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ABSTRACT: This article presents a research summary of prospective elementary teachers' (PTs') mathematical content knowledge in the area of fractions. The authors conducted an extensive review of the research literature and present the findings across three time frames: a historical look (pre-1998), a current perspective (1998–2011), and a look at the horizon (2011–2013). We discuss 43 articles written across these time frames that focus on PTs' fraction knowledge. Consistent across these papers is that PTs' fraction knowledge is relatively strong when it comes to performing procedures, but that they generally lack flexibility in moving away from procedures and using "fraction number sense" and have trouble understanding the meanings behind the procedures or why procedures work. Across the time frames, the trend in the research has moved from looking almost entirely at PTs' understanding of fraction operations, particularly multiplication and division, to a more balanced study of both their knowledge of operations and fraction concepts. What is lacking in the majority of these studies are ways to help improve upon PTs' fraction content knowledge. Findings from this summary suggest the need for a broader study of fractions in both content and methods courses for PTs, as well as research into how PTs' fraction content knowledge develops.

Keywords: fractions, rational number, mathematical knowledge for teaching, mathematical content knowledge, preservice teachers, prospective teachers, elementary, teacher education

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Introduction

Elementary teachers need a "solid understanding of mathematics so that they can teach it as a coherent, reasoned activity and communicate its elegance and power" (Conference Board of the Mathematical Sciences [CBMS], 2001, p. xi). However, research studies on prospective teachers' mathematics knowledge have shown that many possess a limited knowledge of mathematics in key content areas such as number (e.g., Ball, 1990a; Thanheiser, 2009; Tobias, 2013). This is particularly true in the case of fractions, which, along with ratio and proportion, Lamon (2007) calls, "the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science, and one of the most compelling research sites" (p. 629).

The National Mathematics Advisory Panel (2008) affirmed that "proficiency with fractions" is a major goal for K–8 mathematics education because "such proficiency is foundational for algebra and, at the present time, seems to be severely underdeveloped" (p. xvii). Therefore, developing such proficiency in prospective elementary teachers (PTs) is a critical task for mathematics educators. As the authors of *The Mathematical Education of Teachers, Part 1* suggest, "The key to turning even poorly prepared prospective elementary teachers into mathematical thinkers is to work from what they *do* know" (CBMS, 2001, p. 17). Thus, in order to design mathematics courses for prospective teachers that will help them to develop the "solid understanding of mathematics" called for by the Conference Board of Mathematical Sciences (2001), including a deep understanding of and "proficiency with fractions," we must begin by determining what it is that PTs know. In this paper, we discuss the main findings from a research summary of existing studies on prospective elementary teachers' fraction knowledge to identify directions for future research.

Theoretical Framework

In looking at teacher knowledge, we begin by examining the work of Shulman (1986), who proposed three categories of content knowledge for teachers: (a) subject matter content knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge. For Shulman, *subject matter content knowledge* includes knowing a variety of ways in which "the basic concepts and principles of the discipline are organized to incorporate its facts" and "truth or falsehood, validity or invalidity, are established" (p. 9). *Pedagogical content knowledge* refers to the knowledge of useful forms of representations (e.g., analogies, illustrations, explanations) of subject-matter ideas that make it understandable to others, as well as an understanding of the conceptions and preconceptions students bring to the learning processes. The third type of knowledge, *curricular knowledge*, includes knowledge of a "full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances" (p. 10).

Shulman's ideas on pedagogical content knowledge sparked a huge interest in knowledge for teaching, eliciting over a thousand studies (Ball, Thames, & Phelps, 2008) throughout a number of content areas, with a large number of these studies focusing on teachers' knowledge of mathematics (e.g., Ball et al., 2008; Davis & Simmt, 2006; Hiebert, 1986; Ma, 1999). Deborah Ball and her colleagues introduced the term *mathematical* *knowledge for teaching* (MKT) (e.g., Ball & Bass, 2002), which focused on the work that teachers do when teaching mathematics.

Building on Shulman's (1986) categories of knowledge, Ball, Thames, and Phelps (2008) introduced a framework for mathematical knowledge for teaching. This framework broke subject matter knowledge into three categories: *common content knowledge* (CCK), the mathematical knowledge that should be known by everyone; *specialized content knowledge* (SCK), the knowledge of mathematics content that is specific to the work of teachers; and *horizon content knowledge*, which involves understanding how different mathematical topics are related. Pedagogical content knowledge was similarly broken into *knowledge of content and students* (KCS), which dealt with understanding how students relate to different topics; *knowledge of content and teaching* (KCT), which involves the sequencing of topics and the use of representations; and *knowledge of the curriculum* as a whole. While a number of different frameworks look at mathematical knowledge for teaching, we chose to use this framework to ground our study, as it is widely recognizable in the mathematics education field.

Background and Research Questions

This summary work was initiated at a PME-NA Working Group over a four-year period from 2007 to 2010 (Thanheiser et al., 2010). The members of the Working Group all taught specially designed mathematics courses for elementary school teachers in the United States and sought to improve their practice by building on PTs' current knowledge. The Working Group was formed with a goal of summarizing the prior research addressing PTs' content knowledge and its development with the idea that we could improve both our teaching and course design, as well as design further research to extend what we know about PTs' mathematical knowledge. We broke into smaller groups by content area (whole-number concepts and operations, fractions, decimals, geometry and measurement, and algebra) and attempted to summarize the current research in each of these fields.

This paper reports a summary of the research that has been done to this point on prospective elementary teachers' knowledge of fractions. Our goals for the research summary were (a) to identify what we already know about PTs' knowledge of fractions in both the domains of common and specialized content knowledge, as well as knowledge of content and students; and (b) to identify the knowledge gap in the existing research base to help guide future research endeavors. We organize our summary into three categories: (a) a historical look at PTs' fraction understanding, (b) a look at a recent perspective on PTs' knowledge of fractions, and (c) a view of the horizon on what current and future work on PTs' knowledge of fractions may look like and what it should look like.

Background on Fractions in General

Before we discuss what we know about prospective teachers' knowledge of fractions, we must look briefly at the topic of fractions in general, to gain an understanding of what knowledge of fractions would look like from a general perspective.

One research area that encompasses the study of fractions is that of rational number. A rational number is one that can be written in the form a/b where a and b are both integers, and b is not equal to 0; thus, the study of fractions is part of the study of rational numbers. Researchers (e.g., Ball, 1993; Kieren, 1976, 1993; Lamon, 2007, 2012) have tended to agree that in order to gain a deep understanding of rational numbers in general, one must be familiar with many different interpretations of fractions. While researchers have given slightly varying lists of these interpretations, Ball (1993) summarizes that they have tended to agree that fractions "may be interpreted (a) in part-whole terms, where the whole unit may vary; (b) as a number on the number line; (c) as an operator (or scalar) that can shrink or stretch another quantity; (d) as a quotient of two integers; (e) as a rate; and (f) as a ratio" (p. 168), and that in order to have a deep understanding of rational number, students and teachers must be familiar with all of these representations, rather than merely the part-whole area models that are most commonly associated with fractions and most commonly taught in schools. Lamon (2007, 2012), in particular, has emphasized the need for students to be introduced to a variety of fraction interpretations, stating that "students whose instruction has concentrated on part-whole fractions have an impoverished understanding of rational numbers" (2012, p. 256). Thus, one of the important areas of prospective teachers' knowledge of fractions is to have a deep understanding of all of the different interpretations of fraction.

Another research area that has looked at fractions deals with literature on multiplicative structures. Vergnaud (1988) includes rational numbers as part of what he calls the *multiplicative conceptual field*, which, he says, "consists of all situations that can be analyzed as simple and multiple proportion problems and for which one usually needs to multiply or divide . . . [These include] fraction, ratio, rate, rational number, and multiplication and division" (p. 141). The basis of a conceptual field is that it contains a set of situations that are modeled by a similar action. Movement from the additive conceptual field to the field of multiplicative structures has been shown to be difficult for students and teachers (e.g., Fischbein, Deri, Nello, & Marino, 1985; Tirosh & Graeber, 1989). This difficulty is particularly due to a problem Taber (1999) calls the "multiplier effect." Taber describes this effect in this way: "Students seem to select multiplication or division as the operand that will solve the problem depending on their sense of whether the multiplicand is enlarged or reduced by the action of the problem" (p. 2). This problem was described by Fischbein, Deri, Nello, and Marino (1985) in their work with fifth, seventh, and ninth grade students. The students were given a variety of word problems dealing with multiplication and division of rational numbers and asked to write an equation that they would use to solve the problems. In general, when the students thought that the result of the problem should be smaller than the input, they chose to divide; when they thought their result should be larger, they chose to multiply, even though in many instances this was not the correct equation and did not lead to the correct answer.

One aspect of multiplicative structures that can be particularly difficult for students is the concept of division. Division is typically taught using two different interpretations. The *partitive* or *sharing* model involves dividing the total amount by the number of groups in order to find the number in each group (Greer, 1992). The *quotitive*, *measurement*, or *repeated subtraction* model of division involves separating the total number of things by the number in each group to find the number of groups possible (Greer, 1992).

The *partitive* model of division is typically taught to children first, and is called the "primitive" model of division by researchers (Fischbein et al., 1985; Tirosh & Graeber, 1989). This idea is introduced as division through "fair sharing" and can be modeled by giving one object to each person until there are none left. For example, the problem "I have 20 cookies and I want to share them among myself and 4 friends. How many cookies do we each get?" can be modeled by distributing a cookie to each person one at a time until each person has 4 cookies, and there are no cookies left.

The measurement type of division can be modeled by the process of repeated subtraction. The question “I have 20 cookies and I want to give 5 to each of my friends. How many friends can get cookies?” can be modeled by repeatedly taking out groups of 5 from the 20 objects until there are no cookies left, resulting in 4 groups.

Of the two models of division, the *measurement* model is much more easily translated into situations dealing with fractions. We can think of having 5 1/2 pounds of candy, giving 1/2 of a pound to each person, and asking how many people get candy. This situation can be easily modeled by subtracting 1/2 from 5 1/2 until there is nothing left, and we can see that there are 11 groups. Thus, 5 1/2 ÷ 1/2 = 11. However, it gets more complicated when we try to translate the *partitive* model of division into fractional situations. “The fair sharing, or partitive model is a traditional teaching model for division of whole numbers, but it can act as a barrier in the representation of division of fractions” (Rizvi & Lawson, 2007, p. 378). When we look at division of fractions using this model, the original situation that we used with whole numbers does not make sense. We cannot talk about half or a third or three fifths of a person. The partitive situation can be modeled with a word problem, such as “I have 5 1/2 pounds of candy. This is 1/2 of a serving of candy. How much candy is a whole serving?” We still know how much we started with and are trying to determine the size of one group, but the translation of the problem does not always make it seem like it is the same form.

In order to develop proficiency with fractions, one must not only be able to perform operations with them, but must also develop a fraction number sense, which means being able to think of fractions as numbers in a system. Lamon (2012) describes fraction number sense in this manner: “Students should develop an intuition that helps them make appropriate connections, determine size, order, and equivalence, and judge whether answers are or are not reasonable" (p. 136). This makes being able to compare and order fractions an important component of teachers' fraction knowledge.

Lamon (2012) suggests three different strategies for ordering fractions: *same-size parts*, *same number of parts*, and *compare to a benchmark*. These strategies are also suggested in the *Common Core State Standards for Mathematics* (CCSSM) (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA & CCSSO], 2010). In the same-size parts strategy, which is also referred to as the "common denominator" strategy, if two fractions have the same denominator, or size of parts, then they can be compared merely by looking at the numerators. For example, 3/5 > 2/5, because 3 of something is more than 2 of the same thing. In the same number of parts, or "common numerator" strategy, if two fractions have the same numerators, or number of parts, we can compare them by looking at the size of the individual parts. For example, 2/3 > 2/5, because if we break a whole into three equal-sized pieces and break an equivalent whole into five equal-sized pieces, then the thirds will be larger than the fifths.

The third fraction comparison strategy involves comparing two fractions to another "benchmark" fraction, such as 1/2, 1/3, or 1. For example, in comparing 3/7 and 6/11, we know that 3/7 < 1/2, since 3 is less than half of 7, and 6/11 > 1/2, since 6 is more than half of 11. Therefore since 3/7 < 1/2 < 6/11, we can use the transitive property to determine that 3/7 < 6/11.

Now that we have a better understanding of what knowledge of fractions might entail, we can move on to looking at what we know about PTs' fraction knowledge in particular. In order to do this, we need to do an extensive search of the literature to determine what we already know and to look at what we still need to learn.

Methods

The first step of conducting this research summary on what we know about PTs' knowledge of fractions was to identify the existing literature. Thus, we began by looking for articles to fit into the *Current Perspective* section. To maintain the quality of the findings, we began by restricting our search to the peer-reviewed research articles published between 1998 and 2011 to cover the 12-year range prior to our Working Group's meetings. The Working Group chose this time period because it marked the beginning of a renewed interest on teacher knowledge since the publication of Ma's (1999) work that looked at elementary teachers' mathematical knowledge for teaching in the United States and China. This was particularly true in the area of division of fractions, which is the area where the majority of the U.S. teachers struggled.

With key words such as *preservice teachers*, *prospective teachers*, *fraction*, and *rational numbers*, we searched the ERIC, Google Scholar, Dissertation Abstracts, and Rational Number Reasoning databases (gismo.fi.ncsu.edu/database) to find any papers that might fit into our study. The second step required our research team to locate these papers and skim through them to determine if they had a research question focusing on prospective elementary teachers' fraction knowledge. We ended up rejecting a number of papers, because they did not meet this criterion. For example, we found some papers in our searches that focused on prospective teachers' beliefs, rather than their knowledge. Others did not really encompass PTs' knowledge of fractions, but rather included a single example of one PT's thoughts on a problem that happened to have a fraction in it. We carried out careful readings of these documents during the third step. To assist the comparison across these documents, we created a synthesis table with information such as “research questions,” “research design,” “descriptions of participants,” “content foci,” “data collection,” “data analysis,” “findings,” and “implications” for each. Each content group filled in a similar table with information from their respective content areas.

After our initial search, each content group summarized its findings and reported them at a Working Group meeting (Thanheiser et al., 2010). We shared our list of articles and discussed inclusion/exclusion criteria for the journals we had found, focusing on whether the journals published empirical studies and were peer-reviewed. We ended up compiling a list of 23 journals from which at least one group had found articles. We then carefully reviewed each journal for additional articles focusing on PTs’ content knowledge within the given time frame to make sure that we had identified all of the relevant articles for the *Current Perspective* section.

In our search for articles that focused on prospective teachers’ knowledge of fractions that were published prior to 1998, we chose to focus on articles that had been cited in later research. The rationale behind this is that these papers, while older, provided the basis for much of the later research on prospective teachers’ knowledge of fractions. In order to find these studies, we checked the reference sections of all of the articles that we found from our searches for *Current Perspective* articles. In addition, two of the authors of this article were in the process of writing dissertations that related to prospective teachers’ fraction knowledge, so they brought with them a number of articles from literature searches related to this work. While this process may not have identified all of the articles written about PTs' fraction knowledge prior to 1998, we are confident that we have all the articles that provided the basis for future studies.

We conducted a review of recent research, 2011 through the beginning of 2013, to analyze the current and future trends in PTs' understanding of fractions. We conducted a journal search from our list of 23 journals for any articles published in 2012 and the first quarter of 2013. In addition, we manually searched for papers in conference proceedings from the International Group of Psychology of Mathematics Education (PME) and the Psychology of Mathematics Education–North America Chapter (PME-NA) from 2011 and 2012, because we recognized the time lag required for publication and were interested in the directions of future research. We added these articles to our synthesis table and began to organize the articles around different themes.

Results

We organized our findings of 43 papers both into the time frames—pre-1998, 1998–2011, and 2011 and beyond; and around three main components of the theoretical framework outlined by Ball, Thames, and Phelps (2008)—Common Content Knowledge, Specialized Content Knowledge, and Knowledge of Content and Students—and also different instructional interventions designed to help improve this knowledge. We included sections on instructional interventions because we believe that the study of PTs' fraction knowledge encompasses not only what they know, but also how they come to know it. While Ball and her colleagues outlined other aspects of mathematical knowledge for teaching, these did not encompass what we would consider PTs' *content* knowledge, which is the focus of this article, and, thus, we did not frame our discussion around them.

Historical Perspective (Prior to 1998)

In total, we found 12 articles from six different studies, which we felt provided the basis for subsequent work looking at PTs' fraction content knowledge. A summary of articles is included in Table 1.

Table 1

Articles Written Prior to 1998 Dealing With PTs' Fraction Content Knowledge

Authors	Year	Number of PTs Studied	PTs' Level	Country	Methodology
Ball	1990a	252 (217 elementary and 35 mathematics majors)	Point of entry into formal teacher education program	USA	Questionnaire and interviews/observations of a smaller group
Ball	1990b	19 (10 elementary and 9 secondary)	Prior to enrolling in their first education course	USA	Interviews with probing questions
Behr et al.	1997	30	Seniors in a methods course	USA	Videotaped interviews
Borko et al.	1992	1 as the focus (out of a larger group of 8)	During student teaching	USA	Observations of a teaching episode
Eisenhart et al.	1993	1 as the focus (out of a larger group of 8)	During senior year—student teaching and preparation	USA	Observations of teaching episodes
Graeber, Tirosh, & Glover	1989	129	Enrolled in either a content or a methods course	USA	Written test and interviews with 33 of the students

(continued)

Table 1—continued

Authors	Year	Number of PTs Studied	PTs' Level	Country	Methodology
Khoury & Zazkis	1994	124 (100 elementary and 24 secondary mathematics)	After some mathematics content work	USA	Written assessment and clinical interviews
Simon	1993	33	Enrolled in a methods course	USA	Written test and interviews with 8 students
Tirosh & Graeber	1989	136	Enrolled in either a content or a methods course	USA	Written test and interviews with approximately half of the students (n = 71)
Tirosh & Graeber	1990a	21 selected based on pretest data	11 in a content course, 10 in a methods course	USA	Pre- and posttests and interviews with probing questions
Tirosh & Graeber	1990b	136	Enrolled in either a content or a methods course	USA	Written test and interviews with over 85 students
Tirosh & Graeber	1991	80	Enrolled in either a content or a methods course	USA	2 written tests and interviews with 33 of the students

While two of these articles dealt directly with the subject of fractions (Behr et al., 1997; Khoury & Zazkis, 1994), the majority of them focused more on PTs' conceptions of multiplicative structures in general, particularly in the case of multiplication and division, with only portions of these studies focusing on using these operations specifically with fractions. The focus of these papers dealt with the misconceptions that PTs had about multiplication and division in general (e.g., Graeber, Tirosh, & Glover, 1989), PTs' difficulty representing fraction division (e.g., Ball, 1990a, 1990b; Simon, 1993), and the difficulty that one PT had in explaining fraction division to students (Borko et al., 1992; Eisenhart et al., 1993). These articles could all be described as falling under either the CCK or SCK areas of mathematical knowledge for teaching.

Prospective teachers' common fraction knowledge. All of the studies in this time period except for two (Behr et al., 1997; Khoury & Zazkis, 1994) focused on some aspects of PTs' common content knowledge of fractions. These articles focused primarily on aspects of fraction division. Across all of the articles, the vast majority of prospective teachers were able to perform the traditional invert-and-multiply procedure for dividing fractions. However, none of the PTs across the studies were able to explain why this algorithm worked. Ball (1990a) writes: "Although almost all the prospective teachers were able to calculate 1 3/4 ÷ 1/2 correctly, strikingly few were able to represent the meaning underlying the procedure they had learned" (p. 458). This is possibly because PTs do not see the need to understand why they perform the procedures that they do, as long as they work. However, this belief persists into student teaching, when it becomes necessary for some PTs to explain the meanings behind the procedures (Borko et al., 1992; Eisenhart et al, 1993). One student teacher, Ms. Daniels, did not find it necessary to find an explanation of the invert-and-multiply rule either for herself or for the student, even after being unable to answer a question posed to her by a student during a student teaching lesson (Borko et al., 1992).

Tirosh and Graeber and their colleagues' studies (Graeber et al., 1989; Tirosh & Graeber, 1989, 1990a, 1990b, 1991) focus mainly on looking at whether PTs have the same misconceptions about multiplication and division that Fischbein and his colleagues (1985) found in children. They found that PTs do show evidence of Taber's (1999) "multiplier effect," believing that multiplication always makes bigger and division always makes the result smaller (Graeber et al., 1989). Thus, in deciding whether to use multiplication or division to solve a given word problem, they chose multiplication when they believed the answer would be larger than the initial quantities, and division when they believed the answer would be smaller (Tirosh & Graeber, 1991). These misconceptions persisted in interviews when PTs were asked to perform a division problem where the quotient was larger than the dividend; rather than changing their beliefs, they instead determined that they had made a mistake in computation (Tirosh & Glover, 1990b). Thus, like children and adolescents, PTs seem to have a tendency to overgeneralize rules for whole number operations and apply them to fraction operations. Without deliberate attention to attempt to fix these misconceptions, we believe this cycle of both students and teachers struggling with these ideas will continue.

Prospective teachers' specialized fraction knowledge. One key aspect of teaching is being able to design problems for students. While not necessarily a component of common content knowledge, the ability to create realistic problems, especially those in context, is an important part of a teacher's SCK. Ball (1990a, 1990b), Simon (1993), and Tirosh and Graeber (1991) all found that prospective teachers had great difficulty writing word problems that represented division by a fraction. When asked to do this, most PTs either were unable to come up with a problem at all, or suggested a problem that represented a number expression different from what was asked. For example, when asked to create a division problem for 3/4 ÷ 1/4, the most common error among the students in Simon's (1993) study was providing a problem for 3/4 $\times$ 1/4. Students in Ball's (1990a, 1990b) study often gave problems that represented 1 3/4 ÷ 2, when asked to create one for

1 3/4 ÷ 1/2. When discussing their inability to write problems involving fractions, many PTs attributed their problems to the fact that the problems involved fractions, saying, "You don't think in fractions; you think more in whole numbers" (Ball, 1990a, p. 455). Both Ball and Simon attribute the difficulties that the PTs had more to a lack of a full understanding of division, which was exacerbated by introducing fractions into the problems. Simon (1993) and Tirosh and Graeber (1991) did find that the students who used a measurement model of division were more successful than those who attempted to use a partitive model.

Behr, Khoury, Harel, Post, and Lesh (1997) and Khoury and Zazkis (1994) looked at other aspects of PTs' specialized content knowledge. We classify these as specialized knowledge because they go beyond the traditional knowledge that everyone should have. The former was interested in PTs' ability to look at an operator model of fractions, rather than the traditional part-whole model. The latter looked at PTs' abilities to think about fractions and decimals in different bases, to delve into their understandings of place value and how they relate to fractions.

In Behr and colleagues' (1997) study, the researchers investigated PTs' ability to deal with the operator concept of a fraction when finding 3/4 of 8 four-stick bundles. The PTs were asked to do this in more than one way if they could, but the majority of them applied only one solution strategy, which usually focused on what the authors call a duplicator/partition-reducer (DPR) strategy. This strategy revolves around the partitive method of division, which other studies (e.g., Simon, 1993; Tirosh & Graeber, 1991) have found PTs to favor. The second strategy, called stretcher/shrinker (SS), which corresponds to the measurement model of division, was less prevalent.

While all of the 100 prospective elementary teachers in Khoury and Zazkis' (1994) study were able to conclude that 0.2_{three} and 0.2_{five} were unequal, only 26 correctly said that $1/2_{\text{three}}$ was equal to $1/2_{\text{five}}$. Thus, these PTs believed that fractions changed their numeric values under different symbolic representations, rather than realizing that 1/2 was half of a whole and $1_{\text{three}} = 1_{\text{five}}$. These studies (Behr et al., 1997; Khoury & Zazkis, 1994) show that in addition to PTs having an understanding of fraction operations that is not very robust, they also struggle in understanding different interpretations of fractions in general.

Improving prospective teachers' fraction knowledge. While the majority of these early studies do not give suggestions on improving PTs' fraction knowledge, Tirosh and Graeber (1990a) do suggest evoking what they call "cognitive conflict" in order to help PTs with the misconception that division always makes smaller. In interviews with PTs who held this misconception, the prospective teachers were asked to talk about what division meant and think about the terms *dividend*, *divisor*, and *quotient*. The researchers also provided examples, such as 4 ÷ 1/2, which were meant to help PTs question the idea that division always made smaller. Following these interviews, the majority of PTs were able to clear up many of the misconceptions that they held about division, as their pretest performance improved on the posttest.

From our search of literature on PTs' fraction knowledge from research prior to 1998, we find that the majority of studies focus on the understandings or misunderstandings that PTs have with relating multiplication and division to fractions. In general, we found that PTs are familiar and mostly comfortable with performing the algorithms when working with fractions, but struggle when asked to explain why the algorithms work (e.g., Ball, 1990a; Borko et al., 1992), or to create word problems that represent division by a fraction (e.g., Ball, 1990a, 1990b; Simon, 1993). These are both types of tasks that will be necessary for PTs in their work as teachers; thus, helping PTs to improve upon their procedural understandings is an important step in preparing them for the future. In addition, PTs tend to overgeneralize rules for whole numbers, such as "multiplication makes bigger," and attempt to apply them to operations dealing with fractions as well (e.g., Graeber et al., 1989). Creating cognitive conflict about these misconceptions seems to be a way to help PTs question their own faulty understandings and clear up their misconceptions (Tirosh & Graeber, 1990a). As we continue our review into more current articles, the focus shifts somewhat from looking at mostly fraction operations, to a more rounded view of PTs' understandings of fractions.

Current Perspective (1998–2011)

We found 17 journal articles published during the period of 1998–2011 that are included in this review. These studies were conducted in several different countries with groups of prospective teachers ranging in size from 4 to 344. For summary purposes, we have listed the articles in Table 2.

Table 2

Articles Written From 1998–2011 Dealing With PTs' Fraction Content Knowledge

Authors	Year	Number of PTs Studied	PTs' Level	Country	Methodology
Chinnappan	2000	8	First year of education program	Australia	Interview consisting of training and problem solving
Domoney	2002	4	Student teachers in the teacher training program	Great Britain	Task-based interviews
Green, Piel, & Flowers	2008	50 in study 1; 39 in study 2	Study 1: Enrolled in child development course; Study 2 unclear	USA	Pretest, treatment, and posttest
Isiksal & Cakiroglu	2011	17	Final year of their program	Turkey	(1) Questionnaire on PCK; (2) a follow-up interview on multiplication of fractions
Li & Kulm	2008	46	Math methods course/middle school math and science interdisciplinary program	USA	(1) survey for general pedagogical knowledge; (2) a math test for MKT; (3) an assignment on curriculum planning
Lin	2010	48	Integrated content and methods course	USA	Pretest, treatment and control groups, posttest
Luo	2009	127	Mathematics methods course	USA	Written test
Luo, Lo, & Leu	2011	89 USA; 85 Taiwan	Mathematics methods course	Taiwan and USA	Written test
Menon	2009	64	Mathematics methods course	USA	Written test
Newton	2008	85	Mathematics content course for PTs	USA	Pre- and posttests

(continued)

Table 2—continued

Authors	Year	Number of PTs Studied	PTs' Level	Country	Methodology
Rizvi	2004	17	Completed mathematics curriculum studies courses	Australia	Pre-interview, treatment, post-interview
Rizvi & Lawson	2007	17	Primary/lower secondary Bachelor of Education students	Australia	Pretest A, pretest B, treatment, posttest A, posttest B
Son & Crespo	2009	17 elementary, 17 secondary	Mathematics methods course	USA	Written test
Tirosh	2000	30	Mathematics methods course	Israel	Questionnaire, instruction, midterm assignment, final assignment
Toluk-Ucar	2009	50 experimental; 45 control	Mathematics methods course	Turkey	Written test, questionnaire as pre/posttests, math journals
Yang, Reys, & Reys	2008	280	Unclear	Taiwan	Written test
Young & Zientek	2011	344	Enrolled in one of three different mathematics courses required for PTs	USA	Pre/post written tests

As in the historical section, we classified the articles into one or more of the following four categories based on their research questions—prospective teachers' common fraction knowledge, prospective teachers' specialized fraction knowledge, prospective teachers' knowledge of common fraction errors and non-traditional strategies, and improving prospective teachers' fraction knowledge—which we summarize below.

Prospective teachers' common fraction knowledge. Six studies collected data on PT's conceptual and procedural knowledge of fractions. Domoney (2008) investigated whether student teachers who were trained to teach lower-primary age students in Great Britain had the same limited conceptions of fraction, dominated by part-whole constructs. Chinnappan (2000) investigated PTs' ability to transfer their understanding of fractions to a computer environment called JavaBar. Yang, Reys, and Reys (2008) found that while the PTs from Taiwan were fluent in their procedural knowledge when comparing fractions, most of them were not able to use number sense to compare fractions, even when doing so would be more efficient. Young and Zientek (2001) investigated PTs' understanding of fraction operations through four specific problem types: (a) addition with common denominators, (b) addition with relatively prime denominators, (c) multiplication with relative prime denominators, and (d) division of reciprocal fractions. Luo, Lo, and Leu (2011) compared PTs from Taiwan and the U.S. on a variety of fundamental fraction knowledge topics, including part-whole, quotient constructs in different reorientations, as well as their concepts of equivalence and meanings of fraction operations. Newton (2008) conducted a comprehensive survey of PTs' fraction knowledge that included both routine and non-routine problems covering different types of fraction questions typically found in middle school textbooks.

Generally speaking, these studies found PTs were procedurally proficient in fraction addition and subtraction (Newton, 2008; Young & Zientek, 2001). However, their procedures were rule-based and lacked flexibility. For example, 72 out of the 85 PTs in Newton's (2008) study changed both 2/4 and 3/6 to the equivalent fractions of the same denominator to solve the problem 2/4 – 3/6 rather than renaming both to 1/2. This lack of flexibility extended into PTs' work on fraction multiplication as well, as many of the 344 U.S. PTs in Young and Zientek's (2011) study converted fractions into the same denominator when performing fraction multiplication, even though it was not necessary. A good portion of the prospective teachers had difficulty working with fraction multiplication and fraction division procedures in general (Newton, 2008; Young & Zientek, 2001). For example, on a pretest given at the beginning of their mathematics content course, 49 PTs (n = 85) had at least one computation error with multiplication and 45 had at least one with division problems (Newton, 2008). These numbers dropped to 44 and 17, respectively, on a posttest. Although PTs seemed to improve in their fraction division knowledge, some of the fraction multiplication problems persisted despite the semester-long instruction. This was largely due to the wrongful application of the "cross-multiply" procedures, (e.g., they perform $a/b \times c/d = ad/cb$). The same "cross-multiply" pattern also appeared as the most common fraction division procedure error (Newton, 2008; Young & Zientek, 2011).

The dominating rule-based reasoning also showed up in studies examining PTs' ability to compare fractions (Chinnapan, 2000; Domoney, 2002; Yang et al., 2008). In each of these studies, most of the PTs chose procedural methods when comparing fractions, even when applying number sense would have been more efficient. For example, less than half of the 280 Taiwanese PTs used a benchmark of 1 to solve the following fraction comparison problem: "Vicky and Mary each have a ribbon. Vicky used 30/31 of a meter for her ribbon, and Mary used 36/37 of a meter for hers. Who used more tape for their ribbon? Why?" (Yang et al., 2008). Instead, they relied on changing the fractions to decimals, or finding common denominators, which required more difficult calculations than using number sense and also caused nine of the PTs to get an incorrect answer because of a miscalculation.

PTs' performance on conceptual items and items that require deeper understanding of operations was less than satisfactory. Studies conducted by Luo, Lo, and Leu (2011) with PTs in the U.S. and by Domoney (2002) with PTs from the UK found a strong preference for the part-whole meanings of fraction over other meanings such as quotient and ratio. PTs from these two countries also had difficulties working with number lines. For example, when asked to locate the number 3/5 on the number line of 5 units long, with 0–5 being labeled, one PT placed 3/5 on the unit labeled "3" (Domoney, 2002). In addition, none of the four UK PTs interviewed in this study were able to come up with two fractions that summed to 5 on the number lines. However, this difficulty with number lines did not show up in Luo et al.'s study with PTs from Taiwan. The PTs in this study were also found to be strong with the quotient meanings of fraction. This points to possible differences in different countries' methods of teaching fractions. Researching these instructional differences could lead to improved performance in other countries as well, especially with the increased focus on using number lines to represent fractions and their operations in the U.S. *Common Core State Standards for Mathematics* (NGA & CCSSO, 2010).

Finally, Newton (2008) found that PTs' low performance on problem solving, transfer, and flexibility did not improve much after instruction. For example, 40% of the 85 prospective teachers did not appear to recognize the importance of the equal wholes when performing fraction addition when combining one glass of chocolate milk that contains 1/3 of the glass of chocolate syrup, and another glass that is twice as large with 1/4 of the glass of chocolate syrup. It is doubtful that PTs with such understanding of fractions could support their elementary students' learning of fractions in a meaningful way.

Prospective teachers' specialized fraction knowledge. Several studies examined PTs' ability to create diagrams or word problems for given fraction expressions (Li & Kulm, 2008; Luo, 2009; Menon, 2009; Rizvi, 2004; Rizvi & Lawson, 2007; Toluk-Ucar, 2009). The findings of these studies, based on PTs from Australia, Taiwan, Turkey, and the U.S., suggest the majority of PTs are not proficient in this area. This echoes earlier studies (e.g., Ball, 1990a, 1990b; Simon, 1993), which also report PTs' difficulties in creating fraction word problems. These studies have identified a variety of misconceptions behind the poor performance. For example, Luo's (2009) study focused on PTs' ability to represent fraction multiplication expressions. She found that the majority of the PTs used a "multiplication as repeated addition" model that can be problematic when they are not sure how to add a quantity a fraction of a time. Rizvi and Lawson (2007) found a pattern of declining performance from whole number division problems to fraction division problems, when representing division problems either with word problems or diagrams. Toluk-Ucar (2009) found that many Turkish PTs were unable to identify the unit to which each fraction in an expression referred, so when asked to create a word problem for 3/4 – 1/2, they instead wrote one for 3/4 – 3/8 (note: 3/8 is 1/2 of 3/4).

Another type of specialized knowledge is the ability to provide student-accessible justifications to why given rules and procedures work. Li and Kulm (2008) asked 46 prospective middle school teachers in the U.S. how they would explain to students why 2/3 ÷ 2 = 1/3 or 2/3 ÷ 1/6 = 4. About 26% of the participants use pictorial representations to explain the division procedures, and 22% explained using the "flip and multiply" procedure by describing *how* it should be performed. Most of the other PTs were unable to explain either problem, and none of the 46 participants were able to provide an explanation of *why* "flip and multiply" worked.

Prospective teachers' knowledge of common fraction errors and non-traditional strategies. When teachers enter the classroom, they need to have an understanding of student thinking in addition to understanding mathematics content (Ball et al., 2008). With this understanding, teachers can establish classrooms where discussions focus on the validity of students' responses. Knowing how prospective teachers interpret student responses before they enter a classroom can provide a foundation for the types of activities needed in teacher education programs.

Tirosh (2000) set out to investigate PTs' abilities to identify common student mistakes and the possible source of these mistakes, when evaluating fraction division expressions and solving fraction division word problems. She found that while the majority of PTs were fluent in evaluating fraction division expressions, and most were able to identify at least one common student mistake, they were not able to do so with the word problems. In this paper, Tirosh also discussed several class activities specially designed to help strengthen the PTs' fraction knowledge for teaching. One of her activities was later adapted by Li and Kulm (2008) and Son and Crespo (2009) to investigate PTs' KCS. They both asked PTs to evaluate the validity and efficiency of a non-traditional division method: $a/b \div c/d = (a \div c)/(b \div d)$. Only 2 out of 46 participating PTs in Li and Kulm's study stated that this division method was correct. Son and Crespo developed a framework of six levels of reasoning to classify the PTs' responses that was based on validity, generalizability, and efficiency. Eleven out of the 17 elementary PTs were classified at one of the three lowest levels on this scale, because they did not think the division method described above was generalizable. Those who were classified with lower level reasoning tended to use teacher-focused approaches to respond to their students. That is, they would tell or show directly whether the method worked, and they provided little opportunity for students to explain their reasoning.

Isiksal and Cakiroglu (2011) conducted a study to examine PTs' knowledge of student misconceptions and sources of these misconceptions on fraction multiplication. Based on a written test and semistructured interviews with 17 Turkish PTs, they identified five main categories of misconceptions suggested by PTs for students' errors: algorithmically based mistakes, intuitively based mistakes, mistakes based on formal knowledge of fraction operations, misunderstandings of the symbolism with fractions, and misunderstanding the problems. The first three were consistent with findings from Tirosh (2000), while the last two were new findings from this study. For example, one PT pointed out that students may not be able to answer the word problem "Elif bought a bottle of milk. She gave 1/2 of it, which was 1 3/4 lt, to her grandmother. How much did the bottle of milk contain originally?" because they did not understand the key point that half of something is 1 3/4. This PT's description of student error was classified under "misunderstanding the problem."

Improving prospective teachers' fraction knowledge. Several studies have examined the effects of specially designed mathematics courses (Newton, 2008), or special instructional strategies on prospective teachers' knowledge of fractions, for example, the use of manipulatives (Green, Piel, & Flowers, 2008), Web-based instruction (Lin, 2010), and problem-posing activities (Toluk-Ucar, 2009).

Lin (2010) and Toluk-Ucar (2009) used an experimental design to investigate the effect of certain treatments on improving PTs' fraction knowledge. The treatment in Lin's study consisted of 6 weeks (18 hours) of Web-based instruction that included modules from the National Library of Virtual Manipulatives (http://nlvm.usu.edu/en/nav/vlibrary.html) and the National Council of Teachers of Mathematics' *Illuminations*. The treatment in Toluk-Ucar's study included a 6-hour fraction unit over 3 weeks, where problem posing was used as the primary teaching approach. PTs were given different fractions and asked to pose problems where these fractions were answers, and then to justify the validity of their problems to the rest of the class. The PTs were encouraged to use different representations to support their arguments.

While PTs in all of these studies showed significant improvements over the semester course or after the instructional interventions, many PTs still leave their mathematics or methods courses with various deficiencies and misconceptions. For example, 40% of the prospective teachers in Newton's study (2008) did not appear to recognize the importance of the equal wholes when performing fraction addition. This finding suggests that PTs with such an understanding of fractions may need further professional development in order to be able to support their elementary students' learning of fractions in a meaningful way that meets the expectations of the *Common Core State Standards*.

A View of the Horizon

In searching for articles that represented the future trends in research on PTs' fraction content knowledge, we looked at journal articles from 2012 and the first quarter of 2013, as well as conference proceedings from PME and PME-NA for 2011 and 2012. We found a total of 14 articles that focused on PTs' fraction conceptions, which are listed in Table 3.

Table 3

Articles Written Between 2011 and Early 2013 Dealing With PTs' Fraction Content Knowledge

Authors	Year	Number of PTs Studied	PTs' Level	Country	Methodology
Caglayan & Olive	2011	10	Enrolled in an Algebra for Teachers course	USA	Interview
Harvey	2012	13	Graduate students, 5 were in their final month of their teacher education program and had completed a math education course; 8 were in their first month and had not yet completed this course	New Zealand	(1) written questionnaire, then (2) participated in a teaching experiment either individually or in pairs
Ho & Lai	2012	92	First year of the program	Australia	Ten-item test
Kajander & Holm	2011	Over 600	Enrolled in a mathematics methods course	Canada	Pre/posttest
Lin et al.	2013	49 from U.S.; 47 from China	U.S.–third year of program; China–third year of program	China and USA	Test adapted from Cramer, Post, and delMas (2002) given during first week of fall semester
Lo & Grant	2012	16	3 had completed their first required mathematics course; 13 had not yet taken the course	USA	Interviews

(continued)

Table 3—continued

Authors	Year	Number of PTs Studied	PTs' Level	Country	Methodology
Lo & Luo	2012	45	Enrolled in a mathematics methods course	Taiwan	Interview and written questionnaire
McAllister & Beaver	2012	First phase: >100; Second phase: 72	Enrolled in mathematics content courses at two universities: the second phase included 3 groups of students enrolled in a content course that covers fractions and 1 group enrolled in a content course focusing on geometry (having completed the fraction course).	USA	Given 8 fraction operation problems. First phase: asked to solve the problem and write word problems. Second phase: asked only to write story problems
Mochon & Escobar	2011	21	Last semester of formal course work	Mexico	Questionnaire and classroom observations
Muir & Livy	2012	279	Enrolled in a first-year course	Australia	Mathematical Competency, Skills, and Knowledge test
Rosli, Gonzalez, & Capraro	2012	3	Had completed most required coursework	USA	Interviews
Tobias	2013	33	Enrolled in the first mathematics content course (all at least sophomores)	USA	Classroom teaching experiment
Utley & Reeder	2012	42	Enrolled in an intermediate methods course	USA	Pre/posttest
Whitacre & Nickerson	2011	7	Enrolled in a first mathematics content course	USA	Pre/post interview

Seven of the articles focused on fraction concepts (i.e., comparison, equivalence), five on fraction operations, and two focused on both concepts and operations. This is a shift from previous research in which the majority of articles focused on fraction operations and only a small number on fraction concepts. With the more recent publications, research with PTs is starting to include a more comprehensive analysis of their fraction content knowledge.

Prospective teachers' common fraction knowledge. Eight of the studies focused on PTs' common content knowledge of fractions. The focus of these studies varied widely, including fraction comparison (Whitacre & Nickerson, 2011), converting fractions to decimals (Muir & Livy, 2012), fraction meanings (Lo & Grant, 2012; Mochon & Escobar, 2011; Utley & Reeder, 2012), and fraction operations such as multiplication (Caglayan & Olive, 2011) and division (Kajander & Holm, 2011; Lin, Becker, Byun, Yang, & Huang, 2013). In addition, the studies utilized a variety of methods, including one-on-one interviews (Caglayan & Olive, 2011; Lo & Grant, 2012), questionnaires (Lo & Grant, 2012; Mochon & Escobar, 2011), and pre/posttests (Kajander & Holm, 2011; Lin et al., 2013; Muir & Livy, 2012; Utley & Reeder, 2012; Whitacre & Nickerson, 2011).

Studies found that PTs' fraction conceptions are still largely procedurally based (Caglayan & Olive, 2011; Kajander & Holm, 2011; Lin et al., 2013; Muir & Livy, 2012; Whitacre & Nickerson, 2011). For example, Whitacre and Nickerson (2011) found during pre-interviews that PTs tended to favor standard comparison strategies such as using common denominators when solving comparison problems, even when the numbers were cumbersome to work with, but they became more flexible after completing targeted instruction designed to help them reason about fraction size in different ways. For the seven PTs who were asked to solve nine fraction comparison problems, over 73% of their comparisons involved standard strategies on the pretest, compared to 44.4% on the posttest. In addition, Caglayan and Olive (2011) found that when representing fraction multiplication with pattern blocks, PTs could solve the problem but struggled with representing multiplication using the blocks in a meaningful way. When solving $1/2 \times 1/3$, some PTs drew out 1/2 and 1/3 separately with a multiplication sign in between, as opposed to drawing 1/2 of 1/3. Likewise, Kajander and Holm (2011) gave more than 600 PTs a pre/posttest analyzing their knowledge of solving $1\ 3/4 \div 1/2$ and their ability to justify their solution. They found that most PTs relied on procedures to solve the problem and explained the procedural process as their justification.

Others note that while PTs' procedural knowledge is often stronger than their conceptual understandings, it is still not always correct (Lin et al., 2013; Muir & Livy, 2012). Muir and Livy (2012) found that PTs had difficulty when converting fractions to decimals. Only 15% of the 279 PTs in their study could convert 3/7 into a decimal to four places. Errors included rounding incorrectly and dividing 7 by 3 instead of 3 by 7. In a cross-cultural study that included 96 PTs from both the United States and Taiwan, Lin et al. (2013) found that although PTs from both countries were similarly successful when solving fraction division problems, they equally had difficulties explaining fraction division concepts.

Prospective teachers also tended to focus on the part-whole meaning of fractions (Lo & Grant, 2012; Mochon & Escobar, 2011; Utley & Reeder, 2012). Lo and Grant (2012) found that when PTs were asked a series of questions ranging in difficulty, they struggled more when questions could no longer be answered using the part-whole meaning of fractions. For example, when given the picture below (see Figure 1) and asked to find what fraction was represented by D, with the largest outer square representing one unit, more

PTs used guess-and-check strategies than on other questions, because they had no other recourse.

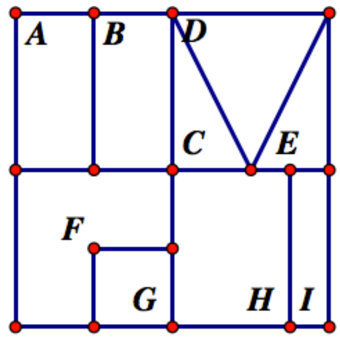

Figure 1. What fraction of the outer square is D? (Lo & Grant, 2012, p. 171)

Lo and Grant (2012) found that questions requiring fraction concepts such as partitioning were conceptually harder for PTs to answer. Predominantly focusing on the part-whole meaning can also affect PTs' ability to understand fractions as quantities (Utley & Reeder, 2012). In a methods course, Utley and Reeder (2012) studied 42 PTs on the topics of fraction benchmarks, sequences, comparison, ordering, and part-whole understanding. They found that PTs struggled with finding the whole, especially when the given fraction was greater than 1. For example, when given a picture of an amount larger than 1 and asked to draw what 1 would look like, only 42.9% of the PTs were able to do this correctly.

Prospective teachers' specialized content knowledge. Four studies focused on PTs' common content knowledge but also added a component of analyzing PTs' specialized content knowledge (Ho & Lai, 2012; Lo & Luo, 2012; McAllister & Beaver, 2012; Rosli et al., 2011). The SCK component required PTs to write word problems for fraction operations and draw pictorial representations of fraction situations. All of the studies dealt with fraction operations with the exception of the work from Rosli et al. (2011) that focused on unitizing.

Underlying the difficulties PTs had with representing operations in a context or in pictorial form was their struggle with understanding the unit (McAllister & Beaver, 2012; Rosli et al., 2011). For example, in a study with three PTs during one-on-one interviews investigating their knowledge of units and unitizing, Rosli et al. (2011) found that PTs had difficulties distinguishing between *how much* and *how many*. When asked how much pizza each person would get when 4 pizzas were shared among 5 people, PTs would answer 4 slices, rather than 4/5 of one pizza. In addition, the PTs struggled with using composite units and being flexible in their thinking. McAllister and Beaver (2012) found in a survey with over 100 PTs that an error that caused PTs to struggle to write appropriate word problems stemmed from their incorrect use of units. When asked to write a word problem for 2/3 + 4/5, one PT posed the question, "Two thirds of the kindergarten class and four fifths of the eighth-grade class mixed together. What fraction of the two classes was mixed?" (McAllister & Beaver, 2012, p. 93). Within this problem, the whole number of students in each of the two classes is unknown; thus, the problem has no answer. In addition, if this problem were solved using 2/3 + 4/5, the answer would be greater than 1, and it is impossible to talk about more than 100% of a class.

Other studies found that PTs have difficulty understanding fraction operations beyond a procedure (Ho & Lai, 2012; Lo & Luo, 2012). In a study with 92 PTs in Australia, Ho and Lai (2012) found that when given the problem $1/3 \times 3/4$, 82.6% of the PTs could solve the problem correctly, but 67.1% of the PTs who provided "justifications" provided an explanation of just the procedure. However, of the 35 PTs who were able to use a context to solve a problem, all were able to provide a pictorial representation for the situation. Lo and Luo (2012) also found similar results in that Taiwanese PTs were able to solve fraction division problems but struggled with writing word problems to represent the situation. When PTs were asked to illustrate a fraction division situation, 40% and 35% of the models they generated were area and linear, respectively. Only 5% of the pictures represented division with a set model.

Improving prospective teachers' fraction knowledge. Two studies gave examples of ways to improve upon PTs' fraction content knowledge. Harvey (2012) suggested using manipulatives as a way to help improve PTs' common content knowledge specifically in the areas of equivalence and comparison. During one-on-one or pair instruction with 13 PTs, Harvey found that an elastic strip that was subdivided into 10, 20, or 25 parts was helpful in developing their understanding of fractions and comparison strategies. For example, three of the PTs who were unable to use a benchmark strategy during a pre-questionnaire were able to do so after instruction using the elastic strip. One PT was able to compare 8/17 and 10/17 by using a benchmark of 1/2 to determine that 8/17 is less than a half and 10/17 is greater than a half. Other PTs used similar methods in determining when fractions were greater than or less than a half. The researchers noted that elastic strips can be useful tools in aiding PTs to keep track of the size of a unit as well as develop their image of number lines.

Whitacre and Nickerson (2011) also found improvements in PTs' number sense and ability to compare fractions after targeted instruction. They designed a sequence of tasks in such a way as to build on and extend PTs' procedural understandings to help them develop a list of fraction comparison strategies, along with agreed-upon names and examples, on which they could draw to solve problems. After completing the tasks, the students in the study improved both in their abilities to correctly solve fraction comparison problems, and in the flexibility of their comparison strategies, getting an average of almost two more questions correct out of nine, and using an average of 2.71 more valid-correct strategies in order to solve the problems.

Prospective teachers' fraction development. Recent reports have also begun to document the ways in which PTs develop an understanding of fractions in a whole classroom setting (Tobias, 2013). Tobias describes how language can confound PTs' understanding of wholes for fractions both less than and greater than 1. For example, when asked to share 4 pizzas equally among 5 people, PTs had difficulties naming the solution of 4/5 in terms of the whole. Some correctly determined the answer to be 4/5 of one pizza, whereas others defined 4/5 to be out of the 5 pizzas, so the question of 4/5 "of what" became important. When developing an understanding of topics, such as fraction language, Tobias notes that PTs' fraction understanding does not develop linearly in that knowledge of one topic may not be fully developed before they start to learn another. For example, PTs started developing the idea that solutions depend on a whole before they developed an understanding of defining an "of what" for fractions, even though the latter idea was introduced to the class first. Likewise, the idea of developing language in terms of what the denominator represents was introduced before the class developed a full understanding of the previous two ideas. Thus, classroom instruction may need to focus on multiple fraction concepts before PTs can fully develop an understanding of one idea.

Conclusions

We began this summary with the intention of determining what we know from research about PTs' knowledge of fractions in the domains of common and specialized content knowledge, and knowledge of content and students. In general, the research we examined indicated that PTs' common content knowledge is relatively strong when it comes to performing procedures, but that they generally lack flexibility in moving away from procedures and using "fraction number sense" (e.g., Newton, 2008; Yang et al., 2008). They also have trouble understanding the meanings behind the procedures or why procedures work (e.g., Borko et al., 1992). PTs seem to favor the part-whole interpretation of fractions, but have trouble with other fraction interpretations such as the operator model (e.g., Behr et al., 1997) and number line models (e.g., Domoney, 2002; Luo et al., 2011).

While prospective teachers' CCK is often adequate to good, many of them have trouble in the areas requiring specialized content knowledge. PTs struggled with representing fractional situations using diagrams and in word problems. Difficulties arose for a number of reasons, including PTs' preference for particular models of multiplication (Luo, 2009) and division (Ball, 1990a, 1990b), which did not lend themselves as easily to working with fractions. PTs also had trouble identifying the unit when trying to represent fraction models (Newton, 2008; Rosli et al., 2011), and with language around fraction ideas, confusing the number of pieces with the fractional part of the whole when these were different things (Rosli et al., 2011; Tobias, 2013).

While knowledge of content and students was not the focus of much of the research on PTs' knowledge of fractions, the studies that were conducted showed that PTs were able to predict some errors that students might make when dealing with fractions; however, they generally attributed these errors to mistakes in following procedures, rather than conceptual errors (Tirosh, 2000). This aligns with findings that PTs' own knowledge is based mostly on following procedures. PTs also had difficulties interpreting non-standard algorithms (Li & Kulm, 2008; Son & Crespo, 2009). This indicates that they may have trouble interpreting their students' solutions to problems.

While the majority of the studies discuss problems that prospective teachers have in working with fractions, few studies have discussed ways to improve PTs' fraction knowledge. Some have suggested special courses (Newton, 2008; Whitacre & Nickerson, 2012) and targeted work with manipulatives, which has seemed to help (Green et al., 2008; Harvey, 2012), but, overall, we do not have enough information on this issue, and we suggest that future research look more at ways to improve PTs' fraction understandings.

In looking at trends in the research on PTs' fraction knowledge, we note that past research has focused primarily on their understanding of fraction operations, predominantly multiplication and division. This is currently starting to shift to include concepts, such as examining PTs' fraction number sense. A trend in all three time frames is that PTs' common content knowledge and/or specialized content knowledge is the focus of the majority of studies. Few have analyzed how to improve PTs' understanding with fractions. Thus, this is still a gap in the research that needs to be filled. In addition, most past research has incorporated quantitative methods that include pre/posttests and/or qualitative methods that include one-on-one interviews. Though this trend is still continuing, there is also research to suggest that future studies will address how PTs learn as they participate in whole-class settings or groups.

Research indicates the need for mathematics courses for PTs to include additional topics such as analyzing student thinking, focusing on standard and non-standard algorithms for solving problems, highlighting concepts that may impact operations such as the role of units, and addressing multiple concepts for PTs to develop one idea. Based on the gaps in the research literature, we suggest future research include more studies on the use of manipulatives with PTs, the role of language with fractions, an understanding of why PTs may have more difficulty with number lines or a linear model over area, and more studies focusing on international comparisons across cultures. By taking into account what we know about prospective teachers' fraction understanding, we can continue to improve our content and methods courses. By also understanding the gaps that still exist, we can design research studies to address these needs. Together these can be used to help us as mathematics educators improve in developing PTs' understanding of the mathematics they are to teach.

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