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**Mathematical Content Knowledge for Teaching Elementary Mathematics:
A Focus on Algebra**

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ABSTRACT: As part of a recent effort to summarize research-based knowledge of prospective elementary school teachers' (PTs) mathematical content knowledge, this paper summarizes research literature on PTs' knowledge of algebra, focusing on the range of years from 1998 through 2012. The 21 papers included in this summary focus on a broad range of topics within algebra, such as (a) producing, representing, and justifying generalizations; (b) interpreting and using algebraic symbols; (c) solving algebraic word problems; and (d) understanding functions. Looking across this body of research, three themes are identified: (1) PTs generally have strong procedural skills and can make mathematically sound generalizations of many different types of patterns; (2) however, PTs tend to struggle to (a) interpret and effectively use algebraic symbols, even those that they have produced themselves; (b) interpret graphical representations; and (c) make connections between representations; and (3) PTs have limited algebraic problem-solving strategies, often relying, inflexibly, on inefficient and/or incorrect computational methods. Suggestions for future research directions are discussed.

Keywords: algebra, algebra content knowledge, mathematical knowledge for teaching, mathematical content knowledge, preservice teachers, prospective teachers, elementary, teacher education

Background and Introduction

In recent decades, algebra has become infamous as a gatekeeper of success in school mathematics (Cai et al., 2005; Jacobs, Franke, Carpenter, Levi, & Battey, 2007; Stephens, 2008). Moses and Cobb (2001) underscored the importance of algebra by making a comparison between people who lack an education in algebra today to “the people who couldn’t read and write in the Industrial Age” (p. 14). Unfortunately, though, it has been documented repeatedly that many students struggle when they reach algebra in middle school or high school (e.g., Kenney & Silver, 1997).

In response to this phenomenon, members of the mathematics education community have called for the inclusion of algebra content in the elementary school curriculum, with the goal of removing the abrupt, often derailing transition from arithmetic to algebra by infusing algebraic ideas into instruction in the elementary and intermediate grades (Kaput, 1998). For example, the National Council of Teachers of Mathematics (NCTM, 2000) suggests incorporating algebra into elementary level curricula:

By viewing algebra as a strand in the curriculum from prekindergarten on, teachers can help students build a solid foundation of understanding and experience as a preparation for more sophisticated work in algebra in the middle grades and high school. (p. 37)

Research also lends support for the notion of including algebraic ideas in elementary school curricula (e.g., Britt & Irwin, 2008; Schliemann et al., 2003). In her research brief on algebra, Kieran (2007) concludes that the current body of research “emphasizes that arithmetic can be conceptualized in algebraic ways” and “this emphasis can be capitalized on to encourage young students to make algebraic generalizations without necessarily using algebraic notation” (p. 1). Thus, algebra can be infused into arithmetic instruction in a way that is appropriate for elementary-aged children.

Accordingly, algebra topics have been included in recent standards documents as an essential component of the elementary mathematics curriculum. The NCTM's *Principles and Standards for School Mathematics* (2000) states that students in all grades should develop their understanding of the following algebraic ideas:

- understanding patterns, relations, and functions;
- representing and analyzing mathematical situations and structures using algebraic symbols;
- using mathematical models to represent and understand quantitative relationships; and
- analyzing change in various contexts. (p. 37)

More recently, in the *Common Core State Standards* (CCSS; National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA & CCSSO], 2010), the Operations and Algebraic Thinking content domain begins in kindergarten and continues through fifth grade, progressing from a focus on understanding properties of, and having flexibility with, the four basic operations, toward a focus on generalizing, describing, and justifying patterns and relationships, and interpreting symbolic expressions.

In light of these standards, it is evident that elementary school teachers are responsible for facilitating their students' development in algebraic concepts, and, therefore, they need to have a deep understanding of the foundations of algebra themselves (Hill, Rowan, & Ball, 2005; Ma, 1999). Moreover, members of the mathematics education community support the notion "that there is a powerful relationship between what a

teacher knows, how she knows it, and what she can do in the context of instruction” (Hill, Blunk, et al., 2008, p. 498).

Thus, the mathematical education of PTs in algebra is of critical importance to the quality of the mathematical education of children. This is reflected in the recently updated recommendations of the Mathematical Education of Teachers II (METII; Conference Board of the Mathematical Sciences, 2012) report, which states that kindergarten through Grade 5 teachers need to be able to “[recognize] the foundations of algebra in elementary mathematics” (p. 26). As an example of this, the report gives the following as an illustrative activity for the mathematical preparation of elementary teachers in algebra: “Explain how to solve equations such as $283 + 19 = x + 18$ by ‘thinking relationally’ (e.g., by recognizing that because 19 is 1 more than 18, x should be 1 more than 283 to make both sides equal) rather than by applying standard algebraic methods” (p. 26). Further, the METII recommends that half of PTs’ mathematical preparation should focus on “number and operations, treated algebraically with attention to properties of operations,” with the other half focused on “additional ideas of algebra (e.g., expressions, equations, sequences, proportional relationships, and linear relationships)” (p. 31), along with geometry and measurement and data.

With these recommendations in mind, it is also critical that the mathematical instruction of PTs is built on their currently held knowledge (Bransford, Brown, & Cocking, 1999). Thus, we need to know what PTs’ currently held knowledge of algebra is, how it changes, and how it develops, so that mathematics educators can appropriately tailor instruction. Accordingly, summarizing the current state of research on PTs’ knowledge of algebra is the goal of this article. In particular, the following questions guide our summary:

1. What research has been conducted on elementary PTs' knowledge of algebra?
2. What is known from this research about elementary PTs' algebra content knowledge?
3. What about elementary PTs' algebra content knowledge remains as-of-yet unstudied?

We address these questions in three sections, organized chronologically according to date of publication:

- First, a historical look, in which we summarize findings of research published in peer-reviewed journals prior to 1998;
- Then, a current perspective, in which we summarize findings of research published in peer-reviewed journals between 1998 and 2011;
- Finally, a view of the horizon, in which we summarize findings of research published in peer-reviewed journals in 2012, along with findings of research published in the 2011 or 2012 proceedings of the meetings of the International Group for the Psychology of Mathematics Education (PME), or in the 2011 or 2012 proceedings of the meetings of the North American Chapter of the Psychology of Mathematics Education (PME-NA).

Methods

Our search for studies to include in this summary closely followed the general methods guiding all content areas for the larger summary project of which this algebra-specific summary is a part (see introductory article to this Special Issue). In particular, we conducted many searches of the ERIC database using combinations of the following search terms: *elementary education, elementary, education, preservice teacher, pre-service teacher,*

prospective teacher, algebraic thinking, algebra, function, symbolic, equation, commutative, associative, distributive, rate of change, patterns, factoring, inequalities, generalization, generalized arithmetic, and graphs.

As we conducted these searches, we realized that there were unique circumstances regarding the search for published literature on PTs' content knowledge of algebra, in particular, that warranted special methodological considerations. Specifically, (a) we needed to establish a definition of *algebra* for this summary, that aligned with the overall purposes of this project; and (b) after conducting our initial searches of the database, we agreed on one exclusionary criterion in addition to those of the larger summary project. We describe both of these considerations below.

Definition of Algebra

Although algebra is now a major component of mathematical standards for grades K–12, what algebra is, exactly, is debated within the mathematics education community. (See Stephens, 2008, for a succinct review of definitions of algebra.) For the purposes of this summary, we put together an inclusive definition of algebra with an eye toward elementary and middle grades content. Drawing on the ideas of many mathematics educators (e.g., Kaput, 1998; Kieran, 1992; National Research Council, 2001), we conceptualized algebra to be content focused on pattern generalization, arithmetical generalization, algebraic symbolization, functions, proportional reasoning, or problem solving when the problems are not amenable to arithmetic strategies. Additionally, we chose to include studies in which the focus was on properties of the arithmetic operations; however, we excluded studies in which the focus was on PTs' understanding a specific type of number (e.g., decimal numbers) in the context of an operation, because these studies are

well-addressed in other articles in this issue. We feel that this conceptualization of algebra satisfies the criteria of being (a) broad and inclusive so as not to unnecessarily exclude studies from the summary, and (b) appropriate for discussing the content knowledge of prospective teachers of children ages 3 through 14.

Inclusion/Exclusion Criteria

This project includes all research published in peer-reviewed journals on the Common and/or Specialized Content Knowledge (Ball, Thames, & Phelps, 2008; Hill, Ball & Schilling, 2008) of elementary PTs. However, as explicated in the introductory article to this Special Issue, our search of the literature adhered to these four exclusionary criteria:

1. We excluded studies that lacked specific attention to algebra.
2. We excluded studies that focused primarily on PTs' perceptions about mathematics or beliefs.
3. We excluded research that focused on describing classroom practice or activities for PT education courses with lack of attention to research design methods.
4. We excluded studies in which the populations primarily consisted of high school PTs, mathematics majors, or inservice elementary teachers.

Additionally, we added to the exclusionary criteria to also exclude studies that focused solely on secondary-level content knowledge of middle grades PTs. To come to this decision, we first revisited the reasoning for including studies on middle grades PTs in the larger summary project. As stated in the introductory article to this Special Issue, "We decided to look at findings from studies of PTs preparing to teach children aged 3–14 to account for cases with combined middle and elementary certifications." The intent of the overall project, then, was to include studies that have a population of middle grades PTs for

the purpose of avoiding inadvertently excluding relevant research on elementary PTs. We decided, though, that some studies that focused on the algebra content knowledge of solely middle grades PTs had a distinctly secondary feel and therefore did not fit with the aim of the larger project to summarize research on elementary PTs. As an example, we excluded a study that focused on prospective middle grades teachers' knowledge of rational functions. Using the *Common Core State Standards* (CCSS) (NGA & CCSSO, 2010) as a guide for determining what counted as secondary level (i.e., high school level) algebra, our fifth exclusionary criterion was as follows:

5. We excluded studies in which both (a) the population was entirely middle grades PTs, and (b) the mathematical knowledge studied was at a secondary school level.

A Historical Look: Prior to 1998

Using the criteria outlined above, we identified only one research study that was published prior to 1998 on PTs' content knowledge of algebra (see Table 1 below). Moreover, a supplementary search through the reference sections of research published between 1998 and 2011 did not yield any additional studies.

Table 1

Information on the Article Published Before 1998 on the Topic of PTs' Algebra Content Knowledge

Authors	Year	Number of PTs Studied	PTs' Level	Country	Data Analyzed
Schmidt & Bednarz	1997	66 elementary, 65 secondary, 33 special education	PTs at the beginning of the certification program	Canada	Written survey

The one study we found (Schmidt & Bednarz, 1997) explored Canadian elementary ($n = 66$), secondary ($n = 65$), and special education ($n = 33$) PTs' strategies for solving arithmetic and algebraic word problems.⁸ The authors define arithmetic problems as those that are amenable to arithmetic solutions, where the solver can work from known information to find the unknown. They define algebraic problems as those that are not amenable to working from known values to find an unknown value, where the solver must directly work with unknown quantities. In the study, PTs' solution strategies for solving algebraic and arithmetic problems were categorized as either algebraic or arithmetic, regardless of which type of problem was being solved. A solution was considered algebraic if it satisfied both of the following criteria: (a) the solution contains at least one equation wherein known and unknown values are related to each other, and (b) the answer is found via transformation of the equation(s) and operating on the unknowns without choosing specific values for the unknowns. All other solutions to either type of problem were

⁸ Schmidt and Bednarz (1997) use the terms *connected* and *disconnected* problems instead of *arithmetic* and *algebraic* problems, respectively. We choose to use the terms *arithmetic* and *algebraic* to align with the terms used in more recent research discussed later in this article.

considered arithmetic. Additionally, arithmetic solutions were categorized as either *guess-and-check* or *manipulating-the-structure*. An example of each type of solution, for each type of problem, appears in Table 2 below.

Table 2

Examples of Arithmetic and Algebraic Problem Types, and Examples of Algebraic and Arithmetic Solution Strategies for Each Problem Type (van Dooren, Verschaffel, & Onghena, 2003)⁹

	Arithmetic Problem	Algebraic Problem
	A primary school with 345 students has a sports day. The students can choose between in-line skating, swimming and a bicycle ride. Twice as many students choose in-line skating as bicycling, and there are 30 fewer students who choose swimming than in-line skating. 120 students want to go swimming. How many chose in-line skating and bicycling?	372 people are working in a large company. There are 4 times as many laborers as clerks, and 18 clerks more than managers. How many laborers, clerks, and managers are there in the company?
Algebraic Solutions	$i = \text{in-line skating}$	Number of managers = x
	$345 = i + i/2 + i - 30$	$x + (x+18) + 4(x+18) = 372$
	$690 = 2i + i + 2i - 60$	$6x = 372 - 90$
	$750 = 5i$	$6x = 282$
	$150 = i$	$x = 47$
	In-line skating: 150, swimming: 120, bicycle riding: 75	There were 65 clerks ($47 + 18$), and 260 laborers (4×65).

(continued)

⁹ The article by Schmidt and Bednarz (1997) is written in French. In order to avoid translation issues, the examples in this table are taken from the van Dooren, Verschaffel, and Onghena (2003) study, which uses the same problem type and strategy type categorizations as Schmidt and Bednarz's study does, but is written in English.

Table 2—continued

Arithmetic Problem	Algebraic Problem
Arithmetic Solutions	“Manipulating the Structure”
<p>Let us assume that the number of students who swim equals the number of in-line skaters. The total augments by 30 to 375. This total is divided into 5 groups:</p> <p>2 groups in-line skaters 2 groups swimmers (in fact 30 fewer) 1 group bicycle riders Each group consists of 75 students. Thus 75 students went for the bicycle ride, 150 chose in-line skating and 120 swimming.</p>	<p>Let us suppose for a moment that there are 18 managers more (i.e. as many managers as clerks). Then the total of people = 390. This total consists of 6 equal parts:</p> <p>4 parts laborers 1 part clerks 1 part managers Each part consists of $390/6 = 65$ people. So there are 65 clerks, 260 laborers, and 47 managers (65-18).</p>
“Generating Numbers”/“Guess and Check”	
<p>120 students went swimming</p> <p>$120 + 30 = 150$ students went in-line skating</p> <p>$150/2 = 75$ students went for the bike ride</p>	<p>Suppose the number of clerks is 80, then there are 62 managers and 320 laborers, which gives 462 people total Too many.</p> <p>Suppose the number of clerks is 60...(Student continues in this way until finding correct numbers.)</p>

Analyses of a written survey that included three arithmetic problems and three algebraic problems suggest that, of the three groups of PTs (elementary, special education, and secondary mathematics), elementary level PTs were the most flexible in their strategy selection, with 58.5% using one of the arithmetic strategies to solve arithmetic problems, and 61.5% using algebraic strategies to solve algebraic problems (Schmidt & Bednarz, 1997). However, follow-up interviews suggest that all groups of PTs displayed difficulties surrounding arithmetic and algebraic problem solving. Most notably, algebraic symbols

(i.e., variables used to translate from a story problem to algebraic equations) were used to stand in for information in the problem (e.g., “ M ” for the amount of money that Marie has), yet the equations did not necessarily correctly describe relationships between quantities. For example, to describe a situation in which Marie had 15,000 more of something than Chantal, one preservice teacher wrote $M + 15,000 = Chantal$, which represents the inverse relationship. This difficulty has been identified in previous research with other populations (cf. the Students and Professors problem in Clement, Narode, & Rosnick, 1981).

Moreover, Schmidt and Bednarz (1997) note that PTs struggled to use their equations to solve problems. For example, one PT assigned variables to all of the unknown quantities in the problem, yet when the PT reached an answer, there was still an x in it. The authors also describe another PT who used variables to stand for unknown numbers but then substituted numbers in place of those variables in order to solve. In this way, even though the PT used algebraic symbolism, the individual seemed to be thinking arithmetically with a “guess-and-check” strategy. In light of these difficulties, the authors express concern regarding the elementary PTs’ preparedness to help students transition from arithmetic to algebraic reasoning.

Apart from this study, there were no studies published about elementary PTs’ content knowledge in algebra prior to 1998. Fortunately, the roughly 14 years that followed (1998 to 2011) saw an increase in the number of studies on elementary PTs’ content knowledge of algebra. This research is summarized in the next section.

A Current Perspective: 1998 to 2011

Our initial search yielded 18 potential articles to be included in our summary of peer-reviewed research papers published on PTs’ content knowledge of algebra between

1998 and 2011. Sixteen of these papers met our inclusion criteria (see Table 3). Because this collection of papers spans a wide range of topics within algebra, we grouped the research into four non-mutually-exclusive content-themed sections, with the understanding that the research could be grouped differently and that the content foci of the different sections clearly overlap. Accordingly, we present findings of the current research in the following four sections: (a) producing, representing, and justifying generalizations; (b) interpreting and using algebraic symbols (in contexts other than generalization tasks); (c) solving algebraic word problems; and (d) understanding functions.

Table 3

Articles Published Between 1998 and 2011 on the Topic of PTs' Algebra Content Knowledge, in Alphabetical Order by First Author

Authors	Year	Number of PTs Studied	PTs' Level	Country	Data Analyzed
Berk, Taber, Carrino, Gorowara, & Poetzl	2009	148	First-year PTs in the second of three mathematics courses for prospective K-8 teachers	USA	Data from written pretest, posttest, delayed posttest, and individual interviews
Billings & Klanderman	2000	19	Undergraduate juniors (3rd year students) in a combination content/methods course; K-8 teachers	USA	Copies of student work, field notes
Briscoe & Stout	2001	106	Undergraduate seniors (final semester of school before student teaching) in a methods course	USA	Transcripts of video presentations, class discussion, and copies of documents produced by students (lab report and overhead transparencies)

(continued)

Table 3—continued

Authors	Year	Number of PTs Studied	PTs' Level	Country	Data Analyzed
Meel	1999	29	PTs at the end of their teaching certification program	USA	Written assessment given prior to unit, in the middle of the course
Nillas	2010	5	Elementary and special education majors; point in program is unclear	USA	Data from 3 written test items
Otto, Everett, & Luera	2008	72	Undergraduate science majors in a required capstone course	USA	Copies of student work, instructors' notes, observations of class activity
Pomerantsev & Korosteleva	2003	119	Elementary and middle grades PTs in various content courses for future educators	USA	Written survey of 5 questions
Prediger	2010	45	Second year middle school PTs, at the beginning of an instructional unit	Germany	Written survey during class, class observation and video data
Richardson, Berenson, & Staley	2009	25	PTs in a methods course, immediately before student teaching	USA	Audio recordings, observational notes, all student work (teaching experiment methodology)
Rivera & Becker	2007	42	PTs in an introductory course for elementary mathematics teachers	USA	Clinical interview data
Schmidt & Bednarz	2002	8, specifically selected based on prior written survey	Undergraduate PTs in an introductory course to a teacher education program	Canada	Data from pair interviews

(continued)

Table 3—continued

Authors	Year	Number of PTs Studied	PTs' Level	Country	Data Analyzed
Stylianou, Smith, & Kaput	2005	28	PTs in a mathematics course for elementary teachers; point in program is unclear	USA	Video of group, task- based pretest and posttest interviews, and video of all class sessions
van Dooren, Verschaffel, & Onghena	2003	45 first year, 52 third year	Comparison of PTs in the beginning and end of teacher education program	Belgium	Data from a written survey
You & Quinn	2010	104	Last stages of a study of PTs in a teacher education program	USA	Data from a 15-item written survey
Zazkis & Lildejahl	2002a	20	PTs in a core course for elementary PTs; after topic was covered	Canada	Clinical interview data
Zazkis & Lildejahl	2002b	36	Unclear	Canada	Student journals of mathematical investigations and follow- up interview data

Producing, Representing, and Justifying Generalizations

In this section, we focus on research that explores PTs' generalizations of patterns or generalizations of physical phenomena. Our search of the literature yielded six such papers. Within this small collection of research, there are studies that focus on a range of aspects of PTs' generalizations, including producing and representing generalizations, connecting those representations, and producing justifications.

The findings of all six studies support the notion that there are many situations in which PTs can produce correct¹⁰ generalizations. With respect to generalizing visual patterns, the findings of Richardson, Berenson, and Staley's (2009) teaching experiment were that 23 of 25 PTs, working in pairs, found a correct explicit rule to describe the perimeter of the n th figure in a train of squares, equilateral triangles, or regular hexagons (see Figure 1 for an example). Similarly, 35 of the 42 PTs working individually during clinical interviews in Rivera and Becker's (2007) study produced a correct generalization for the number of dots in a growing pattern of dots arranged in a square (i.e., n^2 dots for the n th figure, as in Figure 2).

Figure 1. A task from Richardson and colleagues' (2009) teaching experiment (p. 190).

¹⁰ We use the term *correct* to reflect the way generalizations are discussed in the research we reviewed. This research was grounded in the idea that there exist generalizations of patterns that are more correct, natural, and/or mathematically sound than others.

Figure 2. A task from Rivera and Becker's (2007) study (p. 144).¹¹

With respect to generalizations of physical phenomena, 32 of the 49 small groups of three or four PTs across two studies produced correct verbal and symbolic generalizations in the form of an equation, of an observed relationship modeled by a Class 1 lever¹² (Briscoe & Stout, 2001; Otto et al., 2008).

With respect to generalizations of arithmetic sequences of integers, findings from clinical interviews conducted in Zazkis and Liljedahl's (2002a) study of 20 Canadian PTs enrolled in a core mathematics course for elementary PTs, suggest that most PTs can easily

¹¹ In this task, Rivera and Becker (2007) use the word *array* in a way that is different from our understanding of the word. For the pattern in Figure 2, we think of an array as an entire square of dots. Rivera and Becker's use of the word *array* seems to be synonymous with the word *row* or *column*.

¹² In a Class 1 lever, the product of a mass (M_1) and its distance from the fulcrum (D_1) on one end of the lever is equal to the product of a second mass (M_2) and its distance from the fulcrum (D_2) on the other end of the lever; $M_1D_1 = M_2D_2$, or equivalently, $M_1/D_2 = M_2/D_1$.

recognize the underlying structure of arithmetic sequences of multiples (e.g., 7, 14, 21, 28, etc.). For example, PTs were able to produce the 712th element of the sequence by multiplying 712 by the common difference between consecutive elements. Findings of another study of Zazkis and Liljedahl's (2002b) exemplified that PTs can make correct generalizations, usually expressed verbally, in response to a rich visual number pattern (see Figure 3). The 36 Canadian PTs in the study were asked to journal about their mathematical investigations of the numerical pattern for 2 weeks, for periods of at least 30 minutes every other day. Based on this journal data, and follow-up interviews with four of the participants, Zazkis and Liljedahl concluded, "Participants engaged in detecting sameness and differences, in classifying and labeling, in seeking algorithms, in conjecturing and argumentation, in establishing numerical relationships among components or, more generally, in generalizing about data and mathematical relationships" (p. 399)—all demonstrations of algebraic thinking through generalization.

Task prompt: In general, given any whole number, how can one predict where it will appear in this pattern? Explain the strategy that you propose.

Figure 3. A visual arrangement of a sequence of numbers from a task used in one of Zazkis and Liljedahl's studies (2002b, p. 383).

In contrast to these examples of successful generalizations, however, research has also identified problem situations with which PTs struggle to produce a correct generalization, represented symbolically or otherwise. One such situation is generalizing arithmetic sequences of non-multiple integers (e.g., 8, 15, 22, 29, etc.). Of the 20 PTs in Zazkis and Liljedahl's (2002a) study, 9 indicated that they believed multiples of the common difference between cosecutive elements would generate new elements in a non-multiple sequence. For example, the common difference between the elements of the sequence 8, 15, 22, 29, etc., is 7, so the PT might indicate that 7 times 712 is an example of a member of the sequence. Two of the 20 PTs realized that multiples of the common difference were not elements of non-multiple arithmetic sequences, yet they indicated that any non-multiple was a potential element of the sequence. For example, a PT might acknowledge that 70 is not an element of the sequence 8, 15, 22, 29, etc., because 70 is a multiple of 7, but the PTs might also indicate that 75 might be in the sequence because it is not a multiple of 7. Zazkis and Liljedahl conclude that these PTs tend to interpret non-multiple arithmetic sequences as being "sporadic" (p. 116), or lacking any discernable pattern.

Even PTs with seemingly mathematically mature responses to Zazkis and Liljedahl's (2002a) generalization tasks sometimes seemed to lack an understanding of the underlying multiplicative structure of the sequences. For example, one PT correctly generalized the sequence 15, 28, 41, 54, etc., by stating that "the constant difference in the sequence is 13, and any number of the sequence is going to be a multiple of 13 plus 15" (p. 109), yet when he was asked if 1,302 was in the sequence, the PT indicated, incorrectly, that it was not, since "1,300 is . . . a multiple of 13, and that 1,302 is . . . only 2 away from that" (Zazkis &

Liljedahl, 2002a, p. 109). Zazkis and Liljedahl suggest that the PT incorrectly rejected 1302 as an element of the sequence because he was thinking of the sequence formulaically, instead of having a more developed understanding of the invariant multiplicative structure of non-multiple sequences, such as “multiples adjusted” (p. 110). Zazkis and Liljedahl conclude that an individual PTs’ *additive* and *multiplicative schemes* (Vergnaud, 2004) seem to develop dynamically through the identification of differences and invariants in problem situations.

In the other aforementioned study of Zazkis and Liljedahl’s (2002b), the authors found a common tendency among PTs to be satisfied with disjunctive generalizations, instead of looking for generalizations that captured the structure of a pattern as a whole. The authors found that many¹³ PTs in their study searched for patterns in a piecemeal fashion, column by column, instead of recognizing the invariant unit-of-repeat of the pattern, which was 8. Moreover, those PTs that did search for a unit-of-repeat often focused on less mathematically salient units-of-repeat in the pattern, such as 4, 40, or 100, instead of 8. Although some of the PTs’ findings were potentially algebraically useful, the PTs often failed to recognize that potential.

Research (Richardson et al., 2009; Rivera & Becker, 2007) also suggests that PTs struggle to justify their own symbolic generalizations, regardless of the type of pattern they are generalizing. For example, in Richardson and colleagues’ (2009) teaching experiment, 23 of 25 U.S. PTs found a correct explicit rule to describe the perimeter of at least one of the pattern block trains (squares, triangles, or hexagons), but they struggled to justify their

¹³ Exact counts are not given in this article, as the authors’ focus is on describing and exemplifying ways in which PTs approached the generalization task.

generalizations. Based on their teaching experiment data, the authors describe a five-level framework that characterizes PTs' levels of success with justifying a generalization of a train of regular polygons (see Figure 4). At the lowest level, PTs generalize a recursive rule with no justification of the coefficient or y -intercept, relying on observations of numerical growth. As the PTs attain higher levels of justification, they are able to justify various portions of an explicit formula until they reach level 4, where a PT is able to successfully generalize a rule and justify the coefficient and the y -intercept using the model. The framework suggests that PTs gradually develop the ability to justify formulas for linear figural patterns through the process of working in small groups on pattern justification tasks. The authors propose that four features of the tasks used in their teaching experiment contributed to the PTs' development through the levels: (a) the linear and geometric (in a visual sense) nature of the patterns, (b) the use of physical pattern block manipulatives, (c) the isomorphism between the tasks, and (d) the use of tasks that promotes discourse among small groups of PTs, creating communities of ideas.

0	Generalizes a recursive rule with no justification of the coefficient and y -intercept
1	Generalizes an explicit rule with no justification of the coefficient and y -intercept
2	Generalizes an explicit rule with partial or faulty justification of the coefficient and y -intercept
3	Justifies the coefficient and y -intercept and generalizes an explicit rule inconsistently or inefficiently
4	Generalizes an explicit rule and justifies the coefficient and y -intercept

Figure 4. Richardson and colleagues' (2009) five-stage framework for PTs' generalizations of linear figural patterns (p. 197).

In another study of PTs' justifications of geometric patterns, Rivera and Becker (2007) focus on the relationship between justification and the stage of generalizing

wherein explanatory inferences are made.¹⁴ They suggest that the findings of their study point to the relationship between the types of representational cues (either figural or numerical) of the pattern that the PTs use to produce a hypothesized formula, and the PTs' ability to justify the formula. Specifically, PTs relying on visual cues – while generally not able to produce as many strategies as PTs relying on numerical cues – were more often able to justify the viability of their generalization. For example, PTs, who generalized a pattern by focusing on the common difference between the numbers generated by successive figures in the pattern (numerical cues) seemed to be unable to link the generalization back to the geometric pattern. As another example, PTs who used the trial and error until they found a formula that worked with the numbers generated by the pattern, struggled to justify why their generalizations worked with the geometric pattern. By contrast, PTs who generalized geometric patterns by focusing on the figural cues of the pattern were more successful in justifying their generalizations because they were more readily able to connect them to the underlying structure of the pattern.

Findings from Zazkis and Liljedahl's (2002b) analyses of elementary PTs' generalizations in response to the numerical pattern task in Figure 3 above suggest that PTs may struggle to make connections between their symbolic and verbal representations of their own generalizations. Zazkis and Liljedahl noted that, in the rare instances when PTs produced correct verbal and symbolic generalizations, there was often no evidence to suggest that the PTs saw connections between them. For example, one PT in their study

¹⁴ Rivera and Becker (2007) use the term *abduction* to describe PTs' explanatory inferences, and they conceptualize generalization as an abduction–induction process. We use the broad term *generalization* here to create cohesion with the rest of our summary. For a detailed theoretical discussion of the roles of abduction, induction, and deduction in the process of generalizing, we recommend referring to Rivera and Becker's report.

symbolized her generalization of a pattern as $1 + 8r$, yet the same PT later seemed excited to realize that the numbers in the same pattern were “one more than the multiples of eight” (p. 393). Zazkis and Liljedahl point out that this demonstrates a lack of understanding on the PT’s part of the meaning of the algebraic symbols she herself had generated. In light of this, Zazkis and Liljedahl suggest, “Neither the presence of algebraic notation should be taken as an indicator of algebraic thinking, nor the lack of algebraic notation should be judged as an inability to think algebraically.”

Indeed, looking across the studies on PTs’ generalizations, it seems that connections between representations lies at the heart of many PTs’ difficulties related to generalizations. On the whole, the research summarized above lends support to the idea that PTs are usually able to successfully generalize patterns, either verbally or symbolically (Briscoe & Stout, 2001; Otto et al., 2008; Richardson et al., 2009; Rivera & Becker, 2007; Zazkis & Liljedahl, 2002a, 2002b). However, PTs’ struggles interpreting or connecting representations are well-documented. In particular, PTs struggle to connect their symbolic generalizations back to the original patterns (Richardson et al., 2009; Zazkis & Liljedahl, 2002b), particularly when the generalization was produced only from numerical cues instead of visual cues (Rivera & Becker, 2007). PTs also struggle to connect their own verbal generalizations to their own symbolic generalizations, or to leverage their own observations of patterns into more complete or correct generalizations (Zazkis & Liljedahl, 2002a, 2002b).

Further research is needed to document how PTs develop in their ability to interpret and connect representations of their generalizations. Only one of the six above studies (Richardson et al., 2009) compiled a developmental framework for PTs’

justifications of their generalizations, yet this framework is limited, because it is developed out of data taken from observations of small groups of students from only one class, and it applies only to generalizations of linear visual patterns. The other studies were either studies of PTs at one time point, or were more exploratory, documenting examples of PTs' knowledge of generalization. Thus, further research on the ways in which PTs overcome their struggles connecting, interpreting, and justifying generalizations, across various situations, is needed.

Interpreting and Using Algebraic Symbols

In this section, we discuss two studies that focus on PTs' interpretations and procedural use of algebraic symbols (e.g., expressions that include variables, and the equal sign) in contexts other than generalization tasks. Findings from these two studies are summarized below.

Using a framework of three meanings for the equal sign, Prediger's (2010) study explored the collective development of a class of second-year middle school-level PTs in Germany. The three meanings of the equal sign are as follows: An *operational* understanding of the equal sign is defined as a conceptualization of the symbol as a signal to "do something" or as something that separates a problem from its answer. A *relational* meaning, on the other hand, is a conceptualization of the equal sign as a symmetric indicator of equality or a formal equivalence describing equivalent terms. Finally, *specification* refers to the equal sign as a symbol to indicate a definition. Ideally, according to Prediger, PTs will have a flexible understanding of the equal sign that includes all three interpretations.

At the beginning of an instructional unit in Prediger's (2010) study, a class of 45 middle school PTs were asked to determine if and why given chains of equal signs were mathematically correct or incorrect, when presented as examples of children's work (see Figure 5). From written data collected during the first task, Prediger created four representative profiles for PTs' understandings of the equal sign. PTs that fit the first profile¹⁵ reproduced Emily's misconception, stating that Lisa's use of the equal sign was incorrect. PTs that fit the second profile did not seem to try to understand Emily's perspective in their response to the task. PTs that fit the third and fourth profiles seemed to have a mathematically correct understanding of the equal sign, but struggled to fully understand or appropriately respond to Emily's confusion about Lisa's use of the equal sign.

Building off these representative profiles, Prediger (2010) implemented three in-class tasks, in addition to the one above, designed to help PTs progress in their understanding of the equal sign. The first task presented the PTs with a video in which a student understood the equal sign operationally. The teacher in the video explained to the student that mathematicians "get nervous where there is not the same [amount] on the left and the right side [of the equal sign]," to which the student responded that she is "not a mathematician" (Prediger, 2010, p. 88). The PTs were asked to discuss this video, and Prediger suggests that, as a result of this discussion, the PTs explicated the notion of both an operational use and relational use of the equal sign. Moreover, PTs raised the questions "Why can't we allow different notations in different contexts? Why don't we allow chain

¹⁵ Prediger (2010) did not include any counts to indicate the number of PTs that fit each of the four profiles.

notation in arithmetic contexts, and forbid them in algebraic contexts?" (p. 88). To help PTs explore these questions, Prediger designed the second task, wherein the PTs were presented with a pool of examples of the equal sign being used in a variety of ways, and they were asked to consider the purpose of the equal sign in each situation and determine how they were similar or different from the others. In the final task, the PTs were asked to look for changes in the meaning of the equal sign in a sample step-by-step solution to a problem involving perimeter, area, and derivatives.

Situation: In order to study strategies for flexible mental arithmetic, students in grade 5 were asked to solve the following task:
Lisa calculates 24×7 by decomposing:

$$24 \times 7 = 20 \times 7 + 4 \times 7 = 140 + 28 = 168$$

- i) Did she calculate correctly? How would you have done it?
- ii) Calculate 54×6 like Lisa did.

Emily (age 10) is skeptical: "Lisa calculates wrong. 24 times 7 does not equal 20! And what is that after the 20?" Due to her difficulties with the unfamiliar symbolic representation, Emily does not continue with the task although she usually uses the same strategy of decomposing 24×7 into 20×7 and 4×7 .

Questions posed to PTs:

- a) What does Emily mean?
- b) Which view is right?

Figure 5. A response-analysis-type question about the equal sign from Prediger's study (2010, p. 76).

At the end of the course, students were able to choose to complete four of five problems on a final exam. Twenty-seven of the 41 PTs completed a problem that involved an analysis of the uses of the equal sign, and of those, 24 were successful in their analysis, according to Prediger (2010), suggesting that the PTs likely deepened their understanding of the equal sign. Moreover, Prediger's analyses of PTs' responses on an end-of-unit

assessment suggest that the PTs were better able to discern when to use operational or relational definitions of the equal sign after the instructional sequence. Prediger attributes these changes in understanding to the fact that the in-class tasks designed were to help make PTs' implicit knowledge about the equals sign explicit, coupled with tasks that involved comparing and interpreting well-chosen examples of equations presented as artifacts of children's thinking.

Beyond the equal sign, PTs' struggles working procedurally with algebraic expressions and equations are documented in Pomerantsev and Korosteleva's (2003) study. Through a written survey of five questions (see Figure 6) given to two large classes of PTs ($n = 119$) in the U.S., the authors investigated elementary and middle-grades PTs' abilities to recognize the structure of certain algebraic expressions (questions 1, 4, and 5) and to apply rules for cancellation of a common factor (questions 2 and 3). Question 1 was multiple-choice, and the remaining four questions were free-response.

Question 1. What is the name of the expression $4x^2 - 9y^2$?

Choices: (a) difference of squares, (b) difference of products, (c) square of difference.

Question 2. If possible, cancel out the common factor in the expression $\frac{2+x}{2}$.

Question 3. If possible, simplify the expression $\frac{2x+2}{2x}$.

Question 4. Use the statement "If $x^2 = 25$, then $x = \pm 5$ " to solve the equation $(x+1)^2 = 25$.

Question 5. Use the statement "If $2x + 3 = 5$, then $x = 1$ " to solve the equation $2(y+1) + 3 = 5$.

Figure 6. Questions about algebraic expressions, used in Pomerantsev and Korosteleva's (2003) study (p. 2).

While more than 85% of the PTs correctly identified the type of expression in Question 1 as "difference of squares," the reasons they gave on the survey to justify their

answers were often superficial (e.g., “I’ve heard ‘difference of square’ most often in past math classes”). Additionally, few students answered Questions 2 through 5 correctly (see Table 4), suggesting that PTs have weak procedural skills with respect to symbolic expressions and equations. In light of these findings, the authors suggest further research, with the goal of developing effective methods for teaching algebra to elementary school teachers.

Table 4

Percentage of Correct Responses by Question on Pomerantsev and Korosteleva’s (2003) Written Survey

		Q1	Q2	Q3	Q4	Q5
Class 1	(n = 47)	95.8	44.7	29.8	10.7	21.3
Class 2	(n = 72)	87.5	44.4	22.2	5.6	18.1

The two studies summarized in this section (Pomerantsev & Korosteleva, 2003; Prediger, 2010) address different aspects of PTs’ understandings of symbols. The results of Prediger’s (2010) equal sign study is encouraging in that it demonstrates a possible route for broadening and improving PTs’ understandings of the equal sign via carefully designed tasks based on artifacts of children’s thinking. However, it is limited in that the article’s emphasis is more on providing a theory-supporting example of how a class of PTs might develop, rather than on reporting in-depth rigorous research, and it follows only one class of PTs. The findings of Pomerantsev and Korosteleva’s (2003) study is limited as well. The study documents struggles that PTs have in interpreting and using algebraic expressions, but the data are from a single point in time, on a single written survey. Thus, both studies

can be thought of as jumping-off points for the work yet to be done: in-depth qualitative research focused on documenting the details of PTs' understandings of algebraic symbols and the equal sign, and how they develop in those understandings.

Solving Algebraic Word Problems

In the years since Schmidt and Bednarz' (1997) study (described in *A Historical Look* section above), two published papers have focused on elementary PTs' strategies for solving arithmetic and algebraic word problems (Schmidt & Bednarz, 2002; van Dooren et al., 2003). One additional paper focuses on PTs' flexibility of strategy choice when solving proportional reasoning word problems (Berk, Taber, Carrino Gorowara, & Poetzl, 2009). The two former studies explicitly build off of the study by Schmidt and Bednarz (1997). Specifically, van Dooren and colleagues' (2003) study is a near replication of Schmidt and Bednarz' (1997) study, with two differences: (a) their population of PTs are in Belgium, whereas the PTs in the earlier study were in Canada; and (b) the study conducted in Belgium compares PTs at the beginning of their teacher training program to PTs at the end of their program (using a cross-sectional design), whereas the study conducted in Canada includes only PTs at the beginning of their program. Schmidt and Bednarz' (2002) study builds on the findings of their 1997 study, exploring (a) what links between algebraic and arithmetic types of reasoning PTs make (or do not make) in the contexts of algebraic problem solving, (b) characterizations of the types of reasoning PTs use, and (c) potential difficulties in creating a bridge between arithmetic and algebraic reasoning.

van Dooren and colleagues (2003) presented 97 PTs with various arithmetic and algebraic word problems (Schmidt & Bednarz, 1997) via a paper-and-pencil survey. As in Schmidt and Bednarz' (1997) study, PTs' strategies for solving algebraic and arithmetic

problems were categorized as either “algebraic” or “arithmetic,” regardless of which type of problem was being solved. A solution was considered algebraic if it satisfied both of the following criteria: (a) the solution contained at least one equation wherein known and unknown values are related to each other, and (b) the answer is found via transformation of the equation(s) and operating on the unknowns. All other solutions (to either type of problem) were considered arithmetic.

Results of van Dooren and colleagues’ (2003) study differ from the findings of Schmidt and Bednarz’ (1997) study in that that the PTs in the more recent study can be categorized into two groups according to their preferred solution strategies to six algebraic and six arithmetic word problems: (a) those who almost exclusively use arithmetic solution methods regardless of the type of problem, and (b) those who are more flexible in their strategy preference. Specifically, elementary PTs solved 78.8%¹⁶ of the arithmetic problems using arithmetic strategies. By contrast, however, elementary PTs solved 42.5% of the algebraic problems using arithmetic strategies (e.g., “guess-and-check” or “manipulating the structure”), 40.1% of algebraic problems were solved using algebraic strategies, and the rest of the problems (17.5%) were not answered. The authors point out that this finding suggests an opportunity to leverage the existence of the two groups of reasoners in teacher preparation courses, building connections between arithmetic and algebra. For example, they suggest first highlighting the existence of these two groups of reasoners in a PT classroom as a way to start a meaningful, explicit discussion among the PTs about children’s transitions from arithmetic to algebraic thinking.

¹⁶ Van Dooren et al. (2003) largely reported percentages instead of exact counts.

Notably, preferences for strategy selection between first- and third-year students did not differ statistically in the study, although the ability to solve problems correctly was greater for third-year students than for first-year students, largely due to increased proficiency with the “manipulating the structure” strategy (van Dooren et al., 2003). However, just as with the combined group of first- and third-year PTs, there was a subgroup of third-year PTs who tried to use arithmetic strategies to solve algebraic problems, usually with little success in finding a correct solution. The authors expressed their concern about the readiness of this particular subgroup of PTs to prepare elementary school children with the skills that will help them later transition to algebra at the secondary level.

Schmidt and Bednarz’ (2002) exploratory study compared PTs’ algebraic and arithmetic reasoning to illuminate the nature of the similarities and differences between them. In the study, eight Canadian PTs were interviewed in pairs, drawn from a larger pool of preschool or elementary PTs, special education PTs, and secondary mathematics PTs. The pairs were specifically selected to include one PT who tended to reason arithmetically regardless of the type of problem being solved, and one PT who tended to reason algebraically regardless of the type of problem being solved, based on their responses on a preliminary written survey consisting of eight arithmetic and algebraic problems. One of the two elementary PTs included in the interviews answered all four algebraic problems using an arithmetic guess-and-check strategy, and the other elementary PT included in the interviews answered all four algebraic problems by using an arithmetic manipulate-the-structure strategy.

Schmidt and Bednarz' (2002) analyses of the interviews illuminated the nature of the similarities and differences between manipulating-the-structure arithmetic reasoning and algebraic reasoning, and between guess-and-check arithmetic reasoning and algebraic reasoning. In particular, manipulating-the-structure and algebraic reasoners are similar in that they can both successfully solve algebraic problems when there is a known quantity, yet these two types of reasoners differ when there are no known quantities given in the problem statement. For example, consider the following problem: "Luc has \$3.50 less than Michel. Luc doubles his money while Michel increases his amount by \$1.10. Luc then has \$.40 less than Michel. How much did they have to begin with?" (Schmidt & Bednarz, 2002, p. 85). Because this problem does not give a specific amount of money for either Luc or Michel, manipulating-the-structure reasoners had difficulty solving the problem, whereas algebraic reasoners are able to successfully solve the problem by using the given relationships.

An apparent similarity between guess-and-check reasoners and algebraic reasoners is that both rely on manipulating quantities—known or unknown quantities, respectively—yet these two types of reasoners differ in that guess-and-check reasoners are limited by their local, sequential treatment of the elements of a problem via calculations. Algebraic reasoners, on the other hand, tend to account for all of the elements of the problem from the outset of their strategy.

When comparing all three types of reasoners (manipulating-the-structure arithmetic, guess-and-check arithmetic, and algebraic), only the manipulating-the-structure reasoners consistently demonstrated the ability to use the problem context to work with all the elements of a problem; algebraic reasoners tended to verify that their symbolic

procedure was correct but often did not check their reasoning against the problem context. That said, the authors stress that one type of arithmetic reasoning is not definitively superior over the other, and they suggest that future research could further explore whether there is a type of arithmetic reasoning that best prepares students to progress to algebra at the secondary level.

The findings of a study on U.S. elementary PTs' flexibility with solving proportional reasoning word problems echoes the findings of the above research on PTs' strategies for solving word problems, in that PTs tend to enter their training programs with limited flexibility in strategy use (Berk et al., 2009). Specifically, analyses of 148 PTs' solutions of four different types of proportional reasoning problems suggest that PTs either (a) have a limited number of strategies for solving proportional word problems, or (b) are unable to choose strategically among the strategies that they know. Although the PTs in the study demonstrated reasonable proficiency in solving proportional reasoning word problems successfully and accurately, many used cumbersome and/or inefficient solution strategies.

However, Berk and colleagues' (2009) study supports the notion that PTs' problem-solving flexibility in the domain of proportional word problems can improve through the PTs' exposure, discussion, and careful consideration of others' solution strategies. Results of quantitative analyses suggest that PTs achieved gains in strategy flexibility after an instructional intervention on multiple-solution strategies for proportional word problems. Notably, this increase in flexibility occurred along with an improvement in the accuracy of PTs' solutions, and the PTs retained their flexibility at a 6-month follow-up.

Despite the fact that these three studies were conducted in three different countries (Belgium, Canada, and the U.S.), they complement each other in their emphasis on the need

for PTs to have flexibility in solution strategies with respect to algebraic and arithmetic problem solving (Berk et al., 2009; Schmidt & Bednarz, 2002; van Dooren et al., 2003). Moreover, considered together, two studies (Berk et al., 2009; van Dooren et al., 2003) lend tentative support for the idea that PTs can increase their flexibility with strategies and/or their success in solving problems correctly. In order to gain further insight into the ways in which PTs might develop in their problem-solving abilities, more research is needed. Further research is also needed to gain insight into the strengths and weaknesses of PTs' uses of particular strategy types, as Schmidt and Bednarz (2002) have begun to explore.

Understanding Functions

In this section, we discuss the findings of the five studies that explicitly focus on PTs' understandings of functions. Findings from these studies are summarized below.

Research on linear functions suggests that PTs generally have strong procedural skills (Nillas, 2010; You & Quinn, 2010). Quantitative analyses of 104 U.S. PTs' responses to a 15-item survey suggest that PTs perform well on questions designed to test procedural skills related to linear functions, such as calculating a slope (You & Quinn, 2010). This finding is further supported by a qualitative study of the responses of five U.S. PTs on three written test items (Nillas, 2010).

Results of both studies (Nillas, 2010; You & Quinn, 2010), however, suggest that many PTs struggle to (a) interpret the slopes of the graphs of linear functions in real-world contexts, and (b) flexibly translate between multiple representations of a function. PTs seem to have particular difficulty flexibly translating between symbolic and visual representations, and between a representation of a function and its real-world context (You & Quinn, 2010). Nillas (2010) asked PTs to interpret the meaning of an increase in the

slope of a line on a non-scaled graph that showed gallons of gas consumed by Jake's car versus distance traveled by his car (see Figure 7). While two of the five PTs in the study offered correct responses (e.g., "Jake's car would burn less gas per mile . . . having better gas mileage"), three of the five PTs gave the reverse interpretation, stating the car was "using more gas for less distance traveled" or "a gallon of gas takes Jake a shorter distance than before" after the increase in steepness. Moreover, one PT stated that "the cost of gas has increased," despite the fact that no information about the cost of gas was given in the problem.

Test Item 1. Gas Mileage (Linear). The graph below shows gallons of gas consumed by Jake's car compared to distance traveled.

- a. What would it mean in this situation if the line was steeper than shown and the scales on the graph stayed the same?
- b. Put a sensible scale on the axes, and write a question that can be answered from your graph. Show where the answer can be found on your graph.

Figure 7. Graph-related task, as posed on a written test in Nillas's (2010) study (p. 24).

The notion that PTs have difficulty interpreting graphs of functions is further supported by research focusing specifically on PTs' interpretations of graphs involving speed or motion (Billings & Klanderman, 2000; Stylianou, Smith, & Kaput, 2005). Analyses of responses on a pretest given to 28 elementary PTs in the U.S. suggest that many PTs hold

one of two well-documented misconceptions about graphs involving motion:¹⁷ (a) a graph-as-picture misconception, and (b) a slope/height misconception. PTs with the graph-as-picture misconception interpret graphs as though they are aerial pictures of the actual paths traveled. PTs with the slope/height misconception conflate slope with height so that a positive slope is interpreted as an up-hill path, and a negative slope is interpreted as a down-hill path (Stylianou et al., 2005).

Billings and Klanderma's (2000) analyses of the written work and in-class observations of 19 U.S. grade K–8 PTs in a junior-level class focused on algebra reported similar findings; they identified four cognitive difficulties that PTs in their study seemed to have when creating or interpreting graphs where one variable is speed: (a) confusing the concepts of instantaneous speed and average speed; (b) confusing the variables of speed and distance in various ways, for example, graphing a line segment with increasing slope on a time-versus-speed graph to show constant speed; (c) failing to identify the slope of a line segment in a distance-versus-time graph as speed; and (d) difficulty creating an appropriate scale for the axes of a graph involving speed.

To address these cognitive difficulties and misinterpretations of graphs, Stylianou, Smith, and Kaput (2005) suggest the use of specific motion-detection technology, such as Calculator-Based-Rangers (CBRs), during PT education courses. In their study, PTs participated in a 2-week classroom-based, exploratory study wherein PTs completed activities focused on making and interpreting graphical representations of motion using CBRs as data collection devices and as graphing calculators. Based on analyses of a pre- and

¹⁷ For a detailed review of this literature, see Leinhardt, Zaslavsky, and Stein (1990).

posttest, the authors note that the in-class activities appeared to help PTs overcome some of their misconceptions about graphs involving motion (specifically, the graph-as-picture and slope/height misconceptions). Most notably, the PTs seemed to improve in their abilities to interpret graphs as representations of a situation. The PTs also more frequently used graphs as problem-solving aids and as conscription devices to facilitate communication with one another. The researchers attributed these apparent gains in student understanding to the use of the CBRs, coupled with the rich peer-to-peer discussion about graphs that occurred throughout the activities.

One research study (Meel, 1999) explored PTs' definitions of functions via a written survey. In the study were 29 U.S. elementary or early childhood PTs who had chosen to specialize in mathematics and were near the end of their teaching certification program. Results of the study showed that each of the six statements in Figure 8 was indicated as a true definition of function by at least 75% of 29 PTs, with the highest number of PTs (26 out of 29) correctly marking statement B as true. Moreover, statement A was selected as the "best" definition of function by the highest number of PTs (15 out of 29). However, when asked to "define the mathematical concept: function" (p. 4) later in the survey, most PTs produced definitions similar to statement E. Statement E reflects a limited, historical "function-as-formula" understanding of function, which Meel (1999) suggests impedes PTs' understandings of functions.

- (A) A function is a correspondence between two sets that assigns to every element in the first set exactly one element in the second set.
- (B) A function is a dependence relation between two variables (y depends on x).
- (C) A function is a rule which connects the value of x with the value of y .
- (D) A function is a computational process which produces some value of one variable (y) from any given value of another variable (x).
- (E) A function is a formula, algebraic expression, or equation which expresses a certain relation between factors.
- (F) A function is a collection of numbers in a certain order which can be expressed in a graph.

Figure 8. Statements about functions from an item on a survey used in Meel's study (1999, pp. 3–4).

This collection of studies on PTs' understandings of function lends support for the idea that PTs typically have strong procedural skills with respect to linear functions (Nillas, 2010; You & Quinn, 2010), yet they tend to hold a formulaic understanding of function (Meel, 1999), and they tend to exhibit a wide range of struggles with respect to connecting and interpreting representations of functions, with the most-documented struggles relating to graphical representations of functions and story contexts (Billings & Klanderman, 2000; Nillas, 2010; Stylianou et al., 2005; You & Quinn, 2010).

One study lends support to the idea that hand-held graphing technology can be used to help PTs overcome some of their struggles interpreting graphical representations of functions. This is the only study to document pre/post change in PTs' understandings of function, and it is one of only two studies (Billings & Klanderman, 2000; Stylianou et al., 2005) that look at PTs' understandings of function at more than one time point. Further,

neither of these studies attempt to explain how PTs develop in their understandings of functions. Insight into how PTs learn to overcome their struggles with various representations of functions is a necessary next step in this line of research.

A View of the Horizon

In our search of recent literature, we did not find any research published in 2012 in peer-reviewed journals that focused on elementary or middle-grades PTs' content knowledge of algebra. We did, however, identify four such papers in our search of the 2011 and 2012 proceedings of PME and PME-NA (Callahan & Hillen, 2012; Jacobson & Izsák, 2012; Milinkovic, 2012; Mills, 2012; see Table 5 below).

Within these proceedings papers, we found that research is continuing within the topics of the equal sign and proportional reasoning. An exploratory, case-based study suggests that current frameworks of understandings of the equal sign might not be adequate for capturing the understandings of the population of elementary PTs (Mills, 2012). Based on the findings from a brief interview study of one PT, Mills concludes that the PT holds a predominantly operational view of the equal sign, yet the PT also demonstrates some relational understanding of the equal sign. In light of this case, Mills suggests that frameworks for understandings of the equal sign might need to be reworked in order to capture the understandings of PTs that seem to think of the equal sign as both operational and relational.

Table 5

Articles Published in 2012, or in the 2011/2012 PME/PME-NA Proceedings, on the Topic of PTs' Algebra Content Knowledge, in Alphabetical Order by First Author

Authors	Year	Number of PTs Studied	PTs' Level	Country	Data Analyzed
Callahan & Hillen	2012	22	Undergraduate PTs enrolled in a math content course	USA	Transcripts of video of whole-class discussions, field notes, copies of PTs' written classwork, audio recordings of teacher-researchers' weekly discussions
Jacobson & Izsák	2012	28	PTs enrolled in a middle-grades math methods course	USA	Whole-class video data, transcripts of task-based interviews of four pairs of students, written pre- and posttests of all students
Milinkovic	2012	121	Undergraduate PTs in their fourth year of study	Serbia	Written survey data
Mills	2012	1	Undergraduate PT enrolled in her first content course	USA	Task-based interview data

Within the topic of proportional reasoning, a study of 28 middle-grades mathematics PTs enrolled in a methods course suggests that PTs often try to set up and use proportional equations for non-proportional problem scenarios, even when the PTs seem to have a strong understanding of the non-proportional covariance situation (Jacobson &

Izsák, 2012). Analyses of whole-class video, transcripts of task-based interviews of four pairs of students from the class, and written pre- and posttests of all students suggest that PTs who correctly explain relationships between quantities that are not proportional still attempt to use proportion equations to represent the relationships. The authors point out that “these results suggest a sobering assessment of the pedagogical challenge faced by teacher educators,” (p. 635) given that encouraging PTs in understanding the scenarios presented in proportional and non-proportional covariation situations might not help with their ability to judiciously apply or not apply proportion equations to those situations.

In addition to the above-described continuing lines of research on PTs’ understandings of algebra, new lines of research are emerging within the areas of generalized arithmetic and properties of operations. The findings of a study of 22 prospective middle school teachers suggest that PTs struggle to understand a given visual representation of even and odd numbers (Callahan & Hillen, 2012). For example, although PTs were able to describe even numbers as divisible by 2, or having no remainder after dividing by 2, or being a multiple of 2, they struggled to make sense of a geometric representation wherein even numbers were represented as 2-by-whole-number rectangles. In other recent research, the findings of a survey-based study of 121 PTs’ knowledge of representations of multiplication and the Commutative Law of Multiplication suggest that PTs’ choices of representations are linked to problem abstractness (Milinkovic, 2012). Specifically, PTs tended to draw grouping (also commonly referred to as repeated addition) representations for questions using concrete numbers, whereas they tended to draw area representations for questions involving expressions using variables. The author concludes that problem abstractness (in this case, concrete numbers vs. variables) affects

PTs' representational choices, suggesting further research to confirm and explore the significance of these findings.

Conclusion

Looking across the findings of all research on PTs' content knowledge of algebra, we see three overarching themes:

1. Within the content domain of algebra, PTs generally have strong procedural skills and can make mathematically sound generalizations of many different types of patterns (Briscoe & Stout, 2001; Otto et al., 2008; Richardson et al., 2009; Rivera & Becker, 2007; Zazkis & Liljedahl, 2002a, 2002b);
2. However, PTs tend to struggle to (a) interpret and effectively use algebraic symbols, even those that they have produced themselves (Mills, 2012; Pomerantsev & Korosteleva, 2003; Prediger, 2010; Richardson et al., 2009; Rivera & Becker, 2007; Zazkis & Liljedahl, 2002a, 2002b); (b) interpret graphical representations (Billings & Klanderman, 2000; Nillas, 2010; Stylianou et al., 2005; You & Quinn, 2010); and (c) make connections between representations (Billings & Klanderman, 2000; Nillas, 2010; Pomerantsev & Korosteleva, 2003; Prediger, 2010; Richardson et al., 2009; Rivera & Becker, 2007; Stylianou et al., 2005; You & Quinn, 2010; Zazkis & Liljedahl, 2002a, 2002b);
3. Moreover, PTs generally have limited algebraic problem-solving strategies, often relying, inflexibly, on inefficient and/or incorrect computational methods (Berk et al., 2009; Schmidt & Bednarz, 1997, 2002; van Dooren et al., 2003).

Fortunately, though, there is emerging research to suggest that PTs' algebraic thinking and understandings in various areas can develop by focusing on

justification through connections between representations (Richardson et al., 2009; Rivera & Becker, 2007), analysis of children's artifacts (Prediger, 2010), consideration and analyses of multiple solution methods (Berk et al., 2009; van Dooren et al., 2003), and work with hand-held graphing technology (Stylianou et al., 2005).

Notably absent from the themes above are research-based conclusions about how PTs develop in their understandings of various topics within algebra. The few studies that followed the development of PTs' content knowledge of algebra were limited in scope and generalizability. In light of this, we suggest that more research is needed on PTs' development in understanding, interpreting, and connecting representations of various topics within algebra, such as generalizations, functions, and word problems. It is clear from the current research that graphical, symbolic, and contextual representations (and the connections between them) can be points of struggle for many PTs. Accordingly, we also recommend that methodologically rigorous research be devoted to exploring and developing pedagogical innovations for teaching algebra to PTs.

Further, given the recommendations of the METII (Conference Board of the Mathematical Sciences, 2012) and the CCSS (NGA & CCSSO, 2010) that the foundations of algebra should be laid in the elementary grades, we point to the need for continued research that focuses on how PTs come to understand the connections between arithmetic and algebra, including (but not limited to) properties of operations, and judicious and flexible strategy selection in problem solving.

Our summary of the existing body of literature on PTs' understandings of algebra suggests that implications for teacher education courses are tentative and somewhat

scattered. While there are some research-based recommendations for teacher education courses—such as making use of motion-sensor and graphing technology, having PTs analyze children’s artifacts, identifying and leveraging various problem solving strategies of PTs, and encouraging PTs to justify their ideas through connections between representations during visually- or contextually-based tasks—the picture is far from complete. Clear and comprehensive research-based guidance for the development of PT-centered mathematical preparation in algebra for our future educators remains to be established.

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