The development of Calculus in the Kerala School

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Abstract: *The Kerala School of mathematics, founded by Madhava in Southern India, produced many great works in the area of trigonometry during the fifteenth through eighteenth centuries. This paper focuses on Madhava's derivation of the power series for sine and cosine, as well as a series similar to the well-known Taylor Series. The derivations use many calculus related concepts such as summation, rate of change, and interpolation, which suggests that Indian mathematicians had a solid understanding of the basics of calculus long before it was developed in Europe. Other evidence from Indian mathematics up to this point such as interest in infinite series and the use of a base ten decimal system also suggest that it was possible for calculus to have developed in India almost 300 years before its recognized birth in Europe. The issue of whether or not Indian calculus was transported to Europe and influenced European mathematics is not addressed.*

Keywords: Calculus; Madhava; Power series for sine and cosine; Trigonometric series; Kerala School of mathematics; History of mathematics

It is undeniable that the Kerala school of mathematics in India produced some of the greatest mathematical advances not only in India but throughout the world during the fourteenth through seventeenth centuries. Yet many of the advances the scholars and teachers of this school made have been attributed to later European mathematicians who had the abilities to publish and circulate these

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ideas to large audiences. Some of the greatest products of the Kerala school are the series approximations for sine and cosine. The power series of the sine function is sometimes referred to as Newton's Series, and explanations in modern calculus textbooks offer little to no attribution to Madhava. Another series Madhava derived for the sine and cosine functions is similar to the modern Taylor Series, but uses a slightly different approach to find divisor values. The similarities between the various Indian versions of infinite trigonometric series and the later European versions of series lead to the controversial discussion of whether or not the ideas of calculus were transmitted from India to Europe. This paper does not focus on this aspect of the Indian calculus issue. It instead focuses on the techniques Madhava used in developing his infinite series for sine and cosine, and explores how these processes do indeed hint at early versions of calculus.

This examination of Keralese mathematics starts with a brief history of trigonometry, which provides context for many of the functions and ideas used throughout this paper. Then a short look at the mathematical culture of both India and Europe is necessary because it provides background evidence that suggests how calculus developed so much earlier in India. Finally, and most importantly, an exploration of Madhava's derivations for infinite series shows how his and his students' techniques relate to modern calculus concepts.

The history and origins of trigonometry revolve around the science of astronomy; people thousands of years ago noticed the periodicity of the moon cycle and star cycle, and wanted to come up with a way to predict when certain astronomical events would happen. Trigonometry was developed in order to calculate times and positions, and what really established it as a science was the ability it gave people to switch back and forth between angle measures and lengths. Hipparchus is generally said to be the first person to really work with trigonometry as a science. Earlier peoples such as the Egyptians used early trigonometric ideas to calculate the slopes of pyramids, while the Babylonians developed a process of measuring angles based on the rotation of the stars and moon (Van Brummelen, 2009).

Clearly the solar system rotation and the development of a science to monitor this rotation was important to many different cultures. Similarly, there are also mathematical arguments and proofs that are context dependent or situated (Moreno-Armella & Sriraman, 2005).

The work of Ptolemy and Hipparchus gave us the Chord function, which is similar to the modern sine function. It is related to the modern sine function by the equation Crd(2Θ)=2Rsin(Θ). Hipparchus established a method for finding the chord of any arc, and used this to create a table of chord values for every seven and a half degrees. Traditionally a radius of 60 was used since the Greeks used a base 60 number system. But another common way to determine a helpful radius was to divide the circumference into an equal number of parts. Knowing that the total number of degrees in a circle is 360, and that there are 60 minutes in each one of those degrees, there is a total of $360(60) = 21600$ minutes around the circle. Dividing this into parts of size 2π produces about 3438 such parts, and the radius is thus 3438. This value of radius was highly used in Indian mathematics and astronomy because it allowed them to divide the circle into many small, whole number arcs. Indian mathematicians also use angle differences similar to those used by the Greeks in their sine and chord tables (Van Brummelen, 2009).

The similarities between early Greek and Indian trigonometry causes one to think about possible transmissions or communication between the two cultures. Little evidence exists that documents the communication between the two cultures, and there is virtually no evidence that documents any mathematical ideas being transferred. Perhaps a logical explanation is that the cultures may have worked with similar ideas around the same time period (Van Brummelen, 2009). A lack of evidence is also present in the issue of whether calculus was transported from India to Europe, and that is why the problem is not heavily addressed in this paper.

The first use of the Sine function, instead of the Chord function, is found in India. Almost all ancient Indian astronomical works either reference or contain tables of Sine values. The Sine function

was given the name jya-ardha (sometimes ardha-jya) in Sanskrit, meaning "half-chord" (Van Brummelen, 2009, p. 96). This sheds some light on how the function may have developed: "Some early Indian astronomer, repeatedly doubling arcs and halving the resulting chords, must have realized that he could save time simply by tabulating this" (p. 96). This function is still different from the modern sine in that it is greater than the modern sine by a factor of R, the radius of a circle. So jya-ardha(θ)=Rsin(θ). It is now customary to write this jya-ardha as Sin(θ), the capital "S" denoting its difference from the modern function. This is how the Rsin(Θ) will be referred to throughout the paper, while the modern sine function will be referred to as sine or $\sin(\theta)$.

The Kerala school of mathematics, also called the Madhava school, originated in Kerala along the south-west coast of India, and flourished from the 1400' s to the 1700's. This region had a fairly distinct culture because of its location—it was close enough to the coastal trading communities but far enough away so as not to be extremely bothered by political and social events. The main focus of many works from the Madhava school was on trigonometry and other circle properties. This focus stemmed from necessity; Indian astronomers relied heavily on sine tables and trigonometric values to compute the positions of stars and predict when astronomical events such as eclipses would happen. These events were important in scheduling Indian religious holidays, and so more accurate approximations were constantly looked for. Earlier Indian mathematicians had long been interested in ways of computing π and the circumference of a circle, and realized there was no way to produce numbers exactly equal to these values. In the Kerala school, many numerical approximations for π and circumferences were found—some that were accurate up to 17 decimal places—and mathematicians began to realize that using algebraic expressions with a large amount of terms could be used to better approximate values. Thus the interest in series approximations arose, and it was not long before mathematicians began to extend these series into the unknown, that is, into infinity (Divakaran, 2010).

Meanwhile in Europe, the mathematical world was in a period of little activity. Perhaps one of

the last major influential mathematicians before the 1600's was Fibonacci in the early 1200's. Then came the great mathematicians such as Fermat, Newton, and Leibniz, but not until the late 1600's and into the 1700's (Rosenthal, 1951). By this time the Kerala school had already produced almost all the major works it would ever produce, and was on the decline. Also during this time Europe had recently begun to use the base ten decimal system. This system had been in India since at least the third century, partly due to the culture's desire and need to express large numbers in an easy to read way (Plofker, 2009). This decimal place value system makes calculations much easier than with Roman numerals, which were prominent in Europe until the 1500's (Morales, n.d.). The fact that India had this easy to use enumeration system may have helped them advance father with calculations earlier than Europe.

Our discussion on Madhava's ways of finding infinite trigonometric series begins with his derivation of the power series for Sine values. Sankara, one of the students from the Keralese lineage, wrote an explanation of Madhava's process. Since Madhava mainly spoke his teachings instead of writing them, his students would write down Madhava's work and provide explanations. So the derivation of the power series is actually in the words of Sankara, explaining Madhava's methods. To begin with, Sankara uses similar triangles and geometry to explain how the Sine and Cosine values are defined.

(constructed using GeoGebra software)
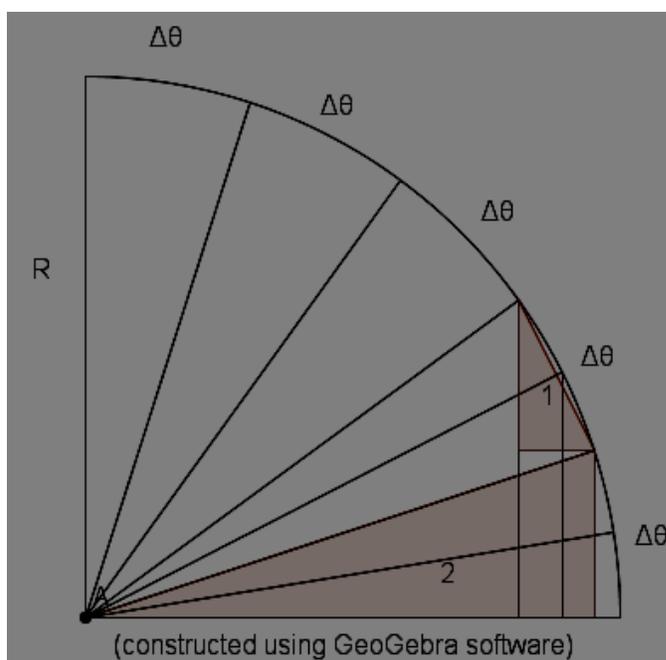
In the following figure, an arc of a circle with radius R is divided into n equal parts. Let the total angle of the arc be θ, so the n equal divisions of this arc are called $\varDelta\theta$. The Sine and Cosine of the ith arc are referred to as Sin_i and Cos_i.

Let the difference between two consecutive Sine values be denoted $\varDelta\text{Sin}_i = \text{Sin}_i - \text{Sin}_{i-1}$, and the difference between two consecutive Cosine values be denoted $\varDelta\text{Cos}_i = \text{Cos}_{i-1} - \text{Cos}_i$. The Sine value of the ith half arc of $\varDelta\Theta$, or $i\varDelta\theta + \frac{(\varDelta\theta)}{2}$, is called $\text{Sin}_{i.5}$, and the Cosine value of $i\varDelta\theta + \frac{(\varDelta\theta)}{2}$ is called $\text{Cos}_{i.5}$.

Then $\varDelta\text{Sin}_{i.5} = \text{Sin}_{i.5} - \text{Sin}_{i.5-1}$, and $\varDelta\text{Cos}_{i.5} = \text{Cos}_{i.5-1} - \text{Cos}_{i.5}$. Reprinting the triangles labeled 1 and 2 from the figure above, we will explore their similarity.

(constructed using GeoGebra software)
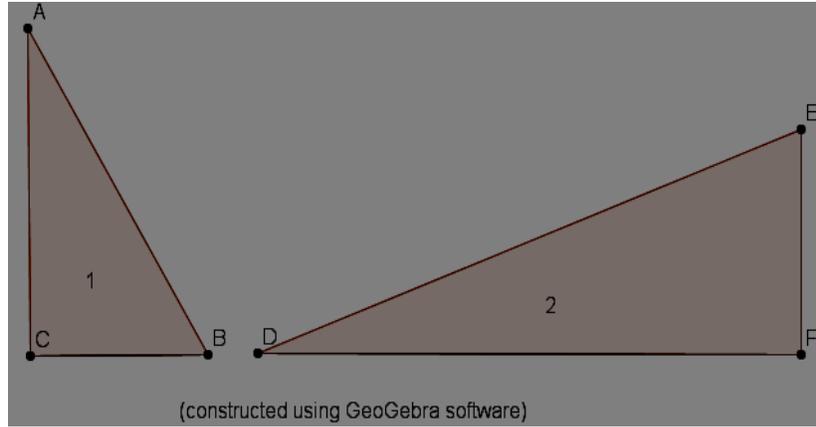
In triangle 1, A is a point that corresponds to the $(i+1)^{st}$ arc and B is the ith arc. Then length AB is the Chord (denoted Crd) of $\Delta\Theta$, and length AC is the difference between Sin_{i+1} and Sin_i which by definition is ΔSin_{i+1}. Side BC is $Cos_i - Cos_{i+1} = \Delta Cos_{i+1}$ (by definition), and angle BAC= $\Delta\Theta$. In triangle DEF, side DF is $Cos_{i.5}$, side EF is $Sin_{i.5}$, and side DE is the radius of the full arc, R. Angle FDE = $\Delta\Theta$, and therefore the two right triangles are similar. So we can set up proportions to write Sines in terms of Cosines and vice versa. So $\frac{AC}{AB} = \frac{DF}{DE} \rightarrow \frac{(\Delta Sin_{i+1})}{(Crd(\Delta\theta))} = \frac{Cos_{i.5}}{R} \rightarrow \Delta Sin_{i+1} = Cos_{i.5}(\frac{(Crd\Delta\theta)}{R})$.

Using the same similar triangle relationship,

$\frac{CB}{AB} = \frac{EF}{DF} \rightarrow \frac{(\Delta Cos_{i+1})}{(Crd\Delta\theta)} = \frac{Sin_{i.5}}{R} \rightarrow \Delta Cos_{i+1} = Sin_{i.5}\frac{(Crd\Delta\theta)}{R}$.

To get expressions for the $i.5^{th}$ Sine and Cosine, the similar triangles used above need to be shifted on the arc. Let point A now denote the $i.5^{th}$ arc and B is the $(i.5-1)^{st}$ arc. Then length AB is still the Chord of $\Delta\Theta$, while AC is now $\Delta Sin_{i.5}$ and CB is now $\Delta Cos_{i.5}$. In triangle DEF, length DF becomes Cos_i, and EF becomes Sin_i, while DE is still the radius R. So, using the above proportions,

$\Delta Sin_{i.5} = Cos_i \frac{(Crd\Delta\theta)}{R}$, and

$\Delta Cos_{i.5} = Sin_i \frac{(Crd\Delta\theta)}{R}$.

Now Sankara defines the second difference of Sine as: $\Delta\Delta Sin_i = \Delta Sin_i - \Delta Sin_{i+1}$.

So, using the expressions from above,

$$\Delta\Delta\mathrm{Sin}_i = \Delta\mathrm{Sin}_i - \Delta\mathrm{Sin}_{i+1} = \Delta\mathrm{Sin}_i - \mathrm{Cos}_{i.5}\frac{(Crd\Delta\theta)}{R}$$

$$= \Delta\mathrm{Sin}_{i+1-1} - \mathrm{Cos}_{i.5}\frac{(Crd\Delta\theta)}{R}$$

$$= \mathrm{Cos}_{i.5-1}\frac{(Crd\Delta\theta)}{R} - \mathrm{Cos}_{i.5}\frac{(Crd\Delta\theta)}{R}$$

$$= \Delta\mathrm{Cos}_{i.5}\frac{(Crd\Delta\theta)}{R}$$

$$= \mathrm{Sin}_i\frac{(Crd\Delta\theta)^2}{R^2}$$

Since Sankara has defined the second difference of some i[th] arc as the difference between two consecutive Sine differences, the difference of the (i+1)[st] Sine can be written as:

$\Delta\mathrm{Sin}_{i+1} = \Delta\mathrm{Sin}_i - \Delta\Delta\mathrm{Sin}_i$, the difference between the first order and second order differences of the ith Sine. By this recursive formula, we can write

$$\Delta\mathrm{Sin}_{i+1}$$
$$= \Delta\mathrm{Sin}_{i-1} - \Delta\Delta\mathrm{Sin}_{i-1} - \Delta\Delta\mathrm{Sin}_i$$
$$= \Delta\mathrm{Sin}_{i-2} - \Delta\Delta\mathrm{Sin}_{i-2} - \Delta\Delta\mathrm{Sin}_{i-1} - \Delta\Delta\mathrm{Sin}_i$$
$$= \vdots$$
$$= \Delta\mathrm{Sin}_1 - \Delta\Delta\mathrm{Sin}_1 - \Delta\Delta\mathrm{Sin}_2 - \Delta\Delta\mathrm{Sin}_3 - \cdots - \Delta\Delta\mathrm{Sin}_{i(1)}$$
$$= \Delta\mathrm{Sin}_1 - \sum_{k=1}^{i}\Delta\Delta\mathrm{Sin}_k$$
$$= \Delta\mathrm{Sin}_1 - \sum_{k=1}^{i}\mathrm{Sin}_k\frac{(Crd\Delta\theta)^2}{R^2}$$
$$= \Delta\mathrm{Sin}_1 - \sum_{k=1}^{i}(\Delta\mathrm{Cos}_{k.5}\frac{(Crd\Delta\theta)}{R})$$

Here is where the a function called the Versine or "reversed sine" (Van Brummelen,2009, p. 96) comes in. The Versine is defined as $\mathrm{Vers}(\theta) = R - \mathrm{Cos}(\theta)$, where R is the radius of the circle. So for our purposes of calculating Sine and Cosine values related to the ith arc, $\mathrm{Vers}_i = R - \mathrm{Cos}_i$, and the difference in Versine values is $\Delta\mathrm{Vers}_i = \mathrm{Vers}_i - \mathrm{Vers}_{i-1}$. Since $\mathrm{Vers}_i = R - \mathrm{Cos}_i$, $\Delta\mathrm{Vers}_i = \Delta\mathrm{Cos}_i$. Now we will progress with the Sine approximation. Noting equation (1), we can write the difference of the ith Sine (as opposed to the (i+1)[st] Sine) as

$$\Delta\mathrm{Sin}_i = \Delta\mathrm{Sin}_1 - \sum_{k=1}^{i-1}(\Delta\mathrm{Cos}_{k.5}\frac{(Crd\Delta\theta)}{R})(2).$$

Now we will write an even more detailed expression for Sin_i into which we can substitute previously found expressions and differences. Since $\Delta Sin_i = Sin_i - Sin_{i-1}$, $\Delta Sin_1 = Sin_1 - Sin_0$, $\Delta Sin_2 = Sin_2 - Sin_1$, $\Delta Sin_3 = Sin_3 - Sin_2$, ..., $\Delta Sin_i = Sin_i - Sin_{i-1}$. Then adding the right hand sides of these identities gives us:

$(Sin_1 - Sin_0) + (Sin_2 - Sin_1) + (Sin_3 - Sin_2) + \cdots + (Sin_i - Sin_{i-1})$. The $Sin_0 = 0$, and the subsequent (i-1) terms cancel out, leaving only Sin_i. Then adding the values on the left hand side gives us $\Delta Sin_1 + \Delta Sin_2 + \Delta Sin_3 + \cdots + \Delta Sin_i$. Thus $Sin_i = \Delta Sin_1 + \Delta Sin_2 + \Delta Sin_3 + \cdots + \Delta Sin_i$ (3).

Rewriting (3) produces $Sin_i = \sum_{k=1}^{i} \Delta Sin_k$. So substituting equation (2) in for ΔSin_k gives:

$$Sin_i = \sum_{k=1}^{i}(\Delta Sin_1 - \sum_{k=1}^{i-1} \Delta Cos_{k.5} \frac{(Crd\Delta\theta)}{R})$$

$$= \sum_{k=1}^{i} \Delta Sin_1 - \sum_{k=1}^{i-1}(\sum_{j=1}^{k} \Delta Cos_{j.5} \frac{(Crd\Delta\theta)}{R})$$

Since ΔSin_1 is a value independent of the index k=1 to i, we can apply the constant rule of summations to get $i\Delta Sin_1$. Then

$$Sin_i = i\Delta Sin_1 - \sum_{k=1}^{i-1}(\sum_{j=1}^{k} \Delta Cos_{j.5} \frac{(Crd\Delta\theta)}{R})$$
$$= i\Delta Sin_1 - \sum_{k=1}^{i-1}(\sum_{j=1}^{k} Sin_j \frac{(Crd\Delta\theta)^2}{R^2}) \quad (4).$$

From here, $i\Delta Sin_1$ is taken as approximately equal to $i\Delta\Theta$, and the arc $\Delta\Theta$ itself is approximated as the "unit arc", making $i\Delta\Theta$ just i. Additionally, the (i-1) sums of the sum of Sines are approximately equal to the i sums of the sum of the arcs. This implies that the (i-1) sum of the sum of the Sines is approximately equal to the "i" sum of the sum of i. So (4) now becomes

$$Sin_i \approx i\Delta\theta - \sum_{k=1}^{i}(\sum_{j=1}^{k} j\Delta\theta \frac{(Crd\Delta\theta)^2}{R^2})$$
$$\sim i - \sum_{k=1}^{i}(\sum_{j=1}^{k} j \frac{(Crd\Delta\theta)^2}{R^2}) \quad (5).$$

Here Sankara also approximates $Crd\Delta\theta$ as 1, since $\Delta\theta \approx 1$. He also substitutes the difference in Cosines back into (5) for j, and substitutes Versine differences for those Cosine differences, getting a

simplified expression for Sin$_i$. Here is a translation of his explanation for these steps:

> Therefore the sum of Sines is assumed from the sum of the numbers having one as their first term and common difference. That, multiplied by the Chord [between] the arc-junctures, is divided by the Radius. The quotient should be the sum of the differences of the Cosines drawn to the centers of those arcs. […] The other sum of Cosine differences, [those] produced to the arc-junctures, [is] the Versine. [But] the two are approximately equal, considering the minuteness of of the arc-division (Plofker, 2009, p. 242).

Now (5) becomes:

$$\text{Sin}_i \approx i - \sum_{k=1}^{i}(\sum_{j=1}^{k} \Delta\text{Cos}_j \frac{(Crd\Delta\theta)}{R^2})$$
$$\sim i - \sum_{k=1}^{i}(\sum_{j=1}^{k} \Delta\text{Vers}_j \frac{(Crd\Delta\theta)}{R^2}) \quad (6)$$
$$= i - \sum_{j=1}^{i} \text{Vers}_j (\frac{1}{R})$$

Sankara then states the rule for sums of powers of integers, and uses that to approximate $\sum_{k=1}^{i-1}(\sum_{j=1}^{k} j(\frac{1}{R^2}))$. If we think about nested sums as products instead, and there are a of these products, and b is the last term (in the a$^{\text{th}}$ sum), then:

$$\sum_{c_a=1}^{b} c_a * \sum_{c_{a-1}=1}^{c_a} c_{a-1} * \cdots * \sum_{c_2=1}^{c_3} c_2 * \sum_{c_1=1}^{c_2} c_1 =$$
$$\frac{[b(b+1)(b+2)\cdots(b+a)]}{(1*2*\cdots*(a+1))} = \frac{((a+b)!)}{((b-1)!(a+1)!)} \qquad \text{(sum of powers of integers rule)}.$$

Using this,

$$\sum_{j_1=1}^{i} j_1 = \frac{(i(i+1))}{(1*2)}$$
$$\sum_{j_2=1}^{i} \sum_{j_1=1}^{j_2} j_1 = \frac{(i(i+1)(i+2))}{(1*2*3)} \qquad .$$
$$\sum_{j_3=1}^{i} \sum_{j_2=1}^{j_3} \sum_{j_1=1}^{j_2} j_1 = \frac{(i(i+1)(i+2)(i+3))}{(1*2*3*4)}$$

These values are further approximated as $\frac{i^2}{2}, \frac{i^3}{6}$, and $\frac{i^4}{24}$, respectively.

Now, using (6) we can rewrite to get:

$$i - \text{Sin}_i \approx \sum_{j=1}^{i} \text{Vers}_j (\frac{1}{R}) \approx \sum_{k=1}^{i} j(\frac{1}{R}) \approx \sum_{j=1}^{i} \frac{j^2}{2} (\frac{1}{R}) \approx \frac{i^3}{6} (\frac{1}{R^2})(7).$$

Here Sankara notices the error in estimating the Versine values based on arcs, and formulates a way to account for the error:

> To remove the inaccuracy [resulting] from producing [the Sine and Versine expressions] from a sum of arcs [instead of Sines], in just this way one should determine the difference of the [other] Sine and [their] arcs, beginning with the next-to-last. And subtract that [difference each] from its arc: [those] are the Sines of each [arc]. Or else therefore, one should subtract the sum of the differences of the Sines and arcs from the sum of the arcs. Thence should be the sum of the Sines. From that, as before, determine the sum of the Versine-differences (Plofker, 2009, p. 245).

So,

$$\text{Vers}_i \approx \sum_{j=1}^{i} j - \frac{j^3}{(6R^2)} \left(\frac{1}{R}\right) = \sum_{j=1}^{i} \frac{j}{R} - \sum_{j=1}^{i} \frac{j^3}{(6R^3)} \approx \frac{i^2}{(2R)} - \frac{i^4}{(24R^3)}.$$

Putting this estimate in for Vers_i in equation (6) then gives us:

$$\text{Sin}_i \approx i - \sum_{j=1}^{i} \left(\frac{j^2}{(2R)} - \frac{j^4}{(24R^3)}\right) \frac{1}{R}$$

$$= i - \sum_{j=1}^{i} \frac{j^2}{2R^2} + \sum_{j=1}^{i} \frac{j^4}{24R^4} \approx i - \frac{i^3}{6} + \frac{i^5}{120} \quad (8)$$

$$= i - \frac{i^3}{(3!R^2)} + \frac{i^5}{(5!R^4)}$$

which is equivalent to the modern power series $\sin(\theta) \approx \theta - \frac{\theta^3}{(3!)} + \frac{\theta^5}{(5!)} - + \cdots$, noting that the modern $\sin(\theta)$ is equal to $\frac{1}{R}\text{Sin}(\theta)$. Subsequent terms of the series are found by using (8) to find more accurate Versine values, which are then used to find more accurate Sine values.

Madhava's ability to produce highly accurate and lengthy sine tables stemmed from his series approximation; once he had derived the first few terms of the series (as shown above), he could compute later terms fairly quickly. He recorded the values of 24 sines in a type of notation called katapayadi, in which numerical values are assigned to the Sanskrit letters. Significant numerical values were actually inscribed in Sanskrit words, but only the letters directly in front of vowels have

numerical meaning, and only the numbers 0 through 9 are used in various patterns. So while the phrases Madhava used in his katapayadi sine table may not make sense as a whole, each line contains two consecutive sine values accurate to about seven decimal places today (Van Brummelen, 2009; Plofker, 2009).

The above derivation of the power series showcases some of the brilliant and clever work of the Kerala school, especially considering that similar work in Europe did not appear until at least 200 years later when Newton and Gregory began work on infinite series in the late 1600's (Rosenthal, 1951). Important steps to note take place in the beginning of the derivation when Madhava uses similar triangle relationships to express the changes in Sine and Cosine values using their counterparts. This process is equivalent to taking the derivative of the functions, because the derivative virtually measures the same rate of change as these expressions. Additionally, the repeated summation Madhava uses can be considered a precursor to integration. The integral takes some function, say B, and divides it into infinitely many sections of infinitely small width. These sections are then summed and used to find the value of function A, whose rate of change was expressed through function B. Madhava uses the summation of extremely small divisions of changes in Sine, Cosine, and arc values to get better estimates for the original functions, which is what the integral is designed to do.

Now we turn to the derivation of another infinite series and examine the background of interpolation in India. The process of interpolation has been known in India since about the beginning of the seventh century when mathematician Brahmagupta wrote out his rules for estimating a function using given or known values of that function (Gupta, 1969). This is the process Madhava evidently used in his derivation of a series for Sine and Cosine around a fixed point, often times called the Taylor Series. The Kerala student Nilakantha quoted Madhava's rule for using second order interpolation to find such a series:

Placing the [sine and cosine] chords nearest to the arc whose sine and cosine chords are

required get the arc difference to be subtracted or added. For making the correction 13751 should be divided by twice the arc difference in minutes and the quotient is to be placed as the divisor. Divide the one [say sine] by this [divisor] and add to or subtract from the other [cosine] according as the arc difference is to be added or subtracted. Double this [result] and do as before [i.e. divide by the divisor]. Add or subtract the result to or from the first sine or cosine to get the desired sine or cosine chords (Gupta, 1969, p. 93).

Here, a radius of 3438 is used, and the "arc difference" referred to is the angle in between the known Sine (or Cosine) value and the desired value. To find the general version of this estimation rather than using specific known and desired values, call the arc difference $\Delta\theta$, so twice the arc difference is $2\Delta\theta$. The value 13751 used in the determination of the "divisor" is four times the radius (3438), making the divisor simply $\frac{2*3438}{(\Delta\theta)}$ or $\frac{2R}{(\Delta\theta)}$. After establishing the divisor, Madhava says to divide the desired function by this value—which is equivalent to multiplying by $\frac{(\Delta\theta)}{2R}$--and add that to or subtract that from the other function. So, say we wanted to know the value of $Sin(\theta + \Delta\theta)$ where θ is the known value and $\Delta\theta$ is the arc difference being added to the known value. First we would divide the known $Sin(\theta)$ by the divisor, which produces $Sin(\theta)\frac{(\Delta\theta)}{2R}$. This is then added to or subtracted from the corresponding Cosine value, $Cos(\theta)$. In this case, $Sin(\theta)\frac{(\Delta\theta)}{2R}$ is subtracted from $Cos(\theta)$. The reason for this is best demonstrated with a diagram.

(Constructed using GeoGebra software)
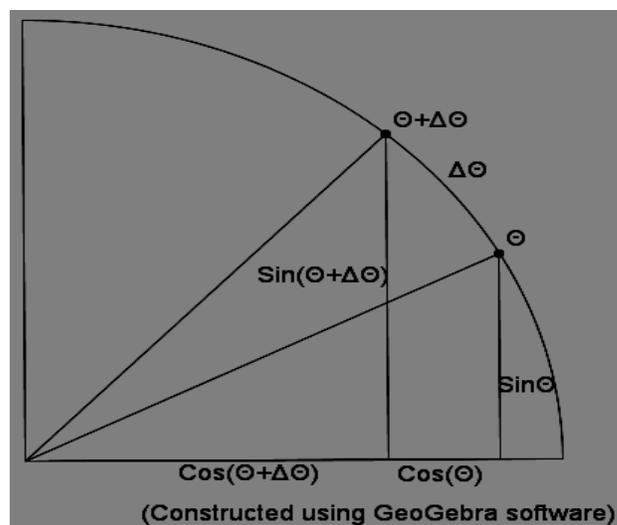
Here, $\text{Sin}(\theta + \varDelta\theta)$ is greater than $\text{Sin}(\theta)$, as indicated by the difference in lengths in the diagram. However, $\text{Cos}(\theta + \varDelta\theta)$ is less than $\text{Cos}(\theta)$, which is also visible in the differences in lengths in the diagram. The rules given by Madhava say to "add to or subtract from the other according as the arc difference is to be added or subtracted" (Gupta, 1969, p. 93), which means that we should subtract $\text{Sin}(\theta)\frac{(\varDelta\theta)}{2R}$ from $\text{Cos}(\theta)$ if $\text{Cos}(\theta + \varDelta\theta)$ is less than $\text{Cos}(\theta)$, and add $\text{Sin}(\theta)\frac{(\varDelta\theta)}{2R}$ if the opposite is true. Since in this case $\text{Cos}(\theta + \varDelta\theta)$ is less than $\text{Cos}(\theta)$, we subtract the stated Sine value. From there, Madhava says to double the expression we have just found, and multiply again by $\frac{(\varDelta\theta)}{2R}$. The last step is to add (or subtract, if a Cosine value is desired) the entire expression from the starting Sine or Cosine value. Following these steps produces the expressions

$$\text{Sin}(\theta + \varDelta\theta) = \text{Sin}(\theta) + \frac{(\varDelta\theta)}{R}\text{Cos}(\theta) - \frac{(\varDelta\theta)^2}{2R^2}\text{Sin}(\theta) \ (1.1) \text{ and}$$

$$\text{Cos}(\theta + \varDelta\theta) = \text{Cos}(\theta) - \frac{(\varDelta\theta)}{R}\text{Sin}(\theta) - \frac{(\varDelta\theta)^2}{2R^2}\text{Cos}(\theta) (1.2) \text{ for Sine and Cosine values, respectively.}$$

Noting that $\varDelta\theta$ is the point about which the series is being evaluated, and also that the appearance of the R in the denominator is due to the difference in definition of Sine and Cosine, it is easy to see that these expressions are the same as their modern Taylor Series approximations. However, as the order increases, the two expressions begin to differ. The third order Taylor Series approximation

for sine becomes

$$\sin(\theta + \Delta\theta) = \sin(\theta) + \cos(\theta)\Delta\theta - \frac{(\sin\theta)}{(2!)}(\Delta\theta)^2 - \frac{(\cos\theta)}{(3!)}(\Delta\theta)^3 (1.3),$$ while   Madhava's

series becomes $\mathrm{Sin}(\theta + \Delta\theta) = \mathrm{Sin}(\theta) + \mathrm{Cos}(\theta)\frac{(\Delta\theta)}{R} - \mathrm{Sin}(\theta)\frac{(\Delta\theta)^2}{2R^2} - \mathrm{Cos}(\theta)\frac{(\Delta\theta)^3}{4R^3}(1.4).$

An extension of the above rules given by Madhava are found in works by a student named

Paramesvara, and explain the process for computing later terms of the series:

> Now this [further] method is set forth. […] Subtract from the Cosine half the quotient from
>
> dividing by the divisor the Sine added to half the quotient from dividing the Cosine by the
>
> divisor. Divide that [difference] by the divisor; the quotient becomes the corrected Sine-portion
>
> (Plofker, 2001, p. 286).

In these rules, the divisor is taken as $\frac{(\Delta\theta)}{R}$, so half the divisor is simply $\frac{(\Delta\theta)}{2R}$, as used in the above

calculations. The Sine-portion also discussed here is the difference between the known Sine value and

the desired Sine value, or $\mathrm{Sin}(\theta + \Delta\theta) - \mathrm{Sin}(\theta)$. Therefore little manipulation is needed to get an

expression for solely the desired Sine value, $\mathrm{Sin}(\theta + \Delta\theta)$. In short, the above rules say to add the

known Sine value, $\mathrm{Sin}(\theta)$to the known Cosine value multiplied by$\frac{(\Delta\theta)}{2R}$. This produces the expression

$\mathrm{Sin}(\theta) + \frac{(\Delta\theta)}{2R}\mathrm{Cos}(\theta)$, which is then multiplied by $\frac{(\Delta\theta)}{2R}$ and subtracted from $\mathrm{Cos}(\theta)$. The entire

expression is then multiplied by $\frac{(\Delta\theta)}{R}$, producing:

$\mathrm{Sin}(\theta + \Delta\theta) - \mathrm{Sin}(\theta) = \frac{(\Delta\theta)}{R}[\mathrm{Cos}(\theta) - \mathrm{Sin}(\theta)\frac{(\Delta\theta)}{2R} - \mathrm{Cos}(\theta)\frac{(\Delta\theta)^2}{4R^2}].$ Then rewriting and simplifying

reduces this expression to:

$\mathrm{Sin}(\theta + \Delta\theta) = \mathrm{Sin}(\theta) + \mathrm{Cos}(\theta)\frac{(\Delta\theta)}{R} - \mathrm{Sin}(\theta)\frac{(\Delta\theta)^2}{2R^2} - \mathrm{Cos}(\theta)\frac{(\Delta\theta)^3}{4R^3},$ the same as equation (1.4) above.

Again, conversion to the modern sine accounts for the R in the denominator, and one can see how

similar this expression is to the Taylor Series expression (1.3). The reason for the difference in denominator in the fourth term is due to the difference in computing the denominators, or correction terms. The Taylor Series relies on the general algorithm of differentiating each term and dividing by the factorialized order of derivative for that term. This is why the fourth term of the Taylor series of $\sin(\theta)$ is a 3!, because the third derivative of $\sin(\theta)$ is computed. Madhava's derivation of this series relies on the process of multiplying each term by the set correction value $\frac{(\Delta\theta)}{2R}$. Therefore applying Madhava's correction term three times produces a denominator of $2^3=8$, and the multiplication of this term by two produces four as the divisor.

Although the processes used to find the denominators in the Taylor and Madhava series are different, the fundamental idea of defining a value using other known functional values is the same. The Taylor Series algorithm allows one to define a complex value or function using a series of successive derivatives, which express the rate of change of the function prior to it in the series. Although Madhava used similar triangle relationships to express the change in one trigonometric function in terms of the other, he still nonetheless used the idea of differentiation. The general goal of the derivative is to measure the instantaneous rate of change of some function, which is what Madhava was doing when he stated that $\Delta\mathrm{Sin}_{i+1} = \mathrm{Cos}_{i.5}\frac{(Crd\Delta\theta)}{R}$ from the previously demonstrated derivation of the power series. These similar triangle relationships and special properties of trigonometric functions allowed Madhava to define changes in Sine values in terms of known Cosine values, and vice versa.

The above accounts of Madhava's techniques for approximating trigonometric functions expose two key ideas about the Kerala school. One is that the members had an exceptional understanding of trigonometry and circular geometry, and the second is that their works contain many of the fundamental aspects of calculus. The longstanding interest Indian mathematicians had in finding infinite representations of complicated values and in finding better approximations of trigonometric

functions helped set the stage for the major advancements Madhava and his followers made. Yet not only did this mathematical culture prime the Kerala school for progress in trigonometry, but it almost made it necessary for scholars and teachers to come up with new strategies to approximate the heavily used trigonometric functions. This promotes the idea that mathematical advancements are made based on what is needed at the time. In Europe, the need to create better navigation and calendar techniques prompted the invention of calculus in Europe, but not until the in the 1600's. But Indian astronomers needed new and more accurate ways to compute Sine tables in the 1400's, which explains why Indian calculus was developed so early. This also explains why it stays within trigonometry, because Indian mathematicians really had no use of generalizing the concepts of calculus for non-trigonometric functions. Perhaps this is why the Indian invention of calculus is rarely recognized even though it predates European calculus by 200-300 years. Whatever the reason for this lack of recognition, it is easy to see how the brilliant techniques used by Indian mathematicians are indeed the techniques of calculus applied to the very specific area of trigonometry.

References

Divakaran, P.P. (2010). Calculus in India: The historical and mathematical context. *Current Science, 99*(3), 293-299.

Gupta, R.C. (1969). Second order interpolation in Indian mathematics up to the fifteenth century. *Indian Journal of History of Science, 4*(1), 86-98.

Mallayya, V. M. (2011). The Indian mathematical tradition with special reference to Kerala: Methodology and motivation. In B.S. Yadav Editor & M. Mohan Editor (Eds.), *Ancient Indian leaps into mathematics* (153-170). New York, NY: Springer.

Morales, L. (n.d.). *Historical counting systems.* Retrieved from http://pierce.ctc.edu

Moreno-Armella, L., & Sriraman, B. (2005). Structural stability and dynamic geometry: Some ideas on situated proofs. *ZDM- The International Journal on Mathematics education*, 37(3), 130-139.

Pearce, I.G. (2002). *Indian mathematics: Redressing the balance.* Retreived from MacTutor History of Mathematics Archive. http://www.history.mcs.st-andrews.ac.uk/index.html

Plofker, K. (2009). Mathematics in India. Princeton, NJ: Princeton University Press.

Plofker, K. (2001). The "error" in the Indian "Taylor series approximation" to the sine. *Historia Mathematica, 28*, 283-295.

Rosenthal, A. (1951). The history of calculus. *American Mathematical Monthly, 58*(2), 75-86.

Van Brummelen, G. (2009). The mathematics of heaven and the earth: The early history of trigonometry. Princeton, NJ: Princeton University Press.