approach. Furthermore, the educational research team had gathered compelling evidence that functions could be introduced early on in the context of the four basic arithmetical operations (Schliemann, Carraher, & Brizuela, 2007).

By focusing on functions as the core concept in the development of middle-school teachers, it may have seemed that we were introducing new, more advanced topics into already-overcrowded middle-school curricula. In fact, we were proposing functions not as additional content but rather as organizers of existing content. To this end, we needed to first challenge the widely held premise that there is no wiggle room in the middle school math curriculum. We had to convince teachers that many topics taught in isolation are in fact different aspects of the same idea. Teaching them together not only leads to a better

understanding but also economizes instruction so it can be devoted to a deeper exploration of topics. For example, rational numbers, ratio and proportion, and linear equations and slope can be combined in a lesson that would help students notice the true meaning of all these notions and their use. Similarly, in any given class, teachers are encouraged to explore problems through multiple representations, especially diagrams, graphs, tables of values, written numeric and algebraic notation, and linguistic constructions.

Reaching Students Through their Teachers

A substantial amount of our work at the onset of the Poincaré project dealt with teaching students rather than their teachers. The “Early Algebra” project carried out research in which the investigators went into the classroom at regular intervals for an extended time and implemented their lessons as a supplement to what was regularly taught in a math class by the classroom teacher. Mathematicians had advised MAT and doctoral students in Math Education but their own teaching was only to undergraduates. While some members of the group participated in the Fulcrum Institute, this was a very different type of experience: Fulcrum was addressed to teachers at all K-12 levels, dealt with science, and teachers came on their own, while this project is targeted to 5-9 Math teachers that work together with their colleagues in their districts.

How could we expect that the Institute may impact student learning if our contacts are solely with the students’ instructors? We address this challenge in several ways.

For one thing, we have chosen topics directly relevant to the middle school curriculum. In our case, these topics were numbers (fractions, rational numbers, integers and divisibility), arithmetic (the basic operations of addition subtraction and multiplication), functions and their representations through graphing and tables, slopes,

solution of equations both linear and polynomial, modeling and applications. As we mentioned earlier, these can be unified under the umbrella of the study of functions. Then numbers become domains for these functions, arithmetic operations become examples of such functions. Slope is an important characteristic of a (nice) function and linear equations can be solved by applying suitable functions to the plane. Modeling and applications are in many ways a scientist's take on functions.

Our challenge then was to first provide the teachers with the background in mathematics they needed to understand these concepts, their interconnections, and their position in the big picture. Then we had to show them specifically how the topics they teach in the classroom relate to this big picture framework. And finally we had to get them ready to develop activities for their students that build on this approach.

The first two goals have been tackled with a series of lessons in written and video format. These lessons increasingly considered together the mathematics and pedagogical aspects of a topic, in an integrated way, rather than separately. Because both mathematical knowledge and its teaching need to be constructed by the learner, special attention was given to the choice of "homework" questions that go beyond confirming that information in the text has been rote learned. The homework questions are designed to trigger discussions and understanding at a deep level and allow multiple approaches. They are based on the lessons and relate the mathematical framework of the courses to the specific topics that are part of the middle school curriculum. Some of these assignments include analyzing a situation that appears in a classroom, presented either through a videotape of such a class or through written work of the students. Exploration, discussion and

appropriate use of technology have been encouraged throughout.

The above last step aims at making sure that the teachers feel confident with the material to the point that they can bring it themselves to their students and that their teaching methods are conducive to learning mathematics with understanding. To encourage these attitudes, right after they have learned about selected topics, teachers either interview their students on the topic or develop a learning activity related to the topic and analyze its implementation. They present their work as written reports often accompanied by video clips. They discuss each other's reports and provide feedback for improving the activities. In the final two weeks of the course, they implement activities in the classroom based on their lesson design or interviews conducted during the term.

Integrating Perspectives

Our initial ideas had to be assessed against the needs of the teachers in the districts. In our preliminary visits to schools, during the grant-writing period, our suggestions to focus the courses around algebra met with considerable enthusiasm. Teachers and administrators alike recognized the need to improve the teaching and learning of algebra. Algebra I and II were regarded as major obstacles to success in high school and preparing students for these courses was seen as a goal for middle school. Generally speaking, strong students take algebra I in middle school, whereas those who lag behind take pre-algebra and leave algebra for high school.

In our discussions with teachers, we tried to determine some specific topics for the courses but they were not clear on what would make a difference in their classrooms. Somehow, they were open to the topics we would choose. Although we had a clear idea of what type of mathematics is important and what type of understandings students should

have by the time they leave the educational system, we were less confident about how to prepare the current teachers to teach in an effective way. Most of the previous work and expertise from educational researchers in the early algebra studies dealt directly with the students rather than their teachers.

Many of the fifth and sixth grade teachers had been trained and licensed to teach elementary school and most of them never expected, when they were in college and during their professional preparation, that they would be mostly math teachers. At the other end of the spectrum, those teaching ninth grade were licensed to teach high school and could find themselves in any given year teaching anything from algebra I, or even pre-algebra, to AP Calculus. Needless to say, the educational background of the teachers was also very diverse. Many teachers had only a bachelor's degree and provisional licensure; some had a master's degree. Majors ranged from mathematics and the sciences to the humanities.

Course Development

Our initial proposal had only course titles and a paragraph description for each course: the first course was to deal with functions and their representations, the second course with transformations and their use in the solution of equations, and the third course with change as modeled by functions. These big ideas served as the basis for the three courses offered to the first cohort of teachers. As described later in this article, this initial proposal has been constantly expanded and adapted, as we implemented course units, examined teachers' work, and carefully considered their suggestions and feedback to course content, structure, activities, and materials. The content and structure of the courses as offered to the first cohort of teachers are described below.

Course 1: Representations

The main goal of Course 1 was to present the idea of function and its multiple representations and uses, especially in modeling arithmetic operations from the middle school curriculum. We wanted to make invertibility a major focus of the course, not only because it united the arithmetical operations, but also because it was fundamental to algebraic operations on equations. It is a crucial and unifying notion that allows one to deal with a multitude of topics, from the relation between addition and subtraction to the fact that one cannot divide by zero or that positive numbers have two square roots.

The course was divided into three units: functions and relations, functions on the real number line, and representation of functions on the plane. Units were divided in weeks, each with a main focus on mathematics, education, or science. Eight of the fourteen weeks of Course 1 focused on the mathematics of functions and relations; two weeks were dedicated to mathematical modeling in science, and four of the fourteen weeks focused on teaching and learning.

Teachers were divided into online teams of six teachers per team, with two instructors (one educator and one mathematician or physicist) as tutors. For each week, teachers were first presented with an exploratory activity. In “math” weeks, the assignment came with a set of notes and videos containing mathematical background. In many cases not much formal knowledge was needed for this first exploration. When this preliminary assignment was completed, more materials and a second set of more complex questions would come up, along with short essays presenting a mathematician’s, a scientist’s, and an educator’s perspective on the main topic. In this second phase, teachers were asked to comment on the work of their online team peers. They were also encouraged to make use

of the general forums where they could post questions and ideas and discuss any topic related to mathematics or classroom practice.

The faculty had invested much time and effort in the course preparation and delivery. However, not everything ran smoothly. At the beginning, in the case of some units, we overestimated the level of mathematical knowledge of our participants and greatly underestimated the amount of time it would take them to complete an assignment. Coordination among the faculty designing different parts of the course was not optimal and integration among the disciplines while present, was not fully achieved.

Despite the above flaws, learning was taking place and enthusiasm towards the program pleasantly surprised us. Even in those units in which we had aimed too high, the teachers were heavily engaged and their effort and cooperation coupled with instructor support led to impressive outcomes and a great sense of achievement.

The teachers were particularly drawn to the “education weeks,” for which they analyzed video of classroom activities or samples of student work produced by the early algebra previous research. Teachers watched and listened carefully and marveled at how much algebra young children were able to grasp. Some teachers modified the activities and used them in their own classrooms.

By the end of the semester, we had learned as much from our teachers as they might have learned from the course. We had the luxury of the summer break between the two courses and we spent most of it preparing Course 2.

Course 2: Transformations

If the Poincare Institute was to have a real impact, teachers should be applying what

they learned from the courses into their classroom. At the same time, in view of the needs of our participant teachers, mathematical content should not be shortchanged.

Taking into account what we witnessed during Course 1, we decided to revise the course structure, organizing Course 2 into five units, each integrating mathematics, science, and educational views. During the first two weeks of each of the first four units, mathematics, modeling applications, and educational insights were to appear together. As planned, in the first week of each unit in Course 2, the teachers explored the topic, discussed models of teaching the unit's specific subject, analyzed students' ideas and challenges in learning the subject, and solved problems relevant to their learning and teaching. In the second week, they were guided to develop a deeper understanding of the mathematical content of the unit, again through notes, videos, problem solving, and online discussions, working on assignments that would require them to think through the questions often from several of these points of view. Then, in the third week of each unit, groups of three to five teachers jointly designed a learning activity for possible future implementation, based on topics from the previous two weeks. For their final, individual project, each teacher implemented in their classroom one of the learning activities they had planned. They videotaped this activity and analyzed his/her teaching and their students' learning in a short individual report, which was posted online, along with selected classroom video clips, and discussed by other teachers.

At the request of teachers we opened Course 2 with a more in depth treatment of fractions and divisibility than what had been presented in Course 1. We then moved to transformations of the line, as a geometric model for arithmetic operations, followed by transformations of the plane. Transformations were then used to analyze graphs of

functions and to present a geometric way of solving equations.

To exemplify our work, let us focus for a moment on the unit on transformations of functions (unit 4 of Course 2). In retrospect, this unit was overly ambitious, insofar as we asked each teacher to work through a number of new ideas as well as practical applications to their classroom. Nevertheless, it was well received.

In the previous unit, the teachers had been examining transformations of the line and of the plane, specifically, translations, dilations, and reflections. (We did not include rotations, which, although interesting, have a more complicated algebraic representation and are less useful for studying graphs of functions and for solving equations.) Through their familiarity with invertibility, the teachers had a rudimentary notion that one could move back and forth between functions. This would be greatly extended in this part of the course.

The transformation of functions unit opened with the story of a train first moving along a track at constant speed, then stopping for a brief period before restarting the journey. Teachers were asked to graph in the Cartesian plane the distance function in terms of time. They then considered variations of the initial trip, such as a train leaving later (but otherwise taking the same trip as the earlier train), or coming from the opposite direction, or moving faster or slower. They were asked to relate the story variant to the initial trip both geometrically and algebraically. They also applied the same type of analysis to other modeling options such as cost functions in terms of weight.

The following week, the teachers worked with the relation between algebraic and geometric presentations in the abstract. They were then presented with a linear equation

interpreted in terms of the intersection of two lines and looked at the types of transformations that preserve solutions and their use in solving the equation. Finally, transformations were used to bring the equation of a parabola to the standard form and this was used to obtain the quadratic formula. Several of these topics were revisited in Course 3 and studied in more depth.

Course 3: Invariance and Change

The Course 2 structure, with three-week units and educational activities explored by the teachers in the third week, was very successful and was therefore utilized for Course 3. However, in week three of each unit the teachers could either develop plans for learning activities (as was the case in Course 2) or interview individual students on problems designed to explore student thinking, their spontaneous solution strategies, and difficulties they would face. Almost all teachers opted to interview students. This then became the basis for the development, implementation, and evaluation of a classroom activity they developed as a final project for Course 3.

The mathematical content of Course 3 began with an analysis of solutions of equations, starting with the meaning of the equal sign, moving from linear equations to quadratic and higher order, and understanding the relation between factoring and roots of an equation. We then explored change with the idea of slope and its meaning. The fourth unit looks at modeling and real life applications and how to teach children to make the connection between the math and word problems. As in Course 2, the final two weeks of Course 3 were dedicated to the development, implementation, and analysis of a learning activity.

Weekly Meetings In Schools

As we mentioned above, the teachers meet after school in their districts once a week. They are free to choose what they want to discuss at their meetings so long as it is related to mathematics and its teaching in their classrooms. Once a month, the faculty pair assigned to that district attends the meeting.

The monthly meeting with Poincaré faculty has been a very useful forum for teachers to express their concerns and suggestions and a good way to further monitor their progress. Some of the teachers have built personal ties with their faculty mentors and are no longer hesitant to contact them when difficulties come up in course material or even in advanced mathematical topics they need to teach. Sometimes, however, especially during the first semester, the weekly meeting became a place to moan about what was wrong in the district. Technology glitches in Course 1 implementation also took a good amount of meeting time. The situation changed dramatically during the second semester, when Course 2 was offered. The main reason was that the new course structure, requiring a group project related to a teaching activity, became an important topic for discussion. All of our participants chose to form a group with other people in their district, most often with those in the same school as themselves. The weekly meetings became then the natural time to plan and discuss these projects. While this has not been the case at all the meetings, we found that, when it happened, it led to very fruitful discussions that helped the teachers develop substantially improved activities or to discuss in depth the thinking and learning of their students. For example, in three districts, after the teachers had submitted their analysis of interviews with individual students on the problem shown in Figure 1, the

monthly meeting with Poincaré faculty was dedicated to the analysis of students' spontaneous ideas about how to represent the problem.

- Elizabeth Excited, Patty Planner, and Carly Catch-up are all cousins. Next year, they would like to send their grandmother on a big vacation for her birthday, but the trip will cost \$3,000. Elizabeth, Patty, and Carly decide that they have one year to raise \$1,000 each.
- Elizabeth starts saving a lot of money on the very first day and realizes that she would like to have some money for herself, too, so each day, she puts less money into her bank account than the day before.
- Patty figures out exactly how much money she will need to save each day to reach \$1,000 in one year and she puts the same amount of money into her account each day.
- Carly begins by saving very little but she realizes that she will not save enough money in time, so each day she puts more money into her account than the day before.
- All three girls saved exactly \$1,000 at the end of the year.
- Draw graphs showing how much money Elizabeth, Patty and Carly had during the year.

Figure 1: The problem students' were asked to represent during individual interviews (Adapted from Yerushalmy and Schwartz, 1995).

In three different districts, during the meetings with Poincaré faculty, the teachers discussed the graphs produced by the students in terms of:

- Use of bar graphs
- Attempts to transition from bar graphs to line graphs
- Representation of savings month by month versus representation of accumulated savings.
- Challenges of representing linear vs. no linear functions.
- Possible intuitive approaches to the representation of step functions.

Teachers discussed students' views as revealed in their interviews, explored the possible origin of students' difficulties, and considered ideas on how to develop learning activities taking into account what teachers found in the interviews. Teachers acknowledged that, even though the children did not know the formal conventions for graphs, many showed interesting and often coherent representations for savings by month

or accumulated savings.

Difficulties identified and discussed were related to:

- What the axes represented.
- The tension between bar graphs and line graphs and syncretism.
- The arriving point for all lines (1-year, \$1,000).
- The tension between the representations of linear vs. non-linear functions.
- The difficulty of representing Elizabeth's savings as starting from the origin (she saves more at the start).

Some teachers then decided to develop a learning activity based on this problem, considering how students' intuitive solutions can be a step towards learning about graphs on non-linear functions.

The participating teachers seemed to enjoy the weekly meetings for a variety of reasons. The most often cited reason for enjoying the meetings was that they allowed them to communicate with the other teachers in the district, understand the continuous progression of the syllabus, form personal bonds with their colleagues, and have a forum for discussion of teaching issues. For many, this was an opportunity they never had before and they seemed to be eager to keep these meetings once their participation in the Poincare Institute was over.

One goal we have, as the second cohort of teachers start taking the courses, is to make sure that teachers from the first cohort will join the new teachers in the weekly meetings, an important aspect to achieve permanent changes in teaching and learning at their districts.

Looking Ahead

Program Revisions

As the first cohort of teachers approached graduation, we started revising the courses for the next cohort, taking into account the written suggestions from our team members, our experience in the first round, some preliminary research results, the needs of participant teachers and their students, and the many suggestions provided by the teachers, online or during our face-to-face meetings in the districts. We began by asking all participant faculty, researchers, postdoctoral fellows, students, or staff members to give us a view of what they would like to do in the second round. Except for a couple of extreme opinions, we were surprised to see that most Poincaré team members recognized the importance of contributions from mathematics, mathematics education, and science. At least to some extent, these two years of working together made mathematicians, educators, and scientists more interested in the work of each other and more appreciative of the role of science and modeling in learning mathematics.

The collaboration process among mathematicians, educators, and physicists at first consisted in individual contributions that were made accessible in a given week. We then evolved into jointly producing course notes which, even though they emphasized one or another perspective, resulted from the collaboration and points of view from the different areas. Administratively, we also improved the process for developing course materials. In revising the courses to be offered to the second cohort of teachers, each unit is produced by a small interdisciplinary team of up to three people. Those in charge of each unit post the first draft of materials for feedback from all course team members, including a teacher from cohort 1. The feedback is compiled by an interdisciplinary editorial board who then asks

the authors to implement the relevant changes. This process of feedback takes place twice, until the editorial board approves the final version of materials.

In terms of content, developing the courses goes beyond the list of topics that we want to cover. The three Poincaré courses are meant to develop habits of mind and foster appreciation for the subject, at least as much or even more than specific topics. We mostly agree on what these habits and ideas should be. We feel we have succeeded in passing some of these to some of our teachers, but we are far from our goal with others.

Among the mathematical abilities that we would like to promote are an awareness of the roles of conjecture and proof. On the one hand, while we do not expect or even desire that teachers be able to write detailed and polished proofs of the sort required of an advanced math major, we believe they should understand that checking a few examples of a result is not sufficient to confirm the truth of a statement that could be applied in much greater generality. On the other hand, playing with a few examples is the only way to get a feeling about the subject that would allow them to, then, formulate a conjecture. We would like teachers to feel sufficiently comfortable with these ideas so that they can model them in their lessons with their students.

We tried to incorporate some ideas about conjectures and proofs during face-to-face workshops offered in the kickoff meetings as well as in notes and assignments. While there seems to be a noticeable awareness of what conjecture and proof are, we are far from having reached our goal. With the second cohort, we will try to further incorporate proofs in the work of each unit of each course, using simple examples to draw the attention of the teachers to the method as much as to the final result. We will also ask the teachers to try

their hands at it, providing help and frameworks as needed.

Something similar could be said for modeling and problem solving, in general. In the first round of courses, we might have been too explicit about modeling, trying to give the teachers words for a variety of phenomena instead of having them work more on developing mathematical models for particular situations. In addition, as assignments were normally related to a topic, those that were only loosely related to a particular mathematical content, or that used many aspects of the content at the same time, have failed to promote deep understanding of modeling and problem solving strategies. We attempted to address this limitation only towards the end of course 3. In planning the second round, we are making a point of offering the teachers a chance to work on these types of modeling and open-ended problems at regular intervals. The biggest obstacles to overcome arise from the fact that some teachers prefer to be sure that they will be able to give the right answer to all of the questions asked and feel uncomfortable when they have to deal with a problem that cannot be solved with the tools they have just learned.

Another aspect that we want to emphasize is “what lies beyond the horizon.” Teachers should be aware that there is a lot more mathematics than what they teach and that, like a work of art, mathematics can sometimes be enjoyed just for the pleasure of it, even without understanding all the details.

Some of the structural aspects of the courses seem to have been working very well in Courses 2 and 3 and are being preserved in future cohort offers. For example, courses will continue to be divided into three-week units. The first two weeks of a unit will include mathematics, education, and science content in an integrated way, and the third week will be a teaching-related exploration of the content covered in the previous two weeks. The

first course will include teaching and learning demonstrations that the teachers will analyze, as a training ground for the other courses. Teachers will interview some children about a topic related to what they learned in the unit and try to understand the students' ways of thinking, or they will design an activity related to the topic that could be used in their classrooms, as both types of activity proved to be useful for cohort 1 teachers.

Two of the issues we want to address are how to foster intense and focused online discussion and how to provide useful feedback to teachers. To be clear, there has been a substantial amount of discussion, often inspired by the lessons or, at other times, by teachers' experience in the classroom. Most of it takes place in a general online discussion forum that is part of the platform for course delivery. A lot of discussion happens also in face-to-face weekly meetings at the schools and during office hours regularly offered to help teachers as they work in the weekly assignments. Since the "third week" activities are teamwork, some discussion is happening as teachers work on the assignments. The regular work for Weeks 1 and 2, however, are posted on-line and can only be viewed by teachers that are members of that team (and by all faculty members). In some of the on-line teams, there is regular discussion of assignments with teachers posting drafts of their answers and helping each other gain a better understanding. Other teams, however, hardly ever discuss their peers' work. We are trying to develop a new model that will insure that discussion on their work happens for all teams and in all weeks of each unit.

In terms of feedback on teachers' responses to the course assignments, we spent a substantial amount of time on a task that teachers might not take so much advantage of because, by the time they receive it, they are already working on the next unit. For the

second cohort, instead of giving feedback once a weekly assignment has been completed, we will provide on-line help to each group while the work is being done and will post some model answers at the end to help teachers decide for themselves if they were on the right track. As before, on-line office hours will still be available but individualized feedback on each participant's submission will be briefer.

Regarding mathematical content, it is not substantially different from the first round, with one exception. In round one, we introduced functions as sets of ordered pairs from the Cartesian product of elements from the domain and co-domain. Although this makes sense, mathematically, it was too abstract a starting point for middle school mathematics. We decided that in the second round we would emphasize, in the beginning, the notion of functional dependency; namely, that output values (the image) were “dependent” on input values (from the domain). This also allowed us to highlight, early on, mappings involving the real numbers.

Presently we start with a study of the real line and incorporate functions as a transition between arithmetic and algebra, skipping our previous attempt with relations. We also agree that an earlier introduction of a variety of functions and a focus on rate of growth would help teachers understand that not everything is linear. The content of the courses offered to teachers in the second cohort is described in the Appendix.

Evaluating the Impact of the Program

Given that our first cohort of teachers has just graduated, a large amount of data remain to be analyzed. The impact of the Poincaré Institute will be analyzed in terms of teachers' and students' evolving understanding of mathematical content and representations and in terms of teachers' implementation of effective teaching activities, as

demonstrated in written assessments designed by the project, videotaped classroom discussions, and course assignments.

Teachers' written assessment data and videotaped lessons have been and will be collected among Poincaré teachers and their colleagues, at the start and end of the five-year project and, for teachers in each of three cohorts, at the start and end of each three-course sequence. Data on student learning are being collected through written assessments designed by the project, state-mandated assessments (MCAS, NECAP), and videotaped classroom discussions. Comparisons between pre-and post-written assessment measures and between participant and non-participant teachers and their students will allow for evaluation of the impact of teachers' progress and of their students' success.

Dependent measures cover the mastery of mathematical content (Numbers, Fractions, Ratios, Proportions, Relations, Linear and Non-Linear Functions, and Algebra Equations), algebra in modeling, and use and interpretation of mathematical representations. Our analysis will focus on willingness to explore problems in depth, considering all potentially relevant aspects before proposing solution methods and answers, use of multiple representations for functions (natural language, tables, number lines, graphs, written notation), and use of algebra as a modeling tool in extra-mathematical contexts. Detailed qualitative analysis of students' questions, answers, argumentation, justifications, solutions, and written work, as they participate in videotaped lessons before and after their teachers are taking courses, will allow further insights into the project's impact on student success.

The Poincaré Institute aims to substantially improve the teaching and learning of

middle school mathematics and the project's research team is working at collecting data that will allow us to show that this is happening. While it is too early to present quantitative data on teachers' and students' progress, we do have some anecdotal evidence and preliminary analyses showing that change is actually happening, if not in how much children are learning, at least in how teachers are teaching.

As we mentioned in the course descriptions, during Course 2, each team of teachers was asked to design four activities related to the content of the course that could be implemented in their classroom. Then, at the end of the course, each individual teacher had to implement one of these activities in his or her classroom, videotape the implementation, and analyze its results.

In most groups, there was a notable progression in the quality of the activities designed over the semester. While the first activity was usually an immediate adaptation of something in a textbook, without much thinking about how it could help students' learn, the last few showed a much richer and careful design, with examples carefully adapted to the goal, and much better use of a variety of approaches and representations. For instance, teachers' learning activity plans show, from the start to the end of Course 2, a clear increase in the number of alternative representations for the math content they proposed to teach, with an average of 2.56 kinds of representations for Unit 1 (with half of the teachers only using one or two kinds of representations), to 4.88 kinds in Unit 4 (with only one plan using fewer than three kinds of representations). Most of all, teachers see a much clearer connection between the algebraic and geometric presentations of a given concept. The teachers, themselves, are very aware that this is something that has permanently changed in their understanding of mathematics and are very happy to discover for themselves and

present to their students this new way of looking at algebra. Here is a teacher's comment in one of the discussion forums for Course 3:

... my biggest walk-away will be the ability to show kids all the great connections between algebra and geometry. The connection between the two when we were working with transformations on the number line and the plane were very enlightening for me and gave me a deeper understanding, which will definitely benefit kids that I work with.

Or from another teacher at the end of Course 2:

My textbook presents equations in chapter with solutions using transformations, no graphs. Graphs of linear equations come in chapter 4. When reading the notes for unit 4 week 2, I had an epiphany: I need not wait for the chapter on linear equations to ask the students to represent their solutions graphically.

Summary

The implementation of Poincaré courses has been generally successful for the first cohort of teachers. As we plan and approach the offer of courses to the second cohort, we hope to improve the collaboration between all Poincaré participants and to correct possible flaws in the design of the different components of the project.

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APPENDIX

Content of the Courses offered to teachers in the Second Cohort

Course 1: From numbers to functions

UNIT 1: *Real numbers*. An introduction to the real line, fractions and their multiple representations, classroom applications and use of numbers in modeling.

UNIT 2: *From numbers to functions*. An introduction to functions: the intuitive idea of function, its use as assignments and as a constraint between two types of quantities, and the formal definition of function. Composition of functions. The vertical line criteria. Use of functions in modeling. Examples include simple arithmetic operations (addition, product) and also functions on objects other than numbers. Special attention to multiple representations of functions (verbal, arrows, tables, algebraic expressions and graphs).

UNIT 3: *Examples of functions*. An expansion of the previous unit focused mostly on examples of functions of one real variable, especially those examples that appear commonly in mathematics and science: linear functions, absolute value, monomials, exponentials and step functions. Some examples of “compound functions” like those obtained from the simpler pieces by composition, addition or product.

UNIT 4: *Division*. The various interpretations and applications of division. Functional approach to ratio and proportion. Division with remainder, decimals and decimal representation of rational numbers. A basic introduction to divisibility for integers and decomposition into product of powers of primes.

Course 2: Transformations and equations

UNIT: 1 *Transformations of the plane*. Functions of two variables, in general, building

on the examples of addition, multiplication and division already introduced. Translations, dilations and reflections on the plane and comparison with similar functions on the line. Compositions and inverses of these functions.

UNIT 2: *Transformations on the graph of functions.* Translations, dilations and reflections acting on the graphs of functions. Interpretation of changes in the data modeled by a function in terms of transformations to the graph. Algebraic representation of transformations for the graph of a function. Solution of linear equations using transformations and the connections between algebraic manipulations and geometric representations.

UNIT 3: *Equations.* Geometric and algebraic representation of equations and their solutions. Parabolas and their equations under transformations. The quadratic formula.

UNIT 4: *Divisibility for integers and polynomials.* Recall of the concept of divisibility for integers. Unique factorization for integers as product of primes. The Euclidean algorithm for the greatest common divisor. Review of basic facts about polynomials. Divisibility for polynomials, unique factorization. The relations between roots and factoring for polynomials. The number of solutions of a polynomial equation.

Course 3: Change and invariance

UNIT 1: *Slope and rate of change.* Slopes as indicators of the rate of change of a function. Average rate of change of a function over an interval and its geometric representation as slope of a secant. Instantaneous rate of change as the limit of an average rate of change over small intervals and its geometric counterpart as slope of a tangent line. Comparison of the growth of linear functions to other types of functions

UNIT 2: *An example-based introduction to the idea of limit.* Decimals with an infinite

number of digits as limits of sequences of some special functions. The idea of limit and of vertical and horizontal asymptotes ($1/x$, exponential). Comparison of the growth behavior of these functions to other types of functions. Applications to arithmetic operations and the middle school classroom (dividing by zero, dividing by large numbers). Approximating solutions to equations.

UNIT 3: *The slope function*. Introduction of the derivative as the function “slope at the point” or rate of change at the point. Comparison of derivatives for different types of functions (constants, linear quadratic, exponentials, $1/x$). Reconstruction of a function given its derivative. Applications to issues relevant to middle school students, to modeling and science.

UNIT 4: *Change and invariance of shapes under transformations*. Transformations that preserve and do not preserve the shape of graphs. Lines through a point and solutions of linear equations.

Feedback to Support Learning in the Leadership Institute for Teachers¹

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Abstract: Feedback is a type of formative assessment used to inform instruction and advance learning. Feedback serves as a mechanism to connect teaching and learning at the student level. Learners receive feedback, formally or informally, as they engage in learning experiences. Within the Leadership Institute for Teachers, a National Science Foundation funded research project, we are exploring feedback as a research-informed process to support learning and improvement for individuals, teams, and university courses. There is an explicit focus on creating a culture of critical thinking and reasoning, taking ownership for learning both individually and collectively, and understanding how to improve teaching and scholarship through an iterative feedback process.

Keywords: Formative assessment, feedback to advance learning, course improvement, mathematics teacher leaders

How do mathematicians, math educators, and teacher leaders utilize feedback to support learning in the Mathematically Connected Communities Leadership Institute for Teachers (MC²-LIFT or “LIFT”)? This article provides an opportunity to understand how feedback is used to improve MC²-LIFT courses, lessons, and learning experiences for the mathematics teacher leader project.

Mathematicians and math educators are engaged in MC²-LIFT, a National Science Foundation (NSF) project focused on developing teacher leaders in mathematics. This project provides opportunities for building content and pedagogical content knowledge

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(Shulman, 1986) for effectively teaching K-12 students mathematics. Six semesters of coursework are designed to build professional knowledge, skills, and dispositions for the teacher leaders. This article (a) introduces our interest in feedback as a research-informed process for improving learning, (b) provides an overview of the LIFT project, (c) and presents specific examples of how and why feedback is used and what we are learning through these processes.

Feedback is a type of formative assessment used to improve instruction and provide mechanisms to support continued learning. Learners receive feedback, formally or informally, as they engage in learning experiences. Feedback can be motivational, evaluative or descriptive and based on standards or learning goals. Within our research project, we are exploring descriptive feedback as a research-informed process to support learning and improvement for individuals, teams, and courses. There is an explicit focus on creating a culture of critical thinking and reasoning, taking responsibility for learning both individually and collectively, and understanding how to support learning as a reflective process, within the LIFT project. These foci afford rich opportunities to provide and receive oral or written feedback on lessons, mathematics writing, classroom videos, and a variety of course experiences to move learning forward.

Provide	Constructive	Feedback
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The course designers utilize a reflective implementation and learning cycle to improve the course experiences and strengthen individual learning. Within this cycle, feedback provides data to assess practices, inform instruction, and to give information that is used to adjust and improve the academic experiences. This feedback process includes receiving input based on learning goals or agreed upon expectations, acting upon the		
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feedback to make revisions, and determining next steps for an individual assignment or perhaps for a lesson within the LIFT courses. A central tenet of the LIFT project is that everyone's ideas contribute to the learning and assessing the impact of those experiences on individual and collective scholarship. An overview of the LIFT project is followed by our exploration into how feedback can be useful in supporting learning and how to solicit that feedback effectively.

Overview of the MC²-LIFT Project

The MC²-LIFT project is a 5-year research partnership between New Mexico State University (NMSU) and southern New Mexico school districts. This collaborative project is funded through the NSF Math and Science Partnership program (NSF #DUE-0928867). Mathematicians, education faculty, and school leaders collaboratively design the MC²-LIFT project. Each LIFT cohort is comprised of about 30 mathematics teacher leaders who develop their knowledge and understanding of K-12 mathematics and the leadership skills for improving teaching and learning.

The goals of the project are:

- (1) Increase teacher leaders' knowledge of K-12 mathematics and expand and enrich pedagogical practices through blended courses that are team-taught by mathematicians and math educators.
- (2) Develop intellectual leaders who understand what students should learn and who can differentiate instruction in their own classrooms and support other teachers to meet the needs of diverse learners.
- (3) Implement LIFT Institute learning in their classrooms and schools with mentoring from the school support team.

- (4) Build and sustain viable partnerships between mathematicians, education faculty, and school districts.

MC²-LIFT provides participating teachers and math coaches with two years of coursework involving intensive summer study, as well as a follow-up academic year program that includes application of their learning to their school or district settings. Each semester as well as during the summer, pairs of courses are designed and team-taught by NMSU mathematicians and educators, blending mathematical concepts with knowledge and skills in pedagogy and leadership. Cohort members work together for two years and have the opportunity to earn a Master of Arts degree in teaching mathematics. Teacher leaders come from elementary, middle, and high schools or serve as math coaches in a school district.

Cohort members in the LIFT program gain a new lens for learning mathematics by studying how concepts progress through the K–12 continuum, connecting within and across grade levels in the LIFT institutes. Cohort members, referred to in this article as *teacher leaders*, are developing a deeper understanding of mathematical concepts through engaging in rigorous math tasks to strengthen mathematical thinking and reasoning, sense making, communication, and math connections. Then, by developing a range of models and strategies to represent mathematical ideas, teacher leaders support other teachers at their respective schools to differentiate their instruction and to meet the needs of diverse learners in their classrooms. The LIFT coursework is developed from the premise that effective mathematics teaching requires a deep understanding of mathematics, pedagogy, and pedagogical content knowledge (Shulman, 1986) to advance K–12 students' learning and achievement.

Principals also engage in professional learning during MC²-LIFT courses to gain an understanding of how to foster a collaborative culture for teaching and learning mathematics in their schools. Principals and teacher leaders are working together to develop a shared vision for the teacher leaders' roles in their classrooms, schools, or districts, communicate expectations for professional learning among school staff, and gauge the progress that their schools are making toward student learning goals. The LIFT school support team helps to connect the university institute experiences to school and classroom practices. LIFT utilizes these school-based team structures for supporting professional learning throughout the year. The school support team provides onsite ongoing mentoring for teacher leaders and utilizes extensive feedback in shaping support at the campus, connecting research and practice, and informing course development.

Feedback Process in LIFT Team Structures

The structure of the LIFT research project includes four teams: Development, School Support Team, Management, and Research. The Development Team designs and facilitates the institute courses; it includes mathematics educators and research mathematicians who collaboratively create and teach courses for LIFT K-12 educators. The entire project is set up to provide feedback and data to each of the four LIFT project teams through iterative feedback loops, utilizing feedback processes and strategies as resources for supporting learning.

Connecting University and School-Based Learning

Teachers need a strong background in mathematics and must understand how to teach math content so students can make sense of the concepts, apply their ideas, and communicate their learning. Teachers utilize research-based pedagogical practices; in

particular, how to facilitate a student-centered classroom with an emphasis on developing conceptual understanding and applying thinking and reasoning skills and practices. A central aspect of the LIFT institute is that facilitators model effective teaching practices that are applicable both at the university and when implemented in K-12 classrooms. For example, lessons have explicit learning goals and instructors model a launch-explore-summary lesson structure and facilitator questioning, rather than lecturing and answering questions.

LIFT goals include course improvement; consequently, feedback is a research-based practice currently under exploration in the project. Course content and pedagogy are studied, analyzed, and possibly modified. Both individual and collective responses are valued in constructing a culture focused on utilizing feedback to support learning. A synthesis of research on feedback is followed by application of feedback within the LIFT courses.

A Research Perspective on Feedback

Assessment is a bridge between teaching and learning. Feedback is usually situated within a context of assessment, specifically, formative assessment that shapes instruction (Wiliam, 2012). Originally, “feedback” was used in engineering to refer to an explicit feedback loop (Weiner, 1948). For engineers, it was the explicit elements needed to move from the current state to the desired state. A feedback process must include a progression for future actions toward directing attention to what is next; it promotes significant thinking. Wiliam (2012) added that the form of feedback is not as important as its effect on learners. It should create cognitively engaging next steps for the recipient, be focused,

relate to the shared learning goals, and increase responsibility for learning by activating students as learning resources through peer feedback.

Evidence for the effectiveness of feedback as a significant activity to improve learning and achievement has been prevalent in the literature (Bangert-Drowns, 1993; Black & Wiliam, 1998; Hattie & Timperley, 2007; Sadler, 1889). Feedback is essential in learning contexts and can serve many purposes, including development of competencies, understanding, motivation, and confidence (Hyland & Hyland, 2001). Hattie and Timperley (2007) indicated that feedback is an important part of communication to support learning if it focuses on attributes of students' work, is descriptive, and is clearly understood and sufficiently detailed. One cited purpose of feedback is to utilize effective communication of timely strategic information to the learner in order to modify thinking and improve learning. Students should have an active role in their own learning; including assessing and monitoring their own progress toward goals to clarify or modify their strategies or reassess their knowledge or skills (McDonald & Boud, 2003; Nicol & Macfarlane-Dick, 2006). When students realize that feedback from teachers, peers and themselves can improve their learning they put in more effort and become more self regulated learners (Brookhart, 2006).

Even though the effects of feedback can be strong, they are variable (Hattie & Timperley, 2007). Negative or judgmental feedback, lack of specificity, lack of clear learning goals, and gratuitous praise did not help learners know how to improve (Brookhart, 2007). Findings from Black and Wiliam's (1998) research indicated that feedback during instruction through formative assessment leads to large achievement gains. Stiggins (2005) focused on assessment to support learning through diagnosing students needs, planning

the next steps, and providing feedback to improve the quality of students' work. This requires understanding how learning develops, determining a student's current level of understanding, and deciding on explicit actions to meet or exceed learning goals.

Educators can determine the current level of a student's understanding within a learning progression of related goals and can communicate to the student the next steps to support learning (Heritage, 2008). Learning can result from students providing feedback and monitoring their work against criteria for success or rubrics to provide guidance for improvement (Brookhart, 2007). Students as peers can learn to provide useful accurate feedback to teachers or each other about the quality and effectiveness of their own work or learning experiences (Leahy, Lyon, Thompson, & Wiliam, 2005). The goal is not to compare students but to provide an explicit process for developing understanding and utilizing models for "learning how to learn" (OECD, 2005). However, Burke (2009) indicated that students should have opportunities to develop strategies and engage in conversations to understand how to use feedback effectively as part of a learning process. Wiliam (2012) reiterated the notion that feedback functions formatively if the information fed back to the learner is used by the learner to improve performance and understanding and moves the learner toward shared goals.

Feedback as a Process to Support Learning

Research on feedback often centers on supporting student learning and achievement within an assessment cycle. In the LIFT research project, everyone is a learner, from teacher leaders to course instructors. Feedback processes are based on the project goals and feedback is utilized to assess, stimulate critical thinking, and inform next steps. In LIFT, feedback is used not just to transmit comments from course instructors to

teacher leaders. Rather, it is a process that includes ongoing dialogue between instructors and teacher leaders. Instructional practices are congruently designed to model, explore, and extend thinking and learning, with the goal of improving both the courses and teaching.

Feedback Examples From LIFT

Both mathematics and education courses incorporate a variety of feedback strategies. There are explicit pause points for reflecting on teaching practices and LIFT teacher learning in the university courses. The LIFT program includes a variety of assessments; the focus here is on strategies within the courses that can be used to improve instruction, not on evaluation. Examples of course feedback strategies include daily written and oral reflections, written feedback on assignments, feedback from teacher leaders on instruction, and peer tutoring or peer feedback. Peer-to-peer feedback is also utilized during performance tasks and presentations. The LIFT teacher leaders engage in structured peer group edits by using reflection questions to make comments on a peer's math work (Leahy et al., 2005). This work is evolving, as it takes time and focused experiences to learn to provide and receive feedback that supports learning effectively.

Education Coursework Daily Feedback. Daily feedback provides a model for giving and receiving feedback. It illustrates to the LIFT teacher leaders that feedback is expected and valued as a learning opportunity. A variety of tools, such as a plus/delta, are used to find out what worked and what could be improved in the day or lesson. Teachers are given class time to complete a feedback form. The data are analyzed and summarized. The synthesis of feedback data is shared with the cohort members at the beginning of the next class together with the modifications and justification for the changes that will occur as a result of the written feedback. For example, one strategy that was used after studying

assessment practices was to ask teacher leaders for an “assessment pulse.” Teacher leaders had a variety of responses to the day’s activities focused on assessment. The course developers read each of the “assessment pulse” responses, noticed themes, issues, or concerns and then shaped the subsequent learning experiences with these ideas in mind. One response by a teacher leader was

My understanding of assessment is much clearer as a result of class discussions. The questions that were used helped to focus the dialogue and make us think below our assumptions. It is important to consider not just the types but also the purposes of assessment and how they support learning. I am curious how I might engage students in an assessment process that supports their continued learning. (LIFT teacher leader, 2012)

Another example of feedback is the Daily Reflection Form. It was used each day of an entire week and included questions such as “What was a big idea of today’s lesson? What did you learn today? What challenges did you encounter? What questions do you have or what would help you to better understand the big idea?” The responses were read by course instructors and used to share collective ideas and make adjustments to instruction. It was a conversational strategy for feedback. The course development team writes questions to individual teacher leaders on their reflection sheets or asks them to share their thinking at that point with a colleague during class, providing an opportunity for dialogue. These daily feedback activities provide opportunities to understand the student’s experiences and learning in relation to course goals and to act upon their written comments and be explicit about any revisions that are made based on their feedback.

Feedback on a Project or Presentation. Feedback on a project or presentation was a course routine. Teacher leaders helped design and apply a rubric, which delineated the criteria for accomplishment on their end-of-course performance task. Teacher leaders utilized the rubric for providing peer feedback as they gave and received descriptive written comments. Each person had time to analyze the feedback and it was used as evidence in his or her final write up for the performance based task. Teacher leaders cited this process as very useful for making revisions to their projects based on peer feedback aligned to the rubric and learning goals before submitting their final work.

Feedback Based on Protocols. Feedback based on protocols was a strategy to provide guidance on effective math lessons. Teacher leaders and mathematicians studied the *Thinking Through the Lesson Protocol* (Smith & Bill, 2004) as a resource for designing and implementing effective math lessons. A mathematician planned a lesson with the protocol in mind. Teacher leaders experienced the math lesson in class and then provided written descriptive feedback to the mathematicians based on the *Thinking Through the Lesson Protocol*. The mathematician read, reflected on, and shared with the teacher leaders what they had learned through this process. This process had an impact on subsequent math lessons in the coursework. Specifically, it influenced the learning targets and summary aspects of the math lessons.

Lesson Study. Feedback from peers, mathematicians, and math educators was used in the formal process of Lesson Study. The Lesson Study cycle included shared lesson design, agreed-upon lesson implementation, and reflection on the lesson and students' learning. Feedback acknowledged the teaching process toward meeting lesson goals and student outcomes and provided guidance for enacting lessons at high levels of cognitive

demand. Peers giving and receiving feedback about successes and improvements of lesson enactment allowed for clear, nonjudgmental communication in a trusting, respectful learning climate. Because the lesson was collaboratively designed, the focus of feedback was on instructional strategies, cognitive demand of math tasks (Smith & Stein, 1998), uses of specific models or representations, or how language and interactions supported or limited students' learning. The feedback process was structured during the debriefing session following the lesson. It was used to guide the next iteration and revisions of the math lesson. The feedback was the central goal of informing the next steps for redesigning and teaching the research lesson based on what students in the classroom understood or what additional opportunities for learning were needed.

Mathematics Coursework

In each institute course, participants were given math tasks and asked to write about their solutions. Initially, the four instructors reading math papers rotated whose papers they read, controlling for variability of instructors' rating standards. After a couple of semesters, it seemed clear that getting written feedback from multiple instructors was not as much of a benefit as had been expected, and it did not facilitate tracking students' progress. Rotating papers may have also hindered developing trust between the participant and the instructor, which led to participants not talking to instructors in order to get clarity on the feedback despite frequent encouragement to do so. Noting this unintended consequence, we then moved to having each participant's papers read by the same instructor for an entire semester. Within this way of organizing the reading of course papers, it became easier for us to push a consistent group of students on developing the

ability to convey reasoning and improve communication of their thoughts. The effect was that the participant's writing became more focused.

To give an example, one participant had been having considerable difficulty in conveying his thinking. We did not give him very useful feedback early on, in part because we did not realize the extent of his confusion on some mathematical topics. By reading his papers only once in a while, it was hard for each instructor to get a clear picture of this student's understanding. Only when one instructor read his papers for an entire semester were we able to give him helpful feedback that allowed him to improve in his ability to explain his reasoning from one assignment to the next. The participant was not clear on several mathematical ideas and had difficulty in putting his ideas on paper. The instructor first focused on correcting the expression of mathematical ideas and then moved on to working with the participant on getting the ideas written clearly. By grading the participant over a full semester, the instructor was able to give increasingly detailed comments, as the participant understood more deeply both the mathematical ideas and how he was describing them in writing. The instructor could also see how the participant's ability to write a coherent introduction and conclusion evolved over time. As the participant got consistent, detailed feedback from one instructor for a semester, his papers improved considerably.

Another change was to incorporate peer feedback. When we began this, we organized the participants into feedback teams and asked them to read drafts of each other's papers and provide feedback. We did not provide much structure to how they should give feedback. After doing this for a couple semesters, we saw that their feedback was more along the lines of cheerleading. For example, participants were giving each other

comments such as “way to go” and “I wish my paper was as good as yours” but not giving descriptive feedback about the mathematics. The participants commented that they were not getting much out of this process. Thereafter, we changed to a structured peer feedback mechanism. For each paper, we posed two or three focus questions to be addressed when someone read a paper and gave feedback. For example, we had participants address whether the mathematical point of the paper was made and whether it was made clearly. Having participants address these questions gave them specific ideas for giving useful feedback. Participants found the new format to be much more useful for revising their writing. In particular, they saw that they could give one another constructive feedback without being critical.

Individual Teacher Leaders Comments on Feedback

The selected written comments made by teacher’s leaders listed below provide insights into their thoughts about feedback within the LIFT courses or their own K-12 classrooms. Notice how the teacher leaders are beginning to understand how to utilize feedback in their own classrooms or they relate to feedback in support of their own learning within the LIFT courses.

- We get feedback in class via peers and from the LIFT instructors (both formally and informally along the way- like with our action research projects). I do something similar in my class through homework, in class feedback, and through one on one interaction.
- I use feedback in my classroom in the same manner that the LIFT facilitators use with use. For example a self-reflection with rationale.

- Through peer editing I had the opportunity to see someone else's perspective. I also got ideas on what I needed to change in my work. This happened through peer editing and the school support team.
- Feedback can be in the form of questioning. The questioning of my thinking and the questioning of my action research project really made me examine my own practices.
- The LIFT feedback processes are developing and refining our understanding of how to learn. I find that as we continue to provide and receive feedback, we get more explicit and focused thinking and open doors for alternative considerations or perspectives ...it both clarifies and stimulates thinking.
- In LIFT, I use feedback to reflect on my own understanding and communication to improve my work. At work- as an educator I offer questions and comments to promote my student's thinking and understanding. I try to be timely, the more immediate and focused the feedback the more impact on learning.
- When we give feedback to our instructors, it is very evident they read and reflect on it and make needed changes to instruction. I try to follow this in my practice because it provides evidence to students that their needs and thoughts are being considered. The feedback process is a dialogue and includes all of us as learners.

Feedback: Our Learning

It takes trust, time, ongoing conversations, and opportunities to develop a shared learning culture. LIFT participants know that their ideas and thoughts are valued. Formal and informal feedback is incorporated in both the instructional and leadership components of MC²-LIFT. Through feedback, adjustments are made in lessons, assignments, and

courses. We have learned that when we solicit feedback from LIFT teacher leaders, we must take explicit action and respond in a timely manner in ways that support the participants' learning.

The innovative processes and structures for feedback ensure opportunities for collaboration, input, and continuous deliberation in order to study and learn in mathematics classrooms at the university and in schools. In many schools and classrooms the general analysis of school data does not impact individual student's thinking and does not advance their learning. Assessment data from a variety of sources needs to get to the level where it guides students' opportunities to learn. Students themselves should understand the role of assessment in learning and actively contribute to a generative assessment process. Effective teaching requires ongoing assessments that provide evidence of students' understanding and a collaborative process for continued learning.

In the LIFT project, teacher leaders' voices are essential in designing the academic experiences and building a culture focused on collective responsibility for learning. Through this process, teacher leaders understand that their ideas matter. We engage in a descriptive feedback process that has the potential to accelerate movement towards shared learning goals. The teacher leaders in the first cohort have provided feedback for the LIFT research project that stimulated revisions to strengthen the courses and the program for the second cohort.

We are continuing to think about feedback as an integral aspect of formative assessment to bridge instruction and lead to robust learning. We began with a focus on the courses but are expanding to other project domains. Perhaps, feedback loops could be strategically planned in advance or built into the project through teaching experiments and

design-based research (Design-Based Research Collective, 2003; Lesh & Sriraman, 2010) in LIFT. We are also curious about relationships of power and identity in socially constructed learning environments, the dynamics of hierarchies or status in classrooms, the role of grading, and how teacher leaders and instructors collaboratively engage in assessment for learning. The LIFT research project will deepen the study of feedback as an assessment process in both the LIFT coursework and the K-12 classrooms of mathematics teacher leaders to better understand how to support mathematics learning.

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Teacher Learning in Lesson Study¹

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Abstract: This article documents teacher learning through participation in lesson study, a form of professional development that originated in Japan and is currently practiced widely in the US. Specifically, the paper shows how teachers in three different lesson study teams 1) expanded their mathematical content knowledge, 2) grew more skillful at eliciting and analyzing student thinking, 3) became more curious about mathematics and about student thinking, 4) emphasized students' autonomous problem-solving, and 5) increasingly used multiple representations for solving mathematics problems. These outcomes were common across three lesson study teams, despite significant differences among the teams' composition, leadership, and content foci.

Keywords: Professional development; Teacher learning; Mathematics education; Lesson study; Mathematics instruction

In this article we report on some outcomes of lesson study as part of a professional development effort to improve mathematics teaching and learning in a large, exurban, diverse elementary and middle school district. In lesson study, a group of teachers identifies a problem from practice on which they would like to make progress in their teaching. Over an extended period of time—several months to a year—the teachers study

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the topic as well as students' perceptions of it, and plan a lesson to address this topic. They bring in other professionals as needed during this process. One member of the group then teaches the lesson while the others observe and record student actions and reactions during the lesson. The group reflects afterwards on the design and teaching of the lesson, its outcomes for student learning, and implications for student learning more generally. The cycle repeats, building teachers' mathematical content knowledge and their shared views of pedagogy simultaneously and over time (Lewis, 2002a; Lewis, Perry & Murata, 2006; Stigler & Hiebert, 1999).

This article documents⁴ the experience of three school-based lesson study teams of teachers in their efforts to address the development of teachers' mathematical content knowledge, pedagogical skill, and leadership capacity through a combination of professional development activities, with an emphasis on the learning that occurs through the lesson study process. The "Noether" Project,⁵ an NSF-funded Math and Science Partnership program, involves 60 teachers from 16 schools (with teams varying in size

⁴ We should note that the descriptions below are mainly extracted from field notes with some video transcriptions. When quotations are extracted from field notes, they may be incomplete in some cases; however, we have endeavored to convey the intent of the message accurately in all instances; the notes were taken by the facilitators as they were participating in the discussions, and were not taken for research purposes at the time.

⁵ All names in this paper are pseudonyms, and we take our Project and school names from some of our favorite mathematicians. Amalie Emmy Noether (1882-1935), in the words of Einstein: "In the judgment of the most competent living mathematicians, Fraulein Noether was the most significant creative mathematical genius thus far produced since the higher education of women began."

<http://www.awmmath.org/noetherbrochure/AboutNoether.html>

Euclid of Alexandria (about 325 BCE-about 265 BCE) was a mathematician and author of the second most printed book in the world, *The Elements*. The contents include plane and spatial geometry, ratios, proportions, and elementary number theory. <http://aleph0.clarku.edu/~djoyce/java/elements/elements.html>

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from four to fourteen, and some teams drawing teachers from multiple sites) who study mathematics and pedagogy in multiple formats. Each year, teachers participate in an intensive two-week summer institute, academic year monthly seminars, self-facilitated monthly collaboration time, and lesson study. All teams meet for ten full days of lesson study during the academic year. The program began with 45 participating teachers and subsequently expanded to include 60. The district is a high needs district with 89% of students eligible for free and reduced lunch. 88% of students are Hispanic, with large numbers of English learners (51%), primarily Spanish-speaking, and many parents have limited academic backgrounds. In each middle school at most one or two teachers have a math credential that qualifies them to teach algebra or single subject mathematics.

The theory of action in the Noether Project is that teachers who participate in lesson study will become increasingly knowledgeable in mathematics and more skillful in teaching mathematics, and this expanded teacher learning will lead to improved student learning. This logic of expected improvement follows recent research (Dudley, 2012) indicating that schools where lesson study is conducted show higher levels of student learning in mathematics relative to comparable schools. By “expanded teacher learning,” we mean teachers’ increased content knowledge, confidence in mathematical skills and abilities to help children learn mathematics, and a growing expertise in teaching mathematics. The Project measures increases in student mathematics learning and proficiency, using standardized test scores as well as outcomes on alternative assessments such as the STAR assessments (California Department of Education, 2010). The alternative assessments gauge student performance on the Standards for Mathematical Practice (Common Core State Standards Initiative, 2010), which represent for the facilitators and instructors a close

approximation of the kinds of student learning in mathematics towards which the Noether Project is working. These Standards for Mathematical Practice figure heavily in teachers' work during the Summer Institutes and academic year seminars, and often constitute learning objectives chosen by Project teams in lesson study.

The facilitators and instructors of this project include mathematicians, mathematics and science educators, and an elementary educator with little formal mathematics background. Administrators play a significant role in support of teachers' participation. At one school, the principal regularly participates in the research lessons and drops in for other phases of lesson study. At another school site the principal and vice principal have observed one research lesson, and at the other, the principal and vice principal made a brief appearance at one lesson study meeting, in addition to maintaining correspondence throughout the year with team members and the lesson study facilitator. Two to three times per year, the project convenes meetings of administrators to discuss project goals and activities, and for teachers to demonstrate the types of work they are doing in the project. In all sites, the principals support teachers' release from the classroom for ten days per year for lesson study sessions.

We have chosen to show lesson study through the prism of three different school-based teams who all participate in the Noether Project within one school district. Throughout the project, teachers have commented repeatedly on the extent to which they are deepening their understanding of mathematics concepts and how students conceptualize mathematics, and report that as a consequence they are enhancing their ability to teach effectively. As we reflected on the work being done by the teachers of these lesson study teams, it became clear that each of the teams' stories exemplifies and

highlights a particular aspect of teacher learning.

The characters of the teams are very different, both by their size and grade levels and by the backgrounds of the teachers involved; and the facilitators have different mathematical and educational backgrounds and work experience. However, the facilitators have in common core goals and beliefs that are central to the lesson study process. The project facilitators share a set of strong convictions regarding mathematics teaching. They believe that key aspects of high quality mathematics teaching include deep content knowledge (Ball & Bass, 2000), and that strong links between conceptual understanding and procedural fluency are essential to learning mathematics (Kilpatrick, Swafford & Findell, Eds., 2001). They also believe that frequent and varied use of formative assessment is central to good mathematics instruction (Black & William, 1998) and that thoughtful listening to students' mathematical ideas is fundamental (Carpenter, Fennema, Franke & Empson, 1999). Furthermore, they are committed to fostering teachers' autonomy regarding their own learning and teaching, and see this as a requirement of good professional development (Little, 2000). The facilitators also share a belief in nurturing students' desire to learn in order to yield long-term improvements (Kilpatrick, Swafford & Findell, Eds., 2001). All three teams follow a fairly standard form of lesson study. By this we mean that participation by teachers is voluntary; that teachers set the goals and topic of work for the lesson study cycle, and that the team studies the topic using curriculum and other supplementary materials extensively to plan a research lesson. All team members observe the research lesson in person and participate in a face-to-face debriefing session afterwards. Outside experts are included at several stages of the cycle. A wide variety of professional development practices are referred to as "lesson study" in the US, but in our

project we have hewed fairly close to the canonical model as implemented in Japan (Lewis, 2002a), with one exception worth noting: Since the US culture of teaching does not normally accommodate a long period of study and planning as is done in Japan, the teams began with shorter cycles of lesson study. However, the teams in the project have been expanding the amount of time for curriculum investigation and planning for each lesson, and consequently reducing the number of research lessons per year. As the teachers and administrators become more comfortable with this extended study and planning time, the time frame more closely approximates the standard Japanese model. In our conclusion we say more about the fundamental ways in which we have followed the Japanese model from the inception of the project and why this matters.

The teams studied mathematical content as well as mathematical practices. One team focused on multiplicative structure for students in the middle grades. Two teams focused on developing student problem solving skills using contextual problems, also in the middle grade. The third team focused on students' argumentation skills in mathematics. Each team participated in extended study of the content area across almost two years, drawing on summer learning institutes and monthly seminars, monthly lesson study sessions, the independent reading of articles and books, and the presence of content and pedagogy experts. The research lessons developed in this process reflect teachers' progress in the areas of mathematics content, mathematical practices, pedagogical skill, and dispositions to teach and do mathematics.

Team "Euclid" is composed of six teachers from two school sites: Two fourth grade teachers and one fourth-fifth grade combination teacher from one site, and a fourth, fifth and second grade teacher from another site. One of the teachers is new to the team and the

project this year, and the second grade teacher was reassigned from his original grade (the project as a whole is geared to fourth to eighth grade teachers). This team's facilitator was also new to the team in Year Two, and is a former secondary mathematics teacher with a master's degree in mathematics education.

Team "Bass" is composed of fourteen teachers of grades four through six, including one special education teacher and the school's instructional coach (who had started the program as a teacher.) Of the fourteen teachers, two joined the project in its second year. The team's facilitator is a professor of mathematics education.

Team "Cohen" includes five teachers, three fifth grade teachers and two teachers of sixth grade gifted and talented students. Four of the teachers work at one school while another joins from a nearby site. Four of the five teachers have been members of this team from its inception, and were joined in the second year by a teacher who had participated on another team during Year One. This team has two facilitators, a mathematics professor and an education professor.

Looking across the three teams, different affordances of lesson study coalesce. Across the three teams, we see themes emerging regarding *why* teachers learn from lesson study, *what* they learn, and *how* teachers learn in the context of lesson study. In Team Euclid, teachers' understanding of the crucial role their own learning, and the value of listening to student thinking, became especially salient. The Team Euclid story told below describes the process through which the team learned to expose student thinking and respond to it, and what motivates the teachers to continue to learn. Theirs is the story of *why* teachers learn.

Team Bass is a large team, and has wide variation in teacher knowledge and

approach to learning. By examining teachers' comments following research lessons, we discern *what* teachers learn in lesson study.

Team Cohen's story illustrates *how* teachers learn. The team has worked to predict what students might think, including any misconceptions, and to design assessments and lessons so that they will highlight anticipated student responses. Teachers' struggles with the task of guiding students to discover mathematical ideas were a key factor leading to their own personal mathematical growth.

Team Euclid

Team Euclid is a case where we can see the development of teachers' internal motivations to learn mathematics and, at the same time, how teachers deepened their ability to understand students' interactions with mathematics. Team Euclid was driven to understand what instruction might look like as guided by the Standards for Mathematical Practice of the Common Core State Standards (Common Core State Standards Initiative, 2010). As teachers learned to reflect more deeply on students' understandings, they gained a deeper appreciation for their own need for content knowledge and meaningful understanding of mathematics. Below we describe a series of turning points in which teachers increased their own understanding and the desire to learn even more.

The Summer Institute introduced teachers to the Common Core State Standards (Common Core State Standards Initiative, 2010), in particular the Standards for Mathematical Practice. In light of these Standards, the team started by asking the following questions:

- What content do we want the students to understand?
- How will we know that they understand the content?

- How do we teach students perseverance, reasoning, modeling, structure and conjecturing when there is so much content to teach? Is there enough time?

First Attempt

The content focus for the first research lesson was writing and evaluating expressions. Teachers were concerned about students' ability to find entry points into contextual math problems. They were interested in incorporating some exploratory aspects into the lesson, and providing a variety of manipulatives for the students to use while solving the problems. Fifth grade students were presented with the following problem:

Sonya spent \$7 and \$9 at Target. She gave the cashier a \$20 bill. Write and simplify an expression to show the change that Sonya should receive.⁶

Students were asked to work with a partner to write and simplify an expression, and be prepared to explain their thinking to the class. When called upon, students would come to the front of the class, show what they did with their selected manipulatives, and briefly explain how they solved the problem. This was the team's first attempt at having students verbalize their solutions, in a classroom where student explanations were not commonly elicited. Students' comments proceeded as follows:

Pair 1: "We knew we had \$20, and 20 minus 7 is 13 and 13 minus 9 is 4."

Pair 2: "We thought it was easier to add 7 and 9 and get 16. And 20 minus 16 is 4."

Both of the above examples indicate a correct solution, as well as two different, but correct, approaches for solving the problem. So, at first glance, it seemed as though the

⁶ This problem was based on one in the textbook *California Math*, Houghton Mifflin, 2009, p.113.

students understood the lesson -- they got *the right answer*. However, further examination of student work indicated that students were still not writing expressions correctly, which was the goal of the lesson. Students were able to compute the arithmetic either mentally or using manipulatives, but while the arithmetic process was correct, the students did not demonstrate an understanding of the underlying structure. For instance, in discussing the lesson afterwards, one teacher reported that one pair of students had the problem written $16 - 20$, and when a student was asked if there was a difference between her expression and that on the board ($20 - 16$), she didn't notice a difference.

Teachers learned that a student *computing a right answer* does not necessarily indicate that the student has an accurate *understanding* of the target content. This prompted the team to return to the content themselves and discuss more deeply the questions that would help them better understand the students' understanding. Teachers considered the responses that were given and what else they could have done to help students deepen their understanding of this concept. With some guidance from the facilitator, the team discussed the missed opportunity for comparing the two strategies written as expressions. For instance, the first pair of students solved the problem by illustrating the expression $20 - 7 - 9$. They then subtracted from left to right, resulting in the answer of \$4. The second pair of students illustrated the expression $20 - (7 + 9)$, though their expression was written as $20 - 16$. Teachers assumed that the students understood the equivalence of these expressions. They concluded that to develop understanding of equivalence of expressions and order of operations, they might have asked questions such as: How did you get to 16 and how is that represented symbolically? How did you show that you added $7+9$ first? How is $20-(7+9)$ different from $20-7+9$?

The Turning Point

Teachers had come to the realization that facilitating a mathematical discussion is a complex process and this spurred them to further learning. Based on their conclusions from the first cycle of lesson study, they began the next cycle with a new question: What are the kinds of questions that I need to ask students to facilitate a productive mathematical discussion?

The team began to utilize more curricular resources in their study and planning. These resources included *Young Mathematicians at Work Constructing Algebra* by Fosnot and Jacob (2010) and *Classroom Discussions: Using Math Talk to Help Students Learn* by Chapin, O'Connor and Anderson (2009). Additionally, this team embraced the use of Mathematics Assessment Resource Service (MARS) assessment tasks and often referred to websites, such as <http://insidemathematics.org>, for such tasks. With an increased focus on questioning and discourse, some teachers began to work on creating a collaborative environment in their classrooms and asking probing questions more regularly. Developing questions was a main focus in the planning of the second research lesson. Teachers wanted to structure opportunities for students to demonstrate their thinking and verbalize their processing while continuing to work on developing problem solving skills. In the next research lesson, the team posed the following question to the class:

The Rodriguez family decided to make tamales and give them to their friends and neighbors as Christmas presents. They made beef, chicken and cheese tamales. They made four dozen tamales and they are going to wrap them in bags of 5. How many bags do they need?

Expected student sticking points included understanding of the term “dozen,” inclusion of extraneous information, and consideration of what to do with the three leftover tamales. As the students worked on the problem independently, the teacher noticed two different solutions arising. Some students had illustrated nine bags of five tamales with three left over, while others had illustrated five bags of nine tamales with three leftover. The teacher determined that this would be an excellent starting point for a class discussion and had students present both of the solutions in order to use comparison as the basis for the discussion, expecting that the students would come to a consensus about the solution. The teacher then used newly developed facilitation skills to re-voice each of the arguments and she gave the students the power to navigate their own learning by asking, “How do *you* decide?” After much debate, a new misunderstanding was revealed: one student read the question to mean that the family had five bags, and did they need them *all*?

The Euclid Team teachers found this lesson eye-opening. Through the teacher’s perseverance in patiently questioning students without providing answers, she demonstrated to the students her interest in their thinking. Teachers realized that without the extended discussion during class, this student misunderstanding—whether linguistic or mathematical—would not have surfaced and thus would never have been identified or addressed.

Continued reflection at the next meeting revealed even more. Teachers’ initial thoughts were that, linguistically, the English learners had difficulty with the translation of “bags of 5.” However, there were some cultural aspects to the question the team had not considered that may have led to the misread of the problem (e.g., tamales often come packaged in dozens). Additionally, “bags of five” in Spanish sounds much like “five bags,”

perhaps another source of confusion in this primarily English learner class. Also, the problem that was created was a measurement type of division problem, as opposed to a partitive type of division problem that is typically more familiar to students.

Teachers came to a deeper understanding both of the students' thinking and of the importance of ongoing reflection on their practice. As a consequence, the teachers' focus in lesson study shifted. From a global concern about students' lack of understanding, the team progressed to a desire to learn about specific aspects of students' understanding and misunderstanding through purposeful questioning.

Transforming thinking

In the third cycle of lesson study, teachers' perspectives and planning questions changed to:

- What can we learn about student understanding of content before we plan the lesson?
- How can we get our students to explain their thinking more clearly? What questions do we ask for this purpose?

To begin answering these questions, the team for the first time developed a pre-assessment to administer to small groups of students at different grade levels. They chose a broad focus--number lines-- and wanted to investigate current student conceptions of number lines; in particular whether students could:

- Identify intervals on a number line
- Construct a number line on their own
- Correctly plot and label rational numbers on a number line given endpoints and intervals

If students could do these things, could they explain how they did them? If not, at what grade level did specific misunderstandings occur?

The team members were in agreement that they wanted to know not only whether students could solve the problems correctly, but also how the students thought about the problems. They determined a very specific process of administering the pre-assessment in order to obtain as much useful data as possible. Students would be brought in in small groups according to grade level; one teacher would lead the students through the pre-assessment, answering questions about the items on the instrument to ensure they were learning about student understanding of the math, as opposed to difficulties with language. A different teacher worked with each age group of students. Teachers planned to pose the following questions to students after they completed the pre-assessment, in order to learn more about how the students thought about the problem:

- Why did you write ____? How did you get that answer?
- Have you seen a number line like this before? Where? What do you know about number lines?
- Which problem was the hardest for you? The easiest? Why?

The assessing teacher would then ask the observers if there were any additional questions that they wanted to ask. Finally, they videotaped each assessment in order to reference it later for clarification if needed.

In the first assessment (of third grade students), it became clear that students did not have a deep understanding of the meaning of the points on the number line. For example, students were asked, “How many units are there between each point?” on the number line below:

One student responded “300.” The teacher asked, “So, how did you decide it was 300?” And the student responded, “I put the 100 and the 200 together and got 300.” Teachers conjectured that the confusion might have stemmed partly from the wording of the problem.

In the second assessment (of fourth grade students), the assessing teacher asked if there were any words that needed to be clarified. This time, one student asked about “units.” The teacher (mistakenly) indicated that “unit” is the space between the markings on a number line. On the assessment, two students wrote, “There are two units between each point.” When asked “Why did you decide to write that?” one student responded, “There’s only two spaces on the line,” thus indicating that the description of the unit had led to a misunderstanding of unit for these students. The team realized there was a need to revise their own idea of what “unit” means, and then decided that the teacher should introduce units by talking about measurement.

Before the last assessment (of fifth grade students), this exchange took place:

Teacher: Let’s look at the question together. You read it. Are there any words that don’t make sense?

Student: Units.

Teacher: Think about how we measure. I could say how many steps to get to the desk, steps can be units; or arm lengths, arm lengths to the pad, how many arm lengths.

Think about a ruler, can you think of how we measure with a ruler?

Student: Inches, centimeters, millimeters

Based on the results of this last assessment, this description worked much better. Four out of the five students wrote that there were 100 units between each pair of points. One student drew ten tick marks between each pair of points and wrote, “I think this way because I am going by tens,” indicating that he understood that ten 10’s make 100, yet not understanding how to label it on a number line (his tenth tick mark should have fallen on the second point).

In reflecting on students’ answers in the pre-assessments, teachers concluded that student understanding of intervals on the number line grew across the three grade levels (though the improvement of the question posed was probably also a factor), perhaps in part because number lines are not included in the third grade curriculum. Other concerns emerged regarding students’ ability to construct a number line and to plot rational numbers accurately on a number line. The team then conjectured that students’ difficulty with placing rational numbers on the number line could either be a reflection of difficulty with rational numbers, or with understanding the ordering of numbers on a number line. The team then decided to experiment with teaching these concepts using Cuisenaire rods as instructional tools.

Although the team’s earlier work was strictly focused on producing and teaching lessons, the team’s work evolved towards investing significant time in deepening their understanding of students’ thinking about content and consequently teachers’ own understanding of that content, before delving into lesson planning.

Team Euclid Conclusion

Over the course of this year, the team has shifted what it means by “need for understanding.” The “need for understanding” no longer means only, “What do our students need to understand?” Over the course of this second year, the question has become, *“What do **we** need to understand about the mathematics and about our students to be able to progress to a desired level of understanding?”*

Team Cohen

Study of the Cohen team is based on analysis of facilitator field notes taken during planning sessions in addition to scripting notes of teacher and student talk during research lessons. This case study shows *how* the Cohen team members learned in lesson study. Through their experience with research lessons, teachers came to realize that students did not fully understand the concepts their lessons were designed to teach. Based on analyses of student (mis)understandings, teachers designed a series of mini-lessons on the same content. Through careful listening to students’ explanations, they ultimately reconceived the content for themselves in a more meaningful way and revised their approach to instruction of this content.

Teacher Content Learning Through Anticipating Student Behavior

In Year One, Cohen Team teachers studied student learning by first looking at benchmark test results and then observing student performance during research lessons. By Year Two, teachers expanded their study of student learning to include both pre-assessments and piloting of draft research lessons with small groups of students. The teachers agreed to focus on teaching through mathematical investigations as part of their annual research goal, and this appears to have altered their approach to pre-assessment.

Where formerly they focused only on skills, they came to also assess students' ability to approach problems and solve them. For example, the Cohen Team decided their first investigative pre-assessment would ask a group of students to respond to the following prompt:

Polly works in a zoo and needs to build pens where animals can live and be safe. The walls of the pens are made out of cubes that are connected together. Polly has 40 cubes and wants to make the largest pen possible, so the animals can move around freely but not get loose. Build the largest area using all 40 cubes. Use the grid paper to show the shape of the pen. Explain to Polly why you believe your pen is the largest one that can be made.

Prior to implementing the assessment, teachers tried to predict how students might respond to the prompt. No longer were they simply thinking about teaching the area formula with already-created shapes, but they were considering how students might design shapes and explore ways to maximize area. Additionally, teachers realized that students often confuse area and perimeter and hypothesized that students would count the centimeter cubes as a part of a shape's area rather than see the cubes as the "fence." They thought students would most likely not plan for dimensions of a shape but would randomly place cubes to see what they could create. Thus, they expected to see some students struggle to use all the required cubes or run out of cubes as they created their shapes. Teachers also predicted that some students would create irregular shapes. Teachers discussed whether they should address these possibilities with students at the start of the lesson but decided to allow students the opportunity to investigate the prompt without any direct instruction in the hopes that students might be able to develop their own insights

into the difference between area and perimeter and how shape might relate to area.

The conversation about what students might do prompted teachers to frame teacher observations during the research lesson. Since the teachers had engaged in discussions about their students' potential interactions with the content, they were able to consider content more specifically as they planned for their observation of student talk and action. Observing teachers were not just going to watch for correct answers and errors. They were going to watch for behaviors that portrayed specific conceptual understandings. They decided that observers would watch to see whether students traced the outside or inside of the cubes that outlined their pens. One observer would attempt to track different approaches used such as including cubes in the total area, narrow vs. wide shapes, and irregular shapes. This information would guide the selection of students for sharing in the class discussion so that different approaches might be viewed and analyzed by the class as a whole.

These considerations are a change from Year One in which observers were assigned to watch students with varied characteristics such as language needs or behavior challenges. Additionally, choosing students to share during the whole-class discussion based upon their approach to the problem is also a change for these teachers as many of them reported typically drawing name cards at random to select students to share, irrespective of the content of student's mathematical work.

A pre-assessment using the "Polly" problem was implemented with five sixth graders. Individual students first worked with 20 cubes to explore the problem. Prior to using 40 cubes with a partner, students were to predict and draw the shape they thought would provide the largest space for elephants to live. Partners then drew as many shapes

as they could, using the cubes and grid paper. Students were able to see one another's drawings and discuss findings.

Each and every student began working on the problem by including the cubes within their area totals. Only one student eventually recognized that the cubes' inclusion in the count made for inaccurate areas. Additionally, the teachers were surprised to find that students had different concepts of "largest." For example, two students said their pen was "largest" because it had included a bend so that each elephant had a private area. Thus teachers learned that a context can get in the way of mathematical understanding, and every aspect of the context needs to be considered carefully in advance.

Student Outcomes as a Basis for Teacher Content Learning

Team Cohen teachers recognized that students, as predicted, did struggle with the concepts of area and perimeter. While it might have seemed easier to address misconceptions directly with students, the teachers wished to maintain an investigative stance in instruction rather than returning to a direct instruction approach. Still, they expressed frustration about how to help struggling students without simply telling them what to do. They wanted to have students reach conclusions about the essence of area rather than hear students repeat back a formula or a definition. As part of this planning process, a visiting math professor taught the group a mini-lesson on the area model for multiplication of fractions, which helped the teachers consider how students might record findings and look for patterns as a way to reflect on learning throughout and after an investigation.

Teachers liked the idea of having students record their findings, look for patterns, and make connections. They thought they might try this approach, and after a few

iterations and related pre-assessments, the team developed a lesson in which students would use Geoboards to create rectangles of assigned sizes. The teacher would not indicate whether students were right or wrong but would record areas and the rectangle dimensions on chart paper. Finally, students would be asked to consider a problem in which a shape with an area of 3 is viewed by a fictional student (Paul), who says its area is actually 8 square units (counting points rather than spaces). Students were to discuss that response, how they thought Paul had reached the conclusion, and what they would say to help Paul see the area in a different way. To assess student understanding at the end of the lesson a short assessment question was developed that asked students to draw as many six-square-unit rectangles as they could, record the area, length and width of each in a table, and describe any pattern they saw in their table.

The Cohen Team teachers were willing to give time for students to develop connections between dimensions and area without direct instruction from the teacher. In fact, the teachers noted that students had already had direct instruction on the formula for area of a rectangle during fourth grade. Realizing that teacher “telling” did not seem to guarantee student understanding, they wanted students to construct their own view of how the length and width of a rectangle connects to its area, and through this gain a better understanding of the concept of area.

Learning about the “Big Ideas” of Content

In order to learn more about how students think about area and perimeter, teachers decided to ask four sixth-grade students to teach four fifth-grade students about the two concepts. The sixth graders were told that the teachers had been struggling with ways to help students understand area and perimeter of shapes. As a way for the teachers to

consider how students learn about area, they wanted the students to think about how they might work with a younger student who didn't know multiplication but wanted to understand area.

The team was intrigued to find that three of the four students independently came up with approaches that started from the whole shape and progressed to the unit rather than moving from the unit to the whole as the teachers had taught their classes. For example, Roberto started by creating three congruent rectangles. He held up the first, which had no grid, and said, "Here is a shape. If we want to know the size of a shape, how might we go about finding out the size? It's not like measuring just a line. We need something else. We might want to divide it into equal spaces (units) and count them." He held up a congruent rectangle on which he'd traced square units from the graph paper. He then showed a third congruent rectangle that he'd cut up into the square units, sliding them apart and then pushing them back together. Roberto's emphasis was on measurement and the need for a way to determine size of spaces, and he implicitly utilized the concept of conservation of area. He emphasized *why* we need a means for measuring space since it's different from measuring a line. He then moved to a Geoboard in which he'd created a rectangle and used different colors of tiles for each row. He planned to have his student find the area of the rectangle by counting tiles.

When actually teaching, Roberto's student had a lot of difficulty. Roberto responded by taking out a row of tiles at a time, trying to deal with area of a smaller region, spreading out the row for the count and then putting the tiles back together and asking his student if the area was still the same (now explicitly checking on conservation of area!). The student truly grappled with the ideas throughout the lesson, and following the lesson he was able

to determine the area of a rectangle.

Esteban created an irregular shape because he wanted to emphasize that any shape can be measured, even one that looks like “a scary sixth grade shape.” He worked with his student to count whole square units and then combine partial square units, documenting the adding of units as they worked and also shading in units as they were counted to be sure not to double count. He commented, “See, when you cut up space into square units, you can count the units easily, and even a strange shape isn’t scary.” When asked how he might help students who are confused by area and perimeter, Esteban quickly defined the irregular shape as the footprint of their school. He cut out a “Fred” character and placed Fred in the school and outside of the school to help his student determine where the area of the school was. He had Fred walk the perimeter of the school. He made the observation that the area--the inside space of the school--is measured in squares while the perimeter is measured in length.

George tried to demonstrate the difference between area and perimeter. He showed a picture of a rectangle outlined in black with a green interior. He made a rectangle of tiles, then placed the tiles on graph paper and outlined them. He commented that the tracing is the perimeter and what’s inside is the area. He then made another rectangle with tiles and stood up tiles around the edge as though they were walls (perimeter). He used the example of carpet or grass and walls and fence as contexts for area and perimeter.

After this “teaching” event, the fifth grade students debriefed the experience with the Cohen Team teachers, followed by a separate debriefing with the sixth grade students. Based upon student recommendations, teachers felt that they would make major changes in their future approach to the teaching of area and perimeter. They stated that they would

teach the concepts together rather than separately, because they realized the connections between area and perimeter. They saw the linear dimensions of a rectangle stemming from the perimeter's linear measure, and they came to believe that students needed to compare and contrast them in order to differentiate between linear and square unit measures.

Team Cohen also discussed the way the student-teachers started with the “big picture” of measurement of space instead of simply defining a unit and moving into counting square units. As one teacher said, “I’ve probably been teaching the concept of area backwards—we always start with the unit of measurement and build on that. The kids today worked from the blank shape and had a focus on the fact that we’re trying to measure space before considering square units as a means for measuring space.”

Teachers also agreed that they’d have a context for area—one that relates more to students’ lives, such as their school building, rather than animal pens. They thought it was important to get kids to talk about where and when they’d interacted with a concept such as area in order to hear students’ present understandings. They also discussed the effects of putting restrictions on students’ use of formulaic language such as “length times width” in order to help students try to define a *concept* rather than rely on surface-level application of a formula.

Team Cohen Conclusion

These changes in the team’s approach to teaching most likely did not result solely from readings or discussions or observations of students. The process of lesson study appears to support risk-taking in implementing new approaches to teaching and learning by providing a collegial and safe environment. Team Cohen teachers used this process to focus both on the specifics of student learning and on the long-term effects of their

instruction on students' content understanding. In turn, this created an intrinsic need to know more about student thinking, and influenced the development of teachers' own content understanding.

Team Bass

As with Team Cohen and Team Euclid, teachers on Team Bass are working on understanding student thinking and, as a result, considering their own understandings of mathematical concepts. For our study of Team Bass, we analyzed the field notes taken during the discussion sessions following each research lesson. The post-lesson discussion session is a post-hoc analysis and discussion about the jointly conceived research lesson that all team members create and observe. Typically the teacher of the lesson speaks first, and team members endeavor to provide evidence with specific data they collected regarding any conclusions or observations they offer about the research lesson. We chose to study this phase of lesson study for Team Bass because teachers' comments in this activity offered a window onto what teachers were thinking about and processing, and their comments are sometimes summative in the sense that they make observations that span the team's efforts together from the beginning of the lesson study cycle through to this point.

Our analysis of the field notes in the post-research lesson discussion sessions was conducted by labeling categories of teachers' comments using each separate teacher turn as the unit of analysis. Each turn was given a label. Turns were mostly considered instances of some kind of thinking or offering; these kinds of thinking or offering were the labels used. So, for example, many teacher turns during this debriefing discussion concerned details about mathematical work that a specific child had done.

Using a version of Yin's (2009) cross-case study method, treating each post-discussion session as a separate case, we created word tables that described teachers' comments, and then worked across them. We sought to identify patterns of practice that emerged across the teachers' turns in this set where trends and patterns were noted and new labels assigned to clusters of related teacher actions. For example, one teacher commented early on in a debriefing, "Didn't Malena skip a step? She took 12 and divided it." This turn was labeled as "student problem-solving specifics" and was later subsumed into the category of "student work and student thinking."

We noticed five categories of teachers' comments during the post-lesson discussion sessions, and we offer these to indicate *what teachers are learning* during lesson study. These five overarching categories are:

- Teachers' instructional moves
- Student work and student thinking
- Understanding the math
- Big ideas about mathematics and learning
- About the lesson study process

We discuss three of these, and provide illustrative examples.

Teachers' Instructional Moves

Teachers in Team Bass offered frequent comments, or posed questions, about actual or possible instructional moves. Some of these were offered as repairs to the planning of the observed lesson, for example, "We might have moved to the whiteboard or a table in the center of the room to show the ways students modeled the problem." This is phrased as a suggestion for how this aspect of the lesson might have been conducted during the

research lesson, but it also represents a tinkering with instructional materials that will serve future lessons, and in this particular case underlines the importance of including all students in the presentation of ideas, and the team's emphasis on modeling. Team Bass had been working this year on modeling in a number of ways: the notion of *mathematical models*, that is, various representations of mathematical ideas; as well as the pedagogical form called *modeling*, where the teacher or a student provides an exemplar for other students to follow.

In other comments regarding this category of teacher learning, teachers' instructional moves, teachers are conducting thought experiments about instruction, playing with possibilities that are prompted by the lesson they observed and considering a range of alternatives and what those instructional alternatives might have generated. The team had been encouraging students to model problems with drawings and other materials, and they also wanted to see how they might best prepare students for the kind of word problems they encounter on standardized tests. The focus had been multiplicative structure, so they formulated the following problem:

We have 4 boxes of pencils. Each box has a dozen pencils in it. If 6 people share all of these pencils equally, how many pencils will each person receive?

After observing the research lesson where students worked on the problem, a succession of teachers' turns included a string of these:

Teacher 1: Would it have been different if we had had "12" instead of a dozen?

Teacher 2: Might we have just presented the 4×12 ?

Principal: Would there have been an advantage to use real pencil boxes?

These comments reveal a care with wording, weighing the use of numeric symbols, and a consideration of various representations that could be used in this problem.

Student Work and Student Thinking

About half of all teacher turns have to do with student work and student thinking. The Japanese, who originated the formal practice we call lesson study, say that lesson study “gives teachers eyes to see students” (Lewis, 2002b). The teacher turns in this category show how this transpires. During the research lesson, teachers are encouraged to collect data on individual students, and these data are shared readily at the debriefing sessions. Teachers share specifics of the mathematical work that individual students did during the lesson, and then often interpret the meaning of their work. Here is a typical comment of this category, on the same problem we discussed above.

Teacher: A girl immediately made one stack of 12 and was about to make another, but then made four stacks one at a time. I realized that you had to destroy the original representation to finish the problem.

The comments often contain highly specific details about what a particular student did, as in the case here, where the actual numbers and methods of problem solving are mentioned, and the sequence of the child’s work in solving the problem.

Notice, too, that in the next sentence, the teacher adds a comment about how watching this student solve the problem led her to realize something about the deployment of models in this problem. Thus, teachers move from specific understandings in the context of this particular problem, to realizations that might be relevant to other problems as well. Here the teacher offers an idea—that the construction of the mathematical model here had to be destroyed in order to finish solving the problem—that may be useful in work on

another problem. We anticipate that such thinking is accessible to teachers when they are alone in their classrooms and outside the framework of the research lesson.

Understanding the Math

While there are only a few teacher turns in this category, over the three post-lesson discussion sessions, teachers' expressions about understanding the mathematics in the lessons are significant. Specifically, teachers say that they did not fully understand the mathematics until they watched students work on the problems during the lesson, or participated in the teachers' analysis of student work during these debriefing sessions. Another research lesson was designed for students to work on the distributive property, and teachers devised the following problem for students:

At science camp, 17 students are doing an experiment, 12 students are taking a hike, and 10 students are in their cabins. There are twice as many students in the dining room as are doing an experiment, on a hike, and in their cabins put together. How many students are in the dining room?

Students were invited to solve the problem in two different ways, which in itself was an innovative practice for this team of teachers: the valuing of eliciting multiple approaches to solving a problem. It is also worth noting that the teachers developed this problem around using as a context the sixth grade camp experience that all students were about to embark upon together. This underscores the teachers' desire for students to use mathematics to describe and model their own experiences as a way of developing "productive disposition" (Kilpatrick, Swafford, & Findell, Eds., 2001) in mathematics.

In listening to students present their solutions to this problem and examining all students' written work during the post-lesson discussion session, a number of teachers

realized that they themselves were not entirely clear on what the distributive property means. It was through this discussion that teachers revealed that they expected to “see the distributive property in students’ solutions,” and by this they meant something resembling Elise’s work:

<p>first way</p> $4(17+10+12) \cdot 2 = 4$	<p>another way</p> $2(17 \cdot 2) + (12 \cdot 2) + (10 \cdot 2) = 4$
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In fact, in the discussion it became clear that teachers understood this precise representation—and only this one—to “*be* the distributive property,” that is, this exact form *is* the property. But they were not sure if Fernando’s work showed the distributive property:

experiment
17, 12 hike
10 cats

another way

$$\begin{array}{r}
 17 + 12 \\
 \downarrow \\
 39 + 39 = 39 \times 2 \\
 \downarrow \\
 78 = 78
 \end{array}$$

$$\begin{array}{r}
 10 \times 2 = 20 \\
 12 \times 2 = 24 \\
 17 \times 2 = 34 \\
 \hline
 78
 \end{array}$$

Even more puzzling was Aric’s work:

<p>first way</p> $ \begin{array}{r} 17 \\ + 12 \\ + 10 \\ \hline 39 \\ \times 2 \\ \hline 78 \end{array} $ <p>78 students</p>	<p>another way</p>
--	--

While these are correct representations of the problem that give correct answers, do they show the distributive property? At one point, just as the group is trying to analyze these examples of student work, one of the teachers says, “Can you use the distributive property for every problem? I don’t really understand the distributive property. The distributive property or the commutative property.” Of course, prior to the research lesson—and prior to this discussion—teachers did not question their understanding of these properties. It was only upon the team’s discussion of which student work constituted use of the distributive property that the teachers began to reconsider their own understanding of what this property really means. At that point the facilitator could see that teachers had been thinking of the distributive property as a formula for computation, rather than a consequence of the underlying structure of the real numbers.

Big Ideas about Mathematics and Learning

Occasionally, teachers offered comments that hint at broader philosophical orientations about mathematics and learning. For example, a teacher who taught one of the research lessons said, “The hardest part at the end was trying not to guide students towards the right answer. I kept having to remind myself to ask students if the representations fit the story problem.” Implicit in her comment is the team’s shared commitment to supporting students’ autonomous problem solving in math class, and her efforts to try to help students in a way that does not spoon-feed answers to them. This orientation towards teaching and learning mathematics is one that is shared, and new, for most members of Team Bass.

Team Bass Conclusion

The analysis offered here is based on the field notes from three post-lesson discussion sessions to give us some insights into what Team Bass teachers may have been learning through their participation in lesson study. These categories are important because they give us a sense of what teachers work on through lesson study and how we can best use this professional development tool to strengthen mathematics instruction in a systemic initiative.

Conclusion

Across these three site-based teams, strong common themes emerge, despite significant differences among the teams' composition, leadership, and content foci. This is particularly surprising because neither the facilitators nor the teacher participants collaborated across teams on the content of the work in lesson study. We attribute this remarkably similar progress across the teams to the lesson study process itself, in conjunction with the shared values of the facilitators. Lesson study groups in this country have modified the structure of lesson study in a number of ways: shortening the time spent in content and curriculum study, videotaping research lessons instead of live observations, skipping the use of knowledgeable others, and abbreviating or eliminating the post-lesson discussion (Yoshida, 2012). We have made a conscious effort to stay faithful to the essential features of the canonical form of lesson study, with minimal adaptations to the local environment. In particular, the lesson study teams mentioned in this paper (and throughout the Noether Project) engage in extended study of content and of student thinking utilizing multiple print and human resources, conduct live research lessons, determine their team goals based on teachers' and their students' needs, invite outside

experts to comment on their work, and devote significant amounts of time to face-to-face post-lesson reflection and discussion.

Some of themes that are common across the three teams are:

- Development of teachers' "mathematical care"
- Elicitation and deep analysis of student thinking
- Developing curiosity about mathematics and about student thinking
- Emphasis on students' autonomous problem-solving
- Increased use of multiple representations for solving problems
- A generous and supportive collegial atmosphere of learning

In all three teams we see teachers developing a significant degree of "mathematical care" in their instruction, that is,

the care with which the teachers consider mathematical choices or options, and with which they attend to children's mathematical thinking and expressions, in the flow of instruction...Mathematical care means that the instructional choices that shape the mathematics in play are treated with heedfulness and attention (Lewis, 2007, p. 144).

As the teachers have engaged in lesson study and are developing "eyes to see students" (Lewis, 2002b), they grow in their ability to question students productively and to devise classroom situations that will reveal student thinking, including students' correct understandings as well as misconceptions. Teachers have become more adept at differentiating conceptual work from rote application of formulas, eliciting student thinking, and analyzing students' ideas about the mathematics.

Teachers' deepening knowledge about mathematics and teaching has stimulated curiosity about their students and mathematical content. Increasingly, we hear teachers express curiosity about mathematical ideas or how their students will react to a particular problem or lesson. With this new perspective, teachers in all three teams are much more likely to widen their instructional efforts to focus on a concept (e.g. area as space, or distributive property as a relationship) rather than a sole focus on algorithms (e.g. length x width for the area of a rectangle, or a particular form of the distributive property).

In all three teams, teachers have moved away from telling students what to do and moved toward developing students' desire and ability to make sense of mathematical situations and to solve problems autonomously. This represents a significant shift for both teachers and students, and progress is slow—but teachers are committed to continuing their work in this direction.

The teams have learned that teaching and learning is not one-size-fits-all. With increased observation of varieties of student representations, and a heightened understanding of concepts, teachers are increasing their interest in representing mathematics in a variety of ways to reach a wider range of students. They now frequently seek multiple representations of mathematical situations, and are becoming adept at devising these on their own.

Underlying the process is a feeling of generosity—teachers being generous with their ideas and their time, gently supporting one another in taking risks, looking for the sense in students' ideas, and sharing successes as a group. Teachers in all three teams are excited about the changes they are making in their learning and teaching. Even small changes the teachers see in their students encourage the teachers to deepen their

commitment to continue their own personal growth in facilitating student learning and we anticipate continued exciting developments in the future.

What is the future of lesson study in this district beyond the grant-funded project? In Japan, lesson study is an ongoing, career-long method of improving instruction for all teachers in elementary schools. Unlike our Japanese counterparts, lesson study is not woven into the fabric of teachers' typical work schedules in the U.S. The Noether Project is creating cultural changes in teachers' approach to teaching and patterns of collaboration, and several of the project schools have begun seeking ways to extend the changes throughout the school. Additionally, the district is committed to assuming increasing financial and leadership responsibility for lesson study throughout the five years of the project. During this time, we intend that lesson study will become well-established as a systematic method of enhancing instruction in the district; the positive outcomes that are becoming apparent in the Noether Project give us reason to hope that at all levels (teachers, site and district administrators) lesson study will come to be seen as indispensable to teachers' continuing professional growth and, therefore, to the students' success.

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**Developing Effective Mathematics Teaching:
Assessing Content and Pedagogical Knowledge, Student-Centered Teaching, and
Student Engagement**

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Abstract: The Mathematics Teacher Transformation Institutes (MTTI) program attempts to develop math teacher leaders in part by providing content, inquiry and leadership courses aimed at making them more effective teachers. We assessed progress by observing teacher leaders' teaching practices, and encouraging them to introduce or extend student-centered pedagogy in their classrooms. We found there was little relationship between our measures of mathematics content knowledge and student-centered pedagogy. But teachers who employed student-centered pedagogy tended to have more highly-engaged math students in their classrooms.

Keywords: effective mathematics teaching; math content knowledge; student-centered teaching; student engagement.

Improving student achievement in mathematics and science has been a concern in the United States of America since the early 1980s when international tests began showing U. S. students falling behind most developed countries in mathematics and science skills. Many U. S. students do not obtain the knowledge and skills, particularly in science, technology, engineering, and mathematics (STEM), which are required for success in the global marketplace of the 21st century (National Academy of Sciences, 2006).

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Educators, educational researchers, and policy makers have not always agreed about the reasons for the failure of U. S. students to perform. Some argue many mathematics teachers have inadequate mathematical content knowledge themselves, and thus are unable to teach their students to the highest level (Ahuja, 2006; Ginsburg, Cooke, Leinwand, Noell & Pollock, 2005). Others (Darling-Hammond, 2007; National Council of Supervisors of Mathematics [NCSM], 2008; National Council of Teachers of Mathematics, 2000; Office of Science and Technology Policy, 2006; U.S. Department of Education, 2004; National Science Board, 2006), in part, relate such an educational failure not only to the lack of qualified teachers with solid content knowledge in STEM, but also to a profound lack of understanding of teaching and learning in grades K-12, which may lead to the use of ineffective teaching practices. For Brown and Borko (1992), and Ball and Bass (2000), understanding content knowledge and methods of inquiry in mathematics are at the core of effective teaching and learning. The use of inquiry-based approaches to instruction, in which students have opportunities to construct their own understanding of basic concepts, is thought by many educational theorists to be most appropriate in developing students' understanding of mathematics and science concepts. Such approaches call for teachers to be able to engage students in critical, in-depth, higher-order thinking through use of manipulatives, technology, cooperative learning and other pedagogical approaches that enable students to construct mathematics concepts on their own through reasoning, verifying, comparing, synthesizing, interpreting, investigating or solving problems, making connections, communicating ideas and constructing arguments (Grouws & Shultz, 1996; National Council of Teachers of Mathematics [NCTM], 2000). These approaches are characteristic of what is often called student-centered teaching as opposed to the so-called

“traditional” approaches in which the predominant view is that mathematics teaching is a show-and-tell as well as a supervision of drills and practice (Davis, 1988). In this view, it is assumed that learning occurs passively when students absorb received knowledge from an all-knowing teacher or expert. This approach is often referred to as “teacher-centered.” The Mathematics Association of America (MAA, 2008) argues that in order to prepare students for the increasingly complex mathematics of this century, a student-centered approach to teaching is more appropriate than the traditional teacher-centered approach. The MAA (2008) asserts the need to develop pedagogies that could be used effectively to facilitate students’ mathematical abilities. In essence the MAA (2008) advocates for an increase in student-centered teaching and learning and a decrease in teacher-centered pedagogy. One assumption is that an increase in student-centered teaching will result in increased student engagement in mathematics and, by implication, this increased engagement will lead, in turn, to increased student achievement. For example, various researchers argue that students are more engaged and achieve more when teachers relate new learning to prior learning, model problems and provide them with a variety of opportunities to apply and use knowledge and skills in different learning situations (Kemp & Hall, 1992; Rosenshine, 2012; Taylor, Pearson, & Walpole, 1999).

Logic Model and Theory of Action for the Project

One of the aims of the Mathematics Teacher Transformation Institutes (MTTI) is to encourage participant teachers to develop both their mathematics content knowledge and a student-centered pedagogy, assuming that these developments will lead to increased student engagement in mathematics. This research aimed to see whether the goal was met, and the assumption was justified.

MTTI is a National Science Foundation (NSF)-funded program designed to support the development of teacher leaders to strengthen mathematics teaching and learning in New York City, especially in Bronx middle and high schools. MTTI developed a three-year three-dimensional program that focuses on deepening participating teachers' content knowledge, broadening their pedagogical repertoire through the process of inquiry, and developing their leadership capacities across a number of domains within the context of a professional community. The model engages teachers in a process of inquiry that does not cease in asking questions and understanding problems, continually revisiting critical issues relative to teaching and learning, designing plans to resolve the issues, implementing the plans, and collecting and analyzing data to assess the effectiveness of the designed plans. As teachers improve their pedagogical skills, they increase their ability to explain terms and concepts to students, interpret students' statements and solutions, engage students in critical, in-depth, higher order thinking (Copeland, 2003; Grouws & Shultz, 1996; Hill, Rowan, & Ball, 2005; National Council of Teachers of Mathematics [NCTM], 2000). Essentially, the aim is to develop teachers' student-centered pedagogy.

MTTI is funded to support two cohorts of 40 teachers with at least four years teaching experience over five years. The first cohort completed the program after three years in June 2011. This paper reports results from the first cohort. The research component of MTTI seeks to broaden the knowledge base on teaching and learning in mathematics through new understanding of: 1) how the study of conceptually-challenging mathematics—particularly in algebra and geometry—benefits teachers; 2) how classroom-based action research contributes to critical and analytical understanding of the

relationships between teaching practices and student learning; and 3) how multi-levels of support prepare teachers with at least four years teaching experience for leadership roles.

MTTI's theory of action, depicted in Figure 1, hypothesizes in essence that teacher background and characteristics, school climate (especially as represented through teacher-teacher interactions) and MTTI experiences will impact participants' teacher-leader practices, one of which is effective teaching. The three main components making up MTTI experiences are math content courses, inquiry-based action research courses (pedagogy), and a leadership course.

MTTI aims to supplement math teachers' content knowledge and help teachers make and sustain fundamental shifts in practice. Our hope is that such changes will result in more effective teaching and teacher leadership. In turn, we hope that effective math teaching will lead to increased student engagement in math.

Figure 1. MTTI's theory of action.

MTTI Project Outline

Improving Teachers' Math Content Knowledge

Two courses aimed at improving MTTI participants' math content knowledge were run throughout the spring and fall semesters of 2009. One of the courses was in math fundamentals and the other in geometry. The math fundamentals course focused on algebra and integrated mathematics. The geometry course was based around geometric proofs, and was related to the New York state standards for geometry. Participants in the geometry course were required to undertake projects related to the topics taught in the course. The courses were taught by members of the Lehman College mathematics faculty.

Action Research Courses

MTTI participants took a two-part course series in classroom-based inquiry including action research. The course series ran for a total of 90 classroom hours. Part 1 of this series took place during spring 2010, "Classroom Inquiry in Middle and High School Mathematics." Part 2, "Mathematics Inquiry Applications," was offered during fall 2010. These courses focused on helping MTTI teachers examine the effectiveness of their pedagogical practices by identifying and describing their students' errors and misconceptions, reviewing literature on research and theories about mathematics teaching and learning, and using alternative assessments and technology. During Part 2, MTTI teachers or teams of teachers used mixed methods to develop and complete Action Research Projects, to assess the performance of their students. As of May 2011, 23 MTTI

teachers developed 29 Action Research projects, involving 1,017 students: 378 from middle schools and 639 from high schools. The course series was taught and coordinated by a member of the Lehman College secondary education department.

Statistics Course

For summer 2010, all MTTI participants were offered a choice of mathematics courses, the last mathematics course they would be taking as part of the program. They could choose either a Statistics (and Probability) course or a second Geometry course. Virtually all of them chose the Statistics course and we offered two sections of the course to accommodate all the participants who wanted the course (and did not offer the Geometry course). The MTTI participants wanted a statistics course for three main reasons: 1) they discovered during the Action Research courses that they did not know the statistics required to complete their projects; 2) many had the opportunity to become involved in their school's self-evaluation and assessment and felt they needed more statistical knowledge to analyze the overwhelming amount of data available to them internally, and their principals were eager for them to serve on these teams; and 3) several were being asked to teach Advanced Placement (AP) Statistics at their high schools. It appears that most of the teachers' preferred the statistics course over the second geometry course for professional reasons other than a desire to improve their mathematical knowledge for teaching students.

Leadership Seminars 1 & 2

The Leadership Seminar 1 began in February 2011; Leadership Seminar 2 began in May 2011. The Director of the New York City Mathematics Project (NYCMP), and the MTTI Director led the seminars. In Fall 2010, they met with the participants three times during

the Action Research course. Because it was important to lay groundwork for further exploration of the Common Core State Standards (2010), the first meeting focused on the Standards. The other two meetings focused on levels of cognitive demand for mathematical tasks as well as case studies from *Implementing Standards-Based Mathematics Instruction* (Stein, Smith, Henningsen, & Silver, 2009).

MTTI Teacher-Consultants

Six MTTI teacher-consultants visited participants in their schools to provide support. The teacher-consultants were retired mathematics teachers with many years' experience, and were drawn from the teacher-consultants who provided a similar service for the NYCMP. The teacher-consultants visited participants twice per month for one half-day on each visit. They supported participants in dealing with pedagogical and leadership issues.

Research Questions

The MTTI project is extremely wide-ranging and made up of several components. However, this paper concentrates on our attempt to answer the following three research questions:

1. Did participating in MTTI increase participants' mathematical and pedagogical knowledge?
2. Did participating in MTTI increase participants' use of student-centered pedagogy in the classroom?
3. Did any increase in either mathematical content knowledge or student-centered pedagogy lead to an increase in student engagement in mathematics?

Method

Math Content Knowledge

Math content knowledge was measured by two sets of pre-post tests developed by the University of Louisville's Center for Research in Mathematics and Science Teacher Education (Bush & Nussbaum, 2004). One of the tests was for Algebra and Ideas, and the other was in Geometry and Measurement. Both tests were set at the middle school level. The tests were part of the Diagnostic Teacher Assessment in Mathematics and Science (DTAMS) instrument that was validated using a sample of 1,600 middle-school teachers (Saderholm, Ronau, Brown, & Collins, 2010). Saderholm and his colleagues determined the equivalency reliability of the pretests and posttests by computing the Pearson product moment correlation. This, they report, was greater than .80. Inter-scorer reliability was also greater than .80. The two Louisville tests were administered before and after the relevant content courses were completed.

Each University of Louisville test contained 20 items. The first 10 items were multiple-choice items and a correct answer scored 1 point. Items 11-20 were open-ended response items each divided into two parts. A correct answer on the first part scored 1 point. A maximum of 2 points were available for answers to the second part, giving a possible score of 40 points. The tests were blinded and scored at the University of Louisville Center for Research in Mathematics and Science Teacher Education by members of the research team under the supervision of the Center's director.

The two MTTI courses, one in math fundamentals and the other in geometry, took place throughout the spring and fall semesters of 2009. Two pre-post tests were administered in association with these courses. These tests are referred to as MTTI tests. The MTTI Algebra and Ideas test dealt with: patterns, functions, and relationships; expressions and formulas; and equations and inequalities. The MTTI Geometry and Measurement test dealt with: two-dimensional geometry; three-dimensional geometry; transformational geometry; and measurement.

These two MTTI tests were designed by MTTI math faculty. The possible score on the MTTI fundamentals test was 100, and the possible score on the MTTI geometry test was 90. The same test was used as both the pretest and the posttest for the MTTI math fundamentals and geometry tests. The MTTI tests were scored by a member of the Lehman College math faculty not associated with the two MTTI courses, based on rubrics developed by the math faculty members who taught the courses.

The questions on the University of Louisville tests assessed participants' general content knowledge. In contrast, the MTTI tests were directly related to the content taught in the two courses.

Math Pedagogical Knowledge

According to our theory of action, the second component of a math teacher's capacity for teacher leadership concerns their mastery of pedagogical practices appropriate both for their students and for the mathematics concepts they teach. Information about this component comes from questions on the Louisville Algebra and Ideas and Geometry and Measurement tests, classroom observations, and teachers' work in the classroom-based inquiry courses.

As mentioned above, the second part of items 16-20 on the Louisville tests measured pedagogical content knowledge and the maximum possible score on these items was 10. An example of a question measuring pedagogical content knowledge is as follows:

Q. 16 A student claims that all squares are congruent to each other because they all have four congruent sides.

- a. Why is this claim incorrect?*
- b. Explain how you would help the student understand the error in his thinking.*

The pedagogical content scores were analyzed separately from the scores on the other questions.

Classroom Observations

Three retired math educators who had previous experience in observing teachers in their classrooms were trained to be observers for the MTTI project. They were trained to use a five-minute time-sampling system in which they were asked to observe for five minute blocks of time and note whether or not any one or more of the pedagogic and/or management behaviors (examples below) was used by the teacher. At the end of training, inter-rater reliability was .71.

Beginning in the fall 2009 term, the observers visited the MTTI teachers' classrooms at least four times each term. Through January of 2011, 265 observations had taken place. The classroom observation protocol ([COP], Lawrenz, Huffman, & Appeldoorn, 2000) contains, among other things, information about types of instructional activities. Some of these activities were judged *a priori* to be indications of student-centered pedagogy, including small group discussions, class discussions, hands-on activities, cooperative learning, student presentations, and use of a learning center or station. Some were

considered *a priori* to indicate teacher-centered pedagogy, including lecturing, lecturing with limited class discussion, modeling problem solving, and demonstrations by the teacher. The exact nature of some activities (e.g. writing work or reading seat work) could not be determined *a priori*. In these cases, the observers used their own judgment whether the activity was student-centered, teacher-centered, or indeterminate.

On average, each observation lasted for about 50 minutes, with most observations being for 45 or 50 minutes. An observation was capped at 60 minutes. The vast majority of observations in high schools were conducted in algebra, integrated math, or geometry classes. A few observations were conducted in advanced math classes, including seven observations in pre-calculus classes and eight observations in calculus classes.

Student Engagement

One of the sections of the observation protocol mentioned concerned the level of Student Engagement (SE) rated as high, medium, or low. During each observation, SE was rated as high when 80% or more of students were engaged, as low when 80% or more of students were off-task, and as mixed otherwise. An engaged student was seen as one who, during the time of the observation, was involved in the lesson in meaningful ways; that is, he/she participated in all classroom activities, collaborated effectively with the teacher and with other students, and was reflective about his/her learning.

The findings from the use of the instruments outlined above for assessing math content knowledge, pedagogical knowledge, and student-centered pedagogy were related to those for student engagement outlined in this section to determine if there was any relationship among the variables.

Results

Math Content Knowledge

Thirty-two participants took both the pretest and posttest versions of the two University of Louisville tests and the MTTI faculty-designed tests. Mean scores on the University of Louisville test of algebra and ideas increased significantly from 25.8 at pretest to 29.8 at posttest. However, mean scores on the University of Louisville test of geometry and measurement did not differ significantly from pretest (22.6) to post-test (20.7) (Tables 1 & 2).

Scores on the MTTI faculty-designed fundamentals test increased significantly from 36.5 at pretest to 48.0 at posttest. Scores on the MTTI geometry course content test also increased significantly from 26.6 at pretest to 36.0 at posttest (Tables 3 & 4).

Table 1

Pre- and post-test means for the Louisville Algebra test

	Mean	Std. Deviation	N
Louisville Algebra Pretest Total/40	25.75	6.309	32
Louisville Algebra Posttest Total/40	29.81	5.544	32

Significant: $t_{(30)} = 4.61, p < .001$

Table 2

Pre- and post-test means for the Louisville Geometry test

	Mean	Std. Deviation	N
Louisville Geometry Pretest Total/40	22.56	7.211	32
Louisville Geometry Posttest Total/40	20.72	6.371	32

Not significant: $F_{(1,31)} = 3.45, p = .073$

Table 3

Pre- and post-test means for the MTTI Fundamentals test

	Mean	Std. Deviation	N
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MTTI Fundamentals Pretest Total/100	36.47	6.567	32
MTTI Fundamentals Posttest Total/100	48.00	5.639	32

Significant: $t_{(29)} = 5.01, p < .001$.

Table 4

Pre- and post-test means for the MTTI Geometry test

	Mean	Std. Deviation	N
MTTI Geometry Pretest Total/90	26.58	6.421	32
MTTI Geometry Posttest Total/90	36.03	5.894	32

Significant: $t_{(30)} = 4.61, p < .001$

Pedagogical Content Knowledge

The average number of correct answers for the five questions of the Louisville Algebra and Ideas test relating to pedagogical content knowledge increased significantly from 4.44 to 5.16 across test administrations. This suggests that MTTI participants' pedagogical content knowledge for algebra and ideas increased following engagement with a course in the fundamentals of mathematics. The mean pedagogical content knowledge scores for the Louisville Geometry and Measurement test declined slightly from pretest (3.90) to posttest (3.55) administrations, but this decrease was not significant (Tables 5 & 6).

Taken together these results indicate that in general participants' math content and pedagogical content knowledge increased from beginning to end of the MTTI course.

Table 5

Pre- and posttest means for the pedagogical items on the Louisville Algebra test

	Mean	Std. Deviation	N
Louisville Algebra Pretest Total/10	4.44	1.722	32
Louisville Algebra Posttest Total/10	5.16	1.629	32

Significant: $t_{(31)} = 2.49, p = .018$.

Table 6

Pre- and posttest means for the pedagogical items on the Louisville Geometry test

	Mean	Std. Deviation	N
Louisville Geometry Pretest Total/10	3.90	2.146	32
Louisville Geometry Posttest Total/10	3.55	2.602	32

Not significant: $t_{(31)} = .706, p = .486$.

As mentioned above, from the classroom observation protocols, instructional activities were coded as teacher-centered, student-centered or indeterminate, at 5-minute intervals. For example, lecture was considered teacher-centered while cooperative learning was considered student-centered. However for some activities (e.g. “writing”), there was insufficient information on the observer’s report to determine the student-centeredness of the activity; these were given a coding of “indeterminate.” For each lesson, the percent of time spent in each of these three categories was then calculated. Across all observations and all teachers and all semesters, the range of time spent was: in teacher-centered activities, 30.2%; in student-centered activities, 30.4%; and in activities that could not be clearly classified as either, 39.4%. There was no significant change across the semester for the percent of time spent in teacher-centered vs. student-centered activities

($\chi^2(10) = 5.29, p = .87$). Thus, it appears that student-centered pedagogy did not increase over the timespan of the MTTI course for Cohort 1.

Student Engagement

In the fall 2009, spring 2010, and fall 2010 semesters, observers assessed the level of student engagement in math class at five-minute intervals. They recorded three possible levels of engagement: low engagement (80% or more of students off-task); medium engagement (mixed engagement); and high engagement (80% or more of students engaged). High engagement increased from fall 2009 to spring 2010. In the spring semester, high engagement had increased significantly from about 40% of observations to 63.5% of observations. In fall 2010 high engagement decreased to 48%. However, across the three semesters low engagement decreased from nine percent in fall 2009 to four percent in fall 2010 (Figure 2). These findings provide some evidence for an increase in high student engagement over the time-span of the MTTI project, and certainly evidence of a decrease in low student engagement.

Figure 2. Level of student engagement by semester.

Student-Engagement, Math Content and Pedagogical Knowledge, and Student-Centered Teaching

Math content knowledge and pedagogical content knowledge did not significantly predict the percentage class time featuring student-centered pedagogy (Tables 7 & 8) or percentage of high student engagement in math class (Tables 9 & 10).

Table 7

Math content and pedagogical content knowledge as measured by the Louisville tests as predictors of student-centered pedagogy.

	Sum of Squares	df	Mean Square	F	Sig.
Regression	205.206	4	51.302	.104	.980 ^a
Residual	8390.215	17	493.542		
Total	8595.422	21			

a. Predictors: (Constant), Geometry Content Knowledge change, Geometry Pedagogical Knowledge Change, Algebra Content Knowledge change, Algebra Pedagogical Knowledge change

b. Dependent Variable: Percent Student Centered Pedagogy

Table 8

Math content knowledge as measured by the MTTI tests as predictors of student-centered pedagogy.

	Sum of Squares	df	Mean Square	F	Sig.
Regression	619.584	2	309.792	.729	.497 ^a
Residual	7228.263	17	425.192		
Total	7847.847	19			

a. Predictors: (Constant), MTTI Geometry change, MTTI Algebra change

b. Dependent Variable: Percent Student Centered Pedagogy

Table 9

Math content and pedagogical content knowledge as measured by the Louisville tests as predictors of high student engagement in math class

	Sum of Squares	df	Mean Square	F	Sig.
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Regression	5659.604	4	1414.901	.837	.520 [□]
Residual	28728.310	17	1689.901		
Total	34387.915	21			

- a. Predictors: (Constant), Louisville Geometry Content Knowledge change, Algebra Content Knowledge change, Algebra Pedagogical Knowledge change, Geometry Pedagogical Knowledge change
b. Dependent Variable: Percent high engagement

Table 10

Math content knowledge as measured by the MTTI tests as predictors of high student engagement in math class.

	Sum of Squares	df	Mean Square	F	Sig.
Regression	5772.912	2	2886.456	1.973	.170 [□]
Residual	24873.178	17	1463.128		
Total	30646.090	19			

- a. Predictors: (Constant), MTTI Geometry change, MTTI Algebra change
b. Dependent Variable: Percent high student engagement

To determine if there was a relationship between student-centered teaching (SCT) and student engagement, we derived two groups of participants; Group A (High SCT) consisted of the six participants who were observed to display the most student-centered teaching techniques as assessed by the classroom observers across both the fall 2009, spring 2010 and fall 2010 semesters; and Group B (Low Student Centered) consisted of the six MTTI participants who exhibited the least student-centered teaching techniques assessed in the same manner across the same time period. For Group A, the mean percentage of time spent in student-centered teaching activities was 48.7% (s.d.=9.0) across all semesters, while for Group B, it was only 15.7% (s.d.=9.2).

We then examined the relationship between student centered teaching and student engagement. We calculated the levels of student engagement for the two groups (high and

low SCT) for each semester and a mean value across semesters. We found that students of Group A (high SCT) teachers were significantly more likely to be highly engaged in their math classes than students of Group B (low SCT) teachers: $\chi^2(1) = 5.81, p = .02$ (See Table 11).

Table 11

Level of student engagement for the High and Low SCT groups

Level of SCT	High Engagement	Mixed Engagement	Low Engagement
High	62.4%	33.4%	4.3%
Low	44.7%	48.7%	6.6%

Discussion

We found that MTI teachers' content knowledge in the fundamentals of mathematics improved significantly following their participation in the program. However, there was no significant relationship between teachers' increase in content knowledge and their use of student-centered teaching or the engagement level of their students in math class. This may have been because the measures we used to assess content knowledge did not adequately tap into participants' pedagogical knowledge. Support for this view comes from additional data from the observations, which show that the classroom observers

rated teachers' mastery of math concepts highly. The observers also reported that participants made extremely few mathematical errors while they were teaching.

It is also worth noting that the University of Louisville tests were tests of general mathematics concepts and pedagogy, while the MTTI math tests were related to the MTTI math courses, but not necessarily to the specific concepts and pedagogy that MTTI teachers were using in their classrooms. The math content of the MTTI courses was determined by the Lehman College mathematics faculty member teaching each course. In general, the content of the math courses was related to the New York State math standards, but it was not related specifically to the content that the teachers were teaching in their classroom. It might not be surprising, therefore, that there was no significant relationship between MTTI teachers' math concept knowledge as measured by the Louisville and MTTI tests and their classroom practices as reported by the observers.

We suggest that the discrepancy between the University of Louisville Geometry and Measurement test results (lack of improvement) and those of the MTTI Geometry test results (significant improvement) may have been due to the lack of fit between the MTTI geometry course, which was designed to correspond to New York State's secondary geometry curriculum, and the items on the Louisville exam.

The content of the Louisville tests had been established with reference to teams of mathematicians, math educators, and math teachers who conducted literature reviews for appropriate content as defined by national recommendations (Saderholm, Ronau, Brown, & Collins, 2010). This resulted in tests that contained content that math experts thought that math teachers generally ought to know and be able to teach, rather than items that

assessed mastery of specific course content or what teachers needed to know to be able to teach particular students.

In addition, fewer MTTI teachers had experience in or were currently teaching geometry compared to algebra. This was in part because, until relatively recently, most emphasis had been placed on algebra by New York State's Board of Regents. Since teachers were being asked to focus more on teaching algebra than geometry, this might explain why the MTTI teachers generally improved more on the Algebra and Fundamentals test than the Geometry tests.

We discovered that teachers who employed a high level of student-centered, inquiry-based pedagogy tended to be more effective as math teachers than those who used a low level of student-centered teaching, at least if effectiveness is assessed by the extent to which their students were engaged in the lesson.

Anecdotally, participants reported that as a result of participation in the classroom-based inquiry (action research) courses, they changed their own teaching practices and saw improvements in motivation toward participating in mathematics on the part of their students. These findings are based on self-report, and in the future we are going to ask teachers to formally assess whether changes in students' motivation to engage actually occur.

For this study, the main variable used for assessing the effectiveness of teaching is level of students' engagement in math class. In part, this was because we had difficulty in gathering pre- and post-test data for state-mandated student tests. To some extent this was because, in order to obtain ethical approval from the New York City Department of Education for the study, we could not track individual students during the period of the

research, nor could MTTI teachers conduct research activities using students in their own classes as participants.

For MTTI Cohort 2, we are able to ask MTTI teachers to collect data from their students as long as those students' identities are not revealed. Therefore, we are in the process of administering math performance tasks to the students of MTTI Cohort 2. These performance tasks reflect the new Common Core State Standards for Mathematics (2010) which are being introduced in New York City schools in the fall 2012 semester. This is in an attempt to obtain student achievement data. We will then be able to look at the relationship, if any, between student-centered pedagogy, student engagement, and student achievement.

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Supporting Middle School Mathematics Specialists' Work: A Case for Learning and Changing Teachers' Perspectives¹

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Abstract: In this paper, we highlight one whole-class discussion that took place in a middle school mathematics *Rational Number and Proportional Reasoning* course, one of the six mathematics courses teachers take to complete our state-wide middle school mathematics specialist program. Statistical measures indicate that teachers made gains in their understanding of concepts and substantial gains in their views of teaching and preparedness. We provide a microanalysis of one of the lessons, to explain, in part, how they might have made this progress. To develop our argument, we coordinate a social analysis with an analysis of the types of specialized mathematical knowledge that teachers might have considered as they engaged in these discussions. As we will illustrate, these types of classroom discussions provided teachers opportunities to consider new visions for mathematics learning and teaching.

Keywords: Proportional Reasoning, Mathematics Specialists, Professional Development, Middle School Mathematics

Professional development initiatives that provide continuing, quality support for middle school teachers have received renewed attention in recent years. For instance, Smith, Silver and Stein (2005) stated that due to students' "lower-than-expected performance on national and international assessments" (p. xi) the National Science Foundation provided financial support for developers to create new middle school mathematics curricula (e.g., MathScape, Connected Mathematics Project & Mathematics in Context). that offered new innovations in teaching and learning mathematics (Reys, Reys &

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Chávez, 2004). Providing new curricula and professional development around implementing these curricula can be catalysts for teachers to further develop (or change) practices, make connections among ideas, and better support student learning (Reys, et al., 2004). However, if teachers do not develop new kinds of practices they may not be able to successfully implement innovative curricula. As Smith, Silver and Stein (2005) state with regard to implementing new middle school curricula,

In short, new curriculum materials are unlikely to have the desired impact on student learning unless classroom instruction shifts from its current focus on routine skills and instead focuses on developing student understanding of important mathematics concepts and proficiency in solving complex problems. (p. xi)

Schifter and Lester (2005) mirror Smith et al.'s (2005) position. Speaking about teachers' participation in the *Developing Mathematical Ideas* programs, they state that if teachers do not "construct new visions for mathematics, mathematics learning and the mathematics classroom" (Schifter & Lester, p. 97), instructors will not be able to implement these curricula in ways that the developers intend.

Schifter and Lester's (2005) position is a useful way to frame our work in our statewide mathematics specialist program for middle school teachers. One of the aims of this work is to help teachers, when needed, to make shifts in their instructional practices so that they can effectively serve as mathematics teacher leaders, who we refer to as mathematics specialists. Our goal is to prepare middle school teachers such that once they successfully complete this program, they will be well positioned to provide ongoing, long-term, classroom-based professional development for fellow teachers in their school buildings.

Throughout the program, we know that the course instructors played a key role in helping teachers reflect more deeply about different aspects of their work (cf. Ball, Thames & Phelps, 2008). For instance, teachers reported that course instructors played a key role in helping them develop deeper understandings in the first two courses (*Numbers & Operations; Rational Numbers and Proportional Reasoning*) (Moffet, Fitzgerald & Smith, 2011). Additionally, teachers made statistically significant gains in their understanding of mathematics content as well as how to better teach these content ideas ($p < 0.05$) (Moffet et al., 2011). Also, they made substantial gains in their perceptions of their understanding of content and teacher preparedness. These findings have prompted us to ask the following questions: What happened during the courses that may have provided opportunities for teachers to make these kinds of shifts? What was the nature of instruction that allowed these changes to occur? How might we better understand the instructors' role in supporting the teachers' understandings of content and their perceptions of themselves as teachers of mathematics? What mathematical ideas for teaching might teachers consider as they engage in these discussions? The purpose of this paper is to unpack one of the lessons in the *Rational Numbers and Proportional Reasoning* course to understand the process by which teachers may have made these shifts in their understandings. We are particularly interested if we can identify instances during the lesson in which teachers had opportunities to consider alternative ways to reason about pedagogical and mathematical ideas. If we can identify such instances, we may gain insight into what and how they may have made these possible shifts in their perceptions and understandings of teaching and content.

To accomplish this task, we provide a microanalysis of one of the lessons in which the participants explored inverse proportions. We chose this lesson because it illustrates how the instructors and teachers established collective ways to reason about proportion problems and, as they did so, created opportunities for teachers to explore their beliefs about and commitments to teaching and learning mathematics for understanding (Shifter & Lester, 2005). Additionally, our example illustrates the some of the challenges that instructors encounter as they attempt to address teachers' more traditional views of mathematics teaching by engaging them in more innovative practices.

In the next sections, we first briefly outline our research efforts. Following this discussion, we highlight constructs that are informing our research about teachers and their work as mathematics specialists—the mathematical knowledge that they need to know to do this work (Ball, Thames & Phelps, 2008). We then analyze the lesson to understand the reasons behind the progress made by the teachers during the course. Finally we offer some comments about the importance of engaging teachers in these types of learning experiences.

Methodology Issues

In this section we outline the methods we used to analysis the classroom episode. Before doing so, we provide background about the mathematics specialist program.

Mathematics Specialist Program

The mathematics specialist program is the result of a concerted effort for over 20 years among stakeholders (university faculty, school district personnel, state professional organizations and the State Department of Education) to provide endorsement programs

for K-8 mathematics specialists. Mathematics specialists are thought to have a particular set of responsibilities in their school buildings:

1. *Support* teachers through coaching, co-teaching, and modeling lessons,
2. *Translate* mathematics standards and research into classroom practice,
3. *Plan and facilitate* in-school practice-based professional development, and
4. *Work collaboratively* with administrators and staff to improve student learning.

(Virginia Mathematics & Science Coalition, n.d.)

There has been a growing interest in supporting mathematics specialists, coaches or instructional leaders in many different states. For instance, states across the country have received federal support to implement and determine the effectiveness of mathematics teacher leader programs (e.g., Nebraska's *Math in the Middle Institute Partnership*, Virginia's *Preparing Virginia Mathematics Specialists*, and Oregon's *Oregon Mathematics Leadership Institute*). These and other programs were developed in part because of the need to provide extensive, on-the-job professional development for teachers of mathematics.

At the same time, several professional documents have called for qualified mathematics specialists to be placed in schools as a resource for improving instruction (e.g., Kilpatrick, Swafford & Findell, 2001; National Council of Teachers of Mathematics (NCTM), 2000; National Mathematics Advisory Panel, 2008; National Council of Supervisors of Mathematics (NCSM), 2008). The NCSM (2008) report is particularly timely in that it provides a framework for the content that mathematics teacher leaders might need to successfully support teachers' daily work.

In our program, teachers are slated to work as mathematics specialists in their districts after they successfully complete a multi-year, 36-39 credits, Masters degree program in mathematics and mathematics education leadership. The program is composed

of three 5-week summer institutes that include six mathematics courses: *Numbers and Operations*, *Algebra and Functions*, *Algebra and Functions 2*, *Statistics and Probability*, *Rational Numbers and Proportional Reasoning*, and *Geometry and Measurement*.

Additionally, each year, teachers enroll in one *Education Leadership* course. They also complete a research in mathematics education course that follows a blended delivery format.

Instructors used activities from different sources to address content in the mathematics courses. For instance, they adapted many of the activities in the Rational Numbers and Proportional Reasoning from the work of Smith, Stein and Silver (2005) and Lamon (2005). The Education Leadership courses were designed so that teachers would explore their own teaching, their role as a math coach and their role as a change agent in the school building and district. In the Education Leadership 1, activities addressed teaching mathematics for understanding, issues that align with reform recommendations. For instance, teachers examined the NCTM (2000) documents and Stein, Smith, Henningsen, and Silver's (2000) work on cognitively demanding tasks. In Education Leadership II & III, teachers learned about coaching and working as a mathematics leader in the school context, respectively. Additionally, these courses were not taught in isolation, per se. When possible, instructors planned instruction so that Education Leadership activities aligned with content addressed in the mathematics courses.

The required mathematics courses address content that is not only covered in the middle school curriculum, but also content that requires teachers to use multiple representations, analyze the work of students, and make connections between procedures and the underlying mathematical ideas. Thus, teachers have a range of experiences that

align with recommendations made by NCTM (2000) and The National Mathematics Advisory Panel (2008).

Throughout the program, course instructors use a problem-centered approach to teach the courses (Yackel & Cobb, 1996). Using this approach, the instructor presents one or more rich problems for which teachers do not readily know the answer. Teachers need to use their understandings to make sense of and solve these problems. They usually work in pairs or small groups to solve the problems together. The key is for them to understand the strategies that they use, and, when possible, to understand the different approaches that other classmates use. Additionally, they are expected to share their methods when the class reconvenes for a whole class discussion. During these discussions, the instructor plays the important role of deciding which ideas to capitalize on and which to place on hold, in addition to which representations might be used to provide teachers opportunities to explore ideas and make connections (Yackel, 2002).

Data and Analysis. The classroom episodes that we use are taken from our classroom data corpus of the two mathematics courses that we studied (we only collected data for two of the courses). Data include observation notes of the lessons, videotape recordings of small group and whole class discussions, digital recordings of small group discussions, digital photos of participants' work during whole class discussions and participants' individual work. Additionally, after viewing each of the lessons, we transcribed lessons to conduct further microanalyses of the entire lesson. As we reviewed our observation notes, we noted that teachers continued to struggle with using pictures, diagrams or manipulatives to illustrate mathematical ideas. We had marked this particular lesson as a potentially pivotal one. Although teachers continued to have various views on if

they might be able to represent and solve problems and, if so, how to actually do it, during this lesson, they reasoned sensibly about proportion ideas as they used manipulatives and diagrams. For this reason, we believe that this whole class discussion was particularly important.

To conduct a microanalysis, we engaged in a process that is similar to that of Glaser and Strauss' (1967) constant comparison method. We first viewed the videotape as we analyzed the transcript of the whole class discussion. As we watched the videotaped lesson, we identified the mathematical ideas that surfaced and clarified the different models that participants used to explain solution methods. We then reanalyzed the transcript of the lesson, line by line, and made conjectures (or refuted conjectures) about how representations emerged as participants engaged in the conversation. As we did so, we also integrated each subsequent participant's contribution to further support our conjectures about if and how the participants used these representations to explain and justify their thinking. As part of this process, we made inferences about the participants' expectations and obligations in relation to their interactions with others' contributions. Through this process, we developed a more general theme about how the participants established ways to reason mathematically using multiple representations.

Theoretical Issues

Our Assumptions

We view classrooms as social settings in which teachers and their students together establish a classroom community (e.g., Ball & Bass, 2003; Cobb & Yacel, 1996). It does not matter how we might characterize the classroom or the teachers' and their students' established ways of acting and participating that are particular to that community or

classroom microculture. Together, the teacher and students constitute what counts as knowing and doing mathematics. When individuals in a social setting, such as in classrooms, agree on ways of acting and participating, we refer to these as taken-as-shared practices (e.g., Cobb & Yackel, 1996; Simon & Blume 1996). Ball and Bass (2003) refer to this notion as public knowledge. Classroom practices are said to be taken-as-shared or public if and only if they are normative, that is, they are agreed upon, and eventually taken for granted by the classroom participants. As such, classroom practices are social constructions that emerge during classroom interactions. This is not to say that individual contributions do not play an important role. Different individuals may participate in these practices in different ways given their understanding of the ideas at hand (Cobb & Yackel, 1996; Ball & Bass, 2003). Although practices are socially accomplished, individuals contribute to and participate in these practices in different ways. Further, their understandings constrain and enable how they might participate in particular practices (e.g., Whitenack & Knipping, 2003)

Background

Teachers had opportunities to solve a range of tasks that were likely different from those that they used in their own classrooms to teach proportional reasoning. Engaging in , what for them were novel activities, posed challenges for many of the teachers. They seemed to address these challenges in different ways. For instance, some teachers embraced the idea of using manipulatives to solve tasks because they began to see that their students might benefit from using manipulatives or diagrams. Others, who had worked in elementary as well as middle school classrooms, were more familiar with using manipulatives to reason about ideas or to represent their thinking. Still others had little

experience with using manipulatives in their classrooms. Additionally, they struggled to use different representations to reason about and to solve tasks. So teachers had varying experiences (and views) about using manipulatives and, more generally, employing multiple representations to reason mathematically. For example, in the lesson we examine below, not all of the teachers successfully used pattern blocks to solve the inverse proportion problem.

Mathematical Knowledge for Teaching

We draw on the work of Ball and her colleagues (e.g., Ball, Lubienski & Mewborn, 2001; Ball, Hill & Bass, 2005; Ball, Thames & Phelps, 2008) to understand the kinds of mathematical knowledge that teachers must have and use when teaching mathematics for understanding. As Ball (2002) asserts, mathematical knowledge for teaching [MKT] is not simply a list of mathematical skills or content that is learned as one participates in traditional mathematics courses. It is a specified type of knowledge teachers must have to effectively teach mathematics.

Ball, Thames, & Phelps (2008) separate MKT into two domains (1) common content knowledge (CCK), mathematical content and skills used in various aspects of work and everyday life—not just in the classroom, and (2) specialized content knowledge (SCK), mathematical content and skills that particularly apply to the teaching profession. Teachers need to draw on both kinds of knowledge in their work with students. With regard to SCK, teachers need to understand the important mathematical concepts that are behind a particular procedure or how to best highlight students' drawings to focus a discussion related to those ideas. With regard to CCK, teachers also need to have a deep understanding of the mathematics that they teach.

What content knowledge do teachers need to know to understand proportional reasoning? Lamon (2005) argues that to reason proportionally, teachers need to reason multiplicatively about the relationships among two or more ratios. Consider, for instance, a problem from Lamon's (2005, p. 99) text: *If 3 pizzas serve 9 people, how many pizzas will I need to serve 108 people?* To solve this problem, the teacher might recognize that the number of people will always be three times the number of pizzas. So 108 pizzas would feed 36 people—one-third of the number of pizzas. Or the teacher could reason that since there are three pizzas for nine children, there are 30 pizzas for 90 children (there are 10 times as many pizzas and children). And she knows that 33 pizzas will feed 99 children. She then adds six more pizzas and 18 more children to arrive at the answer of 36 pizzas for 108 children. Here again the teacher is said to reason proportionally since she relates pizzas and children multiplicatively (Lamon). Additionally, one can explore different relationships among ratios. For instance, two variable quantities can relate directly, or be directly proportional, if their ratio is constant. Our example of pizzas and people above is an example of ratios that are directly proportional since each is equivalent to the same constant, $\frac{1}{3}$ (i.e., each pizza serves three people). By way of contrast, two variable quantities are inversely proportional if their product is constant.

As we analyze the whole class discussion, we will highlight some of the specialized content knowledge that might be in the background during the discussion. We do so to illustrate how closely related specialized knowledge for teaching (e.g., how different manipulatives exploit different aspects of proportional reasoning) and the teachers' solution methods are in this particular lesson. Although it was not the instructors' intent to address specialized knowledge for teaching explicitly during the lesson, these ideas can

naturally surface as teachers reflect on their learning experiences in relation to their own teaching practice.

Using Novel Tasks

One of the challenges that the instructors had was to help teachers understand the ideas that underpin the procedures they routinely use to solve proportional problems. The instructor might use one of several approaches to meet this challenge. He might ask teachers to explain why a particular procedure works. Or he might ask what mathematical ideas surface as teachers use these procedures. Or the instructor might pose tasks that require teachers to use different representations such as manipulatives, diagrams, or pictures, to model and solve problems. This instructional strategy, using models to solve problems, seemed to be an effective way to challenge teachers' understanding and beliefs about teaching for understanding. By requiring teachers to reason about ideas using different models, teachers had opportunities to explore the important ideas that underpin the methods that they used. Teachers did not have ways to readily solve tasks using these representations—these problems were novel ones for teachers. In the lesson that we analyze within this article, teachers did not readily know how to solve an inverse proportions problem using pattern blocks or the area model. As teachers engaged in these types of activities, first working together in small groups and then reconvening in a large group to talk about ideas, they had opportunities to develop deeper understandings of different concepts.

In the next section we analyze parts of one lesson to better illustrate when and under what conditions teachers might have developed new mathematical understandings.

The Inverse Proportion Lesson

During this part of the lesson the participants discussed their solutions for the following problem: *If nine people each work 1.5 hours, how long will it take six people to do this same work?* Teacher S had previously explained that six people would need to do more of the work since there were fewer people doing the work. As the discussion ensued, Teacher C (Tchr C) and Instructor 1 (Instr 1) discussed how Teacher C used blue rhombus and green triangle pattern block shapes to solve the problem. We enter the discussion as Teacher C explained how she used pattern blocks.

Figure 1. Instructor 1 represents Teacher C's represent of the man-hours problem.

Tchr C: I represented it with a rhombus and a triangle? So you have an hour and a half an hour. So you represent it as nine times with a blue and a green...

Instr 1: A rhombus and a triangle. [begins placing blue rhombi and green triangles to for pairs (see Figure 1)].

Tchr C: And I represented it nine times, and I thought that would show all of the time that was spent [inaudible].

As Teacher C explained how she used the blocks, Instructor 1 began making groups of blocks to represent the work that each of the nine people completed. As they engaged in this part of the discussion, teachers had the opportunity to consider how one might use the pattern blocks to solve this problem.

As the discussion continued, Teacher C explained how she would distribute the blocks to show the work that six people needed to do:

Tchr C: For me, that would represent all of the time that it took to do the job. Then I would divide that up into six piles because you only have six people. It is still going to take the same number of hours to do the job. So if you divide that into six equal piles then I should have the amount of time that it would take each person.

Instr 1: [To all the teachers] Well how would I divide nine big things and nine little things into six equal piles?

Tchrs: [Laughter and people talking over one another.] I don't know.

Notice that Teacher C made several comments that related to ideas about inverse proportions. First she explained that the nine blue-rhombus-green-triangle pairs (the number of people/hours of work) represent the total amount of work-hours. She also mentioned that if there were only six people doing the work, they would still need to complete the same number of hours of work. She also explained how she would need to determine the number of man-hours for six people. After Teacher C explained that she divided up the blocks into six piles, Instructor 1 asked the other teachers how they might

divide the pattern blocks. By asking all the teachers this question, Instructor 1 invited others to engage in the discussion. As he did so, he also communicated implicitly that Teacher C's method was a viable approach for solving this proportion problem.

Interestingly, in response to his question, notice too, that teachers talked over one another and some indicated that they did not know how they could divide the blocks to solve the problem.

It is at this point that Instructor 1 and Teacher C talked about how they might redistribute the blocks into six piles to solve the problem.

Instr 1: Everyone gets a green thing....So I will take out six of the blue ...[removes the 6 rhombi] trapezoids and those correspond to people working?

Tchr C: One hour.

Instr 1: One hour. And then I can take out the six of the triangles that correspond to everyone working [removes 6 green triangles]?

Tchr C: Half an hour.

Instr 1: Half an hour. That's what they were doing at the beginning when there were nine of them. That is how much work they had to do [three blue rhombi and three green triangles still presented by the document camera].

Tchr C: And now you have to trade some blues for more greens...so that you can split them all.

As Instructor 1 began distributing the six pairs of blocks, he asked what each block represented. And, each time he asked this question, Teacher C answered his question. As she did so, she and Instructor 1 continued to show how they could distribute these blocks into six equal groups. As further evidence, after distributing the six rhombi, Instructor 1 also explained that the remaining blocks (three rhombi and three triangles) were part of the man-hours they started with. Teacher C, for her part, explained that they also needed to trade rhombi for triangles so they could share all the blocks. So as he and Teacher C

explained what the blocks represented at each pass, they illustrated how they might use the blocks to solve the problem.

Following this exchange, Instructor 1 and Teacher C continued to talk about how they would trade blocks and distribute the remaining three piles of rhombus-triangle pairs equally among the six groups. However, they did not find the actual values of the blocks in each of the six piles. At first we were puzzled as to why Instructor 1 and Teacher C did not actually use the pattern blocks to solve the problem. Further, it was very uncharacteristic of Instructor 1 to explain how he might use the blocks to make six equal groups. Instructor 1 usually expected teachers, not him, to explain their solution methods. So, we suspect that he never planned to solve this problem using the pattern blocks. Instead, he (and Teacher C) demonstrated the problem in order to help teachers see one possible way to use the pattern blocks to reason about this problem.

Examining the Representation

What are some of the specialized content ideas associated with using pattern blocks to solve this problem? Are there any limitations with how one can manipulate quantities when using the blocks? First, we note that the blue, yellow, red and green pattern blocks are related (1 yellow = 6 greens, 1 blue = 2 greens, and 1 red = 2 blues or 3 greens). If the blue rhombus represents 1 hour, then the green triangle represents $\frac{1}{2}$ hour and together they represent $1\frac{1}{2}$ hours. To represent the work of nine people, one could make nine rhombus-triangle pairs, like Instructor 1 and Teacher C did to solve the problem. Trading all the blue rhombi for green triangles, gives 27 triangles. Making six equal piles (i.e., use partitive division) yields four triangles in each pile with three leftover. So each person works 2 hours since triangles are half-hours or $4 \times \frac{1}{2} = 2$. Each person also works

another 15 minutes or $\frac{1}{4}$ hour since and $\frac{1}{2}$ of one green triangle is $\frac{1}{4}$ ($\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$). So each worker will work $2\frac{1}{4}$ hours or 2 hours, 15 minutes. Since this use of blocks requires one to mentally partition the triangles in half, one might wonder if this could be avoided by using a different block to represent the whole. Using the yellow hexagon to represent one hour, and the red trapezoid to represent $\frac{1}{2}$ hour, exchanging both of these for the equivalent number of green triangles (each representing $\frac{1}{6}$ hour) still results in needing fractional blocks to represent the solution. Thus, while pattern blocks are an appropriate way to model inversely proportional problems, the limited “denominations” of blocks can require the solution to involve mentally partitioning blocks.

Figure 2. Teacher Leader represents the man-hours problem using rectangular regions.

Inverse Proportion Lesson—Method 2

Returning to the lesson, as the discussion ensued, Instructor 1 asked Teacher Leader, one of the other instructors, to explain his method to the class. Teacher Leader had used an area model instead of the pattern blocks to solve the problem. So as the discussion continued, Teacher Leader came to the front of the room and explained how he solved the problem using the area model. Teacher Leader explained that he first drew a 9×1.5 rectangle to represent $13\frac{1}{2}$ man-hours. He then divided the rectangle into two smaller rectangles with dimensions, 6×1.5 and 3×1.5 (see Figure 2). Then he split the 3×1.5 rectangle to make two 3×0.75 rectangles. And he placed these two 3×0.75 rectangles, one on top of the other, making a new 6×0.75 rectangle. And finally, he adjoined this new rectangle with the 6×1.5 rectangle to make a 6×2.25 rectangle.

As the discussion continued, Teacher Leader asked the teachers if they understood how he had solved the problem. The following transcript reenters the discussion as Teacher Leader (Tchr Lead) asked the teachers if they followed his approach.

Tchr Lead: ...Does everyone follow what I did?....But when I split this rectangle ($3 \times 1\frac{1}{2}$) in half what is this value right here [points to the side that has length 0.75]? [Draws an arrow pointing to the $3 \times 1\frac{1}{2}$ piece now attached to the 6×1.5 rectangle, see Figure 2].

Tchr X: 0.75.

Tchr Lead: How did you get that?

Tchr X: Half of 1.5.

Tchr Lead: Because remember that is what I did with that area; I split that area in half so it is 0.75 [writes .75 above the 3×1.5 rectangle]. So now I still have the same amount of area, the same amount of work hours [moves his hand over the rectangles] that need to be done. So I kind of have to figure out what that is over here so I have $1\frac{1}{2}$ hours and $\frac{3}{4}$ of an hour, so how many hours would that be?

Tchr X: 2.25.

Tchr Lead: So the men worked 2.25 or $2\frac{1}{4}$ hours [writes these two answers to the right of the new diagram].

As Teacher Leader explained his strategy, he asked the teachers if they understood how he solved the problem. Teacher X, and possibly other teachers, seemed to understand his method. As he continued to explain his diagram, notice for instance that Teacher X provided dimensions of the smaller and larger rectangles. So she and Teacher Leader, together, began to establish this second method for solving the problem.

Examining the Second Representation

Area models (continuous) offer certain advantages over pattern block models (discrete) when representing inversely proportional situations. One can continue to partition area models into smaller and smaller rectangular regions and, in the example above, evenly distribute the $13\frac{1}{2}$ man-hours to each of the six people. Unlike when using the pattern blocks, one can actually rearrange these smaller partitioned pieces. One can also make different choices for how to partition the area. As in our example, Teacher Leader decomposed the rectangle with a side of length nine units into to smaller rectangles with lengths of six and three units. Additionally, the area is preserved because one is simply partitioning the given rectangle and rearranging the different parts to make a rectangle with an area of $6n$ square units.

To summarize, at this point in the lesson, both instructors have illustrated how they (and the teachers) might use two types of models to represent and ultimately solve this problem. Teacher C in our first example and Teacher X in our second example played different but important roles in substantiating that one can use these types of

representations to reason about and to solve proportional problems. The instructors, for their part, asked clarifying questions and highlighted the teachers' explanations.

Interestingly, as the discussion ensued, teachers continued to question whether using these types of representations were useful. Teacher K, for instance, voiced her concern. We reenter the discussion as she commented on Teacher Leader's solution method.

Tchr K: I think...trying to explain it [this method] with... I don't understand... I'm more confused after the explanation. I mean, I know how to get the answer. I just like...the representation of it is really hard for me, for this particular problem. I can explain it. I just think that my students don't understand what I am explaining. But I feel like if I show that or the other example...they would be...and I am so confused by it, that it makes it more difficult.

Although Teacher K understood how to derive the answer, she did not understand how Teacher Leader had arrived at his answer using this representation. Furthermore, she, and possibly other teachers, did not see the relevance of using this type of representation with her students. Teacher Leader and Instructor 1 had some important decisions to make, and quickly, as to how to address Teacher K's comments.

We reenter the discussion as Teacher Leader and Instructor 1 respond to Teacher K's comments.

Tchr Lead: Are there other people that feel that way? [At least one teacher raises her hand.] Were you going to say something?

Tchr S: No. I'm just trying to figure it out.

Tchr G: In my mind, that worked very nicely because it was nine. Because you have the six [inaudible] and all that...it could have been five people. Would it work just the same?

Tchr Lead: Good question.

Instr 1: Let's try it, Teacher Leader.

Notice, in response to Teacher K's comment, Teacher Leader asked if others shared her position. In so doing, he communicated to Teacher K (and the other teachers) that he acknowledged and valued their concerns. Surprisingly, other teachers did not voice similar views. This is not to say that they did not have similar views. They simply did not voice those concerns here. Instead, in response to Teacher Leader's question, Teacher S and Teacher G commented that they were still thinking about Teacher Leader's solution method. In fact, Teacher G asked whether or not this strategy would work for other problems. Notice, too, that in response to Teacher G's question, the instructors and teachers then explored a different problem that was inversely proportional to the original problem. As the discussion continued, with a little bit of calculating, the instructors and the teachers used a similar procedure to determine that it would take five people 2.7 (i.e., $1\frac{1}{2} + 1 + \frac{1}{5}$) hours to do the same work.

In retrospect, we note that Teacher K's comment was an important one. Teacher Leader's subsequent response was equally important. By asking other teachers to respond to Teacher K's comment, he and the teachers had the opportunity to explore if this method worked for other partitionings of the same rectangular region— $13\frac{1}{2}$. As they explored together how they might use similar methods to solve an alternate problem, they collectively established using the area model to solve these types of problems.

Mathematical Knowledge for Teaching

What are some of mathematical ideas needed to use the area model to solve inverse proportions? When one partitions a rectangular region and redistributes the area, one conserves the area of the original region. The region represents the total number of work-hours, and the dimensions of rectangular region represent the number of people and the

numbers of hours each person works. One can also algebraically justify why the area is conserved. To accomplish this task, use the associative and distributive properties to generate different, equivalent expressions that represent different rectangular partitioned regions that sum to an area of $13\frac{1}{2}$ square units. For example, $9 \times 1\frac{1}{2} = (6 + 3) \times 1\frac{1}{2} = (6 \times 1\frac{1}{2}) + (3 \times 1\frac{1}{2})$. The last expression represents the new two rectangular regions with dimensions of $6 \times 1\frac{1}{2}$ and $3 \times 1\frac{1}{2}$. One can also apply the distributive property again to create another equivalent expression: $3 \times 1\frac{1}{2} = [3 \times (\frac{3}{4} + \frac{3}{4})] = (3 \times \frac{3}{4}) + (3 \times \frac{3}{4}) = 3 \times 2 \times \frac{3}{4} = (3 \times 2) \times \frac{3}{4} = 6 \times \frac{3}{4}$. This last expression represents the new rectangular region that is adjoined with $6 \times 1\frac{1}{2}$. So the final string of equivalent expressions is: $9 \times 1\frac{1}{2} = (6 + 3) \times 1\frac{1}{2} = (6 \times 1\frac{1}{2}) + (3 \times 1\frac{1}{2}) = (6 \times 1\frac{1}{2}) + [3 \times (0.75 + \frac{3}{4})] = (6 \times 1\frac{1}{2}) + (3 \times 2 \times \frac{3}{4}) = (6 \times 1\frac{1}{2}) + (6 \times \frac{3}{4}) = (6 \times 2\frac{1}{4}) = 13.5$. By creating this string of equivalent expressions, we have also shown that the products of the values for each ratio are equivalent. Put another way, we have shown that the dimensions of these rectangular regions are inversely proportional since they have the same product. For the sake of brevity, we leave it to the reader to explore how they might partition this same rectangular region to show $9 \times 1\frac{1}{2} = 5 \times (1\frac{1}{2} + 1 + \frac{1}{5})$. (Hint: As one approach, first find the area for $5 \times 1\frac{1}{2}$ and $4 \times 1\frac{1}{2}$. Then somehow redistribute this area for $4 \times 1\frac{1}{2}$ to make a $5 \times 1\frac{1}{5}$ rectangle.) Finally, it is interesting to consider that there are numerous, even infinite numbers of ways to generate rectangular regions with an area of $13\frac{1}{2}$ square units.

Let us now return to the ensuing discussion. Interestingly, after participants solved Teacher G's problem, the discussion returned to exploring how one might use pattern blocks to solve the inverse proportion problem. One of the teachers, Teacher M, initiated this shift in the discussion. Without prompting, she asked if she could show how she solved

the problem using the pattern blocks. We reenter the discussion as Teacher M came to the front of the room and explained her thinking by sharing her work using the document camera.

Figure 3. Teacher M shows how she used pattern blocks to solve the $9 \times 1 \frac{1}{2}$ man-hours problem.

Figure 4. Teacher M trades 3 rhombi for 6 green triangles.

Tchr M: So, the three yellows [hexagon pattern blocks] were the whole. So there is the work that nine people did but we only have six people, so we have this much [removing three blue rhombi from one yellow hexagon but puts them back]...oh, and since it is an hour and a half each of these little blues are an hour and a half, but we had six people so we have this much work left to do [removes six blue rhombi from two yellow hexagons and points to the yellow hexagon, see Figure 3] so if I split that amongst six people [puts six green triangles on the yellow hexagon, see Figure 4]. Then I can see that one blue is the same as two greens. So, these are each an hour and a half [pointing at blue rhombi] so each person works an hour and a half, and also a green which is half of an hour and a half or...

Instr 1: Forty-five minutes.

Tchr M: Yeah...forty-five minutes. So then you can see, this is the same idea, they each work an hour and a half plus forty-five minutes, but less changing [than Teacher C's method] because I started with a whole. The whole was the three yellows, was all the work. Does that make sense?

Instr 1: Very nice. Does everyone understand what she just did? I think this is an illustration where one would get it right...the pattern blocks show us something, right? This solution is one that we and some children could understand. These pattern blocks aren't going to work with Teacher G's modified problem...as well. I mean, you can start off the same [relates problems by talking about pieces]...Okay. I like this. I would like to comment that this is also an example of something where we started off relatively confused with the pieces and when we ended up, we had a nice solution—a nice visual solution, medium [that] our students can understand.

It is interesting that Teacher M asked if she could show her solution method using pattern blocks. Initially, she had struggled with using the blocks. Apparently, she continued to think about the problem as the discussion ensued. She, in fact, explained in some detail why she used different blocks to solve the problem. By using this approach, she only needed to trade six triangles for three blue rhombi. She would still need to do some computing to determine what part of one hour the green triangles represented, but aside from this issue, her method, from her point of view, was more efficient—"less changing" or trading. She only needed to change out three rhombi for six green triangles before she combined one triangle with each of the blue rhombi to make six equal piles. Additionally,

notice how Instructor 1 instantiated her ideas. He actually commented that her method was nice. He also mentioned that Teacher M's approach illustrated how one might use the pattern blocks to solve this problem. In fact, he suggested this was a strategy that students could understand. In so doing, he and Teacher M, continued to establish that using the pattern blocks to reason about inverse proportions was reasonable.

Mathematical Knowledge for Teaching: Comparing Solutions

Are Teacher C's and Teacher M's solution methods mathematically different?

Recall in the first example, Teacher C used the triangle to represent a $\frac{1}{2}$ hour, so the blue rhombus represented one hour of work. Each rhombus-triangle pair represented the work that one person completed. And the nine pairs represented the work that nine people completed for a total of $13\frac{1}{2}$ man-hours. Teacher M used a different unit to show the number of hours each person worked as well as the total number of man-hours. So these two methods are different. Teacher C used the rhombus-triangle pair to represent the work of one person whereas Teacher M used only the rhombus for the same purpose. In other words, they represented to whole differently.

Interestingly Teacher M's approach seemed less cumbersome. Why? Teacher M and Teacher C may have thought about the relationships among the blocks differently. Teacher M, for instance, first represented the total number of man-hours (3 hexagons = 9 blue rhombi—1 hexagon represented the work that three people can do in $1\frac{1}{2}$ hours). Once she had the nine pieces she only needed to trade six green triangles for the three rhombi and then redistribute these pieces. As a consequence of using the relationships among the blocks so that they better fit the problem situation, she was able to more efficiently solve the problem. By way of contrast, Teacher C represented the hours each

person worked with one blue rhombus and one green triangle. So, she needed to trade blue rhombi for green triangles to redistribute the blocks.

At the close of this discussion, the instructors and teachers have contributed in part to constituting that both of these solution methods are reasonable—they can use pattern blocks or the area model to solve these types of problems. Initially using the pattern blocks to derive the solution did not seem viable to the participants. Recall that during the first part of the discussion, for instance, Teacher C and Instructor 1 did not actually solve the problem using the blocks. By the end of this conversation, when Teacher M illustrated how she could use this method, they now had established that using the pattern blocks was a viable approach. Of course, Teacher C's method was equally viable, but because they did not actually solve the problem, teachers may not have been convinced at the beginning of the lesson.

Final Comments

At the close of this discussion, the instructors and teachers began to collectively establish that these approaches were normative, reasonable ways to solve inverse proportion problems. Providing opportunities for middle school teachers to make changes in their views about using multiple representations, is a first and important step in supporting their professional learning about teaching mathematics and supporting teachers' learning. Participants played different parts in advancing discussions. For instance, Teacher M and Teacher C, along with Instructor 1, illustrated how one might use pattern blocks to solve tasks. Also, Teacher G's comment was particularly important in helping teachers consider how they might solve similar problems using Teacher Leader's approach. Additionally, Teacher K's comment about Teacher Leader's approach was

important. Although she may have challenged the idea of using these types of approaches, her concerns, although acknowledged, seemed to fade into the background temporarily as participants, following Teacher G's question, continued to explore how to use the area model to solve a similar problem.

Our goal is to better understand why teachers made the progress that they did by the end of the *Rational Numbers and Proportional Reasoning* course. The pretest-posttest assessment taken by all participants in the institute revealed that all the teachers better understood the content at the end of the course but the assessment does not help us understand how and why the changes were made. Teachers demonstrated that they knew how to solve problems using more traditional paper and pencil methods. However, if they had engaged in more traditional types of activities, they would have had fewer opportunities to explore why those procedures work. And more importantly, they may not have understood the important mathematical ideas that underpin those ideas. Situations such as the ones we illustrated in this lesson, provided teachers with opportunities to explore these ideas more deeply. As teachers represented and solved problems using manipulatives, pictures and diagrams—approaches that were fairly novel for them—they had opportunities to explore the different ideas and concepts.

We suspect that other teachers may ask similar questions as they move through other courses in the mathematics specialist program. Teachers had concerns about how they might support their students' learning using similar instructional practices. As they continue in the program, it will be important for them to have opportunities to address these and other issues around teaching and mathematics. In this particular lesson, there are other questions that might arise naturally. For instance, does using the area model

afford teachers more opportunities to explore proportional relationships with students?

We could imagine that this issue might arise naturally as teachers continued to routinely use these types of models to reason with and about proportions. As such, teachers might explore and possibly expand their understanding of the important mathematical ideas associated with these types of proportional activities. As they do so, they may revise their views about teaching mathematics for understanding. It is critical for teachers to develop these and many other strategies in order to be effective mathematics specialists in their school buildings.

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**A Partnership's Effort to Improve the Teaching of K-12 Mathematics
in Rapid City, South Dakota¹**

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Abstract: Over the span of ten years, a National Science Foundation-funded partnership effort has collected and analyzed multiple forms of evidence, both direct and indirect, about improved teaching of mathematics within Rapid City Area Schools. This article describes the project's impact on K-12 teaching and factors contributing to that impact. The authors argue that improvements in teaching are attributable largely to a robust infrastructure established to support teacher growth. Direct evidence includes classroom observations conducted by the project's external evaluation team. Indirect evidence exists in the form of data on student outcomes: achievement on the state's multiple-choice accountability measure and achievement on project-administered performance assessments.

Keywords: (K-12 mathematics education, teacher professional development, partnership, systemic reform)

Project PRIME (Promoting Reflective Inquiry in Mathematics Education) began in 2002 with funding from the National Science Foundation (NSF). A member of the initial

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³ The authors are indebted to the partnership as a whole and to the many individuals who have made the project flourish. Of particular importance in developing this manuscript were Deann Kertzman (RCAS), Sharon Rendon (RCAS), and Maggie Austin (TIE). Two other individuals of special importance in launching the initiative were Patricia Peel (RCAS, retired) and James Parry (TIE). Finally, thank you to the teachers, to the administrators, and to the families of Rapid City Area Schools.

cohort of NSF-funded Math and Science Partnership programs, *PRIME* was originally funded for five years. The award period has been extended several times and is now slated to conclude in 2013, 11 years after its inception. *Project PRIME* has been working to improve the teaching and learning of K-12 mathematics within Rapid City Area Schools, South Dakota's second largest school district, and to improve the preparation of teachers at Black Hills State University, South Dakota's largest producer of teacher education majors. Project partners include Rapid City Area Schools (RCAS), Black Hills State University (BHSU), Technology and Innovation in Education (TIE), a nonprofit education service provider, and Inverness Research Associates, the external evaluator.

Definition of Effective Teaching

Key elements of effective mathematics teaching as defined by *Project PRIME* include:

- Providing students with rich, meaningful, challenging mathematical tasks;
- Focusing on big mathematical ideas and on connections among them;
- Creating a safe and productive classroom culture -- one that fosters a community of learning;
- Paying attention to conceptual understanding, procedural fluency, student discourse, mathematical representation, and student dispositions; and
- Drawing from a depth of pedagogical content knowledge to recognize patterns of student thinking, anticipate and diagnose misconceptions, and guide the learner in productive directions, especially through asking questions.

PRIME has arrived at these key elements by drawing from the mathematics education literature. Resources used early within the project to develop a common vision among the project's leadership team, district math teacher leaders, building principals, university

faculty, and other project staff included *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001) and *Making Sense: Teaching and Learning Mathematics with Understanding* (Hiebert et al., 1997).

District Profile

Rapid City Area Schools includes 15 elementary schools (kindergarten through grade 5), 5 middle schools (grades 6 through 8), and 3 high schools (grades 9 through 12). It employs approximately 450 teachers of mathematics (including elementary and special education teachers), and it has a K-12 enrollment of approximately 13,000 students. Thirty-seven percent of students qualify for free or reduced-price lunch, and 24% are non-White (15% American Indian, 7% other non-White, 2% two or more races). Rapid City represents the largest off-reservation population of American Indian students in South Dakota.

Project Goals

PRIME's two overarching goals are: 1) to improve student achievement for all K-12 students within Rapid City Area Schools, and 2) to increase and sustain the quality of K-12 teachers of mathematics. Central to goal one of serving *all* students is a commitment to educational equity, seeking in particular to meet the needs of American Indian students and those who are economically disadvantaged. Project sub-goals include reducing the achievement gap between American Indian and non-American Indian students and improving high school graduation rates.

Project Design

At its core, *Project PRIME* is a teacher professional development initiative. The project was initially designed to allow every teacher of mathematics within Rapid City Area

Schools to participate in approximately 100 hours of professional development over the span of five years. Teacher participation has been voluntary throughout the project, but the majority of eligible teachers within the district have now far exceeded the envisioned 100 hours of professional development, with some having completed many hundreds of hours. Some have even earned a master's degree in curriculum and instruction at Black Hills State University, with an emphasis in mathematics education, and received a state-level endorsement as a K-12 Mathematics Specialist. Both the master's degree, with emphasis in mathematics education, and the state endorsement were created as a result of *PRIME*.

When the project began, it was the *partnership* that offered the professional development. Over time, what was once a "project within the district" has become the district's mathematics program. Thus, the language has changed such that it is now the *district* that offers the professional development, but still with support of the partnership. In aggregate, the district currently provides approximately 10,000 to 15,000 hours of mathematics professional development per year⁴. The two primary categories of teacher professional development are 1) district-wide offerings, including graduate-level coursework, and 2) building-based offerings, including classroom coaching and lesson study.

In addition to professional development for teachers, the project has provided professional development for building-level administrators and has supported the adoption and implementation of new instructional materials. Also, throughout its 10-year duration, the project has made abundant and strategic use of student-level, classroom-level, and system-wide data to motivate and sustain change, to highlight successes, to raise

⁴ The accounting is such that if 200 teachers participate in 40 hours of professional development each, then the district has provided a total of 8,000 hours of professional development.

awareness about areas in need of additional attention, and to refine the project design (Sayler & Apaza, 2006).

Project components fit together as a coherent whole, with each element supporting the others. For example, the graduate-level coursework for teachers has helped to build a common vision for quality instruction across the district and to motivate change. New instructional materials have helped teachers to put the common vision into practice. Math teacher leaders have helped classroom teachers to implement new instructional materials and to refine their practice. Administrator training has helped principals to recognize high quality mathematics instruction and to create a supportive building climate.

Graduate-level Coursework

The project has offered a mix of internally and externally developed courses, typically 30 contact hours each, offered for two graduate credits. Central to the coursework has been a strong focus on mathematical knowledge for teaching (Ball & Bass, 2003). Courses have been offered to deepen teacher content knowledge, build pedagogical content knowledge, increase understanding of student thinking, explore and discuss implementation of specific instructional materials, and build leadership capacity.

Courses have typically brought teachers together for a week in the summer or for a few hours per week over the course of a semester. In courses designed to deepen content knowledge, teachers typically have engaged in rich mathematical tasks, working in small groups, seeking multiple solution methods, asking questions of one another, and engaging in whole-class discussion. In courses designed to build understanding of student thinking, teachers have examined K-12 student work, viewed videotapes of students being interviewed about mathematics, and conducted their own interviews. Numerous courses

have also featured discussion of mathematics education articles, books, and K-12 instructional materials. Additional details about the project's coursework are provided in Appendix A.

Classroom Coaching

Building-based math teacher leaders were hired soon after the project began. Math teacher leaders serve as resources, helping classroom teachers to reflect on and refine their instruction, organizing and facilitating study sessions at the building level, and encouraging teachers to participate in the district-wide professional development offerings. As the project has matured, these positions are now all funded with district resources outside of the NSF award. The number of positions fluctuates from year to year and from school building to school building, but in recent years there have typically been 20 to 25 elementary math teacher leaders and 5 secondary math coaches across the district. The titles differ between the elementary and secondary levels, but the duties of math teacher leaders (elementary level) and math coaches (secondary level) are similar. The district has also employed a model in which select secondary classroom teachers retain fulltime teaching duties within their buildings, receive special training, and then provide professional development for their colleagues outside of the duty day and during summers.

Over the duration of the project, coaching in the district has evolved to take a student-centered approach. Student-centered coaching involves: 1) setting specific standards and curriculum based targets for students, and 2) working collaboratively with classroom teachers to ensure these targets are met. In student-centered coaching, a teacher and coach work together to use student evidence to adjust instruction. Student-centered coaching strives to add value to a teacher's work with students; the coach's role is

to think alongside a teacher, rather than to serve as an "expert" who comes in to tell a teacher how to teach. Coaches work in partnership with teachers to improve students' achievement of intended instructional outcomes.

Professional development for the math teacher leaders and coaches has been based, in part, on content-focused coaching (West & Staub, 2003) and cognitive coaching (Costa & Garmston, 2002). A version of lesson study (Gorman, Mark, & Nikula, 2010) has also been employed within the district. Additional details about professional development of math teacher leaders and lesson study are provided in Appendix B.

Administrator Training

During the first few years of *PRIME*, project leaders came to see that principals and other district administrators would benefit from their own professional development to strengthen their support of the teachers within their buildings, as well as math teacher leaders and coaches. Project leaders identified a program called *Lenses on Learning*, developed by Education Development Center (Grant et al., 2003a, 2003b, 2006), and attended training. Once trained, these project leaders then offered *Lenses on Learning* training to RCAS administrators in 15-hour increments (one graduate credit each). All building administrators were required to take the first course in the series (*Lenses on Learning I*) and had options to take the second and third courses. Additional details about administrator training are provided in Appendix C.

Logic Model

PRIME's logic model (Figure 1) starts with teacher professional development. Through professional development, teachers deepen their content knowledge, increase their understanding of student thinking, and come to have improved dispositions about

mathematics. Changes in these teacher attributes lead to improved classroom practice. Improved classroom practice, in turn, produces improved student outcomes. Student outcomes of particular importance to *PRIME* include attitudes and dispositions related to mathematics, academic achievement, and reduced achievement gaps.

Although the project emphasizes teacher professional development as the primary intervention, the project also recognizes the importance of numerous other supports, including quality instructional materials, administrative support, parent and community support, assessments aligned with the instructional materials, and a constructive education policy environment.

Figure 1. Schematic of *Project PRIME*'s Logic Model.

Each arrow within *PRIME*'s logic model has its own research base. Hill et al. (2008), for example, explored the relationship between teacher knowledge and quality of instruction. Ball & Cohen (1996) considered the influence of instructional materials on

teacher knowledge. Campbell & Malkus (2011) studied the impact of math coaches on student achievement. While there exists a sizeable body of research to build upon, this type of work is complex with plenty still to learn. The authors believe this article contributes to the existing body of research as it examines the implementation of multiple project elements in combination with one another across an entire K-12 district and extending over a ten-year period.

All of *PRIME*'s component elements support one another and have been assembled into a coherent improvement effort. Different pieces of the system must work in concert with others. Teachers must be well supported with staff development opportunities. Instructional materials must be of high quality and well aligned with the staff development. Principal and community expectations must be congruent. *PRIME* has attended to dimensions across the system, and all the while, the partnership has paid careful attention to measurable outcomes.

Results

The most direct evidence about the quality of mathematics instruction within Rapid City classrooms and about changes in teacher practice over the project's ten-year duration come from classroom observations. Indirect sources of evidence include student achievement data and measures of teacher knowledge. Indirect evidence about improvements in teaching is presented first, with the balance of the article devoted to changes in teacher practice.

Student Achievement

Two types of student outcome data are shared here: 1) student achievement on the Dakota Standardized Test of Educational Progress (DSTEP), South Dakota's statewide

accountability measure; and 2) student achievement on a project-administered performance assessment, the Balanced Assessment in Mathematics, developed by the Mathematics Assessment Resource Service (MARS).

Dakota Standardized Test of Educational Progress (DSTEP). From the first year of the project through the most recent data available, 2003 through 2011 (Year 1 through Year 9), the percentage of RCAS students scoring at the proficient level or above on the DSTEP increased from 53% to 72% across all grades tested. While that represents significant growth, it essentially mirrors the growth of the rest of the state, which increased from 60% to 78% scoring at the proficient level or above. RCAS has outperformed the state somewhat at elementary grades and underperformed the state somewhat at secondary grades, but in aggregate, growth within RCAS has paralleled the rest of the state on this measure.

A more powerful DSTEP improvement story exists related to the closing of the achievement gap for American Indian students and for those identified as economically disadvantaged. The gap in achievement between American Indian students and non-American Indian students in RCAS in Year 1 was 37 percentage points. By Year 9, that gap had closed to fewer than 22 percentage points. Similarly, the gap for economically disadvantaged students in RCAS dropped from 35 percentage points in Year 1 to 19 percentage points in Year 9. For the rest of South Dakota over the same period, the gaps have decreased, but much less dramatically. Key to closing the achievement gaps within RCAS has been strong growth in performance among American Indian students and those identified as economically disadvantaged. Additional details about student achievement on the DSTEP are provided in Appendix D.

Mathematics Assessment Resource Service (MARS) Tests. To complement DSTEP data, the project introduced Balanced Assessments in Mathematics, developed by Mathematics Assessment Resource Service (MARS). MARS tests are open-response performance assessments that include five in-depth tasks spanning four mathematical strands: number and operations; algebra; geometry and measurement; and data analysis, statistics, and probability. The project considers MARS tests to be well aligned with *PRIME*'s overall vision and approach.⁵

The project administered MARS tests to a sample of 4th and 8th graders in the spring of Year 3 and again in the spring of Year 9. Student achievement on MARS from Year 3 to Year 9 at grade 4 increased from 58% to 77% scoring at the proficient level or above. At grade 8, performance increased from 30% to 42% scoring proficient or above. The growth at grade 4 was statistically significant with Cohen's effect size of 0.4 (medium effect), $p < 0.1$. The growth at grade 8 was statistically significant with Cohen's effect size of 0.5 (medium effect), $p < 0.05$. Additional details about student achievement on MARS tests are provided in Appendix E.

Teacher Knowledge

The project conducted a small study in Years 2 through 4 to examine the impact of its professional development offerings on teacher knowledge (Sayler, Apaza, Austin, & Roth, 2010). A group of 46 RCAS teachers volunteered to take a test of their content and pedagogical content knowledge during Year 2 of the project and again two years later, using parallel forms of the *Learning Mathematics for Teaching (LMT)* measures (Hill & Ball, 2004). The average amount of professional development completed by each of these

⁵ MARS tasks provide students with a real-world context, and student must communicate the process by which they arrive at an answer.

teachers between test administrations was 80 hours. Each participant had completed an average of 60 hours of professional development within the project at the time of the pre-test and 140 hours at the time of the post-test. The teachers in the sample showed statistically significant growth on the *LMT* instrument over the two-year span with a Cohen's effect size of 0.8 (large effect), $p < 0.01$. *LMT* scores are reported as standardized scores with a mean of 0 and standard deviation of 1. The average pre-test score for this sample of teachers was -0.1 ($\sigma = 1.9$), and the average post-test score was 1.7 ($\sigma = 2.7$).

While the teachers in the sample did participate in considerable professional development between the pre and post-test, the study did not examine the relative impact of specific types of professional development (e.g., classes versus coaching). Teacher growth may also be attributable to other project components, outside of professional development, such as the introduction and implementation of new instructional materials.

Teacher Instructional Practice

Direct evidence about the quality of mathematics instruction within Rapid City Area Schools and about changes to instruction over the course of the project comes from classroom observations conducted by the project's external evaluation team, Inverness Research Associates. Inverness collected the first set of classroom observation data in the spring of Year 2, focusing primarily on elementary grades, and including a few observations at middle school. In Year 3, they focused entirely on secondary grades, both middle and high school. In Year 7, they conducted observations across the full span, K-12. In Year 9, for reasons described later, they looked exclusively at middle school. Inverness conducted other evaluation activities in other years, but Years 2, 3, 7 and 9 were times of intensive site visits that included the rating of lessons in randomly selected classrooms.

During each of these intensive site visits, a team of three to seven researchers came to Rapid City for multiple days and observed teaching practice across the district (in addition to conducting other evaluation activities). Researchers visited classrooms in pairs or alone, having made arrangements a few weeks in advance with the teachers to be observed. Prior to observing a lesson, the researcher(s) interviewed the teacher about what was planned. Following the lesson, they asked the teacher to reflect on how it went.

Classroom observation samples. Inverness used a random stratified sampling approach to select teachers for observation. Project staff provided Inverness with a list of teachers who taught mathematics on a regular basis in a whole-class setting and, therefore, were observable. The list of teachers indicated teaching assignment, grade level, building, and number of hours of professional development completed within the project. The population of observable teachers within the district each year was approximately 330: 270 elementary teachers, 30 middle school teachers, and 30 high school teachers. In the early years, Inverness sought a representative sample of classrooms across the district in terms of schools, grade levels, those who had participated in 20 or more hours of professional development, and those who had not. Once Inverness drew the samples, teachers were invited to participate and were assured strict confidentiality. With this assurance, teachers were typically quite willing to be observed.

In later years, the sampling procedure remained similar, but Inverness also did some intentional re-sampling of teachers who had been observed in earlier years. In total, Inverness conducted 74 classroom observations reported in this study: 33 lessons in Years 2 and 3 combined, spanning both elementary and secondary, 27 lessons in Year 7, again

spanning both elementary and secondary, and 14 lessons in Year 9 at middle school grades only.

Classroom observation protocol. Each lesson was rated using a classroom observation protocol developed by Horizon Research, Inc. (2000a) for evaluation of the NSF-funded Local Systemic Change projects. This protocol was designed to align with the National Council of Teachers of Mathematics (2000) *Principles and Standards for School Mathematics* and is congruent with *PRIME*'s definition of effective instruction.

The protocol asks researchers to rate lessons across several dimensions, including lesson design, implementation, mathematics content, and classroom culture. Then the researcher synthesizes subcomponent ratings into an overall "Capsule" rating. Capsule ratings range from Level 1 (*Ineffective Instruction*) to Level 5 (*Exemplary Instruction*). The middle rating is Level 3 (*Beginning Stages Effective Instruction*). Level 3 (and Level 3 only) is subdivided further into increments of 3-Low (3L), 3-Solid (3S), and 3-High (3H). The project considers lessons rated 1 and 2 to be weak, lessons rated 3L and 3S to be competent, and lessons rated 3H, 4, and 5 to be strong. In the results that follow, lessons rated 3H, 4, and 5 are referred to as "highly-rated."

Researcher preparation. Inverness researchers conducting the *PRIME* classroom observations were trained by Horizon Research staff in the use of the classroom observation protocol as part of working on the evaluation of the Local Systemic Change projects. Over the course of a two-day training, researchers viewed and scored videotaped lessons and had to demonstrate sufficient inter-rater reliability on standardized "rating keys" (Horizon Research, Inc., 2000b). Since their initial training, Inverness researchers had observed lessons in pairs on a regular basis and conducted hundreds of classroom

observations across the country using the protocol. Training, pairing, and repeated use of the instrument helped to ensure high inter-rater reliability.

Data analysis. Frequency distributions of classroom observation ratings for different years and different grade bands are displayed graphically in Appendix F. To compare means, rating levels have been equated to numerical ratings. Rating level 3L has been equated to a numerical rating of 2.5, and rating level 3H has been equated to a numerical rating of 3.5. Means are compared using Cohen's effect size. The sample sizes involved are too small and the ratings are not normally distributed such that a t-test can be employed and p-values interpreted. Additionally, rating distributions have been consolidated into percentages of highly-rated lessons (3H, 4, and 5) and compared with national samples (Weiss, Pasley, Smith, Banilower, & Heck, 2003). These comparisons are reported in Appendix F as well.

Elementary Classroom Observation Findings: Year 2 versus Year 7

Elementary instruction was quite strong in Year 2 (the earliest observations), but considerably stronger still by Year 7. Average ratings were 3.3 ($\sigma = 0.8$) in Year 2 and 3.8 ($\sigma = 1.1$) in Year 7. Growth from Year 2 to Year 7 is characterized by an effect size of 0.6 (medium effect). By comparison to the national sample, the elementary lesson ratings are remarkably high. Already in Year 2, elementary instruction exceeded the national sample by a wide margin, and by Year 7, the strength was even more pronounced (see Appendix F).

Secondary Classroom Observation Findings: Year 3 versus Year 7

Classroom observation ratings at the secondary level in Year 3 were markedly lower than those at the elementary level in the same timeframe and showed negligible growth as of Year 7. The average rating at the secondary level in Year 3 was 2.4 ($\sigma = 0.8$), and the

average rating in Year 7 was 2.5 ($\sigma = 1.1$). Growth over this period is characterized by an effect size of 0.1 (between zero effect and small effect). Low observation ratings and lack of growth were troubling, but national comparison data (see Appendix F) indicated that Rapid City was not alone. In fact, RCAS exceeded the national sample for highly-rated lessons at the secondary level in both Year 3 and Year 7, but still RCAS and the project as a whole were highly motivated to improve.

Comparison of Elementary and Secondary Classroom Observation Findings: Year 7

After completion of evaluation activities for Year 7, the external evaluation team met with the project leadership team to present classroom observation findings and discuss program strengths and challenges, drawing on the full range of evaluation components (e.g., staff interviews, student focus groups, meetings with teacher leaders and coaches). The status of the efforts at the elementary and secondary levels were in stark contrast to one another. Elementary was doing great; secondary was not. Inverness noted some progress at the secondary level with pockets of strength, but clearly more work was needed to build a coherent K-12 program.

There were several critical components that contributed to the widespread success at the elementary level. These components represent a complex combination of assets the district had in place prior to *Project PRIME*, assets created through *PRIME*, and assets that were leveraged by the *PRIME* funding. They include:

- a clear vision for elementary mathematics teaching and learning consistent with national standards and research;
- a direct and explicit message from top district administrators about the nature and direction of elementary mathematics;

- the adoption and implementation of high-quality, research-based instructional materials;
- professional development for classroom teachers and ongoing classroom support from teacher leaders focusing on mathematics content, pedagogy, and the specific instructional materials;
- ongoing professional development and support for teacher leaders led by the district's elementary mathematics coordinator; and
- principals who were knowledgeable about and supportive of the mathematics improvement efforts.

In contrast to the strengths found at the elementary level, the external evaluation team found the following at the secondary level:

- lack of a clearly articulated district vision;
- lack of a unified effort to improve mathematics;
- a wide range of instructional materials in use;
- confusion about an inquiry-based approach to teaching mathematics;
- variation in principal understanding of and support for improving secondary mathematics teaching and learning.

These findings resonated with experiences across the full project leadership team. The process of bringing internal project leaders together with the external evaluation team to discuss the collection of assets and challenges was pivotal. The outside perspective and clear articulation of critical issues served to unify and inspire the project team. A truly powerful K-12 system appeared to be within the project's grasp, and project leaders committed themselves to achieve it.

Intensifying *PRIME* at the Secondary Level: Years 8 and 9

The next step was to share the external evaluation findings with additional key stakeholders, including building principals, math teacher leaders and coaches, and the school board. What emerged over the next few months was a plan for an intensive effort at the middle school level, in particular. This was a time of students emerging out of a strong elementary program into an uneven and lackluster middle school program, thus making a focus at the middle grades especially timely and promising. District leaders clarified the district vision and then empowered middle school teachers to develop and implement a path forward. Out of this work came the adoption of new instructional materials, creation of new professional development offerings tailored specifically to middle school teachers, and bolstering of the teacher support system. Among the new teacher supports was the establishment of a dedicated professional development team to lead the implementation of the new instructional materials. This team was comprised of practicing middle school teachers who were implementing the new materials in their own classrooms. Team members met regularly as a group, served as leaders within their buildings, provided support to their grade-level peers, and, in turn, were supported by the district's secondary math coaches and secondary math coordinator.

Middle School Classroom Observation Findings: Year 9

To check progress of the intense effort underway, the project asked Inverness to return in Year 9 and conduct classroom observations exclusively at the middle grades. In the findings that follow, all of the middle school ratings from Years 2, 3, and 7 have been aggregated into a single sample ($N = 17$), and that sample is compared to the ratings from Year 9 ($N = 14$). The middle school data were aggregated across Years 2, 3, and 7 in order

to arrive at a sufficient middle school-only sample size. Aggregating in this way makes sense because of the specific interest in detecting changes subsequent to Year 7 and given that the middle school observations were consistently low in Year 7 and prior. The average lesson rating for the earlier observations was 2.1 ($\sigma = 0.7$), and the average rating for Year 9 was 3.3 ($\sigma = 1.0$). Growth from Year 7-and-prior to Year 9 is characterized by an effect size of 1.4 (large effect).

The fact that classroom observation ratings from Year 7-and-prior had a mean rating of 2.1 affirms the project's intensive focus on the middle school level during Years 8 and 9. The classroom observation findings for Year 9 indicate an astonishing jump in the quality of instruction at middle school and suggest a highly effective effort. Furthermore, the percentage of highly-rated lessons increased from below the national comparison sample to well above.

When the external evaluation team and project leaders met to discuss Year 9 evaluation findings, the following key factors contributing to the progress at the middle school level were noted:

- a clear vision and clear message from the district about the intended nature and direction of the math program at the middle school level, resulting in greater alignment between the elementary and middle school level than seen previously;
- greater district-level and building-level leadership and support for instructional improvements in mathematics at the middle school level than seen previously;
- the adoption of new instructional materials, and the expectation that these instructional materials would be the predominant instructional materials used to teach mathematics at the middle school level;

- the ongoing professional development being provided to teachers; and
- improved principal understanding and support inquiry-based mathematics teaching.

Path Forward: Year 10 and Beyond

Ten years into the project, high school teachers are now making a bold move to shift their instructional materials (see Appendix G for more details about instructional materials). High school teachers and leaders are also making plans to ramp up professional development, following the path of the recent middle school efforts and preparing for enactment of the *Common Core State Standards for Mathematics* (Common Core State Standards Initiative, 2010). Additional classroom observations at the high school level are being conducted in advance of their adoption of new instructional materials to serve as baseline data as the new materials are phased in and as the district's math program transitions beyond the end of the NSF award period. The partnership remains active and committed to support the intensified efforts at the high school level and to sustain the efforts at the elementary and middle school levels.

Relationships between Classroom Observations and Other Project Data

Before concluding, it is worthwhile to note connections between the classroom observation ratings and other project data. Classroom observations provide the most direct evidence of changes in teaching within Rapid City, but student outcome data provide valuable indirect evidence that complements the classroom observations, as do measures of changes in teacher attributes and measures of changes to the system as a whole.

Student achievement on the MARS test at grade 4 serves as a good example. Those data show a pattern that closely parallels the elementary classroom observation data - with

solid performance in Year 3 and even stronger performance in Year 9. The eighth grade MARS results are consistent as well. The low student performance in Year 3 on the MARS test at grade 8 corresponds with low classroom observation ratings at middle school over the course of the project up through Year 7. The performance of the eighth graders on the MARS test in Year 9, while still below the performance of the elementary students, shows strong growth, and, again, that growth is consistent with the dramatically improved classroom observation ratings at the middle grades as of Year 9.

Another connection worthy of consideration is the connection between changes in classroom instruction and the closing of achievement gaps on the DSTEP. The project has been focusing heavily on meeting the needs of *all* learners, and achievement gaps have been shrinking on the DSTEP to a degree not evident across the rest of the state, especially gaps between American Indian students and non-American Indian students and between those identified as economically disadvantaged and those not economically disadvantaged. The reduction of these achievement gaps suggests that significant changes to instruction are occurring within RCAS classrooms and that the changes are paying off, especially for those historically underserved audiences.

From an educational research perspective, it is important to be cautious not to draw overly strong conclusions among these loosely affiliated data sets. The data in many instances have inherent limitations (e.g., teacher observation ratings not tied to student achievement scores). But from the perspective of the *PRIME* partnership seeking to change a complex system, the collection of findings is compelling, and the findings are all the more compelling due to plausible, if not completely definitive, connections between them. A hallmark of *Project PRIME* has been the tracking of system measures as described in this

article, sharing indicators of progress and persistent challenges, attending to multiple components of the logic model concurrently, and exploring connections between independent data sources.

Limitations of the Study

One limitation is the small size of the classroom observation samples. Classroom observations are time consuming and require special expertise to conduct. Nonetheless, even with small sample sizes, the project has derived great benefit from having this direct, external measure of the quality of mathematics instruction and its change over time. A second limitation is that baseline classroom observation data were not collected prior to the start of the project. This precludes determination of the project's full impact over its entire span. A third limitation is that multiple project components (e.g., coursework, coaches, instructional materials) have been implemented concurrently. Project leaders perceive that having a mix of project components has been highly valuable, but having delivered a suite of interventions all at once and with a voluntary participation model, it is difficult to discern the relative impact, relationships, and optimal sequencing of individual components.

Lessons Learned

The project is generating a compelling collection of data that affirms the project's vision for effective mathematics instruction. Having classroom observation data to complement student outcome data has been invaluable – to look for overlap and consistency from one data source to another, to reveal different types of findings that are only evident with one tool or another, and ultimately to help steer the project's implementation.

The project has increased its appreciation for well-designed instructional materials that are implemented consistently from classroom to classroom across the district and that build vertically from kindergarten through high school. The alignment of assessments with the instructional materials is also key. The project is pleased that RCAS students at least mirror their peers statewide on the DSTEP despite less than complete congruence between the test and the project. The MARS instruments serve as better indicators of overall project impact at the student level, but they require additional effort and resources and therefore have been administered only on a limited basis. The MARS instruments are better aligned with the direction the state is headed with the new *Common Core State Standards for Mathematics* (Common Core State Standards Initiative, 2010), however, so the need for MARS testing as a supplement to the DSTEP may soon fade.⁶

We have gained insights into the facets of the project that have been most helpful to teachers at different places on the path to becoming stronger teachers of mathematics – when coaching is perceived to be most helpful, when classes are, and when it matters most to have the right instructional materials. These lessons have been learned in part through teachers' self-reporting (Apaza, 2009) and also corroborated and refined through classroom observations and associated teacher interviews.

We have been reminded time and again that K-12 systemic reform requires great patience. Ten years and counting, the project still has much work to do, sustaining the progress and infrastructure at the elementary and middle grades and intensifying the work at the high school level. Additional effort is also required to fully integrate lessons from the

⁶ This claim is based in part on the fact that the MARS instruments have an open-response format as opposed to the multiple-choice format of the DSTEP. MARS items ask students to communicate their thinking, which is consistent with the Common Core State Standards.

project into the university setting, both for teacher preparation and for regular university mathematics classes. The district has built a strong infrastructure for continued teacher development, and the university partners have built their own capacity, learning vast amounts within the K-12 setting that is informing university transformation, but this is a long journey.

With the recent middle school efforts, we have learned the importance of a consistent and coherent message from top administration about the direction the mathematics program is moving. We have observed a wonderful example of empowering teachers to develop an instructional improvement plan and then supporting them to implement it. As the middle school effort continues and as the high school effort ramps up, instructional leadership and professional development infrastructure remain critical. The district has tremendous promise to achieve an exemplary system across all grades, K-12, but such an accomplishment will require continued nourishment of the infrastructure that has been established and continued support from the partnership. Additional reflections and advice to others engaged in similar endeavors is offered in Appendix H.

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Appendix A: PRIME Coursework

The graduate-level coursework provided to teachers through *Project PRIME* has built on the work of many others. Examples of nationally recognized teacher professional development programs upon which the project has drawn include: Teachers Development Group (*Best Practices* and *Numerical Reasoning*), Mathematics Education Collaborative (*Patterns, Functions, and Algebraic Thinking* and *Building Support for School Mathematics: Working with Parents & the Public*); Education Development Center (*Developing Mathematical Ideas* and *Fostering Algebraic Thinking*); TERC (*Investigations Workshop for Transforming Mathematics: Professional Development Institute* and *Relearning to Teach Arithmetic*), and the Vermont Mathematics Partnership (*Geometry in the Middle Grades*). Other key resources have included the work of Carpenter, Fennema, Loef Franke, Levi, & Empson (1999), Richardson (1998), and Van de Walle (2003).

Instructors for *PRIME* offerings have been drawn from district, university, and other project staff, often trained by outside program developers. In some instances, entire courses have been taught within RCAS by an outside program developer or agent, typically paired with an internal project member.

There has been a shift over time in which almost all of the professional development for teachers has been developed and facilitated by project staff. The philosophy underpinning this work is consistent with the tenets of effective professional development as outlined in the *Standards for Professional Learning* (National Staff Development Council, 2001, 2011) along with the other resources previously cited.

Courses have been designed to improve teacher effectiveness in the classroom in such a way that student learning is positively impacted. The pedagogy and the mathematics tasks have

been chosen in an effort to model desirable practices within K-12 classrooms. While most of *PRIME*'s coursework was developed prior to publication of the *Common Core Standards for School Mathematics* (2010), there exists good alignment with the *Common Core* and, in particular, with the *Standards for Mathematical Practice*.

The following mathematics task and facilitator notes provide a taste of *Project PRIME* coursework. This particular task, the Garden Problem, is one of a series of tasks designed to move teachers through the process of understanding patterns used in early elementary grades and how these and similar pattern problems can be used in higher grades to develop a deep understanding of linear functions. This particular pattern was found in a *MathScape* middle school unit published by McGraw-Hill (2005), but any number of pattern problems would work just as well. The facilitator notes, written by the designer of the course, are a description of the questions to be used with a whole series of pattern problems for developing an understanding of linear functions (see facilitator notes that follow the Garden Problem).

After the facilitator notes are titles and descriptions of ten graduate-level courses developed by *PRIME*. Each course is 30 contact hours and is offered for two graduate credits. Taken together, these ten courses qualify a teacher for a K-12 Mathematics Specialist endorsement from the South Dakota Department of Education.

Sample Mathematics Task for Teachers: THE GARDEN PROBLEM

Explain your thinking for all parts of this problem. Here are three sizes of gardens framed with a single “row” of tiles. Build these three gardens using two colors of color tiles.

Garden 1 **Garden 2** **Garden 3**

- Using color tiles, build and then draw the next two steps in the pattern. How many border tiles (the white tiles) would you need for *Garden 4* and for *Garden 5*? Explain how you know. Begin a table that shows the number of tiles used for the border of each garden.
- How many tiles would you need to make a border around gardens of each of these lengths? Explain.
 - Garden 10
 - Garden 100
- What patterns do you notice in the models/drawings? In the table?
- Explain how you would figure out the number of tiles you would need for a garden of any length?
- How does your rule relate to the model (show geometrically why your rule makes sense)?
- Graph the values in your table on a coordinate grid. Use the horizontal axis (x -axis) to show the input (garden) number and the vertical axis (y -axis) to show the number of tiles in the border for that step (the output).
- Tell how you would find the length of the garden if you knew only the number of tiles in the border. Use your method to find the length of the garden if the following numbers of tiles are used for the border. Explain your thinking.
 - 68 tiles
 - 152 tiles
 - 512 tiles

STOP here for whole group discussion.

There were a number of methods for visualizing the ways in which the pattern was growing:

• $2n + 6$

• $2(n + 2) + 2$

• $2(n + 3)$

• $3(n + 2) - n$

8. Are these expressions equivalent? How do you know?
9. Theoretically, what would the step before Garden 1 (the “zero” step) look like? (Think about how the garden is “growing” in each step; go backwards to think about the “zero” step.) Add this information to your input/output table. Does it “match” the other patterns in the table? Add this point to your graph.
10. Using the expression that is in simplest form, $2n + 6$, compare your table, your graph, and the expression.
 - a. Where does the “2” in the expression “show up” in your table? In your graph? In the model?
 - b. Where does the “6” show up in your table? In your graph? In the model?

STOP here for whole group discussion.

FACILITATOR NOTES

General Instructions and Questions for Pattern Problems

- All content should emerge via small group work and whole group presentations.
 - Begin with 2-3 minutes of individual think time and then work together in small groups.
 - End with whole group processing.
1. Build or draw the next two steps in the pattern.
 2. Describe what the 10th step will look like.
 3. How many _____ (tiles, cubes, toothpicks, etc.) in the 10th step?
 4. Record your findings in a table (relate the step # to the # of _____ in that step).
 5. What patterns do you notice in the models/drawings? In the table?
Note: Patterns out of context are open to interpretation. For example 2,4,6,8... could be 2,4,6,8,10,12... or 2,4,6,8,2,4,6,8... or 2,4,6,8,6,4,2,4,6,8... etc.
 6. Write a rule in words describing how the pattern is growing.
 - Recursive rule (as participants describe this pattern, “label their thinking” by explaining how this is called recursion or the recursive pattern. What is the disadvantage of the recursive rule? You always have to know the step before to use it.
 - General rule for any step number
 7. How many _____ in the 100th step? How do you know?
 8. How could you figure out the number of _____ in any step of the pattern? (the “*n*th” step)?
This may be the recursive pattern, the general rule in words, and/or the general rule written as an expression or equation (i.e. relating the step number to the number of _____). After whole group processing of The Garden Problem, participants should be looking beyond the recursive rule for the general rule. Later, we will be relating the constant rate of change in linear function tables (the recursive rule) to the slope of the line on the graph and to the $y = mx + b$ form of an equation.
 9. How does your rule relate to the model (show geometrically why your rule makes sense)?
 10. Can you see a different way to visualize the pattern? If so, write a different algebraic expression that matches it and show geometrically why it makes sense. **Different methods will emerge during the whole group discussion.**
 11. Write your rule for the “*n*th” step using an algebraic expression or equation.
Have participants share different solution methods with the whole group (put on overheads, chart paper, etc: some ways to record the different approaches). Make sure it becomes clear to the whole group how each expression relates to the concrete model or drawing. Some participants may not have an algebraic expression for the first pattern problem they do. This will also emerge as participants share in whole group.

***Refrain from simplifying these expressions at this point. We want the expressions to relate to the model. See next step.

12. Are your expressions equivalent? How do you know?

Check several steps to see if each expression would work. Simplify the expressions. Discuss simplest form.

13. What would the “zero” step look like? Add this information to your table.

Eventually we will relate this step to the *y-intercept* in the graph and in the $y = mx + b$ form of an equation.

14. Graph the values in your table on a coordinate grid. Use the horizontal axis (*x*-axis) to show the step number and the vertical axis (*y*-axis) to show the number of _____.

Have a short discussion of independent and dependent variables. Ask participants if anyone can explain; if not, facilitator may explain. They will be using just Quadrant I for the pattern problems, so use centimeter grid paper. They will use pre-printed coordinate grid paper with all four Quadrants when we get to linear functions and slope.

15. Does it make sense to connect the points?

No, not in the context of this problem. However, you may want to see the “shape” of the graph or the “trend”. Connect the points recognizing that there is no half-step, quarter-step, etc. just to see the shape of the graph. Alternatively, connect the points with a dotted line to show that you recognize that the ordered pairs are discrete points.

Note: Sometimes students think that you must connect the points in the order given; if the values in the table weren’t “in order” their graphs would be incorrect. Hopefully, this won’t be an issue for our participants, but be aware of the possibility that it may come up.

16. What representations have we used so far?

Concrete models, pictures, words, tables, graphs, symbols (expressions/equations).

17. What patterns do you notice in the graph? How do these patterns relate to the model? The table? The expression?

By the end of the series of pattern problems, participants will be looking for the slope and the *y*-intercept in all four of the representations and seeing the connections among the four.

Note: pattern and real-world problems will also be used to develop concepts of quadratic and exponential functions.

K-12 MATHEMATICS SPECIALIST ENDORSEMENT COURSEWORK

ED 601: Foundations and Issues of Mathematics Education (2 credits)

This course provides an introduction to K-12 mathematics content and process standards, makes the case for using an inquiry-oriented approach in classrooms, and looks at current research. Participants will gain an understanding of the components needed to create a learning environment that encourages and supports *all children* in building understandings, making connections, reasoning, and solving problems as described in *Principles and Standards for School Mathematics*, published by the National Council of Teachers of Mathematics. (Fulfills South Dakota Department of Education Standards 3b 3e 4a 4d [Administrative Rule of SD 24:15:06:39])

ED 611: Algebraic Reasoning for K-12 Educators (2 credits)

This course is designed for K-12 educators to deepen their understanding of algebraic concepts that build from kindergarten through high school. Consistent with the *Principles and Standards for School Mathematics*, published by the National Council of Teachers of Mathematics, the course emphasizes patterns and functions; representation and analysis of mathematical situations; using models and symbols to represent quantitative relationships; and analyzing change. Instruction revolves around rich mathematical tasks and includes explicit attention to questioning, conjectures, and justification. Participants reflect on the benefits and challenges of this kind of learning environment and consider implications for their own teaching. (Fulfills SD Standards 3a 3b 3d 4c)

ED 621: Geometry & Measurement for K-12 Educators (2 credits)

This course is designed for K-12 educators to deepen their understanding of geometry and measurement concepts that build from kindergarten through high school. Consistent with the *Principles and Standards for School Mathematics*, published by the National Council of Teachers of Mathematics, this course emphasizes characteristics of two- and three-dimensional shapes; spatial relationships and reasoning; transformations and symmetry; units, systems, and processes of measurement; and applying techniques, tools and formulas to determine measurement. Instruction revolves around rich mathematical tasks and includes explicit attention to questioning, conjectures, and justification. Participants reflect on the benefits and challenges of this kind of learning environment and consider implications for their own teaching. (Fulfills SD Standards 3a 3b 3d 4c)

ED 631: Data Analysis & Probability for K-12 Educators (2 credits)

This course is designed for K-12 educators to deepen their understanding of data analysis and probability concepts that build from kindergarten through high school. Consistent with the *Principles and Standards for School Mathematics*, published by the National Council of Teachers of Mathematics, this course emphasizes methods of collecting, organizing, and displaying data; using appropriate statistical methods to analyze data; evaluating inferences and predictions that are based on data; and understanding and applying basic concepts of probability. Instruction revolves around rich mathematical tasks and includes explicit attention to questioning, conjectures, and justification. Participants reflect on the benefits and challenges of this kind of learning environment and consider implications for their own teaching. (Fulfills SD Standards 3a 3b 3d 4c)

ED 641: Understanding Student Thinking in Numbers and Operations (2 credits)

This course is designed to deepen teachers' awareness of ways that students come to understand whole numbers, rational numbers, and operations. Emphasis is placed on common student difficulties and on how teachers can help to move students from a procedural approach to conceptual understanding.

(Fulfills SD Standards 3a 3b 3d 4a 4b 4c 4d)

ED 651: Understanding Student Thinking in Algebra (2 credits)

Based on recent research in mathematics education, this course provides opportunities for educators to deepen their understanding of how K-12 students develop algebraic reasoning. The course focuses on conceptual and procedural understanding of the key algebraic ideas of equality, variables and equations, patterns and functions, proportional reasoning, symbolic representation, and inductive and deductive reasoning.

(Fulfills SD Standards 3a 3b 3d 4a 4b 4c 4d)

ED 661: Understanding Student Thinking in Geometry & Measurement (2 credits)

This course is designed to help teachers think through major ideas within the areas of K-12 geometry and measurement and to use recent research to examine how students develop their ideas. The course is also designed to raise awareness of common student misconceptions and to deepen teachers' knowledge of effective instructional practices.

(Fulfills SD Standards 3a 3b 3d 4a 4b 4c 4d)

ED 671: Assessment for School Mathematics (2 credits)

This course supports educators in assessing what K-12 students know, what they can do, how they think mathematically, and their attitudes toward mathematics. Current assessment practices, from informal questioning to standardized testing, are explored, and the use of assessment information to guide instruction is emphasized. The course also considers national data and examines connections between staff development, classroom practice, and student outcomes, thereby laying a foundation for discussions about the future direction of local, state, and national mathematics improvement efforts.

(Fulfills SD Standards 3e 4a 4b)

ED 741: Historical Development of Mathematical Concepts (2 credits)

This course traces the origins and development of key concepts in the history of mathematics starting with early Egyptians, Babylonians, and Mayans and continuing to current times.

Emphasis is given to the impact of mathematical discoveries on the civilizations that gave rise to them and to the impact of these discoveries on subsequent mathematical thought.

(Fulfills SD Standard 3c)

ED 751: Leadership in School Mathematics (2 credits)

This course focuses on how to provide effective professional development for K-12 teachers of mathematics and how to support meaningful change within an educational system. Lessons are drawn from research in mathematics education as well as research about improving schools.

Topics include creation of a demonstration classroom, engaging key stakeholders (e.g., parents, administrators, and community members), forming and facilitating study groups, peer coaching, mentoring, and curriculum review. (Fulfills SD Standard 4e)

Appendix B: Other PRIME Professional Development

Professional Development for Math Leaders

The very first professional development experience for Mathematics Teacher Leaders (Math Leaders or MTLs) was a weeklong training in 2003 to build a clear understanding of the philosophy and vision for the instructional change they were going to be supporting in the mathematics program for Rapid City Area Schools. The training focused specifically on the research articulated in *Adding it Up* (Kilpatrick, Swafford, & Findell, 2001) and *Making Sense* (Hiebert et al., 1997). The initial training also provided an opportunity for the group of Math Leaders, along with district administrators and other project partners, to work together to define roles and responsibilities of the MTLs. This training began building a collaborative work group that would continue to meet throughout the life of the project.

Mathematics Teacher Leaders meet one half-day per week to support their own professional growth. These study sessions have focused on three major areas: 1) coaching, 2) mathematics content with pedagogy, and 3) district work. The balance of time spent on these three areas is adjusted based on the needs of the district and of the Math Leaders at a particular time. Below are specific examples of study or work in each of these three areas.

Study to improve coaching skills. A majority of study time has focused on current research in the emerging field of mathematics coaching. The following books have served as guides:

- Content-focused coaching (West & Staub, 2003)
- The math coach field guide (Felux & Snow, 2006)
- Cognitive coaching (Costa & Garmston, 2002)
- The PRIME leadership framework: Principles and indicators for mathematics

education leaders (National Council of Supervisors of Mathematics, 2008)

- Cultivating a math coaching practice: A guide for K-8 math educators (Morse, 2009)
- Student-centered coaching: A guide for K-8 coaches and principals (Sweeney, 2011)

On-line resources from these authors have also been accessed for current articles.

In the past few years, MTLs have been asked to provide evidence of practicing the coaching strategies found in these guides. Evidence and documentation of coaching are then shared and discussed to assist all MTLs in growing as coaches. In Year 10, for example, after completing *Cognitive Coaching* training, several MTLs shared videotaped segments of themselves engaged in authentic coaching sessions and reflected on these sessions with their peers.

Study to improve mathematics content knowledge with pedagogy. Staff from Black Hills State University have supported district staff in offering some of the mathematics content classes from the K-12 Math Specialist endorsement sequence. Math Leaders have also had opportunities to participate in the specialist classes as they are offered across the district to classroom teachers. Three MTLs and the district's elementary mathematics coordinator have completed the full sequence of the K-12 Math Specialist endorsement.

In a usual year, about one third of MTL sessions involve mathematics content and pedagogy study. Complementing the K-12 Mathematics Specialist coursework, the *Developing Mathematical Ideas* (DMI) series (Schifter, Bastable, & Russell, 2000-2007) has served as a key resource. DMI sessions have typically been facilitated by district and university staff working together. Two MTLs attended national training to become certified DMI facilitators and teach DMI at the district level as well.

With South Dakota's adoption of the *Common core state standards for mathematics*

(2010), much of the recent math content and pedagogy study has focused on understanding the mathematics in each standard and the connection between standards and domains.

District Work. Over the years the MTL group has written district curriculum, standards-based report cards, and revisions to both. Pacing guides, assessments, and screeners have been developed, adapted, and implemented as well through this group of building-based MTLs.

Lesson Study

A form of lesson study called the *Learning Lab Initiative* has been initiated by the district Math Coordinators and Math Leaders. Learning labs provide a setting and forum for educators to observe student learning and instruction in a colleague's classroom and reflect on practice in their own classrooms. Learning labs have focused on using formative assessment, supporting student discourse, and the use of a simple learning cycle. The learning cycle involves launching a task, monitoring and supporting student learning, and debriefing the mathematics of the lesson. An additional purpose of the learning labs has been to increase collaboration, dialogue, and reflection among teachers.

Those who designed the learning lab process recognized the importance of coaching and of follow-up over time as professional development components. Learning labs consist of three learning experiences: coaching for the host teacher, the learning lab event, and follow-up study sessions. This total learning lab experience is consistent with the Gorman, Mark, and Nikula (2010) model of lesson study that includes a cycle of planning, teaching, observing, and reflecting on a lesson.

During the coaching experience, a facilitator (a coach) meets with the lab host (a classroom teacher) to discuss a focus for the coaching cycle. Throughout the cycle, the facilitator provides support and resources to refine instructional strategies and to assist the host in

preparing for the learning lab event. The half-day learning lab event utilizes a protocol that includes a pre-brief, classroom observation, and debrief. In addition, monthly study sessions are held afterwards for the purpose of collaborating and further reflecting on the learning lab process.

Learning lab teams have been diverse in grade levels and schools. Each cohort has had multiple grades and brought together teachers from buildings that serve diverse student populations. Each cohort has studied together for a semester with four or five study sessions and three of four classroom lab observations. At the start of each lab cycle, each cohort has considered problems of practice or areas of instruction to improve and, based on the work of Wiggins & McTighe (2005), has formulated an overarching student-based essential question. Study sessions and student-centered debriefing of lessons are viewed through the lens of this essential question. Lastly, all lessons taught and discussed have been "in-sequence lessons" from district-adopted instructional materials. No new lessons have been created for the labs. The goal is to improve teacher practice in using the adopted materials. This is part of staying the course and providing consistency for students.

Appendix C: PRIME Administrator Training

In the second year of the project, *PRIME* was invited by Education Development Center to receive training in the *Lenses on Learning* professional development program. *Lenses on Learning* is designed to help administrators as instructional leaders in their schools and districts, to think through the ideas that underlie standard-based reform mathematics and to relate those ideas to their own work of supporting the reform efforts. Two project staff members attended the two-week training in the three modules that comprised the program at that time.

During the first school year after *PRIME* staff were trained, all three of these modules were offered within RCAS on an invitational bases. More than half of the elementary building principals attended at least two of the three modules, as well as several district-level administrators. In the second year, the district required all building administrators to attend Module One of the training, and the majority of school administrators were able to comply. All three modules were offered each year for the next two years. In the fourth year after *Lenses on Learning* training began in the district, an additional module was released by Education Development Center with a specific focus on supervision and more secondary examples. This new module was offered to all building administrators and was well attended by both elementary and secondary principals.

Sometimes the trainings were held in a location away from the district in order to avoid distractions and allow principals to focus. On the whole, the trainings have been well received.

As one elementary principal recalls,

In contrast to how I had been taught as a student, these initial sessions allowed us to actually experience a problem-solving approach to mathematics. We were given a problem, and we were encouraged to think and collaborate. I learned that the approaches that I had developed as an adult to solve math problems were strategies that are actually taught to students today. I remember thinking that if I had been taught math

in these active, engaging, sense-making ways that I would likely be more confident and competent mathematically as an adult.

Lenses on Learning trainings have continued to be offered as new administrators have been added to the district.

Appendix D: Student Achievement—DSTEP Results

The Dakota Standardized Test of Educational Progress (DSTEP) is a multiple-choice test administered each spring at grades 3 through 8 and grade 11. It is a strong measure of procedural fluency, but less strong in measuring conceptual understanding, communication, representation, and numerous other strands of mathematical proficiency that the project values. Regardless of how well the DSTEP is aligned with *PRIME*'s overall vision and approach, it is the statewide accountability measure and holds high importance for project leaders and other key stakeholders. Student scores are reported in terms of 4 performance levels: below basic, basic, proficient, and advanced.

From the first year of the project through the most recent DSTEP data available, 2003 through 2011 (Year 1 through Year 9), the percentage of RCAS students scoring at the proficient level or above increased from 53% to 72% across all grades tested. While that represents significant growth, it essentially mirrors the growth of the rest of the state, which increased from 60% to 78% scoring at the proficient level or above. RCAS has outperformed the state somewhat at elementary grades and underperformed the state somewhat at secondary grades, but on the whole, the magnitude of growth within RCAS has tracked the rest of the state on this measure. What accounts for the overall growth in student achievement as measured by the DSTEP over the past nine years may well be increased attention statewide to mathematics during these years, with extensive professional development opportunities available both within and outside of RCAS. The growth may also be due to changes in the test instrument, changes in proficiency cutoff scores, and related measurement artifacts.

A more powerful DSTEP story exists related to the closing of the achievement gap for American Indian students and for those identified as economically disadvantaged. The gap in

achievement between American Indian students and non-American Indian students in RCAS in Year 1 was 37 percentage points. By Year 9, that gap had closed to fewer than 22 percentage points (Figure 2). Similarly, the gap for economically disadvantaged students in RCAS dropped from 35 percentage points in Year 1 to 19 percentage points in Year 9. For the rest of South Dakota over the same period, the gaps did not close nearly as dramatically: from 37 to 35 percentage points for American Indian students and from 26 to 24 percentage points for economically disadvantaged students.

Figure 2. Closing of the achievement gap between American Indian and non-American Indian students in Rapid City, comparing Year 1 to Year 9.

Key to closing the achievement gaps has been strong growth in performance among American Indian students and those identified as economically disadvantaged. Figure 3 shows the growth in achievement of American Indian students within RCAS compared to those across the rest of the state. From Year 1 to Year 9, the percentage of American Indian students within RCAS scoring at or above the proficient level increased by 31 percentage points. Across the rest of the state, the increase was considerably less at 20 percentage points. The growth for

economically disadvantaged students within RCAS showed a similar increase of 29 percentage points, compared to only 19 percentage points outside of RCAS. These data suggest that *Project PRIME* is having relatively greater impact on students historically underserved in mathematics.

Figure 3. Growth in achievement among American Indian students in Rapid City Area Schools compared to American Indian students across the rest of South Dakota, comparing Year 1 to Year 9.

Appendix E: Student Achievement—MARS Results

To complement DSTEP data, the project introduced Balanced Assessments in Mathematics, developed by Mathematics Assessment Resource Service (MARS). MARS tests are open-response performance assessments to be completed within approximately 40 minutes. Each test includes five in-depth tasks spanning four mathematical strands: number and operations; algebra; geometry and measurement; and data analysis, statistics, and probability. The project considers MARS tests to be well aligned with *PRIME*'s overall vision and approach.

The project administered MARS tests to a sample of 4th and 8th graders in the spring of Year 3 and again in the spring of Year 9. At grade 4, one randomly selected class per elementary school building was tested. At grade 8, one randomly selected class per 8th grade mathematics teacher was tested. This protocol yielded sample sizes of approximately 200 to 300 students per grade level per year from the full population of approximately 1,000 students per grade level. Tests were scored using detailed rubrics that accompany the tests. Raw scores were converted to performance levels, Level 1 through Level 4, according to prescribed cutoffs. The project interprets Level 3 to be proficient and Level 4 to be advanced, akin to DSTEP performance levels.

Figure 4 shows increased student achievement on MARS from Year 3 to Year 9 at both grade 4 and grade 8. The growth at grade 4 was statistically significant with Cohen's effect size of 0.4 (medium effect), $p < 0.1$. The growth at grade 8 was statistically significant with Cohen's effect size of 0.5 (medium effect), $p < 0.5$.

Figure 4. Growth in student achievement as measured using MARS tests, comparing Year 3 to Year 9.

Appendix F: PRIME Classroom Observation Results

Frequency distributions of classroom observation ratings for different years and different grade bands are displayed graphically below. To compare means, rating levels have been equated to numerical ratings. Rating level 3L has been equated to a numerical rating of 2.5, and rating level 3H has been equated to a numerical rating of 3.5. Means are compared using Cohen's effect size. The sample sizes involved are too small and the ratings are not normally distributed such that a t-test can be employed and p-values interpreted.

Comparison with National Sample. In 2003, Horizon Research, Inc. completed a study providing a snapshot of K-12 classroom instruction in mathematics across the United States (Weiss et al., 2003). This study serves as a national comparison for *Project PRIME*'s classroom observation ratings. The sample sizes for the national study at each grade band are as follows: elementary $N = 57$, middle school $N = 66$, and high school $N = 61$. The percentage of highly-rated lessons nationally at each grade band is shown below in comparison to the percentage of highly-rated lessons observed in Rapid City Area Schools.

Elementary Classroom Observation Findings: Year 2 versus Year 7

Classroom observation ratings at the elementary level are shown for Year 2 ($N = 20$) and Year 7 ($N = 14$). Average ratings were 3.3 ($\sigma = 0.8$) in Year 2 and 3.8 ($\sigma = 1.1$) in Year 7. Growth from Year 2 to Year 7 is characterized by an effect size of 0.6 (medium effect).

Figure 5. Distribution of classroom observation ratings at elementary grades, comparing Year 2 with Year 7.

Figure 6 consolidates the distributions shown in Figure 5 into percentages of highly-rated lessons (3H, 4, and 5). The percentage of highly-rated lessons at the elementary level within the national sample is shown as well.

Figure 6. Percentage of highly-rated lessons within elementary classrooms, comparing Year 2 with Year 7 and with national sample.

These classroom observations indicate solid teaching at the elementary grades within RCAS as of Year 2 (the earliest observations) and stronger still as of Year 7. By comparison to the national sample, the elementary lesson ratings are remarkably high. Already in Year 2, elementary instruction exceeded the national sample by a wide margin, and by Year 7, the strength was even more pronounced.

Secondary Classroom Observation Findings: Year 3 versus Year 7

Figure 7 displays classroom observation distributions at the secondary level for Years 2 and 3 combined (N = 13) and for Year 7 (N = 14). Ten of the 13 secondary observations in Years 2 and 3 occurred in Year 3. For simplicity in reporting from here forward, that sample will be designated as Year 3. The average rating in Year 3 was 2.4 ($\sigma = 0.8$), and the average rating in Year 7 was 2.5 ($\sigma = 1.1$). Growth over this period is characterized by an effect size of 0.1 (between zero effect and small effect).

Figure 7. Distribution of classroom observation ratings at secondary grades, comparing Year 3 with Year 7.

Figure 8 shows the data of Figure 7 consolidated into percentage of highly-rated lessons at the secondary level for Year 3 and Year 7. The corresponding percentage from the national sample is also shown.

Figure 8. Percentage of highly-rated lessons within secondary classrooms, comparing Year 3 with Year 7 and with national sample.

Classroom observation ratings at the secondary level in Year 3 were markedly lower than those at the elementary level in the same timeframe (specifically, Year 2) and showed negligible growth as of Year 7. The percentages of highly-rated lessons shown in Figure 8 also show a lack of growth, but the national comparison data indicate that Rapid City was not alone. In fact RCAS exceeded the national sample for highly-rated lessons at the secondary level in both Year 3 and Year 7.

Middle School Classroom Observation Findings: Year 9

In the findings that follow, all of the middle school ratings from Years 2, 3, and 7 have been aggregated into a single sample ($N = 17$), and that sample is compared to the ratings from Year 9 ($N = 14$). The middle school data were aggregated across Years 2, 3, and 7 in order to

arrive at a sufficient middle school-only sample size. Aggregating in this way makes sense because of the specific interest in detecting changes subsequent to Year 7 and given that the middle school observations were consistently low in Year 7 and prior. The frequency distribution associated with the earlier observations (Year 7 and prior) is compared to Year 9 observations in Figure 9. The average lesson rating for the earlier observations was 2.1 ($\sigma = 0.7$), and the average rating for Year 9 was 3.3 ($\sigma = 1.0$). Growth from Year 7 and prior to Year 9 is characterized by an effect size of 1.4 (large effect). □

Figure 9. Distribution of classroom observation ratings at middle grades, comparing Year 7 and prior with Year 9.

Finally in Figure 10, the percentage of highly-rated lessons at the middle school level for Year 7 and prior is shown together with the percentage of highly-rated lessons in Year 9 and with national comparison data specific to middle grades.

Figure 10. Percentage of highly-rated lessons within middle school classrooms, comparing Years 7 and prior with Year 9 and with national sample.

Appendix G: PRIME Instructional Materials

Concurrent with *PRIME*'s launch in Year 1, RCAS adopted and began transitioning to the use of new instructional materials: *Investigations in Number, Data, and Space* (developed by TERC) at the elementary grades and *MathScape* (developed by Education Development Center) at the middle grades. Both sets of instructional materials are student-centered, inquiry-oriented, and consistent with the project's vision. At the high school level, the landscape of instructional materials was more complicated and varied in the first few years, including a mix of more traditional, teacher-centered textbooks together with pilot testing of *Discovering Algebra*, *Discovering Geometry*, and *College Preparatory Mathematics*.

Over time, the elementary program transitioned to *Investigations II*, but throughout the project, some version of *Investigations* has been in use consistently across the district. The same level of consistency was lacking at the middle grades throughout the first seven years of the project, with many teachers never transitioning fully to *MathScape*. In the eighth year of the project, the district switched to *Connected Mathematics Project II (CMP II)* as the formally adopted middle school instructional materials. As of the ninth year of the project, *CMP II* was being used much more consistently than *MathScape* materials had been previously (external evaluation findings, 2011).

At the high school level, the district moved steadily toward *College Preparatory Mathematics* as the prevailing instructional materials, particularly for freshman and sophomore-level algebra and geometry. Following the introduction of new instructional materials at middle school in Year 9, however, the district made a decision in Year 10 to seek new materials at the high school level. In particular, they sought materials aligned with the integrated pathway within the *Common Core State Standards for Mathematics* (Common Core State Standards Initiative,

2010), that are student-centered and inquiry-oriented, and that build well on *CMP II. Core-Plus Mathematics* has been selected for introduction in Year 11.

Appendix H: Advice to Others

With the hope that the design and implementation of *Project PRIME* might inform other efforts in other districts, we present here the reflections of co-principal investigator and co-author of this paper Dr. Susie Roth, Director of Staff Development, Rapid City Area Schools.

I have learned so much by being involved with *Project PRIME*, particularly with regard to project design, the importance of vision and direction, and the necessity for strong leadership at multiple levels. My learning is based more on what we did *not* do than what we *did* do, and has been the result of my reflection, ongoing study, and collaboration with others.

First, when launching an initiative such as *PRIME*, time needs to be devoted to designing and communicating numerous elements of the initiative. People want to know why the project is being launched. If care is not taken to thoroughly develop the rationale, research, and explanation, teachers can develop the misperception that they are being criticized for their past approach to teaching mathematics, and this can create defensiveness and impede implementation. Project designers also need to determine and clarify key concepts of the project, the resources and professional development that will support the project, and how the initiative will proceed. Building clarity about participation and commitments supports people in knowing who is involved and what their roles and responsibilities are.

I've also learned more about the vital importance of developing and maintaining a clear, consistent, articulated vision. This involves setting a unified direction and continually moving forward, and sometimes this is an inch-by-inch process. A shared understanding of specific practices brings clarity to developing this vision. Linking the work to a shared purpose brings meaning and significance to the work. When those involved believe in the vision and assume responsibility for the part they play in achieving that vision, the progress a district can make, even in a year or two, is quite remarkable.

Finally, leadership is critical at all levels. *Project PRIME* has been a true partnership, and I have valued the contributions of Black Hills State University, Technology and Innovation in Education, and Inverness. Central office staff, building principals, coordinators, and coaches all are necessary to influence others and take action, and the leadership capacity of all levels to lead an initiative must be developed. When these leaders are passionate about their work and support one another, they are able to persevere when confronted with the inevitable challenges and difficulties of trying to bring about substantive change. And the difficult journey is worth the effort!

**Mathematical Habits of Mind for Teaching:
Using Language in Algebra Classrooms¹**

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ABSTRACT: The notion of mathematical knowledge for teaching has been studied by many researchers, especially at the elementary grades. Our understandings of this notion parallel much of what we have read in the literature, but are based on our particular experiences over the past 20 years, as mathematicians engaged in doing mathematics with secondary teachers. As part of the work of Focus on Mathematics, Phase II MSP, we are developing, in collaboration with others in the field, a research program with the ultimate goal of understanding the connections between secondary teachers' mathematical knowledge for teaching and secondary students' mathematical understanding and achievement. We are in the early stages of a focused research study investigating the research question: *What are the mathematical habits of mind that high school teachers use in their professional lives and how can we measure them?* The main focus of this paper is the discussion of the habit of using mathematical language, and particularly how this habit plays out in a classroom setting.

Keywords: Mathematical habits of mind, mathematical language, algebra

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Our Philosophy and Approach

Building on two decades of prior work, the *Focus on Mathematics* (FoM) Math and Science Partnership program (MSP) has, over the last decade, developed and refined a distinctive framework for a mathematics-centered approach to developing teacher leaders, and it has built a mathematical community based on that framework. The FoM approach involves teachers, mathematicians, and educators working together in professional development activities. The common thread running through this tightly connected set of activities is an explicit focus on mathematical habits of mind.

We define *mathematical habits of mind* (MHoM) to be the web of specialized ways of approaching mathematical problems and thinking about mathematical concepts that resemble the ways employed by mathematicians (Cuoco, Goldenberg, & Mark, 1997, 2010; Goldenberg, Mark, & Cuoco, 2010; Mark, Cuoco, Goldenberg, & Sword, 2010). These habits are not about particular definitions, theorems, or algorithms that one might find in a textbook; instead, they are about the thinking, mental habits, and research techniques that mathematicians employ to develop such definitions, theorems, or algorithms. Some examples of MHoM follow:

- Discovering the structure that is not apparent at first by experimenting and seeking regularity and/or coherence.
- Choosing a useful representation—or purposefully toggling among various representations—of a mathematical concept or object.
- Purposefully transforming and/or interpreting algebraic expressions (e.g., rewriting $x^2 - 6x + 10$ as $(x - 3)^2 + 1$ to reveal its minimum value).

- Using mathematical language to express ideas, assumptions, observations, definitions, or conjectures.

Our work over the past decade has convinced us of the importance of MHoM for students and for *teachers* of mathematics, particularly at the secondary level. These habits foster the development and use of general purpose tools that make connections among various topics and techniques of secondary school mathematics content; they can bring parsimony, focus, and coherence to teachers' mathematical thinking and, in turn, to their work with students. In this sense, we envision MHoM as a critical component of mathematical knowledge for teaching (Hill, Rowan & Ball, 2005) at the secondary level (i.e., the knowledge necessary to carry out the work of teaching mathematics).

We begin this paper by describing the mathematical community that we have built and the impact that it has had on our teachers, in particular, the impact on teachers' mathematical understanding and instructional practices. Then we discuss the research that grew out of our desire to study scientifically how MHoM might be an indicator of teacher effectiveness. Lastly, we shed light on one habit that emerged prominently in our research—using mathematical language. We examine how a teacher might use this habit in a classroom, possible implications for student learning, and how use of the habit relates to teachers' use of other mathematical habits in the classroom.

We end this section with a few remarks. Although we describe our research on MHoM, the emphasis of this paper is not on our study, on its particular outcomes, or on the measurement instruments in development. Instead, we intend to illustrate, using examples, our motivation for why we think these mathematical habits are important. Hence, the main focus of the paper is the discussion of the habit of *using mathematical language*, and

particularly how this habit plays out in a classroom setting. We include a detailed discussion of the FoM MSP, partly to situate our work within the MSP context in this special issue of *The Mathematics Enthusiast*. We also want to provide background for the research that emerged from and is motivated by our ongoing MSP work with secondary teachers. Indeed, our study of teachers' MHoM and corresponding instrument development arose from our desire to measure progress in and continue to improve our work with our own FoM teachers.

Focus on Mathematics

Focus on Mathematics (NSF DUE 0314692) is a targeted MSP funded by the National Science Foundation since 2003. Our partnership is devoted to improving student achievement in mathematics through programs that provide teachers with solid content-based professional development sustained by *mathematical learning communities* in which mathematicians, educators, administrators, and teachers work together to put mathematics at the core of secondary mathematics education.

The original FoM district partners include the Massachusetts school systems of Arlington, Chelsea, Lawrence, Waltham, and Watertown. These systems range from suburban to urban, with middle and high school student populations from 1,300 to 6,000. Over the years, FoM has offered a variety of professional opportunities for teachers, including: (a) a public colloquium series devoted to mathematics and education; (b) partnership-wide mathematics seminars; (c) week-long summer institutes for teachers; (d) online problem-solving courses; and (e) a new *Mathematics for Teaching* Masters Program at Boston University. Two activities deserve special mention.

- PROMYS for Teachers summer institute, a six-week intensive immersion in mathematics, engages participants in experiencing mathematics as mathematicians do, solving problems and pursuing research projects appropriate for them. Each summer, the institute combines teachers from multiple districts, Grades 5–12.
- Academic-year study groups are district-based—often building-based—groups that meet biweekly for two to three hours over the course of a year. Though focused on *doing* mathematics (rather than being taught its results or how to teach it)—again, experiencing mathematics as a mathematician would—these trade the intensity and immersion of the summer institute for long-term, ongoing study.

These mathematical learning communities with core involvement of mathematicians are designed to help teachers develop the mathematical habits of mind that are central to the discipline of mathematics. Our teachers have responded enthusiastically, with comments such as:

- “[The study group] is the best ‘professional development’ that I have been involved in throughout my 35-year teaching career. I guess the best testament for the success of Focus on Mathematics comes from the continued attendance of so many teachers. We continue to talk about the topics discussed at our study groups long after the weekly session is over” (Cuoco, Harvey, Kerins, Matsuura, & Stevens, 2011).
- “The [Masters] program has expanded my knowledge of mathematics and deepened my understanding of how children learn mathematics, but—more importantly—I am now connected to people who are as passionate about children learning and doing mathematics as I am” (Cuoco, Harvey, Kerins, Matsuura, & Stevens, 2011).

To study the impact of FoM's professional development programs on teachers' professional lives, the Program Evaluation Research Group at Lesley University (FoM's evaluators) collected and analyzed teacher and student data over five years (Lee, Baldassari, Leblang, & Osche, 2009) and conducted case studies of teachers (Baldassari, Lee, & Torres, 2009). Below are those findings most strongly informing our current work:

- *Teacher beliefs and attitudes about the nature of mathematics:* In interviews, teachers reported understanding the structure of mathematics in greater depth—how topics and ideas are connected and how they are developed through the grade levels. Teachers referred to developing a more complete picture or understanding of mathematics as a system and understanding the connections between different threads within it (Lee, Baldassari, & Leblang, 2006; Lee, Baldassari, Leblang, Osche, & Hoyer-Winfield, 2007).
- *Teacher changes in instructional practice:* Many of the instructional changes teachers reported stem from the ways in which they experienced learning through FoM (Lee et al., 2006). When teachers developed a deeper understanding of mathematics, their confidence often increased and they developed more flexibility in their teaching and the ability to adjust lessons based on student responses.

Through our work in FoM, we have seen that MHoM is indeed a collection of *habits* teachers can acquire, rather than some static you-have-it-or-you-don't way of thinking. And teachers report to us that developing these habits has had a tremendous effect on their teaching. We have collected ample anecdotal evidence, but recognize the need for scientifically-based evidence to establish that these teachers have indeed learned MHoM

and that these habits have had a positive impact on their teaching practices. We also recognize the need to study student outcomes affected by teachers' uses of MHoM.

Mathematical Habits of Mind for Teaching Research Study

Focus on Mathematics, Phase II: Learning Cultures for High Student Achievement (NSF DUE 0928735) is an MSP project that began in 2009. In FoM-II, we continued to refine our mathematical learning communities and began an exploratory research study focused on teachers' mathematical habits of mind.

As a basis for beginning the research study, we used the theoretical frameworks developed by Clarke and Hollingsworth (2002) for their "Interconnected Model of Teacher Professional Growth," which is characterized by networks of "growth pathways" among four "change domains" in teachers' professional lives—the *external* domain (E), the personal domain (K) (of *knowledge*, beliefs and attitudes), and the domains of *practice* (P) and *salient outcomes* (S). Significant, from our point of view, is the Clarke-Hollingsworth theory of *professional growth* (as distinct from simple *change*), which they represent as "an inevitable and continuing process of learning" (p. 947). They aptly distinguish their framework from others: "The key shift is one of agency: from programs that change teachers to teachers as active learners shaping their professional growth through reflective participation in professional development programs and in practice" (Clarke & Hollingsworth, 2002, p. 948). The agency of teachers in their own professional growth characterizes virtually all FoM programs, so we see the Clarke-Hollingsworth model of professional growth as well suited for our purposes.

We illustrate our use of the Clarke-Hollingsworth framework with an example. Shown in Figure 1 is a change environment diagram for "Ms. Crew," a middle school

teacher and active member of the FoM learning community. The diagram represents the change domains as four boxes, labeled E, K, P, and S, as explained above. The solid arrows refer to growths due to *enactment*, while the dashed arrows depict those due to *reflection*. The loop on the box E refers to interaction between study groups and the immersion.

Figure 1. Schematic diagram of Ms. Crew's change environment

This particular diagram depicts activity related to Ms. Crew's research on Pythagorean Triples and shows how this activity led to her growth, both mathematically and as a teacher. Each arrow represents a growth in Ms. Crew that occurred as a result of a change in her professional life. For example, arrow 6 depicts how her increased belief about herself (a change in box K, the personal domain) leads to Ms. Crew encouraging her students to perform more explorations (a change in box P, the domains of practice). Moreover, arrow 6 is solid, because the change in her classroom is due an enactment, i.e., a particular course of action that she took as a teacher. The arrows are numbered in chronological order, so arrow 1 denotes a growth in Ms. Crew that occurred before that depicted by arrow 2, and so on. The dashed arrow from box E to K has multiple numbers

(as does the solid arrow from K to E). Here, the dashed arrow may be interpreted as three separate arrows (arrow 1, arrow 3, and arrow 5)—we simply condensed them into one arrow to save space in the diagram.

Ms. Crew first encountered the concept of Pythagorean Triples while studying Gaussian integers during her summer immersion experience. The topic left such an impression on her (reflective arrow 1) that she pursued it (enactive arrow 2) as a research project under the guidance of an FoM mathematician. Through months of hard work—familiarizing herself with Pythagorean Triples through dozens of examples, making careful data recording and analysis, discovering beautiful patterns, coming up with interesting conjectures (some were true, some were false), and finally writing down clear and concise propositions and proving them—she came to understand (reflective arrow 3) features of Pythagorean triples that would have been beyond her conception before this experience. Ms. Crew produced an independent research paper and a one-hour mathematics talk for her peers (enactive arrow 4).

Neither the summer immersion experience nor the independent research project was easy for Ms. Crew, who came into our program with a rather weak mathematics background. But completing this project had a significant effect on her mathematical self-confidence (reflective arrow 5). The loops of this upward spiral between domains K and E repeated many times. Amongst her peers, Ms. Crew became one of the leaders in her study group (4). In her curriculum planning, she now has more belief in her decision-making abilities (5). And in her classroom, she engages her students in performing mathematical exploration (6). This new classroom atmosphere, as well as her new attitude towards mathematics, led to more curiosity and questions from her students (7, 8). And while she

may not be able to answer all of them on the spot, she now welcomes mathematical dialogs and uncertainty in her classroom (9, 10). All of this represents significant professional growth and Ms. Crew's change diagram enables us to see the elements of that growth at a glance.

Looking at Ms. Crew's change diagram, one cannot fail to notice the intense activity taking place around the node K, which includes growth in Ms. Crew's knowledge of mathematics. But it seems to us that more is involved than simply knowing mathematics as a body of knowledge. Ms. Crew is learning mathematics in a certain way. Her beliefs about the nature of mathematics are changing. She is acquiring certain mathematical habits of mind and she is finding these habits useful for her work in the classroom and also for leadership roles in the school.

Applying this framework of teacher change, we began to build for ourselves a theoretical understanding of how MHoM plays a role in the work of teaching. Recognizing the need for a scientific approach to test the theory, and indeed investigate the ways in which MHoM is an indicator of teacher effectiveness, we conducted an exploratory study titled *Mathematical Habits of Mind for Teaching* that centers on the following question:

What are the mathematical habits of mind that secondary teachers use in their profession and how can we measure them?

To investigate this question, we developed a detailed definition of MHoM and have been building the following two instruments:

- *A paper and pencil (P&P) assessment* that measures how teachers engage MHoM when doing mathematics for themselves.

- An *observation protocol* measuring the nature and degree of teachers' uses of MHoM in their teaching practice.

We emphasize that both instruments are needed, because in our work with teachers, we have seen those who have very strong MHoM for themselves but do not necessarily employ the same mathematical habits in their teaching practices.

Our current work fits into a larger research agenda that we are developing in collaboration with leaders in the field, with the ultimate goal of understanding the connections between secondary teachers' mathematical knowledge for teaching and secondary students' mathematical understanding and achievement.

Operationalizing MHoM

To operationalize the MHoM concept, we relied on our own experiences as mathematicians doing mathematics with secondary teachers (Stevens, 2001). We also studied existing literature—in particular, Dewey's (1916) and Dewey and Small's (1897) earlier treatments of habits and habits of mind, the Study of Instructional Improvement (SII) and the Learning Mathematics for Teaching (LMT) projects to develop measures of mathematical knowledge for teaching (MKT) for elementary teachers (Ball & Bass, 2000; Ball, Hill, & Bass, 2005; Hill, Schilling, & Ball, 2004; Hill, Ball, & Schilling, 2008), and the description by Cuoco et al. of mathematical habits of mind (1997, 2010). And we consulted the national standards, i.e., the NCTM Principles and Standards for School Mathematics (National Council of Teachers of Mathematics [NCTM], 2000) and the Common Core Standards for Mathematical Practice (National Governors Association Center for Best Practices and the Council of Chief State School Officers [NGA Center & CCSSO], 2010). But above all, we went into the classrooms of FoM teachers, where we observed a broad

sampling of MHoM strengths. Some teachers exhibited precise use of language and careful reasoning skills; others had strong exploration skills, were good at designing mathematical experiments, or showed special strength at generalizing from concrete examples.

From these various sources, we began to compile a list of habits that constitute MHoM. As the list grew, we identified four broad and overlapping categories into which our mathematical habits naturally fell:

- Seeking, using, and describing mathematical structure
- Using mathematical language
- Performing purposeful experiments
- Applying mathematical reasoning

Indeed, these are categories of mathematical practices that are ubiquitous in the discipline. And in order to conduct a fine-grained study of these categories, we teased apart multiple habits within each category that we wanted to measure, some of which were identified earlier. That being said, we primarily envision MHoM as being comprised of the four categories, with the list of habits within each category providing more detail and texture to these four. By no means is our list final. In fact, we consider it an evolving document that we will continue to revise as we obtain more data using our instruments. From our data, we will learn which habits are more prominently used by secondary teachers, both when *doing* and *teaching* mathematics.

Paper and Pencil (P&P) Assessment

We developed a pilot P&P assessment that measures how secondary teachers use MHoM while doing mathematics. This assessment contains seven open-ended problems and is designed to be completed in one hour. In particular, we developed problems that

most teachers have the requisite knowledge to solve, or at least begin to solve. And what we are assessing is *how* they go about solving it. It is the *choice* of their approach that we are interested in, as opposed to whether or not they have the necessary knowledge/skills to solve it. Each item is designed to reveal what habits and tools teachers choose to use in familiar contexts. To date, we have gone through several rounds of design, pilot-test, data analysis, and revision of this instrument. For our latest pilot-test in the summer of 2011, we administered the P&P assessment to 43 secondary mathematics teachers participating in the NSF-funded study *Changing Curriculum, Changing Practice* (NSF DRL 1019945). We will carry out another field test with approximately 50 teachers in the summer of 2012.

To gather initial data on the role that teachers' approach to solving mathematics problems plays in their approach to mathematics instruction, we asked a follow-up question to some of our P&P assessment problems: *What strategies would you want your students to develop for a problem like this?* Our 43 respondents almost unanimously reported that they want their students to approach the problems exactly as they did themselves. (Note: A few teachers wanted their students to appreciate a variety of approaches.) This finding provides initial evidence that teachers' own mathematical work may be indicative of how they choose to explain/formulate the subject matter for their students. Recognizing the need for further study of this hypothesis, we began to create an observation protocol.

Observation Protocol

We are in the process of designing an observation protocol and coding scheme that measure the nature and degree of teachers' uses of MHoM in their classroom instruction. To develop the instrument, we conducted live and videotaped observations of two to three

consecutive mathematics lessons collected from a total of 30 secondary teachers to identify teacher behaviors that reflect the uses of a particular mathematical habit. In addition, we developed a simple protocol for pre- and post- interviews with teachers we videotape. We also collected classroom artifacts (lesson plans, in-class worksheets, homework, and assignments) from each classroom we observed.

An important feature of our observation protocol is that it measures how teachers *use* MHoM in their instruction. Thus teachers are coded not for *possessing* certain mathematical habits in the abstract, but for *choosing* to bring them to bear in a classroom setting. To develop such an instrument, we are currently studying our videos and slicing these lessons into small episodes—i.e., short instructional segments lasting 30 seconds to 4 minutes. In each episode, we determine whether there were behavioral indicators that reflected teachers' uses of MHoM, and we create codes that generalize and characterize these teacher classroom behaviors. We emphasize that our current focus is on *teacher* behaviors and uses of MHoM in the classroom. We are still a step away from connecting teaching practices centered on MHoM to students' development of MHoM and to student achievement—partly because we do not yet have the instruments to assess these habits in students—but impacting students, of course, is our ultimate goal.

Later, we describe three teachers from whom we gathered video data for our observation protocol development. Specifically, we will discuss how they apply the habit of using mathematical language in their classroom instruction. We will also consider how teacher use of this particular habit may affect student understanding.

Relevant Literature and Related Work

The theory of mathematical habits of mind is philosophically grounded in Dewey's (1916) and Dewey and Small's (1897) earlier treatments of habits and habits of mind. Their seminal work has since encouraged educators (Duckworth, 1996; Meier, 1995) and education researchers (Kuhn, 2005; Resnick, 1987; Tishman, Perkins, & Jay, 1995) to further operationalize the concept of habits of mind—that is, to respond to the general question: *What do habits of mind look like in the context of learning?* Not as evident in the literature are the habits of mind that promote successful learning in specific disciplines. In the case of mathematics, the question that has gained research attention within the last decade is: *What do habits of mind look like in the context of learning and doing mathematics?* While addressing this question is not an unfamiliar task (Hardy, 1940; Polya, 1954a, 1954b, 1962), what is less familiar is the task of gathering evidence of mathematical habits of mind from *teachers* of mathematics. We began this work in our FoM-II study; we are in the long-term process of developing valid and reliable instruments that will allow us to more rigorously investigate the relationship between teachers' own MHoM, their uses of MHoM in their teaching practice, and student achievement.

As mentioned earlier, we envision MHoM as an integral component of MKT at the secondary level. The notion of MKT has been studied by many researchers (Ball, 1991; Ball, Thames, & Phelps, 2008; Heid, 2008; Heid & Zembat, 2008; Heid, Lunt, Portnoy, & Zembat, 2006; Hill et al., 2008; Kilpatrick, Blume, & Allen, 2006; Leinhardt & Smith, 1985; Ma, 1999; Stylianides & Ball, 2008). Our understandings of this notion parallel much of what we have read in the literature, but are based on our particular experiences over the past 20 years, as mathematicians engaged in doing mathematics with secondary teachers.

As mathematicians working in schools and professional development, we have come to understand some of the ways in which teachers know and understand mathematics.

These fit into four large and overlapping categories:

- (1) *Teachers know mathematics as a scholar*: They have a solid grounding in classical mathematics, including its major results, its history of ideas, and its connections to precollege mathematics.
- (2) *Teachers know mathematics as an educator*: They understand the thinking that underlies major branches of mathematics and how this thinking develops in learners.
- (3) *Teachers know mathematics as a mathematician*: They have experienced a sustained immersion in mathematics that includes performing experiments and grappling with problems, building abstractions from the experiments, and developing theories that bring coherence to the abstractions.
- (4) *Teachers know mathematics as a teacher*: They are expert in uses of mathematics that are specific to the profession, including the ability to “think deeply of simple things” (Jackson, 2001, p. 696), the craft of task design, and the “mining” of student ideas.

The first two of these ways of knowing mathematics are common to most pre-service and in-service professional development programs. FoM has paid particular attention to the last two, which typically receive less emphasis. We have become convinced that (3) greatly enriches and enhances the other ways of knowing mathematics and that many teachers who go through such an experience develop the habits of mind used by many mathematicians. Furthermore, we have seen that participation in a mathematical learning

community helps such teachers “bring it home” in the sense that they create strategies for helping their students develop the mathematical habits that they themselves have found so transformative.

Other researchers are developing instruments to assess secondary teachers’ content knowledge and use of mathematics in their classrooms (Bush et al., 2005; Ferrini-Mundy, Senk, McCrory, & Schmidt, 2005; Horizon Research, Inc., 2000; Measures of Effective Teaching Project, 2010; Piburn & Sawada, 2000; Reinholz et al., 2011; Shechtman, Roschelle, Haertel, Knudsen, & Vahey, 2006; Thompson, Carlson, Teuscher, & Wilson, n.d.). In developing our own instruments, we have drawn insight from all of these projects. But we have most closely followed the model developed by Ball and Hill—specifically, their MKT assessment and Mathematical Quality of Instruction (MQI) protocol for documenting MKT in elementary teachers (Hill et al., 2005; Learning Mathematics for Teaching, 2006). Their instruments measure “specialized” mathematical knowledge, that is, knowledge that teachers use, as distinct from the mathematical knowledge held by the general public or used in other professions, whose components include representation of mathematical ideas, careful use of reasoning and explanation, and understanding unique solution approaches. These skills resemble the kinds of mathematical habits that we are interested in studying at the secondary level.

The collective efforts of the field will all contribute to what we know about MKT, but there are important differences between our instruments and those of others. The differences are listed below.

- A focus on MHoM—the methods and ways of thinking through which mathematics is created—rather than on specific results (Cuoco et al., 1997). It is impossible, even

in three or four years of high school mathematics aligned with the Common Core, to equip students with all of the facts they will need for college and career readiness. But learning to think in characteristically mathematical ways is a ticket to success in fields ranging from business, finance, STEM-related disciplines, and even building trades.

- The core involvement, at every level, of mathematicians who have thought deeply about the implications of their own habits of mind for precollege mathematics curricula, teaching, and learning (Bass, 2011; Schmidt, Huang, & Cogan, 2002).

Our instruments are, therefore, aimed at discerning the extent to which secondary classrooms are centered on the practice of *doing* mathematics rather than on the special-purpose methods that often plague secondary curricula (Cuoco, 2008). In our work with teachers, we have seen how expert teachers use core mathematical habits of mind in their profession—in class, in lesson planning, and in curricular sequencing. And, as the Common Core becomes the nationally accepted definition of school mathematics, teachers will be expected to make the development of mathematical habits an explicit part of their teaching and learning agenda. Our work, therefore, makes a unique contribution to the field's increasing level of attention to secondary mathematics teaching.

Using Mathematical Language

In this section, we will focus on a specific mathematical habit—*using mathematical language*—and examine how teachers use this core habit in their instructional practice. We will also consider its potential implications for student learning, and how this habit may work in conjunction with other mathematical habits in the classroom.

In particular, we will discuss examples of three teachers whose Algebra 1 classrooms we observed in our research study. We will begin with Mr. Hart, who uses mathematical language to encapsulate the experiences, observations, and discoveries of his students. Second, we will look at Ms. Graham, who uses precise and operationalizable language as a way of promoting conceptual understanding and ease of problem-solving. And third, we will describe an example of a teacher, Mr. Braun, whose choice of language can interfere with students' engagement in activities designed to promote other MHoM.

All three of these teachers have shown evidence of strong MHoM in their own doing of mathematics. Mr. Hart has held formal and informal leadership roles in a number of FoM's mathematical learning communities; and in those roles, he has exhibited strong MHoM. The other two teachers performed well on our P&P assessment. The names of these teachers have been altered to protect their identities.

Mr. Hart

We consider Mr. Hart, an Algebra 1 teacher who uses mathematical language to encapsulate the underlying structure that students discovered through experimentation. The mathematical topic of the day is recursive rules. The class begins with students working on the following warm-up problem.

A function follows [this rule] for integer valued inputs: The output for a given input is $\frac{3}{2}$ greater than the previous output. Make a table that matches the description. Can you make more than one table?

Note that the rule is incomplete, because it is missing the base case. Students experiment with this rule, creating input/output tables and trying to derive closed-form equations.

Because of their different choices of base cases, they come up with different functions defined by expressions of the form $f(x) = \frac{3}{2}x + b$. Students conclude that the graphs of these functions are parallel lines with different y -intercepts. Mr. Hart also asks, “So what’s the part where you get to be creative in making these tables?” He then explains, “So you get to pick one number, and then everything else is decided by the part that I gave you [in the warm-up]. But there’s still an awful lot of different numbers.” Here, he is foreshadowing the *need* to fix the base case.

Then Mr. Hart formally introduces the notions of recursive rule and base case to summarize students’ experiences and to capture the underlying structure they observed when working on the warm-up problem. He says,

A recursive rule, that’s just the description that tells us how to get from an output—to an output from the previous ones. So basically, what we were doing. Now as you saw, there’s another piece that’s not really enough information. It’s just me telling you how to get from one, to the next, to the next. To have a complete rule, we also need to know where to start. Because otherwise, we won’t know if we have the rule that—the first rule, the second rule, the third rule, or some other rule completely.

(Video transcript, February 14, 2011.)

Next, the class studies the function described by the following table:

n	$f(n)$
0	3
1	8
2	13
3	18

4	23
5	28
6	33

In this table of data, students recognize the +5 pattern, i.e., “You add 5 to the output.”

Through discussion, Mr. Hart guides them to articulate the relationship more precisely:

$f(5) = f(4) + 5$. Using this concrete example, students are able to derive a general equation:

$$f(n) = f(n-1) + 5.$$

To make sense of this recursive rule, Mr. Hart points out that the equation

$f(n) = f(n-1) + 5$ “lets us relate any output to a previous one.” In essence, it is the symbolic representation of what he told students in the warm-up problem. Then he describes the need for the base case, saying, “But that wasn’t quite enough because lots of you wrote down different rules. And [Student 1] had one, [Student 2] had a different one, [Student 3] had a different one probably, and so on. So we need something else to sort of fix it in place.”

Here, a student interrupts and proposes a closed-form rule: $f(n) = 5n + 3$. There are now two ways to describe the function at hand, namely the (still incomplete) recursive rule $f(n) = f(n-1) + 5$ and the closed form rule $f(n) = 5n + 3$. He says, “[The recursive rule] tells us how to work our way down the table. If I know one value, I know 23, I can find the next one really easily. Now this one’s [points to the closed-form rule] nice too because it lets me work across the table. If I know the input, I can say the output really quickly.” In this short episode, Mr. Hart uses the symbolic representation of each rule to discuss its underlying structure.

Mr. Hart returns to the equation written on the board (i.e., $f(n) = f(n-1) + 5$) and says, “But still, this—this rule almost tells me the whole table, but it doesn’t quite because

I'm missing one critical piece of information." A student chimes in, "Well, you don't know what you started with." Mr. Hart responds with, "That's a good point. Yeah, so like [Student]'s saying this 3 in the table, that's where we're starting. So we kind of need to know that. So the way (pause) a good way that we can sort of keep track of this and write our rule..." Almost 20 minutes into the lesson, Mr. Hart finally introduces the complete notation

$$f(n) = \begin{cases} 3 & \text{if } n = 0, \\ f(n-1) + 5 & \text{if } n > 0. \end{cases}$$

He explains this new equation by saying, "So this formula captures exactly what we did. The key part is the recursive part that we had written down already. And this just adds that last bit, the base case, so we can summarize it into one compact rule."

Instead of being a starting point, this notation is the *culmination* of the structures that students discovered through their experimentation and the follow-up discussion. Students readily make sense of the new notation and the accompanying ideas that it encapsulates, because the experience gained through their "struggles" allows them to connect the new language to already-established ideas.

Mr. Hart uses the structure that students found through their experiments to motivate the language needed to describe their observed results. For instance, students' experiments with the warm-up problem, in which they propose different functions that all satisfy the given rule, make the *need* for the base case come alive for them. Indeed, his mathematical habits of mind allow Mr. Hart to create a learning environment where students build new knowledge from their experiences (NCTM, 2000).

Ms. Graham

Through Ms. Graham, we look at how an Algebra 1 teacher uses precise and operationalizable language as a way of promoting ease of problem-solving. More specifically, she helps students make sense of the objective of the given problem and, subsequently, provides insight into how to proceed.

In this episode, a student asks about the following question:

Determine if $r = -2$ is a solution to $6r + 2 = 12 + r$.

Ms. Graham asks, “Did we not understand what they were asking?” The student confirms, “Yeah, obviously there’s an easier way to do it, but I just didn’t know how.” Then the following dialogue occurs, in which Ms. Graham presses for the meaning of the word “solution”:

Teacher (T): All right. When we use the word “solution,” all right, we’ve talked a lot about what a solution is. What does “solution” mean?

Student (S): Like, does—it—when it works.

T: When you said “it works,” what do you mean? Because I think you’re on the right track.

S: Like, does it make sense?

T: Be a little more specific.

S: I don’t know how, like...

T: What does “solution” mean, anyone know? All right.

New student (SN): The answer?

T: “The answer.” We talked about this a lot. What’s a solution to an equation?

SN: Something that can go into make an equation work.

T: Something that makes the equation true, OK?

As we will see later in Mr. Braun's example, "works" is often used by students and teachers to describe what it means for a number to be a solution to an equation. Ms. Graham does not settle for this nor other oft-used phrases such as "it makes sense" and "the answer." The language used by students does not help them unravel the problem to understand what they are being asked to do. Only after the operational definition of "solution" has been given can Ms. Graham continue with an explanation of how to proceed.

T: We're stating that $6r+2$ will be equal to $12+r$. And they're asking, "Is $r=-2$ a solution?" So you got to test it out, just as I asked you to test out that one that we just did. So $6r+2=12+r$. Substitute in $r=-2$. So 6 times -2 plus 2—does that have the same value as 12 plus -2 ? And we have to test. All right? We're asking ourselves the question of, does this equal that? [Points to each side of the equation.] OK?

Then Ms. Graham leads the class through the process of substituting $r=-2$ into the equation and concluding that it is not a solution, since $r=-2$ yields unequal values of -10 and 10 for the two sides of the equation. The student who originally inquired about this question says, "Ok. Now I get it." The definition of "solution" provided by Ms. Graham—namely, "something that makes the equation true" is operational (i.e., students can *use* this definition to understand and accomplish the task posed by the given question). Indeed, once the definition has been given, substituting $r=-2$ and checking if it makes the equation true is a natural next step.

Ms. Graham concludes this episode by foreshadowing what students will be learning next, by providing them with another definition:

T: We're getting to the point where we're going to ask you, "What is the value of r that makes the equation true?" And that's called *solving the equation*.

Throughout the lesson, Ms. Graham consistently uses language carefully. She corrects a student who writes $82 + 8 = 90 \div 3 = 30 - 5 = 25$, calling it a "run-on sentence in math." When a student describes two sides of an equation by saying, "It's equals," Ms. Graham immediately responds, "They're equal to each other." She repeatedly tells students to check their answer after solving an equation, reminding them what "solution" means. She is also precise in her instructions (e.g., asking the students to "write an expression for the right side of the equation, so that you've got an equation that works and is true when $x = 3$ ").

Mr. Braun

One of the issues we have encountered in the development of our observation protocol is, "What counts as evidence of *non-use* of MHoM?" In the case of the habit of using mathematical language, we do see moments in which teachers choose less careful language. For example, a teacher might choose to use informal language. Sometimes there is evidence that the teacher is making this choice because the informal language seems more accessible to students. But such choices—if not made carefully—can lead to student confusion.

In the following example, Mr. Braun is setting up an investigation that aims to lay the foundation that the graph of an equation is a representation of the solution set of the equation (Education Development Center, Inc., 2009b). To launch the investigation, Mr. Braun writes the equation $3x + 2y = 12$ on the overhead projector and asks students, "What's the answer?" He then describes some of the solutions students offer as "that works" or "that doesn't work." The following is an excerpt from the launch of the investigation. There are two things to note. First, Mr. Braun is modeling how students

might experiment with numbers as a way of making sense of the relationship between graphs and equations. Second, observe how frequently he uses the word “works.”

T: $3x + 2y = 12$. What’s the answer?

SN: It’s complicated.

T: Oh, no. What do you think?

SN: 1 and 2?

T: You think I can use 1 and 2?

S: x is 1 and y is 2.

T: x is 1 and y is 2. How would I find out if [name] is right? I could put in the numbers that he gave me, so I’m going to put in 1 for x and I’m going to put in 2 for y , and do I get 12, like I’m supposed to? What’s 3×1 ?

Students (Ss): 3.

T: What’s 2×2 ?

Ss: 4.

T: What’s $3 + 4$?

Ss: 7.

T: Did I get 12?

Ss: No.

T: Man, [name], that’s a bummer. OK, so—

SN: Oh, I know it.

T: —that was something that didn’t work. It’s not bad to find out things that don’t work. Sometimes, you’re going to be asked in these investigations to find things that don’t work, so remember how we did that.

At this point, the teacher continues to take student guesses for x and y . Students make guesses and one student suggests $x = 2$ and $y = 3$. Mr. Braun tries that suggestion, and sees that indeed, $3(2) + 2(3) = 12$.

T: OK, so we found out that 1 and 2 did not work; we found out that 2 and 3 did work.

Do you think there are any more things that don't work?

SN: Yes.

T: A lot more things that don't work. OK, do you think there are any more things that do work?

S: Yes.

T: Can you think of another thing that does work? [...]

SN: $3(3)$...

T: OK, if I put a three there, OK.

S: And then, the $2y$ is 2, $2(1)$.

T: 2×1 . OK, this is 9, right? Plus 2, makes 11 instead of 12. So, we found another thing that doesn't work. So, I—[name], you must have been right, there were more things that do not work. Can you find anything else that does work?

SN: 4 and 1.

T: You think 4 and 1 works? Where do I put my 4, for x or for y ?

S: For x , yeah.

T: OK, so I put in $3(4) + 2(1)$, that gives me $12 + 2 = 14$. We found another thing that doesn't work.

S: Actually, put 3 for y , plus 1.5.

T: [...] $2(1.5)$, what are we going to get?

Ss: It's 3.

T: 3, and we had 9. Is $3 + 9 = 12$?

Ss: Yes.

T: Hey, look at that. All right, now, that's the kind of thing I want you to do. You're just going to try some things. Some of them will work; some of them won't work.

Mr. Braun has modeled a detailed investigation of looking for points that satisfy the equation $3x + 2y = 12$, using the word "works" as a substitute for "satisfies the equation."

He uses the phrases "works" and "doesn't work" repeatedly. He then hands out a worksheet for investigation that includes the problems:

Each point in the following table satisfies the equation $x + y = 5$.

a) Complete the table.

x	y	(x, y)
1	4	(1, 4)
2		
-3		
	0	
$\frac{1}{2}$		
	-2	
	$-\frac{11}{3}$	

b) Graph the (x, y) coordinates that satisfy the equation $x + y = 5$. [Grid supplied.]

c) What shape is the graph?

and

Use the equation $2x + 3y = 12$.

- a) Find five points that satisfy the equation.
- b) Find five points that do not satisfy the equation.

Students begin the investigation. Some do not know what it means for a point to “satisfy an equation.” Mr. Braun had created the worksheet based on problems in an Algebra 1 textbook—in the book, students are reminded that “If a point’s coordinates make an equation true, the point ‘satisfies the equation’” (Education Development Center, Inc., 2009a, p. 251). Mr. Braun had left that reminder off of his worksheet, and some of the students get stuck. For example:

S: ... Please!

T: You just told me, though. [Laughter] What are we trying to do? What’s it asking you to do?

S: Find this point...

T: OK, what does “satisfy” mean? That’s the same equation we played with at the beginning of class, right?

S: I don’t know.

T: It is, right? We didn’t say “satisfy” and “not satisfy”; what were the words that we used?

S: I don’t know. I don’t know.

T: When [name] gave us 3 and 1.5, what did we say?

S: Decimal?

T: Well, we said they were decimals, we sighed at [name], but beside that, what else did we say? What does this side equal?

S: x ? y ? What?

T: What's 3×3 ?

S: 9.

T: What's 2×1.5 ?

S: 3.

T: What's $9 + 3$?

S: 12.

T: So, what did we say? "[Name]'s solution..."

S: Works?

T: Works! "Works" is another word for "satisfies." If you want to sound smart, you say, "It satisfies the equation." OK? All right.

Similarly, another student asks:

S: I don't understand what it's asking us! [Laughter]

T: All right, fair enough. It says, "Sketch a graph of all the (x, y) coordinates that satisfy"—work—"in this equation," and here's my equation.

On one hand, this is not a big deal. The teacher can travel from group to group, reminding them what "satisfies the equation" means, but he usually simply says that "it means 'works.'" However, "works" as a description is not operational. When students are solving problems, they repeatedly ask about the phrase "satisfies the equation." Rather than offer the operationalizable definition: "if a point's coordinates make an equation true, the point satisfies the equation," Mr. Braun returns to the phrase "works."

It is worth noting that the following day, Mr. Braun poses a warm-up question to his class: "What does it mean to be a solution?" Although he does not specifically address the

definition of a point satisfying an equation (and the issue continues to persist for students), he does start working on unpacking that language for students.

Common Themes in the Examples

Several observations and questions emerge for us in these examples. First, what strikes us again and again is the complexity of teachers' uses of MHoM. These habits cannot be deployed independently in the classroom any more than they can be when teachers (and mathematicians) do mathematics for themselves. In fact, we saw that the habit of using mathematical language can either complement or get in the way of student experimentation and inquiry, depending on how the teacher uses the habit. In Mr. Hart's class, the precise definition of recursive function is motivated by the structure that his students discovered through experimentation. And, in turn, Mr. Hart plans to use this function notation as an investigative tool to explore further topics (e.g., the connection between linear and exponential functions). Mr. Braun also brings experimentation into his classroom. Indeed, his students conduct an investigation to explore the relationship between an equation and its graph. However, some students have difficulty beginning the investigation, because they do not understand the language they encounter in the task. Here, an operational definition of the phrase "satisfies the equation" may have led them to understand the problem statements and given them insight into how to proceed.

Throughout these examples, we also saw how the use of mathematical language can support students' understanding. In Ms. Graham's class, we see how she pushes her students to clearly state the meaning of the word "solution." And its definition becomes a vehicle that facilitates the problem-solving process. In contrast, we see Mr. Braun whose students encounter the phrase, "satisfy the equation." Instead of providing a usable

definition, he offers an alternative, namely “works.” We believe Mr. Braun is well-intentioned here. Specifically, there is evidence that he is trying to make the language less intimidating for students by offering a more informal phrase. Indeed, he says, “‘Works’ is another word for ‘satisfies.’ If you want to sound smart, you say, ‘It satisfies the equation.’” But as discussed earlier, “works” is a phrase that is difficult to operationalize. It leads to confusion for his students, because they do not know how to *use* it. One of the mathematical practices advocated by the Common Core is attending to precision. The Common Core states that, “Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning” (NGA Center & CCSSO, 2010, p. 7). That “usability” of language is an important part of communicating precisely, and one that seems especially important for teachers.

In particular, the careful use of mathematical language not only helps clarify ideas for students, as it did in Ms. Graham’s class, but it helps them understand the mathematics itself in a deeper way. We see this in Mr. Hart’s lesson, where the recursive formula for $f(n)$ captures the properties of the function that students found through their investigations. Indeed, this formula is both a product and a reflection of their experiences. In our work with FoM teachers, we have found that encapsulating various insights into precise language—as we saw in Mr. Hart’s class—helps one better understand the ideas themselves.

Mr. Hart also recognizes the power of precise language to drive further investigations. Later in the school year, these students will use function notation to study transformations of functions (e.g., stretches, shrinks, and translations). He adds, “I think

that will be a place where students will really appreciate the function notation in representing those transformations more easily.”

Mr. Hart concludes the post-interview by describing how today’s lesson is part of a bigger unit and how it sets the foundation for later lessons. He plans to use these recursive rules as a vehicle for better understanding their closed-form counterparts. In a future lesson, students will investigate the connection between linear and exponential functions. “I want my students to see that recursively, exponential functions are very, very similar in their representation to linear functions. I think that will provide a nice foundation for studying exponents,” he says. Here, Mr. Hart is using the language of recursive functions to shed light on the connections between their corresponding closed-form representations.

Our own goals in watching these videos have been to better understand teachers’ uses of MHoM, and to learn about how we might measure that use. Part of our desire to measure the use stems from our desire to understand (eventually) the link between teachers’ uses of MHoM and learning outcomes for students, particularly if we can measure students’ uses of MHoM or students’ facility with Common Core’s Mathematical Practices, which include significant overlap with MHoM. Within the context of the examples in this paper, might teachers’ use of language have an impact on student achievement? Even to begin to answer such a question, we must have some objective way of deciding whether or not a given teacher is using clear, usable, and precise language. This, too, is complex. Establishing what counts as “clear, usable, and precise” language depends very much on the classroom context. Mr. Braun uses the word “works” so consistently in his classroom discussion, that if it did not cause confusion, surely we would want to “rate” that as totally acceptable language, taken as shared by the whole classroom.

Impact and Next Steps

We began our research work partly because we wanted to assess the effects of our own MSP professional development programs using tools that were consistent with the goals of our MSP, and partly because we wanted to understand the MHoM of secondary teachers better. We did not find instruments that measured teachers' MHoM—either when doing mathematics for themselves or teaching mathematics in their classrooms—in existence in the field, so we began to create our own. Although we expected to learn from the data gathered using our instruments, we did not anticipate the immediate implications that our research would have on the professional development programs in our MSP. For example, based on what we had learned from our research, we piloted the *Mathematical Habits of Mind Shadow Seminar* in the summer of 2011, geared toward teacher participants returning to PROMYS for Teachers (our summer immersion program) for a second summer. Through discussions, readings, curriculum analyses, and lesson designs, the goal of this seminar was to explore (a) the ways in which secondary teachers know and use MHoM in their profession, and (b) the effects that a learning environment that stresses MHoM might have on secondary students. We will continue to offer and refine this course as part of our summer immersion program for teachers.

We also did not anticipate the potential for impact on the field. While development and validation of truly reliable tools is beyond the scope of the current FoM-II study, we have been laying the groundwork for our MHoM instruments—the P&P assessment and the observation protocol—over the last few years. This exploratory phase of instrument development also coincided with the emergence of the Common Core State Standards and its adoption by 45 states (NGA Center & CCSSO, 2010). Our MHoM construct is closely

aligned with the Common Core, especially its Standards for Mathematical Practice, and there is considerable overlap in the two. For example, both place importance on seeking and using mathematical structure, uses of precision, and the act of abstracting regularity from repeated actions. As we presented our preliminary findings at national conferences (Matsuura, Cuoco, Stevens, & Sword, 2011; Matsuura, Sword, Cuoco, Stevens, & Faux, 2011), we received several requests to use our instruments, even though they were in the pilot phase of development. One district leader wanted to diagnose the preparedness of her teachers to teach from a curriculum based on the Common Core. Others wanted to use the instruments as pre- and post- measures for evaluating professional development programs aligned to the Common Core. We have become abundantly aware of the national need for *valid and reliable* instruments to measure teachers' knowledge and use of MHoM/Mathematical Practices, as well as guidelines for acceptable use of such instruments. Thus, in the next phase of our research, we plan to subject our pilot instruments to rigorous scientific testing. The examples in this paper are exemplars of those that provide both the content basis for the P&P assessment and the behavioral indicators for the observation protocol.

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Making Explicit the Commonalities of MSP Projects: Learning from Doing

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Abstract: The seven projects discussed in the preceding articles are funded by the National Science Foundation (NSF) Math and Science Partnership (MSP) program (Hamos et al., 2009), which began in 2002. One of the main goals of the MSP program is to build capacity and integrate the work of higher education, especially its STEM disciplinary faculty, with that of K-12 to strengthen and reform mathematics and science education (Hamos et al., 2009). Thus, the MSP program brought together three sets of people (disciplinary faculty, teacher educators, and school system personnel) who do not usually work together to reform the mathematics and science education of teachers. For many of the MSP partnerships this was the first time that members of these groups were purposefully working together to develop mechanisms designed to 1) increase both preservice and inservice teachers' mathematical content knowledge for teaching; 2) provide teachers with the opportunity to learn mathematics in the manner in which their students should learn mathematics in order to develop habits of mind similar to those of mathematicians, such as making conjectures and testing them out, modeling contextual situations with mathematics, and persevering in solving problems; and 3) engage all of the partners in collaborative opportunities focused on student learning and assessment. Accordingly, the seven partnerships discussed throughout this issue and other partnerships chose coursework at universities, some combination of coursework and professional development, and/or study groups as the mechanisms to accomplish the objectives of the MSP program.

As principal investigators of a Targeted MSP, we can empathize with the leaders of the seven partnerships discussed in this special issue of the *Mathematics Enthusiast*. The project with which we are affiliated is the East Alabama Partnership for the Improvement of Mathematics Education (also known as Transforming East Alabama Mathematics or TEAM-Math), which was formed in November 2002 to improve mathematics education in 14 school districts in East Alabama with the support of Auburn University, Tuskegee University and other partners. Together, the districts in this partnership serve roughly

59,000 students. TEAM-Math received major funding from the NSF MSP program in 2003, along with a number of other internal and external grants.

The mission for this partnership is: “To enable all students to understand, utilize, communicate, and appreciate mathematics as a tool in everyday situations in order to become life-long learners and productive citizens by Transforming East Alabama Mathematics” (TEAM-Math, 2003). A central goal of the partnership is to ensure that *all* students, including African-American and other historically underserved groups, receive high-quality mathematics education. This requires a comprehensive set of strategies addressing all aspects of the educational system. Thus, the partnership has been working to systemically change what is happening in mathematics education across the east Alabama region. TEAM-Math’s design includes five primary components: (1) curriculum alignment, (2) teacher leader development, (3) intensive professional development, (4) outreach to stakeholders, especially parents, and (5) improvement of teacher education. In our 10 years of existence we have impacted over 1700 K-12 teachers of mathematics in the partner schools.

We believe that involvement in professional development will lead to change in teacher attitudes toward and use of reform practices (i.e., those consistent with the recommendations of *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 2000), which in turn will positively influence student motivation, ultimately leading to improved achievement in mathematics. Previous analyses of TEAM-Math project data (e.g., Woolley, Strutchens, Gilbert, & Martin, 2010) showed that students who reported greater teacher use of reform practices, higher teacher expectations, and higher teacher standards, demonstrated higher levels of confidence and

interest in mathematics and lower levels of anxiety as it relates to mathematics. Moreover, students with more desirable levels of motivation to learn mathematics performed better in mathematics, including standardized test scores and self-reported grades in mathematics. There was also a direct relationship between teachers' uses of reform practices and expectations and students' performance in mathematics (Woolley et al., 2010).

The teaching practices advocated by TEAM-Math are consistent with the findings of research focused on classroom strategies for enhancing students' motivation (e.g., Stipek et al., 1998; Turner & Patrick, 2004). However, an obstacle to implementation of reform practices is teachers' own beliefs about mathematics teaching (e.g., Ross, McDougall, & Hogaboam-Gray, 2002). TEAM-Math professional development activities are designed to affect teachers' beliefs about the nature of mathematics as a problem-solving activity and about what it means to learn mathematics, based on national standards (NCTM, 2000, 2006), state standards (Alabama State Department of Education, 2003), and research on teaching and learning. Teachers are given opportunities to develop a variety of instructional strategies for students to explore curriculum content, a wide selection of sense-making activities or processes through which students can come to understand and "own" information and ideas, and many options through which students can demonstrate or exhibit what they have learned (Tomlinson, 1995; Haberman, 1992; Senk & Thompson, 2003). Teachers are provided an opportunity to enhance content knowledge through examination of exemplary curriculum materials and solutions to tasks teachers find mathematically challenging. In order to address variable expectations and levels of support for different groups of students as stated in Equity Principle (NCTM, 2000), teachers were

challenged to reconsider their beliefs about who can be successful in mathematics.

The structure of TEAM-Math's professional development was based on best practices (Loucks-Horsley et al., 2003; Borasi & Fonzi, 2002). A cohort-based model was used, where teachers at a school entered the professional development as a group. Qualitative analyses of participating schools have shown the importance of developing a supportive environment—including administrators and teacher leaders—in encouraging teacher participation in project activities (Strutchens, Henry, & Martin, 2009). Together, teachers from a school experienced a two-week and a one-week summer institute, quarterly follow-up meetings on Saturday mornings throughout the school year, other special workshops and events, and school-based activities focused on developing professional communities of practice (Wenger, 1999).

Professional Learning Communities

Even though we specifically discussed developing professional communities of practice within the schools, we developed professional learning communities across the TEAM-Math partnership without explicitly naming what we were doing. Professional Learning Communities (PLCs) have been characterized as having shared missions, visions and values; typically involving collective inquiry, collaborative teams, action orientation/experimentation, continuous improvement and a results orientation that focuses on student learning (DuFour, 2004; Hord, 2008). Fulton, Doerr, and Britton (2010) identified five dimensions that practitioners and researchers consistently identify as important for success in Science, Technology, Engineering, and Mathematics (STEM) PLCs:

- 1) *Common vision and shared values* emerge from a collaboratively defined understanding of what constitutes worthwhile student learning, with all members of the PLC working

together on related problems. 2) *Collective responsibility* requires participants to contribute and share their expertise, and a sense of accountability for the student learning that is being supported. 3) *Leadership support* is the support of principals and other school leaders, who give school faculty space and dedicated time to meet. Continuity over time is important, since it takes time for trust to be built and more time to build a common language, norms, and protocols that work for the particular PLC. 4) *Good facilitation* contains three types of facilitator roles: knowledge facilitation to direct participants to information or strategies; process facilitation to attend to the structure and interaction of the group; and focus facilitation to keep the group on target. 5) The *use of data and student work* is central to the effectiveness of the PLC. Because the work of the PLC is focused on student learning, members of the PLC need to become comfortable with working with a variety of authentic measures for gauging changes in student learning and teaching effectiveness. Observing each other's teaching and providing feedback loops and protocols for reflecting on practice are also often used as key elements in the work of the PLC (Fulton et al., 2010).

Within the structure of TEAM-Math, several PLCs were formed. We had a core leadership group that met biweekly to discuss how we were going to meet the goals of the MSP. In the first set of meetings we noticed we were not all speaking the same language so we decided to create a seminar series to help us all to get on the same page. During the seminars, mathematicians, mathematics teacher educators, graduate students, and other project leaders who are available meet to discuss issues related to teaching and learning.

These seminars (which are still on-going) enable mathematics teacher educators, mathematicians, and school leaders to develop a common vision for the partnership and

help us to have a united professional development focus for the teachers. For our initial phase of the partnership, beyond the leadership core, we had a professional development committee; a presenter team, which was subdivided by grade bands, but met as a whole group in preparation for institutes and quarterly meetings; a teacher preparation committee; a stakeholder committee; and an evaluation committee. Each of these committees contained mathematics teacher educators, mathematicians, and school partners (teachers and/or administrators). Furthermore each of these committees was a PLC. We also had a teacher leader PLC that contained teacher leaders from all of the schools that were a part of the partnership, which met quarterly.

In like manner, most of the seven partnerships featured in this journal issue have PLCs that are intentional and ones that evolve as the projects grow. For example, Focus on Mathematics (Matsuura, Sword, Piecham, Stevens, & Cuoco, 2012) is devoted to improving student achievement in mathematics through programs that provide teachers with solid content-based professional development sustained by *mathematical learning communities* in which mathematicians, educators, administrators, and teachers work together to put mathematics at the core of secondary mathematics education. On the other hand, Kinzer, Bradley, and Morandi (2012) in describing project LIFT never explicitly talk about the development of learning communities, but in the work that they do, learning communities are implicit. In addition to having different forms of PLCs, the partnerships have other components in common. In the following sections we discuss those components.

General Logic Model

In looking across the seven projects, a general logic model seems to either explicitly or implicitly drive their MSP work. First, there is a focus on improving teachers'

mathematical content knowledge, leading to an improvement of teachers' instructional practices, which ultimately leads to improvement in student learning; see Figure 1. Note, however, there is substantial variation in how these areas are conceptualized, and a few projects include additional emphases. We will briefly describe the different perspectives taken by the seven projects.

Figure 1. General logic model for the projects.

Despite the variation among the programs in the manner in which professional development was provided, all included a major emphasis on improving teachers' mathematical content knowledge as a primary cause of change. But within that emphasis on mathematical content knowledge, there was substantial variation in the type of mathematical content knowledge targeted. Nonetheless, several themes were prevalent. All of the projects either explicitly or implicitly focused on helping teachers to develop pedagogical content knowledge (e.g., Shulman, 1986) or the mathematical knowledge for teaching (e.g., Ball & Bass, 2000) – that is, content knowledge that is interwoven with what teachers actually need to know and be able to do to support student learning. A number of projects focused on developing general themes or approaches that would be useful in looking across the curriculum (e.g., functions as a connecting theme [Teixidor-i-Bigas, Schliemann, & Carraher, 2012]) or specific conceptual areas central to the curriculum (e.g., rational number and proportional reasoning [Whitenack & Ellington, 2012].) Other

projects focused on developing a greater appreciation for what it means to do mathematics – for example, mathematical habits of mind (cf. Matsuura, Sword, Piecham, Stevens, & Cuoco; 2012; Teixidor-i-Bigas, Schliemann, & Carraher, 2012). Across all these approaches, there was a clear focus on the need for teachers to develop a deeper understanding of mathematics beyond merely increasing their knowledge of the discipline.

The projects further differed in the degree to which their professional development explicitly addressed changes in instruction. While some projects provided explicit definitions of effective teaching (e.g., Sayler, Apaza, Kapust, Roth, Carroll, Tambe, & St. John, 2012) or student outcomes, in other cases the target was more implicit. However, considering both the explicit targets along with implicit targets gleaned from descriptions of projects' work and their findings, the general theme across the projects is that students were expected to “engage in critical, in-depth higher order thinking” (cf. Gningue, Peach, & Schroder, 2012) that would promote students' development of conceptual understanding, beyond attaining procedural skill. They also imply a focus on helping students develop ways of thinking about mathematics, sometimes called processes (NCTM, 2000) or mathematical practices (CCSS, 2010). Teachers were either implicitly or explicitly expected to use instructional methods that would support the development of that kind of knowledge, becoming more student-centered, with a focus on responding to student thinking, effectively questioning students, and building classroom discourse.

Indeed, all of these aims seem quite aligned with the national consensus around school mathematics over the past decade as expressed in NCTM's standards documents, particularly *Principles and Standards for School Mathematics* (NCTM, 2000). Although the *Common Core State Standards* (Common Core State Standards Initiative [CCSSI], 2010)

postdated all of these projects, commonalities can also be seen in the emphasis on conceptual development as well as the mathematical practices. Thus, these projects can continue to provide important insights about improving mathematics education in the coming years. In fact, we have found that new activities of the TEAM-Math project have rather seamlessly transitioned to a focus on Common Core State Standards for Mathematics (CCSSM) (CCSSI, 2010); for example, we conducted a textbook review (TEAM-Math, 2012) that built on our previous work with curriculum alignment.

Finally, while the general logic model seems relevant across the projects, we would be remiss in not mentioning how some projects expanded upon this model. For example, several projects described the importance of engaging administrators in building an environment that supports change (e.g., Kinzer, Bradley, & Morandi, 2012; Lewis, Fischman, Riggs, & Wasserman, 2012; Sayler et al., 2012). Likewise, several projects focused on developing teacher leaders who could support improvement efforts at the school-level (e.g., Gningue, Peach, & Schroder, 2012; Kinzer, Bradley, & Morandi, 2012; Whitenack & Ellington, 2012). Our experience fully matches with the observation by Sayler et al. (2012) that “a robust infrastructure established to support teacher growth.” We found that that support systems within a school significantly impacted teacher engagement (cf. Strutchens, Martin, & Henry, 2009). This implies that the proposed logic model may be embedded in a larger context of system change; see, for example, the expanded logic models used by Sayler et al. (2012) and by Gningue, Peach, and Schroder (2012).

Measures and Findings

Not surprisingly, the projects used a wide range of measures to assess progress in reaching their targets. In considering changes in teachers’ content knowledge, projects

used previously-developed instruments (cf. University of Louisville, 2012), their own instruments, performance tasks, and classroom observations. In considering changes in teacher's instructional practices, projects primarily used classroom observation protocols (some designed by the state or other projects) or in-depth analyses of transcripts of classrooms. Only a few projects directly measured changes in student learning, primarily relying upon state assessments, probably a reasonable target given that these assessments are the primary targets for the K-12 partners.

Given the variety of methodologies, grain sizes, and levels of development of the analyses presented in these papers, it would be nearly impossible to provide any synthesis of the findings. We shall, however, provide a few general observations. First, projects tended to get better results when using instruments or protocols that they designed than when using more general assessments, instruments, or protocols. This is probably not surprising, since the more general measures are likely to be less aligned with project aims, particularly when considering state assessments that may focus more on procedural understanding. (Note that this may change as states implement common assessments designed by the two assessment consortia based on CCSSM.) On the other hand, self-designed measures may be less refined than external measures, lack the psychometric grounding, and may be viewed as less credible. The struggles of identifying or developing measures useful in describing progress will continue to be a challenge for projects such as these. Nonetheless, several projects were able to report informative findings supporting the effectiveness of the approaches they took.

Second, several projects engaged in more qualitative analyses of their progress, looking at what happened within a course being conducted by the project or within classes

conducted by participants in the project. These sorts of analyses were better able to capture the richness of the work being done by the projects and to lend insight not only into what happened, but why it happened. A number of important insights can be gleaned from these analyses. However, in some cases, a more careful description of their methodology and data analysis methods would help their findings rise above what could be interpreted as anecdotal evidence to a more scholarly level.

Reflections

We close with reflections that may be useful to those planning projects with related aims and approaches. First, it is imperative that projects be designed with knowledge generation as a key component. As the MSP movement has progressed, the inclusion of clear research plans has been increasingly emphasized in the National Science Foundation Request for Proposals (RFPs) for the MSP program. This perspective has to be part of the “DNA” of a project, not merely an add-on designed to satisfy the RFP. We suggest that to the degree possible, MSPs and other projects begin with a clear logic model, identifying measures that will be useful in tracking their progress. As TEAM-Math evolved, we recognized that our initial measures were difficult to collect in a reliable manner, leading to on-going difficulties throughout the life of the project. Moreover, as the project’s understanding of its mission is refined, so the logic model and measures can be updated accordingly. For example, at its onset TEAM-Math did not adequately recognize the important role played by guidance counselors in influencing students’ participation in mathematics across the grades, leading us to later include them both in our logic model and in the data we were collecting.

Second, to help ensure that adequate attention is being paid to the project’s research

agenda, we suggest that someone on the leadership team might be given a primary responsibility for tracking the research effort, related to but apart from project evaluation. Efforts should be made to identify workable research designs that can fit into the life of the project in a way that generates knowledge usable by others without dramatically adding to what can seem an already overwhelming agenda. For example, as described in an earlier section, many of the projects engaged their participants in PLCs. The work of these PLCs might be “mined” not only to better understand the progress of the projects but also to generate knowledge that will be more generally useful. Indeed, considering the project leadership team as a PLC could provide an opportunity to explicitly track data on emerging understandings across the various stakeholders regarding what is needed to produce changes in teacher knowledge, in their understanding of teaching, and in student performance.

In summary, we applaud the efforts of these projects to generate knowledge that can inform others, beyond simply evaluating one’s efforts for internal use. We fully appreciate how difficult it can be to simultaneously carry out a large project and capture what is happening in that project in a manner that will be useful to others. The reports in this collection illustrate a number of creative ways of meeting that challenge and will provide numerous useful insights for others engaged in similar efforts.

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