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The Miracle of Applied Mathematics

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ABSTRACT: We provide an outline of a college-level lecture on systems of differential equations that is intended to offer an alternative, integrative view of the subject. Particular emphasis is placed on the fact that very elementary concepts and insights can be employed to gain access to a very wide range of important applications, including the Schrödinger equation, the Klein-Gordon equation, the Laplace equation, and the Poisson equation.

Keywords: applicability of mathematics, systems of differential equations, partial differential equations, approximate solutions

1 Introduction

In the teaching profession, as in many other professions, an initially bright aspiration is all too often overcome by chronic, nagging frustration. The original enthusiasm that once inspired a person's choice of vocation will tend to dissipate as so-called real-world pressures take their toll, depleting mental energy and putting tight constraints on free creative acts.

Ideally, a college classroom ought to be a place where genuine communication engages minds and changes lives by stimulating joyful curiosity and honest exploration. But in an academic environment in which the publish-or-perish mentality is prevalent and extreme over-specialization corrupts the educational mission, these worthy goals will often seem remote or even wholly out of reach. One way to counter such adversities is to be very intentional in setting aside some time for topical excursions that broaden the view and thereby re-energize the minds of students and teachers alike. This is not to say, of course, that large chunks of course content ought to be sacrificed to random pursuits of topical tangents. But it is to say that the cultivation of a certain versatility in subject matter selection may help to re-invigorate a teaching effort which, perhaps, has come to be burdened over time by a general loss of creative momentum.

To give the reader a concrete example of how such versatility can be practiced without straying all too far from standard course material, I would like to provide an outline of a lecture that I have given in each of the past three years in my course on differential equations at John Brown University. The students taking this course are mostly engineering majors in their second year and have successfully completed Calculus I and II. They are used to placing mathematics in an application-oriented problem-solving context but have never been exposed to the full theoretical rigor that mathematics majors can be expected to practice. Consequently, some of the steps in the calculations that follow will be skipped over relatively lightly without providing concise analytical proofs. This lack of rigor, though, can easily be overcome, as any reader familiar with the fundamental principles of real analysis can simply look up the relevant arguments in standard analysis textbooks.

In order not to cause any misunderstandings, it also needs to be emphasized that—at the time when the lecture is given—the students in attendance are already well familiar with the theory of linear systems of differential equations and have studied in particular the representation of solutions by means of exponential matrices. So the purpose here is not to communicate new mathematical content but rather to offer a hopefully interesting alternative view.

2 Outline of the Lecture

One of the most important problems in the study of differential equations is the computation of solutions of linear systems of the form

$$\boxed{\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{f}(t)}, \quad (1)$$

where \mathbf{A} is a constant $n \times n$ -matrix and \mathbf{f} a continuous vector-valued function. The range of applications to which such systems are relevant is very wide indeed. From electric networks, multiple overflow systems, and mechanical linkages of masses pulled by springs, the spectrum readily extends to chemical bonds in solid-state matter, to the entire classical theory of quantum mechanics by way of the Schrödinger equation, and even more generally, to just about any theory or field that is described by linear partial differential equations.

Given this very broad applicability, it is truly amazing to realize that the problem of solving equation (1) can be reduced to the following three elementary tasks:

Task 1: drawing tangents to a graph.

Task 2: rearranging beads of two different colors. (2)

Task 3: drawing balls at random from two bowls.

How is it possible that these three basic tasks combine to give us access to such a vast array of physical phenomena? What is the link from beads to Schrödinger's equation, from random draws to circuits, or from tangents to a graph to masses pulled by springs? The most direct reply that we can give is to observe that in Task 1 we merely face the problem of computing a derivative: we take the limit of the secant slope

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}$$

and write, as usual, the following equation:

$$\boxed{\frac{d}{dx}f(x) := f'(x) := \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}}. \quad (3)$$

However, the fact that this equation is so solidly familiar need not *per se* diminish our sense of genuine astonishment. For absent our prior knowledge of the calculus we scarcely would suspect that such a simple formula for converting secant into tangent slopes might somehow enable us to understand, let's say, how currents in a network oscillate or quantum waves disperse.

That said, we now will turn to our second task of rearranging beads. To do so we suppose that we are given five colored beads—two blue ones and three yellow ones. Denoting these beads by two

different letters, say a and b , we proceed to inquire in how many ways these letters can be lined up in a row. How many patterns can these letters form when placed adjacent to each other? The answer is determined most efficiently by indexing the letters with numbers so as to render them pairwise distinct. This yields the following symbols:

$$a_1, a_2, b_1, b_2, b_3.$$

Since the number of arrangements (or permutations) of these five distinct symbols is $5!$, it follows that the number of patterns formed by two letters a and three letters b (without the indices) is

$$\binom{5}{2} = \frac{5!}{2!3!} = 10$$

because the product $2!3!$ in the denominator equals the number of ways in which the indices alone can be permuted without affecting the arrangement of the letters a and b . At this point in the presentation the students in the audience can be expected to infer (or recall) quite readily that, in general, the number of arrangements of k letters a and $n - k$ letters b in a row of length n is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

However, in my experience at least, most students are very hard pressed when asked to remember or to deduce how the combinatorial argument just given relates to the binomial expression $(a + b)^n$. To help them better see the link I commonly consider as an example the case where $n = 3$ or $n = 4$. In expanding the power $(a + b)^3$ (or $(a + b)^4$) into 2^3 (or 2^4) terms, it is not difficult to make the students realize that the number of summands in this expansion that are equal to $a^k b^{n-k}$ is precisely the number of arrangements of k letters a and $n - k$ letters b in a row of length n . Thus we find that

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}. \quad (4)$$

Having arrived at this *binomial formula*, we now apply it to the term $(1 + x/n)^n$. This yields

$$\left(1 + \frac{x}{n}\right)^n = \sum_{k=0}^n \binom{n}{k} \left(\frac{x}{n}\right)^k = \sum_{k=0}^n \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{x^k}{k!} = \sum_{k=0}^n \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \frac{x^k}{k!},$$

and with a bit of faith (or analytical insight) we may therefore infer that

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = \sum_{k=0}^{\infty} \left(1 - \frac{1}{\infty}\right) \cdots \left(1 - \frac{k-1}{\infty}\right) \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

With this result in place we pick an arbitrary $a \in \mathbb{R}$ and set

$$g(x) := \lim_{n \rightarrow \infty} \left(1 + \frac{ax}{n}\right)^n = \sum_{k=0}^{\infty} \frac{a^k x^k}{k!} (= e^{ax}).$$

To proceed, we return to Task 1 and ask how the tangents to the graph of g are to be drawn. Using (3) together with some basic trust in the interchangeability of limits and sums, we find that

$$g'(x) = \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} = \sum_{k=0}^{\infty} \frac{a^k}{k!} \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^k - x^k}{\Delta x}.$$

Since

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^k - x^k}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left(\sum_{i=0}^k \binom{k}{i} (\Delta x)^i x^{k-i} - x^k \right) \quad (\text{by (4)}) \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \sum_{i=1}^k \binom{k}{i} (\Delta x)^i x^{k-i} = \lim_{\Delta x \rightarrow 0} \sum_{i=1}^k \binom{k}{i} (\Delta x)^{i-1} x^{k-i} = \binom{k}{1} x^{k-1} = \frac{k! x^{k-1}}{(k-1)!} = kx^{k-1}, \end{aligned}$$

it follows that

$$g'(x) = \sum_{k=0}^{\infty} \frac{ka^k x^{k-1}}{k!} = a \sum_{k=1}^{\infty} \frac{a^{k-1} x^{k-1}}{(k-1)!} = a \sum_{k=0}^{\infty} \frac{a^k x^k}{k!},$$

and therefore,

$$\boxed{g'(x) = ag(x)}. \tag{5}$$

At this point of our discussion we shift gears once again and direct our attention at Task 3. That is to say, we consider two bowls—one white and one black—that each contain five balls marked with either a zero or a one (see Figure 1). Whenever we draw a ball at random from one of these two

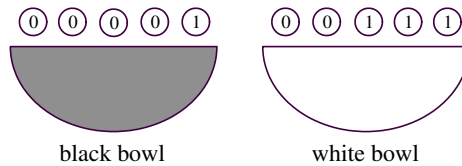


Figure 1: a probability experiment.

bowls, we record the number written on it and put it back into the bowl from which we took it. If the number recorded was a zero we will draw next a ball from the white bowl, and otherwise, if the number was a one, we will draw a ball from the black bowl. Given this rule and given the distribution of balls shown in Figure 1, we readily find the following array of transitional probabilities:

$$\mathbf{P} := \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} 2/5 & 3/5 \\ 4/5 & 1/5 \end{pmatrix}.$$

A natural problem that we may consider concerning this experiment is the computation of a longterm expectation: in what relative proportion do we expect to draw zeros and ones as the number of draws increases to infinity? The first step in approaching this problem is the calculation of higher-order

transitional probabilities. When we ask, for instance, how likely it is to draw a zero given that the next to last draw was a zero as well, we easily realize that the probability in question is

$$p_{00}(2) = p_{00}p_{00} + p_{01}p_{10}.$$

That is to say, the probability $p_{00}(2)$ is the 00-element of the twofold matrix product of \mathbf{P} with itself. More generally and by direct analogy, the n -step transitional probabilities $p_{ij}(n)$ are the elements of the n -fold product of \mathbf{P} with itself:

$$\begin{pmatrix} p_{00}(n) & p_{01}(n) \\ p_{10}(n) & p_{11}(n) \end{pmatrix} = \mathbf{P}^n.$$

Furthermore, as we increase the number of bowls in synchrony with the number of digits written on the balls, we are led to introduce analogous computational schemes for probability arrays \mathbf{P} of higher dimensions. So what we see happening here is the natural emergence of the operation of *matrix multiplication* for square-shaped $m \times m$ numerical arrays.

To be sure, there are various other paths along which the notion of a matrix product can be plausibly encountered. But even so it is surprising to notice that a simple probability experiment involving nothing more than the drawing of balls of two different kinds from two different bowls has led us to define an algebraic operation that allows us to extend the concept of a power of a number to numerical arrays. By implication, if we further define the matrix sum and scalar product componentwise, we may in fact replace the parameter a in the definition of g by an $n \times n$ matrix:

$$\mathbf{g}(t) := \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!} (= e^{\mathbf{A}t}),$$

where $\mathbf{A}^0 := \mathbf{Id}$. Moreover, as we now ask how the tangent lines to the component functions of \mathbf{g} are to be drawn, we find—in much the same way as in the derivation of (5)—that

$$\mathbf{g}'(t) = \mathbf{A}\mathbf{g}(t).$$

Multiplying both sides of this equation by a constant vector \mathbf{C} (in the manner suggested by the multiplication of \mathbf{A} with itself) and setting

$$\mathbf{y}(t) := \mathbf{g}(t)\mathbf{C}, \tag{6}$$

it follows that \mathbf{y} is a solution of the homogeneous equation

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t). \tag{7}$$

By the standards of contemporary mathematics it seems fair to assert that the conceptual distance which we just traveled in demonstrating that (6) is a solution of (7) was very small indeed. Thus we rightly feel astonished as we now come to realize that we are given here the key to one of the most fundamental and most important theories in all of modern physics: the Schrödinger description of quantum mechanics. According to Schrödinger, physical entities such as electrons, atoms, molecules, or even larger-scale macroscopic systems are universally described by a wave-function ψ that mathematically encodes a given system's total information content. For a particle of mass m with potential energy $V(x)$ the Schrödinger equation that ψ satisfies assumes the following form:

$$\boxed{-\frac{\hbar}{i} \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi.}$$

In order to solve this equation *approximately* (which is always sufficient from an empirical point of view), we replace the continuous variable x by discrete points $\dots x_{-1}, x_0, x_1, x_2, \dots, x_k, \dots$, where the separation $x_{k+1} - x_k = \Delta x$ is thought to be extremely small. Since particles are commonly localized within a certain finite space, we may assume that there exists an $n \in \mathbb{N}$ such that $\psi(t, x_k) = 0$ for all $k \in \mathbb{Z} \setminus \{1, \dots, n\}$. Setting

$$y_k(t) := \psi(t, x_k)$$

and

$$V_k := V(x_k)$$

and observing that

$$\begin{aligned} \frac{\partial^2 \psi(t, x_k)}{\partial x^2} &\approx \frac{\partial}{\partial x} \frac{\psi(t, x_{k+1}) - \psi(t, x_k)}{\Delta x} \\ &\approx \frac{(\psi(t, x_{k+2}) - \psi(t, x_{k+1})) / \Delta x - (\psi(t, x_{k+1}) - \psi(t, x_k)) / \Delta x}{\Delta x} \\ &= \frac{y_{k+2}(t) - 2y_{k+1}(t) + y_k(t)}{\Delta x^2}, \end{aligned}$$

we may apply the Schrödinger equation to infer that

$$y'_k(t) \approx \frac{i\hbar(y_{k+2}(t) - 2y_{k+1}(t) + y_k(t))}{2m\Delta x^2} - \frac{iV_k y_k(t)}{\hbar}.$$

Letting the index k vary from 1 to n and setting

$$\mathbf{A} := \begin{pmatrix} \frac{i\hbar}{2m\Delta x^2} - \frac{iV_1}{\hbar} & \frac{-i\hbar}{m\Delta x^2} & \frac{i\hbar}{2m\Delta x^2} & 0 & 0 & \dots & 0 \\ 0 & \frac{i\hbar}{2m\Delta x^2} - \frac{iV_2}{\hbar} & \frac{-i\hbar}{m\Delta x^2} & \frac{i\hbar}{2m\Delta x^2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix},$$

we may write the corresponding system of differential equations in the following matrix form:

$$\mathbf{y}'(t) = \begin{pmatrix} y_1'(t) \\ \vdots \\ y_n'(t) \end{pmatrix} \approx \mathbf{A} \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} = \mathbf{A}\mathbf{y}(t).$$

As we saw, the general solution of this equation is $\mathbf{y}(t) = \mathbf{g}(t)\mathbf{C}$.

A similar approach can also be applied to linear partial differential equations that are of a higher order in t . To see why this is so, we may consider for example the Klein-Gordon equation of elementary relativistic quantum physics:

$$\boxed{\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} + m^2 \psi = 0.}$$

Setting again

$$y_k(t) := \psi(t, x_k),$$

we readily find that

$$y_k''(t) \approx \frac{y_{k+2}(t) - 2y_{k+1}(t) + y_k(t)}{\Delta x^2} - m^2 y_k(t).$$

In order to reduce this system of equations to first order, we introduce the substitutions

$$\begin{aligned} u_1(t) &:= y_1(t), \\ u_2(t) &:= y_1'(t), \\ &\vdots \\ u_{2n-1}(t) &:= y_n(t), \\ u_{2n}(t) &:= y_n'(t). \end{aligned} \tag{8}$$

This yields

$$\mathbf{u}'(t) = \begin{pmatrix} u_1'(t) \\ \vdots \\ u_{2n}'(t) \end{pmatrix} \approx \mathbf{A} \begin{pmatrix} u_1(t) \\ \vdots \\ u_{2n}(t) \end{pmatrix} = \mathbf{A}\mathbf{u}(t),$$

where

$$\mathbf{A} := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \frac{1}{\Delta x^2} - m^2 & 0 & \frac{-2}{\Delta x^2} & 0 & \frac{1}{\Delta x^2} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{\Delta x^2} - m^2 & 0 & \frac{-2}{\Delta x^2} & 0 & \frac{1}{\Delta x^2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}. \tag{9}$$

Evidently, the methods employed in the preceding two examples can also be easily used to approximately solve numerous other partial differential equations such as the classical wave equation

$$\frac{\partial^2 \psi}{\partial t^2} - v^2 \frac{\partial^2 \psi}{\partial x^2} = 0,$$

the density preservation equation

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0,$$

the heat equation

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2},$$

or the Laplace equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0.$$

Having reached this conclusion, it only remains for us to discuss the problem of finding the general solution of the inhomogeneous equation (1). As it turns out, even in solving this more general equation we do not stray far from the conceptual context so far established. To see why this is so, we need to recall that the standard method for generating a solution of (1) is to vary the constant vector \mathbf{C} in the definition of \mathbf{y} . That is to say, we set

$$\mathbf{y}(t) := \mathbf{g}(t)\mathbf{C}(t) \tag{10}$$

and try to determine $\mathbf{C}(t)$ from the requirement that \mathbf{y} be a solution of (1). This yields—by way of the product rule—the following vector equation:

$$\mathbf{g}(t)\mathbf{C}'(t) = \mathbf{f}(t).$$

In order to eliminate the factor $\mathbf{g}(t)$ on the left, we need to demonstrate first that $\mathbf{g}(s+t) = \mathbf{g}(s)\mathbf{g}(t)$ for all $s, t \in \mathbb{R}$. To do so, we multiply term by term (with a bit of faith) the series that represents $\mathbf{g}(s)$ with the one that represents $\mathbf{g}(t)$. Using the binomial formula (i.e., the task of rearranging beads), we easily find that the product thus obtained is indeed equal to $\mathbf{g}(s+t)$. Setting $s := -t$ and observing that $\mathbf{g}(0) = \mathbf{Id}$, it follows that

$$\mathbf{C}'(t) = \mathbf{g}(-t)\mathbf{f}(t).$$

At this point any student of the calculus is bound to recognize that $\mathbf{C}(t)$ can be obtained by means of antidifferentiation:

$$\mathbf{C}(t) = \mathbf{C}(0) + \int_0^t \mathbf{g}(-\tau)\mathbf{f}(\tau) d\tau. \tag{11}$$

However, the fact that this deduction is so frequently performed does not at all imply that its essential nature is universally well understood. In other words, it usually takes quite a bit of reminding for students to recall that what is involved in this case is nothing more than a simple cancellation of

terms in a sum:

$$\begin{aligned} \int_0^t \mathbf{C}'(\tau) d\tau &\approx \sum_{k=0}^{n-1} \mathbf{C}'(\tau_k) \Delta\tau \approx \sum_{k=0}^{n-1} \frac{\mathbf{C}(\tau_k + \Delta\tau) - \mathbf{C}(\tau_k)}{\Delta\tau} \Delta\tau \\ &= \sum_{k=0}^{n-1} (\mathbf{C}(\tau_{k+1}) - \mathbf{C}(\tau_k)) = \mathbf{C}(t) - \mathbf{C}(0). \end{aligned}$$

Finally, with this explanation added, we may combine (10) and (11) to conclude that

$$\mathbf{y}(t) = \mathbf{g}(t)\mathbf{C}(0) + \mathbf{g}(t) \int_0^t \mathbf{g}(-\tau)\mathbf{f}(\tau) d\tau$$

is a solution of the inhomogeneous linear system given in (1). As is well known, this formula allows us to determine, for example, the solutions of multiple-spring mechanical systems that are subject to external forces and electric networks that are controlled by external voltages. But it also allows us to approximately solve inhomogeneous partial differential equations such as, for instance, the Poisson equation:

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \rho(x, y)}. \quad (12)$$

For as we set

$$\mathbf{A} := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \frac{-1}{\Delta y^2} & 0 & \frac{2}{\Delta y^2} & 0 & \frac{-1}{\Delta y^2} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \frac{-1}{\Delta y^2} & 0 & \frac{2}{\Delta y^2} & 0 & \frac{-1}{\Delta y^2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

in direct analogy to (9),

$$\rho_k(x) := \rho(x, y_k),$$

and

$$\mathbf{f}(x) := \begin{pmatrix} 0 \\ \rho_1(x) \\ 0 \\ \rho_2(x) \\ \vdots \\ 0 \\ \rho_n(x) \end{pmatrix},$$

we find that the $2n$ -dimensional first-order system that approximately represents equation (12) is

$$\mathbf{u}'(x) = \mathbf{A}\mathbf{u}(x) + \mathbf{f}(x)$$

where the components of \mathbf{u} , of course, are defined in a manner analogous to (8).

In the light of this concluding example, we may rightly assert that the potential range of applications of equation (1) is very extensive indeed. By implication, we may say that the three basic tasks, listed in (2), provide us access to a very vast array of physical phenomena—and that, no doubt, is truly astounding!

Suggested Further Readings:

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