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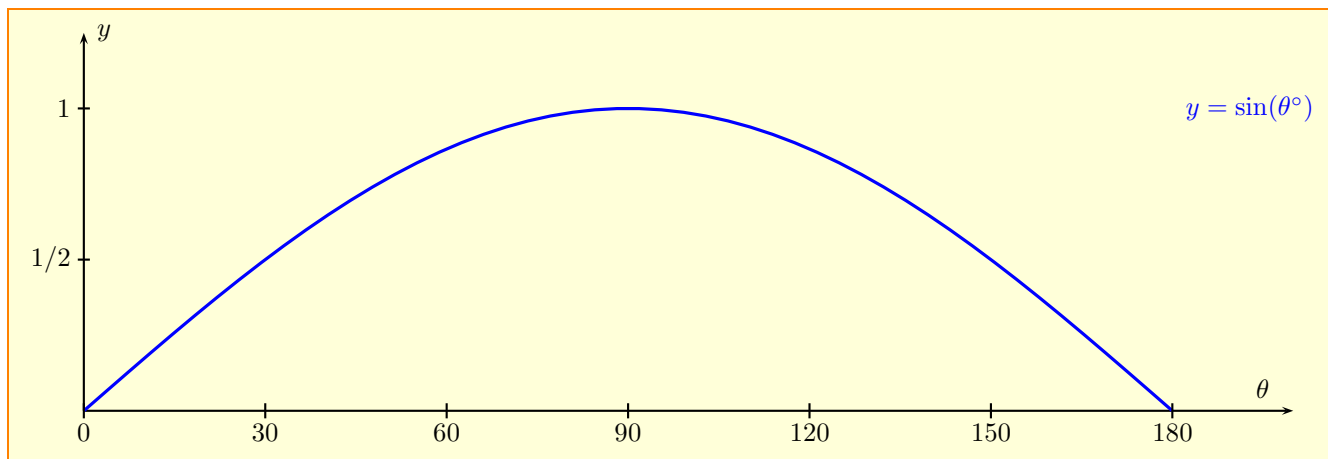
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Bhaskara's approximation for the sine

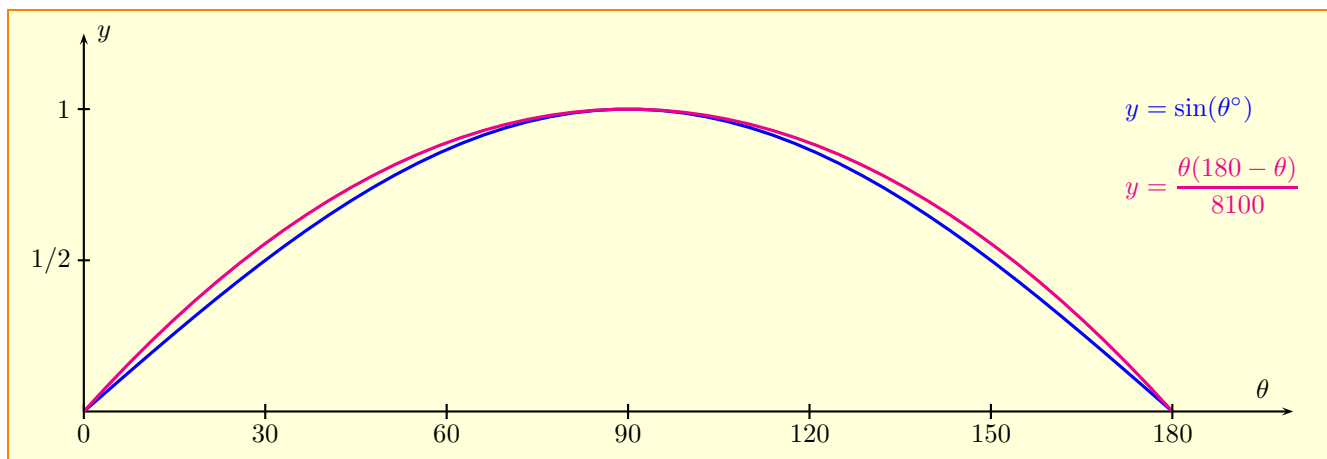
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The 7th century Indian mathematician Bhaskara (c.600 – c.680) obtained a remarkable approximation for the sine function. Many subsequent ancient authors have given versions of this rule, but none provided a proof or described how the result was obtained. Grover [1] provides a possible explanation, but I think the rule can be explained more clearly. Rather than give the rule first, we will derive it, and then discuss its accuracy, and explore some alternative approximations. Our derivation is simply an exercise in modeling.

Approximating the Sine: A Possible Derivation of Bhaskara's Approximation. Our goal is to approximate the sine function on the interval $[0^\circ, 180^\circ]$ (I have seen the rule formulated for radians, in conjunction with certain approximations of π ; we will see that none of this is necessary when we simply work with degrees. I will be careful to use degrees in the notation, so that we can translate some of the formulas for use with radians without ambiguity). Graph of the function $y = \sin(\theta^\circ)$ over the interval $[0, 180]$:



We will use that the above graph is symmetric with respect to $\theta = 90$. It is not hard to find a polynomial with the same symmetry which takes values 0 for $\theta = 0$ and $\theta = 180$: $p(\theta) = \theta(180 - \theta)$. The value of p at $\theta = 90$ is $p(90) = 90(180 - 90) = 8100$, so the polynomial $p/8100$ has the same symmetry about $\theta = 90$, and the same values as $\sin(\theta^\circ)$ at the points $\theta = 0, 90, 180$. This is all as in Grover [1], who states that Bhaskara referred to the quantity $\theta(180 - \theta)$ as 'prathama' (Grover does not provide the meaning of this word; it is Sanskrit for "first"), from which we can infer that this is also in Bhaskara's work. The quadratic polynomial $p(\theta)/8100 = \theta(180 - \theta)/8100$ can be viewed as a crude (first) approximation of $\sin(\theta^\circ)$ on the interval $[0, 180]$. The following graph shows how the two functions compare:



Note that $p(30) = \frac{30(180-30)}{8100} = \frac{5}{9}$. To get a better approximation, Bhaskara must have interpolated also the value $\sin(30^\circ) = 1/2$. To do this, consider the following function

$$f(\theta) = \frac{ap(\theta)}{1 + bp(\theta)}, \quad 0 \leq \theta \leq 180.$$

Clearly f has the same symmetry as p about $\theta = 90$, and $f(0) = f(180) = 0$. To get value 1 at $\theta = 90$ it is required that $\frac{a}{1+b} = 1$, that is, $a = 1 + b$, so

$$f(\theta) = \frac{(1+b)p(\theta)}{1 + bp(\theta)}, \quad 0 \leq \theta \leq 180.$$

To get value $1/2$ at $\theta = 30$ it is required that

$$\frac{1}{2} = f(30) = \frac{(1+b)p(30)}{1 + bp(30)} = \frac{(1+b)\frac{4}{9}}{1 + b\frac{4}{9}},$$

which is easily solved to yield $b = -\frac{1}{5}$. It follows that

$$f(\theta) = \frac{\frac{4}{5}p(\theta)}{1 - \frac{1}{5}p(\theta)} = \frac{\frac{4}{5}\frac{\theta(180-\theta)}{8100}}{1 - \frac{1}{5}\frac{\theta(180-\theta)}{8100}} = \frac{4\theta(180-\theta)}{40500 - \theta(180-\theta)}.$$

This gives Bhaskara's approximation formula for the sine function.

Bhaskara's Approximation Formula: $\sin(\theta^\circ) \approx \frac{4\theta(180-\theta)}{40500 - \theta(180-\theta)}, \quad \text{for } 0 \leq \theta \leq 180.$

Alternative Derivation. There is an even simpler argument to obtain Bhaskara's formula. I have no doubt now that Bhaskara must have reasoned as follows.

The polynomial $p(\theta) = \theta(180 - \theta)/8100$ is symmetric with respect to 90, and agrees with the values of the sine for 0, 90, and 180 degrees, but not for 30 degrees (and 150 degrees). In fact we have $p(30) = \frac{4500}{8100} = \frac{5}{9}$. Now, it is easy to construct a polynomial q symmetric about $\theta = 90$ which takes value $\frac{10}{9} = 1 + \frac{1}{9}$ at 30 (the reason for this value will become clear momentarily) and value 1 at 90: the polynomial

$$q(\theta) = 1 + \frac{(\theta - 90)^2}{9(30 - 90)^2} = 1 + \frac{(\theta - 90)^2}{32400}$$

will do the job. Since $p(90) = 1$ and $q(90) = 1$, we have $\frac{p(90)}{q(90)} = 1$. Note that

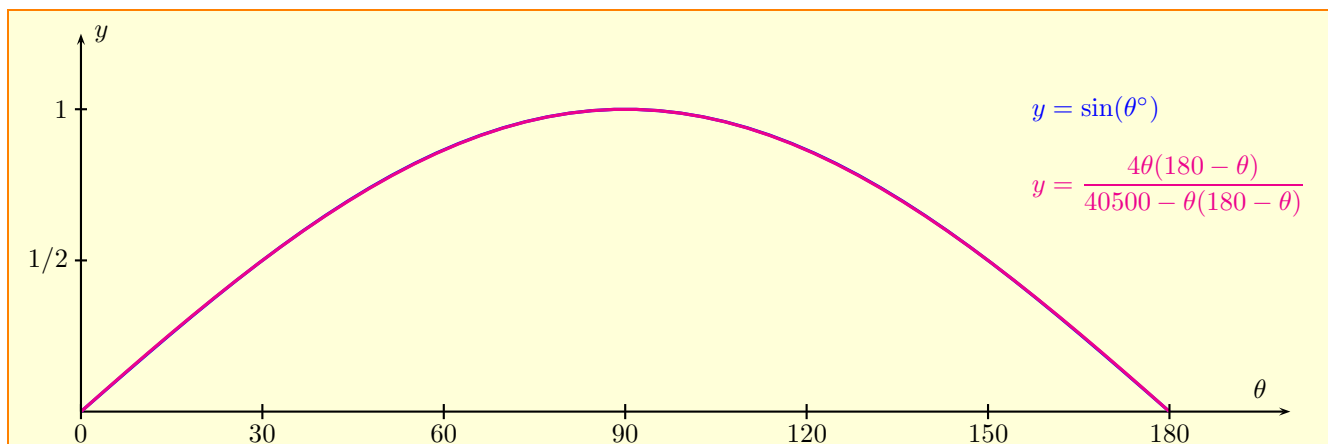
$$\frac{p(30)}{q(30)} = \frac{\frac{5}{9}}{\frac{10}{9}} = \frac{1}{2}.$$

Also $\frac{p(0)}{q(0)} = 0$. So the following function is symmetric about 90 and has the same values as the sine at 0, 30, and 90 degrees (as well as 150 and 180 degrees by symmetry):

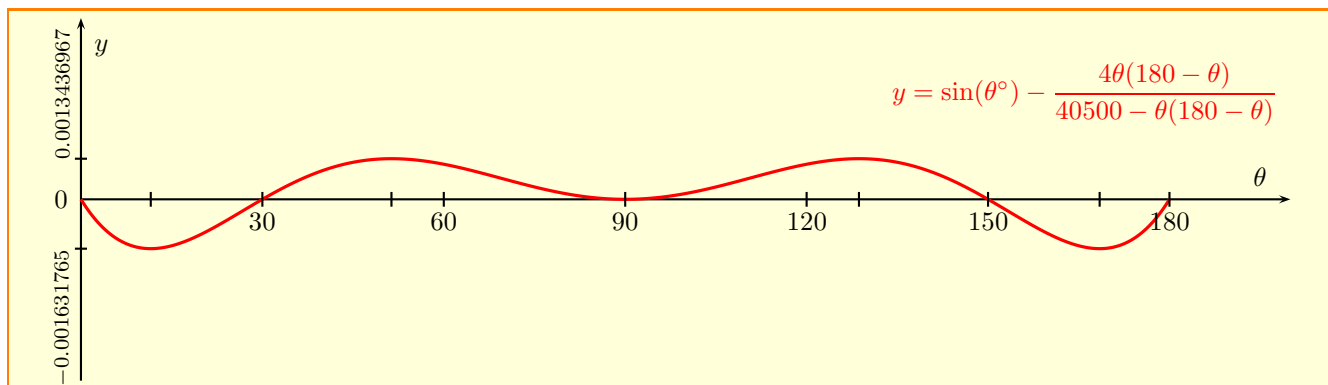
$$r(\theta) = \frac{\frac{\theta(180 - \theta)}{8100}}{1 + \frac{(\theta - 90)^2}{32400}} = \frac{4\theta(180 - \theta)}{32400 + (\theta - 90)^2} = \frac{4\theta(180 - \theta)}{40500 - \theta(180 - \theta)},$$

which is Bhaskara's rational function! The simplicity of this argument makes it likely that the formula was derived along these lines.

Accuracy of Bhaskara's Approximation. To see how good of an approximation this is we plot the above rational function in the same window as the sine function:



Since the two graphs are not distinguishable, we plot the difference on a larger scale:



The error is bounded by 0.00165 over the entire interval (in the above graph the minimum of about -0.001631765 occurs near $\theta = 11.543848$ and $\theta = 168.4561524$, and the maximum of about 0.0013436967 occurs near $\theta = 51.34589377$ and $\theta = 128.6541062$) so this is a very good approximation.

An Approximation Formula for the Cosine. It is fairly easy to see that we get the same result if we approximate $y = \cos(\theta^\circ)$ on the interval $[-90, 90]$, starting with the quadratic polynomial $q(\theta) = \frac{8100 - \theta^2}{8100}$. Of course this can also be seen using the identity $\cos(\theta^\circ) = \sin(\theta^\circ + 90^\circ)$. Either way, Bhaskara's approximation for the sine has the following analog for the cosine:

$$\begin{aligned}\cos(\theta^\circ) &\approx \frac{4(90 + \theta)(180 - (90 + \theta))}{40500 - (90 + \theta)(180 - (90 + \theta))} \\ &= \frac{4(90 + \theta)(90 - \theta)}{40500 - (90 + \theta)(90 - \theta)} \\ &= \frac{32400 - 4\theta^2}{32400 + \theta^2}.\end{aligned}$$

We can rewrite the above rational function as

$$\begin{aligned}\frac{32400 - 4\theta^2}{32400 + \theta^2} &= \frac{32400 + \theta^2 - 5\theta^2}{32400 + \theta^2} = 1 - \frac{5\theta^2}{32400} \frac{1}{1 + \theta^2/32400} \\ &= 1 - 5 \left(\frac{\theta}{180}\right)^2 \frac{1}{1 + (\theta/180)^2} \\ &= 1 - 5 \left(\frac{\theta}{180}\right)^2 \sum_{n=0}^{\infty} (-1)^n \left(\frac{\theta}{180}\right)^{2n} \\ &= 1 - 5 \left(\frac{\theta}{180}\right)^2 + 5 \left(\frac{\theta}{180}\right)^4 - 5 \left(\frac{\theta}{180}\right)^6 + \dots.\end{aligned}$$

Now, if $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, then $\cos x = \cos(\theta^\circ)$, where $\theta = \frac{180x}{\pi}$, so the above analog of Bhaskara's sine approximation for the cosine gives the approximation

$$\begin{aligned}\cos x &\approx 1 - 5 \left(\frac{x}{\pi}\right)^2 + 5 \left(\frac{x}{\pi}\right)^4 - 5 \left(\frac{x}{\pi}\right)^6 + \dots \\ &= 1 - \frac{5}{\pi^2}x^2 + \frac{5}{\pi^4}x^4 - \frac{5}{\pi^6}x^6 + \dots.\end{aligned}$$

So Bhaskara's approximation does not agree with even a second order approximation unless $\pi^2 = 10$. This agrees with the approximation of π by $\sqrt{10}$ in use at Bhaskara's time; although this was not the best approximation known at the time, this approximation was popular in India, perhaps because it fits so nicely with the above approximation rules for sines and cosines.

Approximating the Sine using Polynomials. The above method is not the only way to improve the first order approximation of $\theta(180 - \theta)/8100$. If we put $p = \theta(180 - \theta)$ as above, then we could also fit $y = \sin(\theta^\circ)$ to the function $ap + bp^2$, a second order polynomial in p (thus a 4th order polynomial in θ). We need the following values for p :

θ	p	y
0	0	0
30	4500	1/2
90	8100	1

The following polynomial in p fits the above data points:

$$\frac{p(p - 4500)}{8100(8100 - 4500)} \cdot 1 + \frac{p(p - 8100)}{4500(4500 - 8100)} \cdot \frac{1}{2}.$$

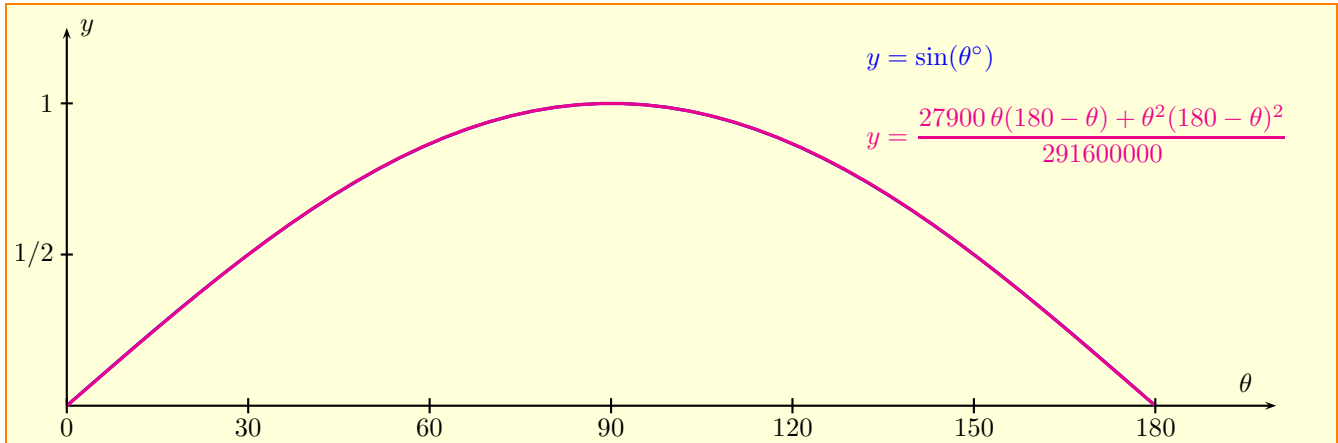
The above expression simplifies to

$$\frac{27900p + p^2}{291600000}.$$

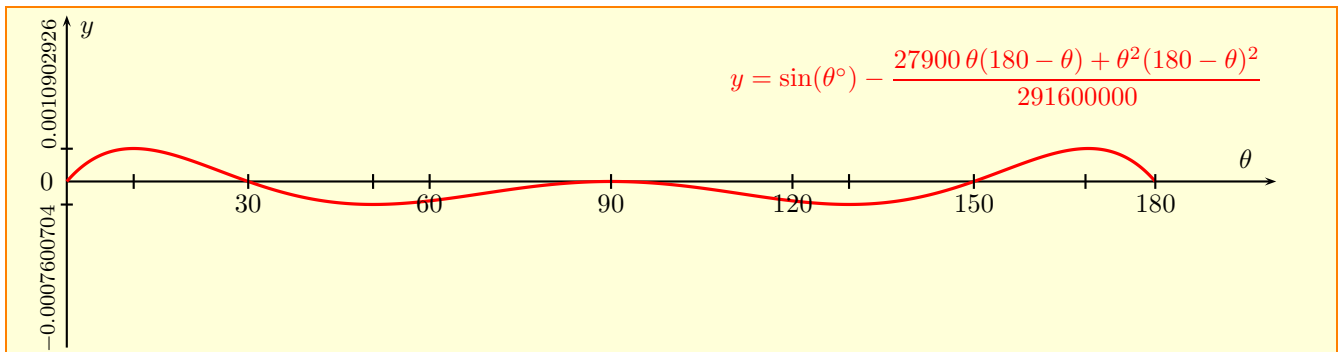
Now substitute $p = \theta(180 - \theta)$ into the above expression to get the function¹

$$g(\theta) = \frac{27900\theta(180 - \theta) + \theta^2(180 - \theta)^2}{291600000}.$$

To see how good of an approximation this is we plot the above polynomial in the same window as the sine function:



Since the two graphs are not distinguishable, we plot the difference on a larger scale:



The error is bounded by 0.0011 over the entire interval (in the above graph the minimum of about -0.0007600704 occurs near $\theta = 50.64638193$ and $\theta = 129.3536181$, and the maximum of about 0.0010902926 occurs near $\theta = 11.063325$ and $\theta = 168.936675$) so the approximation

$$\sin(\theta^\circ) \approx \frac{27900\theta(180 - \theta) + \theta^2(180 - \theta)^2}{291600000}$$

¹That the same result is obtained by substituting $\theta = 90$ and $\theta = 30$ into the function $a\theta(180 - \theta) + b\theta^2(180 - \theta)^2$ to obtain the system of equations

$$\begin{aligned} 8100a + 8100^2b &= 1, \\ 4500a + 4500^2b &= 1/2, \end{aligned}$$

is left as an exercise for the reader.

is better than Bhaskara's. The corresponding approximation for the cosine is

$$\begin{aligned}\cos(\theta^\circ) = \sin(90^\circ + \theta^\circ) &\approx \frac{27900(90^2 - \theta^2) + (90^2 - \theta^2)^2}{291600000} \\ &= 1 - \frac{49}{324000}\theta^2 + \frac{1}{291600000}\theta^4 \\ &= 1 - \frac{49}{10}\left(\frac{\theta}{180}\right)^2 + \frac{18}{5}\left(\frac{\theta}{180}\right)^4.\end{aligned}$$

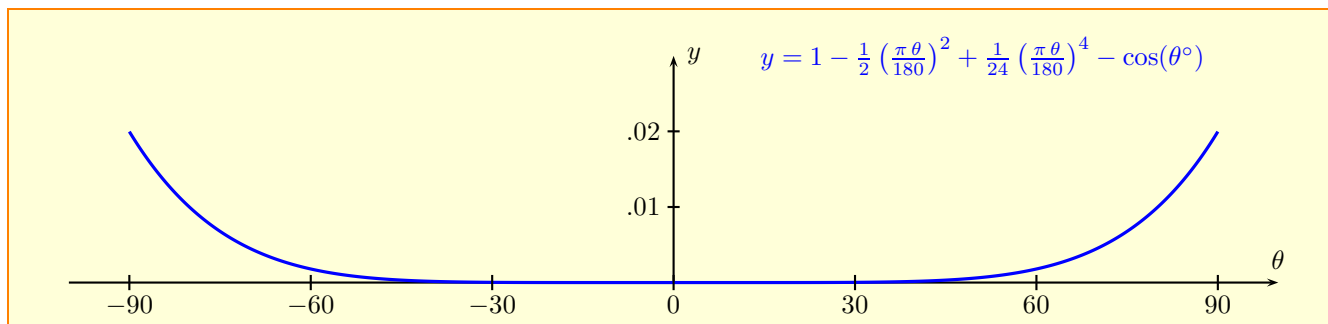
This corresponds to the approximation

$$\cos x \approx 1 - \frac{49}{10\pi^2}x^2 + \frac{18}{5\pi^4}x^4,$$

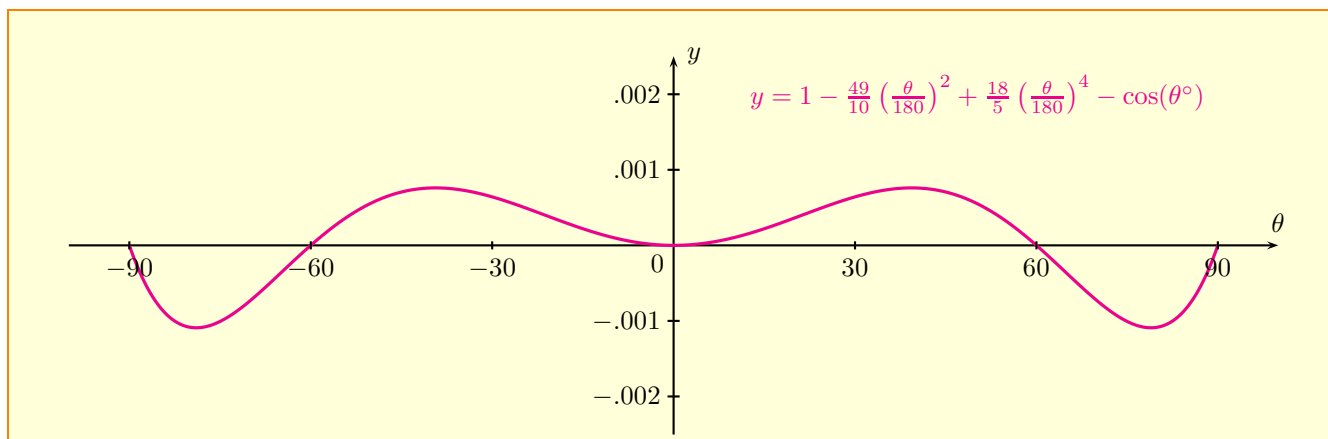
for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. The above approximation is correct for order two if π is assumed to be $\sqrt{2(49/10)} = \sqrt{9.8} \approx 3.1305$, which is closer to the true value of π than $\sqrt{10}$. The 4th order Taylor approximation of the cosine (and the conversion from radians to degrees) gives

$$\cos(\theta^\circ) = \cos\left(\frac{\pi\theta}{180}\right) \approx 1 - \frac{1}{2}\left(\frac{\pi\theta}{180}\right)^2 + \frac{1}{24}\left(\frac{\pi\theta}{180}\right)^4.$$

Graph of the difference of the above 4th order Taylor polynomial and the cosine:



Compare this with the above approximation:



The absolute value of the maximum error over the entire interval is almost 20 times as large for the Taylor approximation.

In spite of the fact that the approximations to π associated with these sine and cosine approximations are lousy at best, it is amazing that the convergence of the functions over the appropriate intervals is so good. Obviously we can get a better approximation by interpolating more values. Noting that Bhaskara's rational approximation and the above polynomial approximation have errors generally in opposite direction (see the graphs of the errors), a better approximation is obtained by taking the average of the two approximations. I will leave it as an exercise for the interested reader to plot the resulting error, which has absolute value less than 0.0003 (the difference between the sine and the approximating function² has maximum less than 0.0002926893092 and minimum more than -0.000272872444).

Approximating the Sine using More Special Values. We can also obtain better approximation formulas by interpolating more values of the sine. For example, using $\sin(60^\circ) = \sqrt{3}/2$, we need the following values for p :

θ	p	y
0	0	0
30	4500	1/2
60	7200	$\sqrt{3}/2$
90	8100	1

The following polynomial in p fits the above data points:

$$\frac{p(p - 4500)(p - 7200)}{8100(8100 - 4500)(8100 - 7200)} \cdot 1 + \frac{p(p - 7200)(p - 8100)}{4500(4500 - 7200)(4500 - 8100)} \cdot \frac{1}{2} + \frac{p(p - 4500)(p - 8100)}{7200(7200 - 4500)(7200 - 8100)} \cdot \frac{\sqrt{3}}{2},$$

which simplifies to

$$\frac{2p(13p - 69300)(p - 7200) - 15p(p - 4500)(p - 8100)\sqrt{3}}{52488000000}.$$

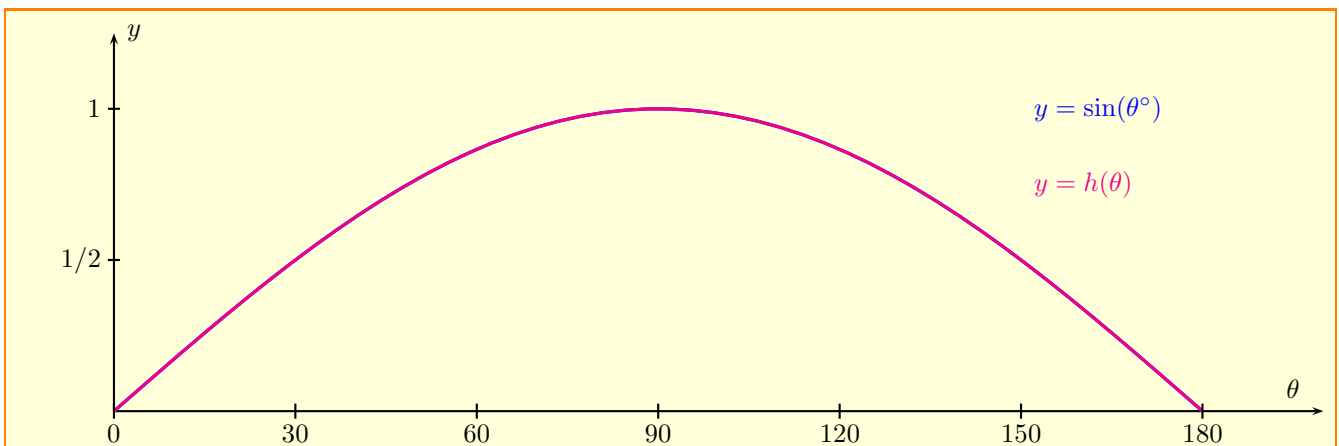
This gives the polynomial

$$h(\theta) = \frac{2\theta(180 - \theta)(13\theta(180 - \theta) - 69300)(\theta(180 - \theta) - 7200) - 15\theta(180 - \theta)(\theta(180 - \theta) - 4500)(\theta(180 - \theta) - 8100)\sqrt{3}}{52488000000},$$

which simplifies to

$$h(\theta) = \frac{(26 - 15\sqrt{3})\theta(180 - \theta)(810000(1657 + 930\sqrt{3}) + 324000(19 + 15\sqrt{3})\theta - 1800(1 + 15\sqrt{3})\theta^2 - 360\theta^3 + \theta^4)}{524880000000}$$

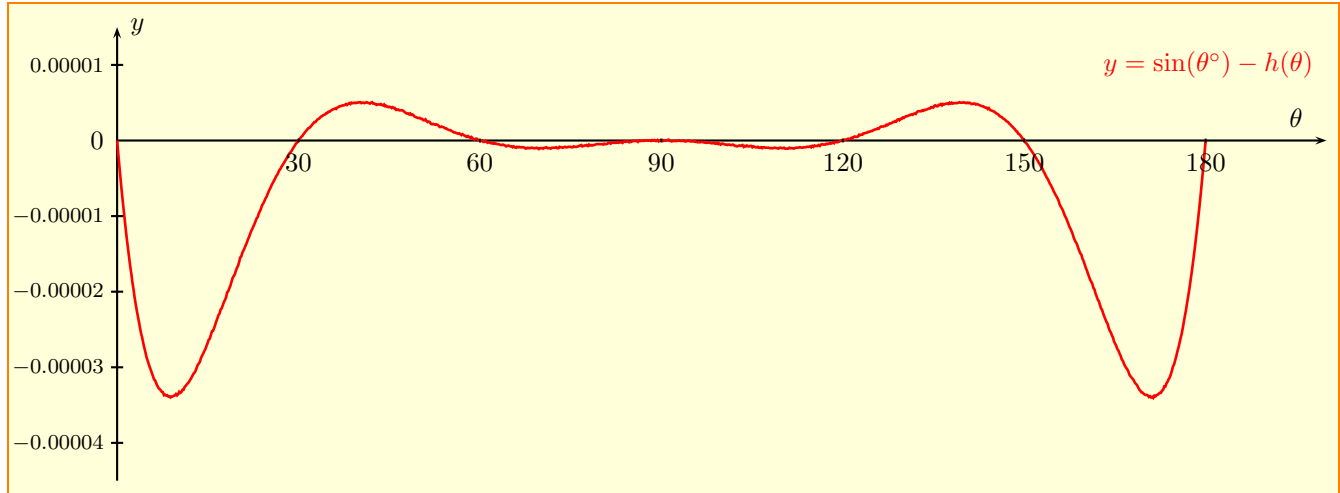
Graph of this polynomial together with the sine over the interval $[0, 180]$:



²The approximating function is

$$\frac{2\theta(180 - \theta)}{40500 - \theta(180 - \theta)} + \frac{31\theta(180 - \theta)}{648000} + \frac{\theta^2(180 - \theta)^2}{583200000}.$$

Since the two graphs are not distinguishable, we plot the difference on a larger scale:



The error in the approximation

$$\sin(\theta^\circ) \approx \frac{(26 - 15\sqrt{3})\theta(180 - \theta)(810000(1657 + 930\sqrt{3}) + 324000(19 + 15\sqrt{3})\theta - 1800(1 + 15\sqrt{3})\theta^2 - 360\theta^3 + \theta^4)}{524880000000}$$

is less than 0.000035 (in fact, the absolute value of the difference of the left-hand side and the right-hand side has absolute maximum of about 0.000033873 for $x \approx 8.859147374$), for all $0 \leq \theta \leq 180$.

Reference

- [1] S. Grover, History of Development of Mathematics in India, Atma Ram & Sons, Delhi and Lucknow, 1994.

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