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ABSTRACT: In the last 25 years a small collection of reports of studies focused on gaining insight into PTs' knowledge of decimals has been published. Three themes are used to frame findings from papers published prior to 1998. Additional findings from papers published between 1998 and 2011 are discussed. Direction for future research that can contribute to the development of curriculum and instruction in mathematics teacher education is shared.

Keywords: decimals, mathematical knowledge for teaching, mathematical content knowledge, preservice teachers, prospective teachers, elementary, teacher education

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Introduction

The historical evolution of decimals as a representation of quantity rests largely on the development of place value and the use of zero in the numeration system. Far more difficult than using the notational system is understanding the quantities represented with the system (Irwin, 2001) in context. Of particular difficulty are decimal fractions (decimals), rational numbers "which originate by subdivision of each unit interval into 10, then 100, 1000, etc., equal segments" (Courant & Robbins, 1996, p. 61). Research on children's conceptions of decimals illustrates a series of conceptual hurdles involved in interpreting and using the notational system (Resnick et al., 1989; Sackur-Grisvald & Leonard, 1985). Because children build their understandings of decimals from their existing or coemergent understandings of multidigit whole numbers and fractions, they tend to over-apply concepts for these more familiar objects when the numerals being discussed are decimals. Findings from studies of children's understandings encouraged researchers to begin to explore prospective teachers' (PTs') understandings of decimal notations (Putt, 1995; Thipkong & Davis, 1991). Such studies unearthed parallels between categories of reasoning used by children and reasoning used by PTs, encouraging researchers to identify teachers' misconceptions as a source of children's faulty reasoning.

Research on PTs' knowledge of decimal fractions has focused on exploring how decimals are interpreted and used in computation, and how mathematics educators might challenge existing beliefs about the use of decimal fractions. In this report, we focus primarily on terminating decimals that are included in primary school curriculum. A very small collection of reports focused on PTs' knowledge of decimals has been published over the last 25 years, but findings point to the importance of place value in PTs' understanding and application of decimals.

Approaches and Orientations

In the sections that follow, we have summarized historical influences in the study of PTs' knowledge of decimals, findings of published peer-reviewed papers from 1998 to 2011, and additional insights drawn from more recent work. Our approach to identification of articles was consistent with the method described in the introductory article of this Special Issue. In addition, our perspective on decimal understanding influenced our interpretations of the articles. We share this perspective to enable readers to gain insight into our interpretations.

Our view of decimal is informed by explorations of PTs' understandings (D'Ambrosio & Kastberg, 2012; Kastberg & D'Ambrosio, 2011) of decimals using a framework including units, relationships between units, and additivity. As Courant and Robbins (1996) suggest, decimal units in the place value system involve repeatedly "subdividing" an individual unit into 10 parts. So if we begin with 1, then subdividing this unit into 10 parts produces 10 subunits 0.1. This action creates the opportunity for the development of relationships between 1 and 0.1, namely, that 1 is 10 times 0.1 and 0.1 is one tenth of 1. While this example involves adjacent units in the set of place value units {..., 10, 1, 0.1, 0.01, ... }, any two units in the set can be thought of as related multiplicatively. Finally, sums of multiples of the units can be used to create new decimals, an idea that is represented in expanded notation. For example, if we compare 0.606 and 0.66 using the additive structure, we can see that 0.606 = 0.6 + 0.006 and 0.66 = 0.6 + 0.06. This understanding and understanding of multiples of the units 0.001 and 0.01 allow us to quickly determine that 0.606 is less than 0.66. Understanding decimals as linear combinations of place value units allows us to compose and decompose decimals to quickly compare them. While there are certainly other views of decimals, it was this view that we held and used to make sense of the findings reported in the research.

The limited number of existing studies encouraged us to create a "conceptual review" (Kennedy, 2007, p. 139) of the historical research rather than a systematic review. Such a review focuses on "gaining new insights into an issue" (p. 139) rather than providing an answer to a specific research question. Our approach was iterative in that we each read a portion of the papers and developed central ideas that we drew from the papers. We then shared the ideas we drew from our readings, read the balance of the papers, and again met to revisit initial perspectives on the papers. A final set of three themes emerged and were refined as we developed our perspective on the papers over time. Because there were only a few papers published since 1998, they were treated more as individual cases informed by drawing from the existing literature and extending the insights researchers had historically provided.

A Historical Look: Decimal Fraction Prior to 1998

In this overview, we discuss the themes we found in research exploring PTs' difficulties with decimals: PTs' interpretations of decimals, PTs' use of concepts and associated beliefs, and changing PTs' concept through cognitive conflict.

Interpretation of Decimals

Prior to 1998, nine research reports were published whose focus was PTs' difficulties with decimals (Graeber & Tirosh, 1988; Graeber, Tirosh, & Glover, 1989; Khoury & Zazkis, 1994; Putt, 1995; Thipkong & Davis, 1991; Tirosh & Graeber, 1989, 1990a, 1990b;

Zazkis & Khoury, 1993). The reports document difficulties PTs have with decimal tasks, including comparing and ordering decimals as well as representing decimals (Thipkong & Davis, 1991), and suggest that the origin of such difficulties stems from the ways in which PTs interpret decimal notation. Authors identified PTs' misconceptions and hypothesized about the origins of these misconceptions. For example, Putt (1995) asked PTs to order a collection of decimals between zero and one (0.606, 0.0666, 0.6, 0.66 and 0.060). In studies of children's approaches to decimals, ordering decimals had been shown to be more cognitively demanding than simply comparing two decimals (Sackur-Grisvald & Leonard, 1985). The PTs in Putt's study also found ordering the collection difficult, but Putt noted that the errors suggested a varied collection of reasons for the PTs' difficulty. Among these reasons were the longer-is-larger and shorter-is-larger misconceptions originally identified in research on children's approaches to decimals (Resnick et al., 1989; Sackur-Grisvald & Leonard, 1985). Learners who use the longer-is-larger misconception apply whole number reasoning to decimals and would identify 0.125 as greater than 0.25 since 0.125 is longer. The shorter-is-larger misconception stems from the application of an early understanding of place value. Positions to the right of the radix point decrease in value, so learners who identify 0.1 as greater than 0.12 do so since tenths are greater than hundredths. In addition, Putt's interviews with participants revealed that that some PTs interpreted decimals as negative numbers, so, when asked to compare a decimal and zero, these PTs' chose zero as larger than a decimal.

The origin of PTs' difficulties seems to be a lack of understanding of place value units and the relationships between units. Strategies used by PTs illustrate that they made comparisons using procedures learned in elementary school, such as appending zeros and treating the quantities like whole numbers, or converting each decimal to a fraction and finding a common denominator that allowed the numerators to be compared as whole numbers. Putt (1995) suggested that some PTs had difficulty understanding that 0.7 and 0.70 are equivalent. In particular, they seemed to struggle to interpret decimals as composites of multiples of units. These difficulties shed light on PTs' interpretation of decimals.

Khoury and Zazkis (1994; Zazkis & Khoury, 1993) proposed that explorations of PTs' concepts rather than their ability to apply rules could be conducted using quantities in bases other than ten, for example, converting 12.34_{five} to base ten. They reasoned that such tasks would encourage the participants to use a general place-value structure rather than rules or procedures. Zazkis and Khoury found that PTs related the fractional part of a number to the base in the number in non-standard ways. For example, in 12.34_{five}, some PTs suggested that the 3 was in the 0.5 position and the 4 was in the 0.05 position, reasoning that is aligned with the consistent use of 1 in decimal notation for tenths (0.1) and hundredths (0.01). Other PTs ignored the fractional part of the number, noting that decimals exist only in base ten (Zazkis & Khoury, 1993). The digits after the decimal were unchanged, while the integer part of the number was converted using a conventional strategy.

Khoury and Zazkis (1994) investigated PTs' "concepts of invariance of fractional number under different symbolic representation" (p. 203). This work explored the students' ability to reason in situations where the quantities were different, but the representations were similar ("Is $(0.2)_{\text{three}} = (0.2)_{\text{five}}$?" [p. 193]) and when the quantities were the same and the representations were similar ("Is the number 'one-half" in base three equal to the number 'one-half' in base five?" [p. 193]). Sixty-three of the 100 elementary PTs correctly answered the first problem using place-value charts and computations such as "$(0.2)_{\text{three}} = 2 \times 1/3$" to generate fractions in base ten they could compare (p. 194). While these students provided correct answers, their reasoning during interviews often revealed attention to place-value syntax rather than quantity value. Some students overgeneralized reasoning derived from their experience with base-ten place-value units to reason about values of the positions to the right of the radix point. The values were identified as 1/5, 1/50, 1/500 (p. 195), a finding consistent with reasoning the authors identified in their prior work (Zazkis & Khoury, 1993). Investigating one half in different bases ("Is the number 'one-half' in base three equal to the number 'one-half' in base five?" [p. 197]) was far more difficult for the students, with only 26% of elementary PTs concluding the two representations for the second task referred to the same quantity. Drawing from the computational strategies used by the students, the authors concluded that PTs' "knowledge of place value and rational numbers is more syntactical than conceptual" (p. 203).

Thipkong and Davis (1991) used line and area models to assess PTs' understanding of decimals. PTs were asked to place given decimals on a number line and represent decimals with an area model, in the form of a square as a unit. They were also asked to reverse this reasoning and identify decimals from positions on number lines and identify the decimal being represented using a square as a unit. PTs had the greatest difficulty when units on the number line were subdivided into subunits other than 10. For example, when asked to mark 1.4 on a number line with subunits of 8, 42% of PTs represented 0.4 as 4 of the 8 subunits in a unit. PTs performed better when they were asked to represent 1.4 using an area model with 8 subunits in the unit. Only 17% counted 4 subunits as 0.4. More than 80% of PTs were successful in representing more familiar decimals, such as 0.5. These findings suggest that PTs can reason about and represent familiar decimals using models with subunits other than 10, but may struggle with less familiar decimals. The authors suggest that models may be useful in supporting PTs to build "relationships of the parts of the unit to the unit" (p. 98).

Findings from this collection of studies suggest that while some PTs master computational strategies that allow them to compare decimals, convert between bases, and represent familiar decimals, others have difficulty. Sources of this difficulty seem to be in building meaning for and interpreting decimal notation. Notations are designed to represent different linear combinations of quantities using a system of units, yet explorations of place-value systems in bases other than ten reveal that PTs attend to the patterns in the base-ten notation rather than the quantities they are meant to represent. For such students, since 0.1 and 0.01 are units in the base-ten system, 0.5 and 0.05 are incorrectly assumed to be units in the base-five system. Research contains evidence that use of the number line and area models to represent decimals were effective tools in revealing overgeneralizations based on subunits of 10 used in non-base-ten systems. In addition, these findings point to the importance of exploring units and relationships in contexts where base ten is not used, such as time.

Use of Concept and Associated Beliefs

Perhaps the most complete investigation of PTs' understandings of decimals and their use is the collection of investigations conducted by Graeber and Tirosh (1988; Graeber et al., 1989; Tirosh & Graeber, 1989, 1990a, 1990b). Following the work of

Fishbein et al. (1985), the authors conjectured that PTs might hold misconceptions regarding multiplication and division that are held by children. Fishbein and his colleagues identified a collection of generalizations, primitive models, children used to predict the results of multiplication and division, such as multiplication makes larger or division makes smaller. The work of Graeber and Tirosh exploring PTs' conceptions of multiplication and division is relevant because it includes a discussion of how PTs' primitive models of multiplication influence their performance writing expressions for word problems. In particular, the authors found that "nonintegral operators, especially operators less than 1, proved troublesome to preservice teachers" (Graeber & Tirosh, 1988, p. 264), a finding confirmed later by Thipkong and Davis (1991). Performance on word problems, meant to be modeled with multiplication or division strategies, was impacted by the presence of decimals. For example, 41% of the 129 students studied incorrectly modeled the following problem.

> One kilogram of detergent is used in making 15 kilograms of soap. How much soap can be made from .75 kilograms of detergent? (Graeber & Tirosh, 1988, p. 264)

The most common incorrect expression, given by 17% of the 129 PTs, was 15 ÷ .75. The authors concluded that the source of the students' difficulty was not the "presence of the decimal," but rather "the role (operator) the decimal plays in these word problems" (p. 265). In particular, when the decimal in the word problem conformed to the primitive model, the PTs performed very well.

The authors found that the PTs were perfectly capable of performing operations with decimals and were generally able to identify statements such as "In a multiplication problem, the product is greater than either factor" (p. 270) as false. Despite the knowledge held in writing expressions from word problems or writing word problems for particular expressions, the PTs enacted their implicit beliefs, including that “multiplication makes larger” and “division makes smaller.” This finding has implications for teacher educators as they work with PTs. Exploring generalizations such as “In a multiplication problem, the product is greater than either factor” (p. 270) with subsets of the real numbers may encourage PTs to revise whole-number reasoning to build ideas about operations with decimals. While PTs identify this statement as false, their reasoning from interview data reveals that counterexamples they use to support their reasoning are drawn from their experiences with whole numbers and algorithms. PTs used procedures to reason that you cannot “divide by a decimal” (p. 273) because you change it to a whole number by moving the decimal point before dividing. The authors suggest that the use of the algorithm “may support their misbeliefs about the relative size of the quotient and the dividend” (p. 274) in a division problem.

Tirosh and Graeber (1989) noted that one source of difficulty with division involved the primacy of the partitive model. Decimal quantities as divisors violate the primitive model that dictates that, in division, one is partitioning a whole rather than finding the number of units of a given size in a given whole. So, while understanding the decimals was admittedly difficult for the PTs (Graeber et al., 1989), the primitive models of operations and the sets of numbers that were allowed to perform various roles in a computation were central to the difficulties in performance when decimals were involved.

Changing the Concept Through Cognitive Conflict

Tirosh and Graeber (Graeber & Tirosh, 1988; Tirosh & Graeber, 1990a) recommended various instructional techniques and activities meant to help PTs connect their explicit reasoning and implicit beliefs. The method highlighted as having the most promise involved the use of "conflict teaching" (Bell, 1983, cited in Tirosh & Graeber, 1990a). The authors used the technique to encourage participants' conscious consideration of the statement "In a division problem, the quotient must be less than the dividend" (Graeber & Tirosh, 1988, p. 275) in light of computational evidence to the contrary. All but 1 of the 21 participants interviewed for this study "realized that a conflict existed between their belief about the relative size of the dividend and the quotient and their computation with decimals" (pp. 275–276). The realization impacted the participants' performance providing correct expressions for word problems involving decimals. In particular, the authors share, "When the conflict approach is carefully applied, pre-service teachers may form a more accurate conception about the relative size of the quotient and the dividend and improve their performance in writing expressions for multiplication and division word problems" (Tirosh & Graeber, 1990a, p. 107).

The one-to-one interviews the authors used to change PTs' beliefs about the impact of decimals on the product or quotient were viewed as inefficient. Instead, Tirosh and Graeber (1990a) suggested that modifications or whole-class activities based on building connections between algorithms, beliefs about operations and subsets involved in operations, and word problems could draw on the strengths of the conflict approach.

A Current Perspective: Decimal Fraction from 1998 to 2011

Between 1998 and 2011 there were three reports of studies whose focus was the development of PTs' knowledge of decimals (Stacey et al., 2001; Widjaja, Stacey, & Steinle, 2008, 2011). Widjaja and her colleagues (2008, 2011) explored the density of rational numbers and the representation of negative decimals to gain insights into misconceptions about decimals that might be hidden in more familiar contexts. The work of Stacey et al.

(2001) is the only study that explores the relationship between PTs' performance on decimal comparison tasks and their identification of decimal comparison tasks that would be difficult for children.

Noting that the whole numbers do not have the density property, Widjaja et al. (2008) justified their exploration PTs' notions of the density property of the rational numbers as a mechanism to unearth misconceptions about decimals. The density property is described by the authors as "the property that between any two decimals, there are infinitely many other decimals" (p. 118). Based on a pre- and posttest, the authors described four incorrect strategies used to identify decimals between two given decimal quantities. Using whole-number reasoning, some students noted that there were no decimals between given decimals, for example, 3.14 and 3.15. Reasoning with only one additional decimal place, other PTs suggested there were a finite number of possibilities between decimals such as 3.14 and 3.15. These students developed lists of possibilities, such as 3.141, 3.142, … , 3.149 (p. 125). Another subset of PTs relied on the "rounding rule" (Stacey, 2005; Steinle & Stacey, 2004), viewing decimals such as 0.799 and 0.80 as the same—"0.80 is the result of rounding 0.799" (p. 125)—so there are no decimals in between them. Some PTs subtracted the two given decimals to find the number of decimals in between. Widjaja et al. (2008) attribute PTs' challenges with the density of decimals to the lack of opportunity to work with decimals that are not the same length. In addition, PTs must understand that the discreteness of whole numbers does not apply to decimals.

Widjaja et al. (2011) returned to interpretations of decimals, this time focusing on PTs' placement of negative decimals on a number line. The significance of this work lies in the power it has to provide insight into PTs' understanding of decimals as a number system. Widjaja et al. identified two misconceptions that involved "interpreting the negative part of the number line" (p. 86) that explain incorrect responses given by PTs who were asked to identify the position of decimals, including numbers such as –0.35 and –1.65 on a number line with –1, 0, and 1 marked (subunits of 10). To explain the PTs' misconceptions, Widjaja et al. described two rays students used: the "positive number ray," beginning at zero and continuing to the right, and a "negative number ray," also beginning at zero (some students identified zero as negative zero, and continued to the right with negative integers identified [p. 86]). The first misconception involved students using the "separate negative number ray misconception" (p. 86). Students with this misconception overlap the positive and negative number rays so that the positive number ray is laid on top of the negative number ray. This action creates values –0, –1, and –2. The value 0 on the positive number ray is coincident with –2 on the negative number ray. When these rays, both extending to the right, are overlapped, the number line begins with negative zero and the sequence of integers –1, 0, 1, 2 are marked. Using this approach, –0.5 would be viewed as less than –1.2, since –0.5 is between –0 and –1 and –1.2 is between –1 and 0. The second misconception involved students "translating positive intervals" to positions "between integers" in the negative region (p. 88). Students with the "translating positive intervals misconception" (p. 88) can correctly locate positive and negative integers on a number line. It is negative decimals that give them difficulty. Students translating positive intervals "know that 1.2 is to the left of 1.3 and assume that the same relationship holds for negative numbers so that -1.2 is to the left of -1.3" (p. 88). For these students, –1.2 is interpreted as positioned at –0.8. The authors identified variants of this thinking that result from placing the translated positive intervals in different positions on the number line. This exploration of the landscape of the number line adds a dimension to the work of Thipkong and Davis (1991), suggesting not only that some students find decimals between zero and 1 difficult to represent, but also that understanding number lines can be a challenge. PTs' approaches illustrate the importance of models in understanding decimals, but also suggest that meanings for models, such as number lines, must be developed with PTs rather than assumed to exist in the minds of PTs.

Only one study examined PTs' knowledge of decimals and application of this knowledge to tasks common in teaching (Stacey et al., 2001). PTs' knowledge of decimal concepts and an awareness of common misconceptions are essential components of mathematical knowledge for teaching. Stacey et al. (2001) asked 553 PTs and 25 practicing teachers to complete a collection of decimal comparisons. The Decimal Comparison Task (DCT), created by Stacey and colleagues, contains groups of decimal comparisons designed to identify known misconceptions (Moloney & Stacey, 1997; Stacey & Steinle, 1998). Participants in the Stacey et al. (2001) study were asked to look at pairs of decimals and indicate which one was "larger." PTs were also asked to identify comparison items that were likely to be difficult for children and provide possible rationales for the cause of children's difficulties. The researchers sought to broaden their understanding of PTs' ideas about the difficulty of the decimals for children and the ability of PTs to recognize gaps in their own understanding.

PTs demonstrated a moderate awareness of their own difficulties with decimals, and this awareness made them more sensitive to possible difficulties for children. Correlations existed between the errors made by PTs and difficulties they identified for children. Fifty-seven percent of PTs' errors on the comparison tasks correlated with possible difficulties for children identified by PTs. Four common misconceptions were identified by PTs: "length, comparison with zero, presence of a zero digit, and similarity" (p. 217). PTs were aware that children might think that longer decimals have a larger value than shorter ones. In addition, they seemed to know that comparing a decimal to zero or to a decimal containing a zero between non-zero digits can be challenging. PTs further were aware that decimals that had a collection of digits in common were harder to compare (e.g., 8.245 and 8.24563 [p. 216]). PTs showed very little awareness of the shorter is larger misconception.

PTs identified two aspects of lengths of decimals that could make decimal comparison tasks difficult for children. These aspects were long decimals and decimals of unequal length (p. 218), difficulties likely connected to the longer is larger misconception. Regarding comparisons with zero, PTs commented that children may apply whole number thinking and conclude that zero is larger or may think that decimals are negative numbers. The presence of zero in the tenths place could make decimal comparison more difficult for students, but PTs did not elaborate on this reasoning. PTs provided no discussion of zero making the value smaller, but their perception of children's difficulty with the presence of a zero digit is mostly likely connected to the longer-is-larger misconception. Similar decimal numbers were identified by PTs as a potential source of difficulty for children because they may not know or recognize the effect of the additional digits in the third and fourth decimal place. Within these four categories, PTs expressed an awareness of the longer-is-larger misconception, but there were fewer comments about the shorter-is-larger misconception, a misconception more common in older students.

The Horizon, Future Directions, and Considerations

Research at the horizon is much more difficult to find. No reports of studies focused on PTs' understanding of decimals were published in recent proceedings reviewed by the research team. Rather than speculate regarding why such evidence was missing, in this section we focus on future directions for researchers exploring PTs' knowledge of decimals and considerations they should attend to as they plan research agendas.

As mathematics educators move forward in their exploration of PTs' understanding and use of decimals, three questions shape suggestions for future research. How do PTs develop decimal concepts? How can mathematics educators support the development of PTs' decimal concepts? How do PTs use their concept of decimals in activities that approximate the work of teaching?

Current findings illustrate how PTs interpret, represent, and use decimals. What is less clear is how these concepts develop. Of significant importance to mathematics educators planning and developing mathematics courses for PTs has been the identification of difficulties with decimals, yet also of importance is an understanding of how these concepts might develop. Studies of children's development are of use in building ideas about adult development (McClain, 2003), yet adults' prior experiences with decimals and their facility with them can allow them to share correct answers that reveal little about underlying concepts. As the reports discussed have shown, it is difficult to develop tasks and activities that take seriously the existing constraints of PTs' productive computational and procedural approaches to tasks. Adult tasks should be built (a) with PTs' existing ways of operating in mind, and (b) to create cognitive conflict. Such tasks will stand in contrast to situations in which PTs are told by teachers and researchers that they may not use their most productive procedural approaches (e.g., converting decimals to fractions with a common denominator to compare them) on given tasks. Mathematics educators must challenge themselves to explore PTs' development of decimal understanding that does not rely on PTs' compliance with authority. Tasks that cannot be solved effectively, efficiently, or correctly with procedures will challenge PTs to develop new understandings of decimals.

Efforts to challenge PTs' existing understandings have proven to be labor-intensive, as suggested in the work of Tirosh and Graeber (1990a). Thus far, studies have included collections of paper-and-pencil tasks given to a group of PTs and interviews of a subset of the participants to explore reasoning underlying approaches identified. While these studies have provided tasks that can serve as the basis for instructional materials, developmental research (Gravemeijer, 1994) with the goal of creating instructional materials and pedagogical approaches is needed. Mathematics content courses for PTs must provide opportunities to build decimal concepts, while honoring and identifying the power of efficient approaches PTs use, such as appending zeros to compare decimals.

Only one research study included explorations of PTs' use of decimals in approximations of practice (Grossman, 2011). Approximations of practice allow PTs to create insights through activities they will have to perform as practicing teachers. Stacey et al. (2001) found that although PTs correctly identified the longer-is-larger misconception as a challenge for children, they were less likely to identify the shorter-is-larger misconception. These and other findings from Stacey et al. stand alone. Research exploring how PTs make sense of decimal tasks and, in turn, what they identify as challenging for learners, provides insight into the evolution of their mathematics, including their understandings of children's mathematics (Steffe, 1994).

As the horizon of PTs' decimal understanding is constructed by research findings, we also suggest two additional considerations that should be attended to by researchers. First, we question what concepts should be studied. Programs of study in mathematics teacher education are necessarily limited in scope. Time and content limits are significant considerations mathematics teacher educators attend to as they plan opportunities for PTs to learn. In order to serve mathematics teacher educators charged with supporting PTs, it is critical that researchers move beyond asking what can be studied to what should be studied to serve the important goals of teacher education. For example, Wajaja and her colleagues' (2008, 2011) work exploring PTs' representations of negative decimals initially may seem less significant to mathematics teacher educators working with PTs. Yet, this work not only builds from the existing literature, but also contributes new insights regarding difficulties with positive decimal quantities. Widjaja, Stacey, and Steinle (2011) illustrate that understanding the "twisted geography" (p. 80) of the number line for negative decimals unearthed further evidence that "decimals with a zero integer part (e.g. 0.35) are conceptually harder than decimals with a non-zero integer part, and so would benefit from special attention in teaching" (p. 90). Many concepts could be explored by researchers. Deciding which concepts demand the most attention from the research community should involve discussions with mathematics teacher education colleagues and the identification of the grand challenges in conceptual development they see as central to their teaching of PTs.

Second, we encourage researchers who study concept development to provide insights regarding how such concepts might efficiently be developed with PTs. Tirosh and Graeber (1990a), in their studies of cognitive conflict as a method of developing understandings of decimals, recognized the inefficiency of their individual interviews as a method for teaching PTs and suggested alternatives. Activity sequences designed as part of research that takes many instructional hours to implement, but only supports the development of one concept, must be critically examined. Researchers should attend to the institutional constraints mathematics teacher educators face as they work with PTs and should provide insights into how research findings could be translated into practices that efficiently and productively support concept development. One such example is the examination of non-terminating repeating decimals (Burroughs & Yopp, 2010; Weller, Arnon, & Dubinsky, 2009, 2011). While our review focused on terminating decimals, these studies provide insights that allow mathematics educators to speculate about reasoning PTs might use to build from the set of points everywhere dense to a "correspondence between all the points on the number axis and all the finite and infinite decimal fractions" (Courant & Robbins, 1996, p. 63). The challenge for mathematics teacher educators may be whether the collection of ideas associated with such reasoning is critical for PTs and to their future work with children. This example illustrates the importance of creating not only new knowledge about PTs' development and use of concepts, but the significance of such findings for mathematics teacher education.

In this review spanning 25 years, we found a very small collection of reports focusing on PTs' decimal content knowledge. Yet mathematics teacher educators planning instruction for PTs need research results to inform their practices. Of critical importance are studies that (a) result in understandings of how PTs develop decimal concepts, including understanding notations and representations of decimal quantities; (b) generate instructional activities and methods that mathematics teacher educators can implement with PTs to create opportunities to understand decimal concepts and representation; and (c) explore and describe how PTs use decimal concepts in tasks that approximate the work of teaching. Such studies support the development of curriculum and instructional methods that expand opportunities for PTs to the learn content critical to their future work as teachers.

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