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Pursuing Coherence among Proportionality, Linearity, and Similarity: Two Pathways from Pre-service Teachers' Geometric Representations*

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Abstract: The importance of using multiple representations of a mathematical concept and connecting the representations has been discussed in learning and teaching mathematics. The Common Core State Standards further the discussion with an emphasis on focus and coherence in teaching mathematical concepts across grades. Preservice teachers in our problem solving class were asked to use geometric representations to solve a problem that required proportional reasoning. They were also asked to sequence the works of their peers as well as their own from a developmental perspective. Sequencing geometric representations with various levels was challenging because it required showing a coherent understanding of proportionality, linearity, and similarity. In this article, we present two pathways of developing proportional reasoning and discuss how proportionality, linearity, and similarity can be developed coherently. We also discuss the significance of engaging preservice teachers in others' thinking and having them sequence others' works in the journey of pursuing focus and coherence in teaching.

Keywords: coherence, proportional reasoning, geometric representations, developmental perspective.

Introduction

Arithmetic and geometry, which have their own code, means, and symbol system, are complementary (Otte, 1990). Mathematical objects disclose their essence in different forms. A single form of representation does not represent the essence of a mathematical object comprehensively. Multiple representations of a mathematical concept help students build a rich connection around the concept and develop insights into the concept (Even & Lappan, 1994). Thus instruction ought to be designed to allow students to create and use various representations and relate them (NCTM, 2000). For instance, students should be encouraged to present their understanding of a proportional relationship not only numerically (e.g. a/b=c/d) but also geometrically (e.g. a straight line passing through the origin or similar triangles).

It is also important to help students recognize connections among mathematical ideas. Building a connection among different ideas and topics aids the development of mathematical maturity (Lester et al., 1994). Away from the traditional view of mathematics as a set of isolated facts and procedures that causes difficulties in learning mathematics (Carpenter & Lehrer, 1999; Hiebert, 2003), students should

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learn how various mathematical ideas interconnect and build on one another and thus develop a coherent understanding (Common Core State Standards Initiative (CCSSI), 2010). When it comes to proportion, it is imperative to help students understand proportionality, linearity, and similarity as a coherent whole. Isolating proportionality from other subjects and lacking the visualization of it prevent students from seeing proportionality as a concept that connects many topics they learn (Streefland, 1985).

If preservice teachers do not see proportionality and its related geometric ideas as a coherent whole, they would have little chance to guide their future students toward a comprehensive understanding of proportion. With that in mind, we asked preservice teachers in our problem solving class to use geometric representations to solve a problem that requires proportional reasoning. The students' representations varied in terms of the degree of sophistication of their thinking. We also asked them to sequence different representations of their peers' as well as their own from a developmental perspective. In this article, we describe how our task helped preservice teachers be aware of (1) the importance of seeing proportionality and its related ideas as a whole and (2) the significance of sequencing the works of others and their own from a developmental perspective.

Proportionality, Linearity, and Similarity

Proportional reasoning includes comparing ratios or establishing an equivalent relationship between ratios (Tourniaire & Pulos, 1985). It has played an important role in the development of mathematics in history (Radford, 1996). In school mathematics, proportion is the capstone of the elementary school curriculum and the cornerstone of algebra and beyond (Lesh, Post, & Behr, 1988). It is also considered a unifying theme in a sense that it involves using numbers, graphs, and equations to think about quantities and their relationships (NCTM, 2000).

Among the connections between algebraic and geometric aspects of proportional reasoning, linearity that presents a common ratio to make a line passing through the origin is particularly essential (Karplus, Pulos, & Stage, 1983; NCTM, 2000). Those who think proportionally have a sense of covariation so that they can analyze the quantities that vary together and determine the relationship that remains unchanged (Lamon, 1999). A proportional relationship between two quantities can be formalized using an equation y=kx, which is represented geometrically with a straight line through the origin (see Figure 1).

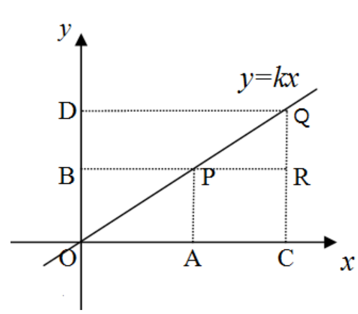


Figure 1. Proportionality and linearity

Proportionality and similarity have a deep connection in nature. In Book VI of *Elements*, Euclid defined similarity based on proportion (Heath, 1956). Similar triangles visualize a proportional relationship. In Figure 1, ΔOAP, ΔOCQ, and ΔPRQ are similar and constitute a line through the origin that represents the proportionality between x and y. As there is a straight line where a proportion is involved, there are similar triangles where a straight line is drawn. Formally, similarity is defined as a conformal isometry, which is an isometry of a metric space with itself equipped with a rescaled metric (Givental, 2007). The ratio of the rescaled unit to the original unit determines all the proportional relationships in similar figures (Clairaut, 1741; Freudenthal, 1983).

It has been reported that students hardly develop the comprehensive understanding of proportionality, linearity, and similarity (De Bock, Van Dooren, Janssens, & Verschaffel, 2002; De Bock, Verschaffel, & Janssens 1998; Hart, 1984; Van Dooren, De Bock , Depaepe , Janssens, & Verschaffel, 2003; Vollrath, 1977). In our problem solving class for preservice teachers, we noticed that their understanding of proportionality and its related geometric ideas is one-way or disconnected rather than comprehensive or unified. Given a proportion problem, they could set up an equation a/b=c/d and solve it by the cross-multiply rule. However, most students had difficulty using geometric representations to solve a proportion problem when asked to do so. If students are constrained by one way when solving proportion problems, they may have little chance to understand proportionality as a concept involving linearity or similarity.

Task, Constraint, and Expectations

Problem and Constraint

The coffee problem below was posed to preservice teachers in our problem solving class. They were expected to demonstrate their solutions using dynamic features of Geometer's Sketchpad (GSP):

> [Coffee Problem]
>
> Ali bought 2lbs of her favorite mixture of French Vanilla and Columbian Supreme coffee. Amanda bought what she thought was 1lb of Ali’s favorite mixture and combined it with Ali’s 2lbs. It turns out Amanda’s mixture was not Ali’s favorite. Ali had to mix an additional 3/4 lb of French Vanilla with the entire mixture to make it perfect. What percentage or fraction of Ali’s favorite mixture is French Vanilla and Columbian Supreme, and what percentage or fraction of Amanda’s 1 lb mixture is each coffee type?

The coffee problem could be solved by setting up a proportion $x/2 = (y+¾) / 1¾$ where x stands for the amount of French Vanilla in Ali’s 2lbs mixture and y the amount of French Vanilla in Amanda’s 1 lb mixture. Yet, the constraint of using geometric representations to solve the problem did not allow simply setting up an equation and using the cross-multiply rule. As they engaged in the coffee problem with the constraint, our preservice teachers experienced the process of learning through confusion or frustration. This kind of experience has been found to help teachers be better able to provide guidance to their students (Shifter & Szymaszek, 2003).

Expectations of the Problem

The coffee problem encouraged preservice teachers to think about a situation where a non-proportional relationship and a proportional relationship are related. It is crucial in the development of proportional reasoning to develop an ability to differentiate proportional situations from non-proportional situations. Differentiating situations in terms of proportionality requires an ability to compare ratios.

Recognizing two different ways to compare ratios, *within ratios* and *between ratios*, helps students identify what the ratios being compared represent. The *within ratio* is a ratio of two measures within a situation, whereas the *between ratio* is a ratio of corresponding measures between different situations (Van de Walle, Karp, & Bay-Williams, 2010). In the posed situation, CS_{2lb}:FV_{2lb} and $CS_{1¾lb}$:$FV_{1¾lb}$ are within ratios, and FV_{2lb}:$FV_{1¾lb}$ and CS_{2lb}: $CS_{1¾lb}$ are between ratios, where FV_{xlb} is the amount of French Vanilla in x lbs mixture and CS_{xlb} the amount of Columbian Supreme in x lbs mixture. Our preservice teachers were expected to show flexibility in using within ratios and between ratios and interpret the ratios in a meaningful way in the problem context. They were also expected to demonstrate an understanding of linearity as they investigated relationships among the quantities using geometric representations.

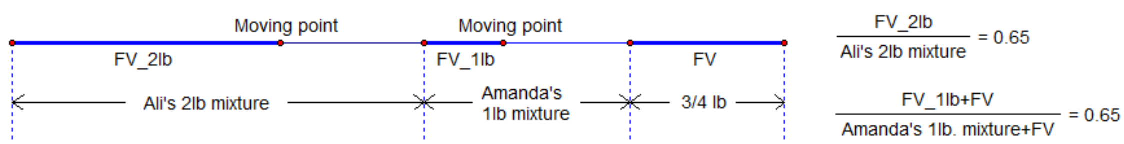
When asked to present their understanding of the coffee problem using geometric representations, our preservice teachers came up with various representations showing different levels of proportional reasoning. They were encouraged to enhance their understanding by building on others' ideas and gaining insights into the task of selecting and sequencing the works of others as well as their own.

Two Developmental Pathways of Proportional Reasoning

We present two pathways of developing proportional reasoning using the geometric representations generated by preservice teachers in our problem solving class. Two fictitious students, Sue and Kara, are used to help readers envision a possible developmental pathway of individual students. Each stage includes a composite of one or multiple preservice teachers' works. The pedagogical significance of sequencing the works of others is discussed in the next section.

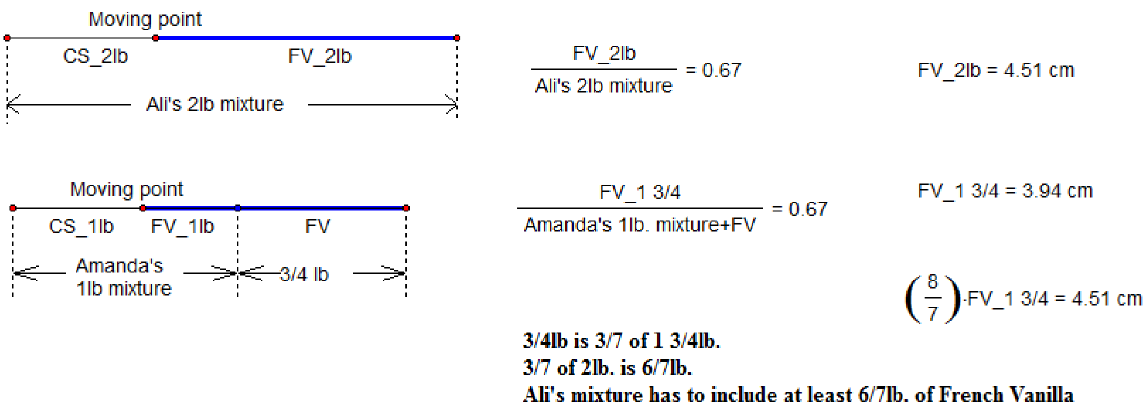
Sue's Pathway of Proportional Reasoning

[Stage 1]



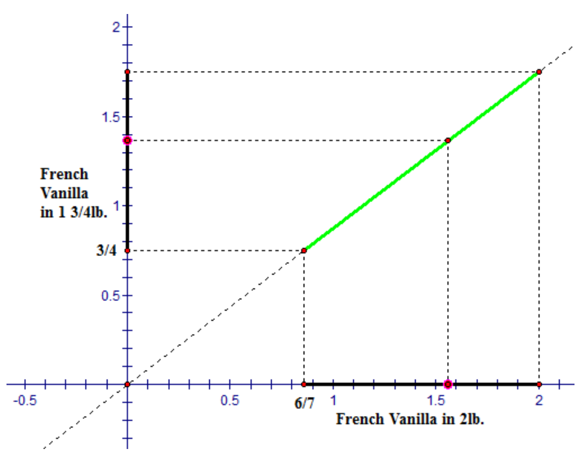
Stage 1 shows Sue's early stage of developing a sense of covariation. Sue drew a line segment consisting of three parts: Ali's 2lbs mixture, Amanda's 1lb mixture, and ¾lb French Vanilla added to the compound of the 2lbs and 1lb mixtures to get Ali's favorite mixture. She set a moving point on the part of Ali's mixture and another point on the part of Amanda's. Once the point on Ali's part was set, she adjusted the point on Amanda's part so that the ratio of the amount of French Vanilla in the 1¾lbs mixture (made up of Amanda's 1lb mixture and ¾lb French Vanilla) to 1¾ is equal to the ratio of the amount of French Vanilla in Ali's 2lbs mixture to 2. She attended to making the ratios generated by GSP equal as she adjusted the moving points. However, she was yet to find a relationship between the amounts of French Vanilla in Ali and Amanda's mixtures.

[Stage 2]



Stage 2 shows how Sue became explicit about the covariation between the amounts of French Vanilla in different mixtures. Sue split the entire 3¾lbs segment produced in Stage 1 into two segments, Ali's 2lbs mixture and the 1¾lbs mixture. Each of the split segments was to represent Ali's favorite mixture. She aligned the two segments so that she could easily compare the amounts of French Vanilla and the amounts of Columbian Supreme in the mixtures. She moved the moving points accordingly but independently as she tried to make equal the ratios of the amount of French Vanilla in each mixture to the amount of the mixture. Sue noticed the split segments have the ratio 8:7 in lbs and set up an equation $FV_{2lb} = 8/7 \times FV_{1¾lb}$. She also noticed Ali's mixture needs to include at least 6/7lb French Vanilla, which is 3/7 of 2lbs, from the information that ¾ lb is 3/7 of 1¾lbs.

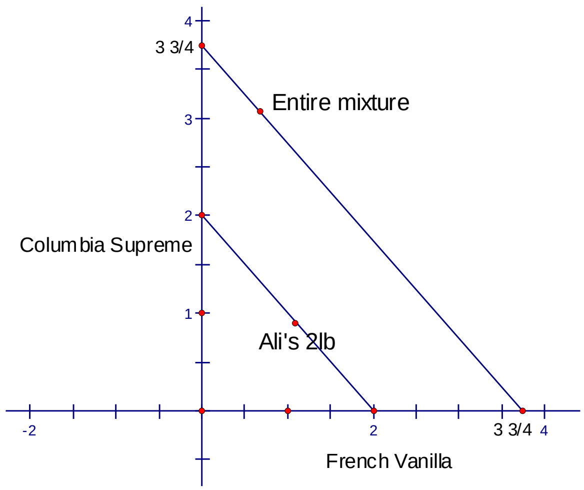
[Stage 3]



Stage 3 shows Sue's attempt to find a relationship between the amounts of French Vanilla in Ali's 2lbs mixture and the 1¾ lbs mixture. Instead of independently manipulating the ratios generated by GSP, Sue shifted her attention to drawing a graph to show a relationship between the amounts of French Vanilla in each mixture. She used the x-axis to represent the amount of French Vanilla in Ali's 2lbs mixture and the y-axis the amount of French Vanilla in the 1¾lbs mixture. Keeping in mind that Ali's mixture needs to include at least 6/7lb French Vanilla, she moved the moving point on the y-axis from ¾ to 1¾ while she moved the moving point on the x-axis from 6/7 to 2. She traced the moving point with a change of 1lb in y-coordinate and a change of 8/7 in x-coordinate, which creates a line segment with the slope of 7/8 (see the line segment in Stage 3). The fact that the extension of the line segment passes through the origin verifies the proportionality between the amounts of French Vanilla in the two mixtures.

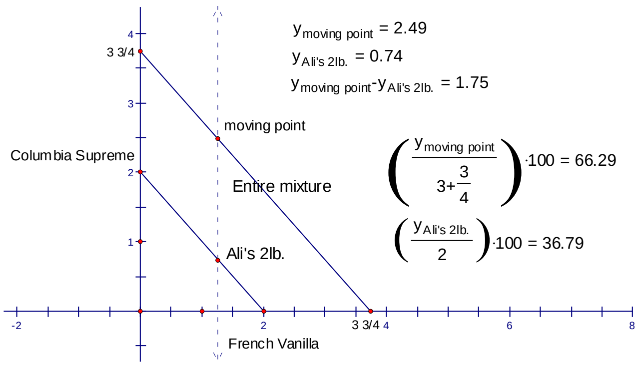
Kara's Pathway of Proportional Reasoning

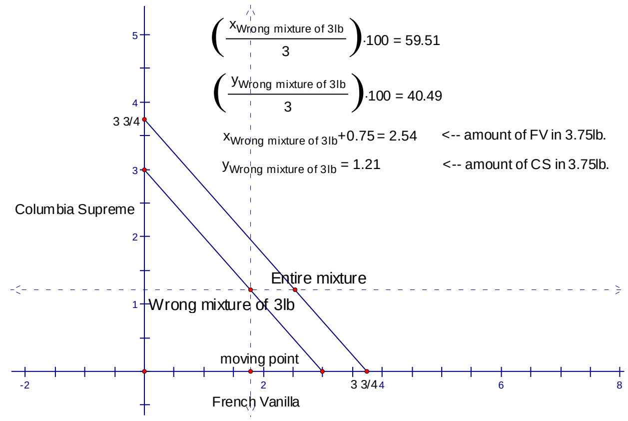
[Stage 1]



Stage 1 shows Kara's attempt to represent her understanding of the coffee problem on a coordinate system. Kara labeled the x-axis and y-axis with French Vanilla and Columbia Supreme, respectively. She drew two line segments, one from (0, 2) to (2, 0) and the other from (0, 3¾) to (3¾, 0). Then she constructed moving points on each line segment independently. The line segment passing through 2 on each axis represents all possible pairs of the amounts of French Vanilla and Columbia Supreme in Ali's 2lb mixture. That is, the sum of the x and y-coordinates on the line segment is 2. So, the line segment passing through 2 on each axis is called 2lbs-mixture segment. The same idea of labeling applies to all other line segments.

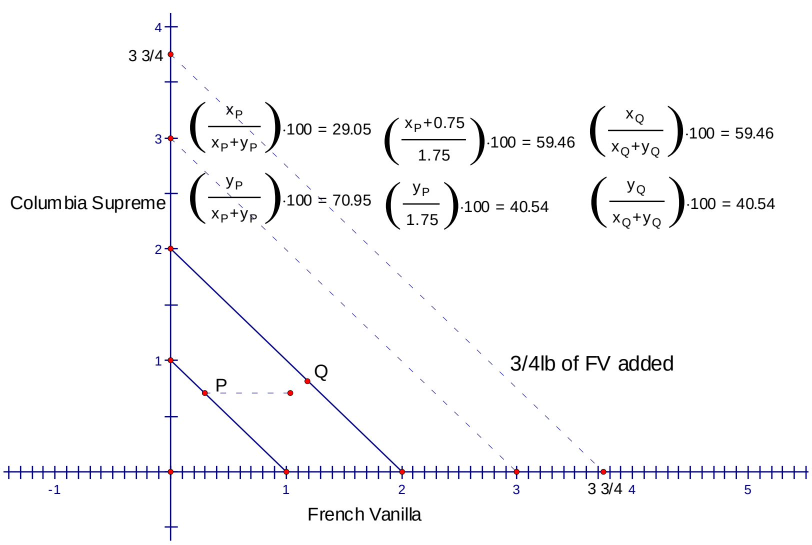
[Stage 2]





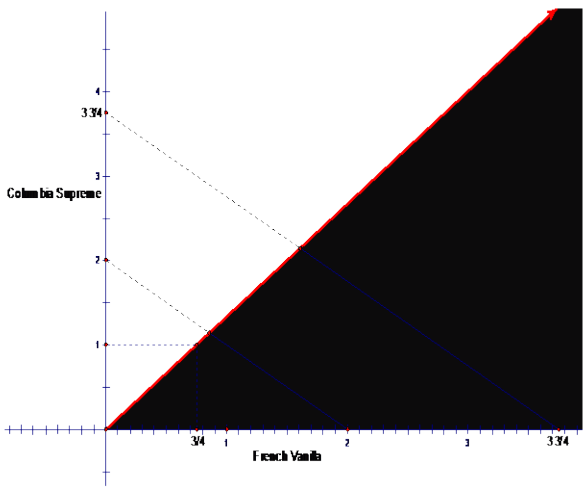
Stage 2 shows Kara's early stage of developing a sense of covariation and linearity. Kara reset the moving points used in Stage 1. She constructed a moving point on the 3 ¾lbs-mixture segment and drew a vertical line through the moving point (See the left of Stage 2). Then she found the intersection of the vertical line and the 2lbs-mixture segment and checked to see if there is a relationship between the points on each line segment. She found that the percents of the amounts of Columbia Supreme in each mixture differ from each other when she moved the moving point on the 3¾lbs-mixture segment along the segment. This implies that the 3¾lbs mixture with Columbia Supreme as much as the y-coordinate of the moving point is not Ali's favorite mixture. After a while, Kara erased the 2lbs-mixture segment and drew another line segment from (0, 3) to (3, 0), which represents the compound of Ali and Amanda's mixtures (See the right of Stage 2). Note that none of the points on the 3lb-mixture segment is Ali's favorite. She constructed a moving point on the x-axis instead of the 3lb-mixture segment and computed the percents of each type of coffee in the 3lbs compound. She illustrated that ¾lb French Vanilla was added to a specific 3lbs compound to make it Ali's favorite mixture.

[Stage 3_1]

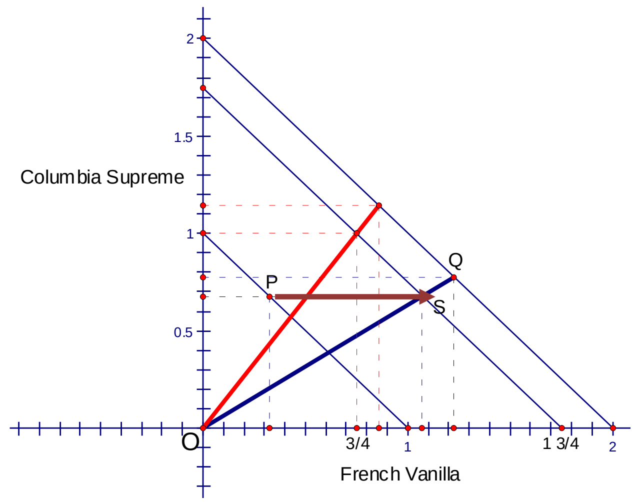


Stage 3 shows how Kara advanced her sense of covariation and linearity between the amounts of each type of coffee in different mixtures. Stage 3_1 presents six different kinds of percent Kara found. Three numbers in the top row indicate the percent of the amount of French Vanilla in Amanda's 1lb mixture, the 1¾lbs mixture, and Ali's 2lbs mixture, respectively. The three numbers in the bottom row indicate the percent of the amount of Columbia Supreme in each mixture. In addition to the line segments used in Stage 2, Kara drew two other line segments, one from (0, 1) to (1, 0) and the other from (0, 2) to (2, 0) (see the graph in Stage 3_1). Then she constructed two moving points, P on the 1lb-mixture segment and Q on the 2lbs-mixture segment, and computed the percents of each type of coffee in each mixture. The percents of each type of coffee in a particular 1¾lbs mixture were obtained by horizontally translating P by ¾ to the right. Once she has fixed P on the 1lb-mixture segment, she adjusted Q so that the percents of each type of coffee in the 1¾lbs mixture are equal to the percents in Ali's 2 lbs mixture. As she moved P along the 1lb-mixture segment, Kara observed that while P can be located anywhere on the 1lb-mixture segment, Q cannot be on the 2-lbs mixture segment.

[Stage 3_2]



[Stage 3_3]



To be more specific about possible locations of Q in Stage 3_1, Kara imagined an extreme case in which Amanda purchased just Columbia Supreme. In the extreme case, the ratio of the amounts of Columbia Supreme to French Vanilla in the 1¾lbs mixture becomes 4/3. Taking 4/3 as a slope, she drew a line passing through the origin (Stage 3_2). She reasoned the ratio of the amounts of Columbia Supreme to French Vanilla in Ali's favorite mixture should be less than 4/3 since "Ali's mixture should be between the line with the slope of 4/3 and the *x*-axis." Then she drew two line segments, Ali's 2lbs-mixture segment and the entire 3¾lbs-mixture segment, only in the region she just identified.

Using the ideas shown in Stage 3_1 and 3_2, Kara established a relationship between the amounts of each type of coffee in Ali's 2lbs mixture and Amanda's 1lb mixture. Given P on the 1lb-mixture segment, the slope of OP represents the ratio of the amounts of Columbia Supreme to French Vanilla in the mixture. Each coordinate of S, which is a horizontal translation of P by ¾ (see the arrow in Stage 3_3), represents the amount of each type of coffee in the 1¾lbs mixture, and so does each coordinate of Q in Ali's 2lbs mixture. The line passing through the origin and (¾, 1) determines the ranges for S and Q.

Discussion

This section consists of two parts. We first discuss how each of the pathways can be extended to better show a coherent progression of proportional reasoning. Then we discuss the significance of engaging preservice teachers in others' thinking and sequencing the works of others in pursuing focus and coherence in teaching.

Extension of Sue and Kara's Pathways

The pathways of proportional reasoning presented in the previous section could be extended with more explicit ideas of linearity and similarity. These extensions would help students recognize the connections among proportionality, linearity, and similarity.

Sue's progression and its extension. Over three stages, Sue gradually developed two conceptual aspects of proportional reasoning, covariation and linearity. The idea of covariation emerged when she realized the amount of French Vanilla in the 1¾lbs mixture changes as the amount of French Vanilla in Ali's 2lbs mixture changes (Stage 1). This sense of covariation diverted her attention to the ratio of the amounts of the mixtures and thus to matching it with the ratio of the amounts of French Vanilla in the mixtures (Stage 2). However, it was not until Stage 3 that Sue began to notice a linear relationship between the amounts of French Vanilla in different mixtures. When she attempted to find all possible amounts of French Vanilla in the 2lbs mixture, Sue became aware that the 2lbs mixture must contain at least 6/7lb of French Vanilla, which is 3/7 of 2lbs. She then found all possible amounts of French Vanilla in Ali's and the 1¾lbs mixtures. This awareness of possible values related in different mixtures led her to recognize a linear relationship between the amounts of French Vanilla in two Ali's favorite mixtures.

Sue's pathway could be extended as it showed the use of similarities to solve the problem. Figure 2 is a schematized version of the diagram in Sue's stage 2 for this very purpose. Starting with segments AB and CD that represent two mixtures of Ali's favorite, Sue could construct a center of dilation P by finding the intersection of $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$. It would allow her to determine the amount of French Vanilla in the 1¾lbs mixture when given the amount of French Vanilla in the 2lbs mixture. Since ΔPCD is similar to ΔPAB and CD:AB=7:8, the ratio of PD to PB is 7:8. ND:MB and CN:AM also results in 7:8. Note that ND and MB stand for the amounts of French Vanilla in the 1¾lbs mixture and Ali's 2lbs mixture, respectively. Using the scale factor 7/8 between similar figures generated from ΔPCD and ΔPAB, we could find the amount of French Vanilla in the 1lb mixture Amanda bought (NF) when given the amount of French Vanilla in the 2lbs mixture (MB).

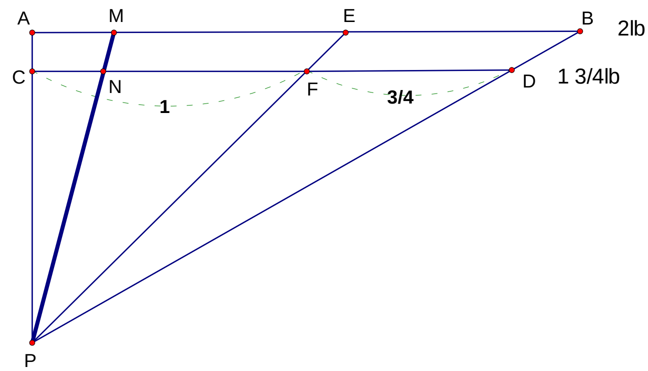


Figure 2. Extension of Sue's reasoning about covariation and linearity

Kara's progression and its extension. Over three stages, Kara developed a conceptual understanding of proportional reasoning related to graphical representations. Kara illustrated all possible pairs of the amounts of each type of coffee in different mixtures using line segments on a coordinate system (Stage 1). She then attempted to find a relationship between the amounts of French Vanilla as well as Columbia Supreme in two Ali's favorite mixtures. But she failed to do so due to a lack of sense of covariation (Stage 2_1) and linearity (Stage 2_2).

During Stage 3, Kara began to compare ratios generated from different mixtures. Being aware of the fact that both the 1¾lbs mixture and the 2lbs mixture are Ali's favorite, Kara manipulated the moving points on each mixture accordingly so that the ratio of the amount of French Vanilla in the 1¾ lbs mixture to 1¾ is equal to the ratio of the amount of French Vanilla in the 2lbs mixture to 2. Yet, she did not look further to find a relationship between the corresponding points (Stage 3_1). After spending time on computing part-to-whole ratios, Kara began to pay attention to a part-to-part ratio such as a ratio of the amounts of French Vanilla and Columbia Supreme in the 2lbs mixture (Stage 3_2). She drew a line with the slope 4/3, which is the ratio of the amounts of French Vanilla and Columbia Supreme in an extreme case where Amanda purchased only Columbia Supreme. Then she figured out a possible range of the ratio in the 2lbs mixture. Shifting her attention to a part-to-part ratio was crucial in that it enabled Kara to realize the ratio of the amounts of French Vanilla and Columbia Supreme should remain the same regardless of the amounts of Ali's favorite mixture (Stage 3_3).

Kara's pathway also could be extended so that it involved the idea of similarity. Figure 3 illustrates two kinds of similarity. One similarity comes from two different amounts of Ali's favorite mixture. Q and R in Figure 3 represent the 1¾lbs and 2lbs mixtures of Ali's favorite, respectively. If we draw a line connecting Q and R, the line passes through the origin because the ratio of x-coordinate to y-coordinate of R is equal to the ratio of x-coordinate to y-coordinate of Q. The similarity between ΔOQT and ΔORU represents a proportional relationship between two different amounts of Ali's favorite mixture. Another, less explicit, similarity comes from Amanda's 1lb mixture and Ali's 2lbs mixture represented by P and R, respectively. If we make P movable along the 1lb-mixture segment and draw a line connecting P and the corresponding point R on the 2lbs-mixture segment, the varying line always passes through N(-6, 0), and ΔNPS and ΔNRU are similar.

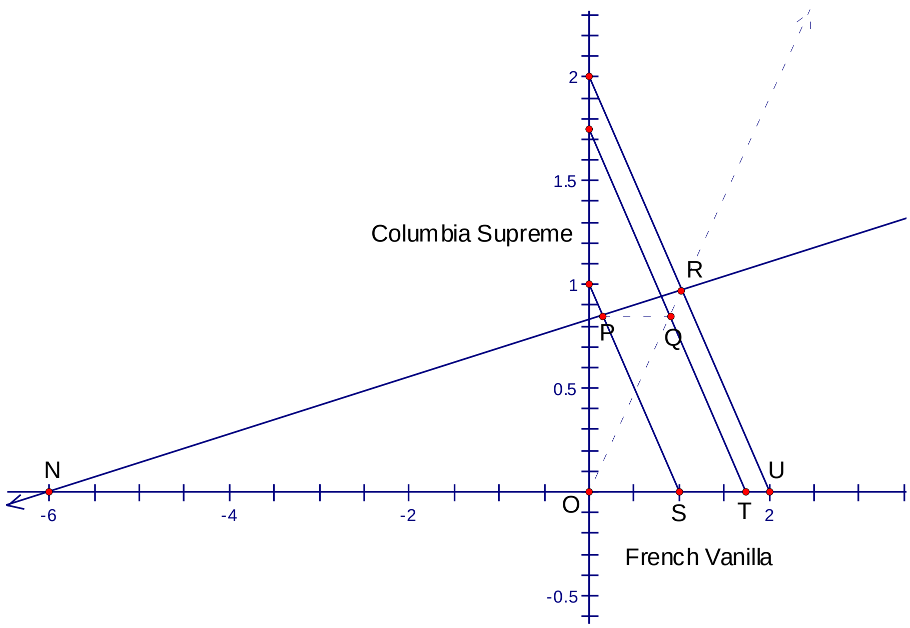


Figure 3. Extension of Kara's reasoning about linearity

Experiencing proportion and related ideas as a whole. Teachers are expected to have a profound understanding of fundamental mathematics (Ma, 1999) and teach mathematics with focus and coherence among mathematical ideas (CCSSI, 2010). If a teacher did not have his or her own experience of pursuing coherence among the ideas s/he has learned, s/he would be unaware of the importance of looking for a coherent whole in the learning of mathematics. As Even and Lappan (1994) argued, preservice teachers' own experience as learners furnish the data they use to make sense of what mathematics is and how it should be taught.

After completing the coffee problem with all the required activities, our preservice teachers had a chance to think over the implications of their experience for their future teaching. They discussed the issues of the importance of "drawing a diagram" as a problem solving strategy, the elements that constitute a mathematical diagram, a way to help students draw a mathematically meaningful diagram instead of an aesthetically pleasing picture, and a way to help students evaluate their strategy. One preservice teacher said that now that she realized a solution using a diagram or geometric representation can complement arithmetic or algebraic solutions, she would like to solve the problems which she originally solved numerically using diagrams. Another preservice teacher confessed that since she realized the difficulty of teaching the "drawing a diagram" strategy to kids in a meaningful way, she felt a definite need to study and think more about it. Taking the discussed issues and confessions into consideration, we suggest that preservice teachers be provided opportunities to contemplate their future teaching after they have personalized school mathematics through their own mathematical inquiry.

Significance of Engaging Preservice Teachers in Sequencing Others' Works

Analyzing mathematical tasks in the consideration of students' learning enhances teachers' understanding of mathematics for teaching and their knowledge of students (Doyle, 1983; Stein, Grove, & Henningsen, 1996). Task analysis also helps teachers make a task worthwhile and create and maintain an appropriate level of cognitive demand for the task (Stein et al., 1996). As they make a transition in their perspective of a mathematical task from a curricular material being presented to a task being implemented by students in the classroom, teachers become aware of students and attend to the mathematical thinking the task demands of students in the process of solving the task. The consideration of students in problem solving calls teachers to increase their understanding of students' way of thinking at various points of learning. It is desirable for teachers to develop knowledge of a mathematical task from both dimensions, a cognitive demand of the task and students' developmental path of a mathematical concept involved in solving the task (Sztajn, Confrey, Wilson, & Edgington, 2012).

In order to help increase the knowledge described above, we provided preservice teachers an opportunity to understand others' ways of thinking and to think about a possible pathway that shows how their thoughts might grow. In particular, they were asked to sequence the works of their peers as well as their own as they discerned a progression of a mathematical concept they have been working on. Sequencing students' works is a teacher's purposeful choice about the order in which students' works are to be shared. It has been considered an act of teachers to maximize the chances of achieving their mathematical goals (Stein, Engle, Smith, & Hughes, 2008). The practice of sequencing others' works including their own appeared to be beneficial to our preservice teachers in that they would reflect on whether the depth of their own mathematical knowledge is sufficient to help others advance their current understanding to the next level.

When first exposed to a variety of mathematical representations of their peers, our preservice teachers struggled to figure out which representation or approach they should take in order to advance their own representation. Limited in their thinking, they took others' work as irrelevant to or to some extent away from the ideas that they could build on. It appeared daunting to extend their horizon to see a range of understanding involved in the problem that requires proportional reasoning. This seemingly daunting situation, however, created for preservice teachers an environment conducive to (1) building a rich web of connections among different approaches and various levels of understanding, (2) identifying potential conceptual challenges that their future students may encounter, and (3) learning to use the identified conceptual challenges to help develop proportional reasoning. The use of dynamic geometric representations helped make more explicit where a way of thinking could be located on the continuum of a developmental pathway of proportional reasoning.

Over the course of solving the coffee problem, the focus was shifted from deepening preservice teachers' understanding of a mathematical concept to helping them build up a didactical perspective collaboratively. This shift of the focus reflects the idea that teachers' understanding of how their students are thinking should be incorporated with their knowledge of how students would or could progress their thinking (van den Kieboom & Magiera, 2012). To preservice teachers, sequencing the works of their peers as well as their own seems equally significant to sequencing children's work. In method courses, they learn strategies to help children learn while they sequence children's works. In content courses, they can develop awareness of the importance of seeing their own work from a developmental perspective as an inquirer. Engaging preservice teachers in the practice of sequencing as collaborative inquirers, we educated their awareness of the significance of a developmental perspective in teaching and learning mathematics. Our accomplishment may resonate with Gattegno (1987)'s argument that only awareness is educable.

References

Carpenter, T., & Lehrer, R. (1999). Teaching and learning mathematics with understanding. In E. Fennema & T. A. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 19 - 32). Mahwah, NJ: Lawrence Erlbaum Associates.

Clairaut, A. C. (1741/2006). *Élements de Geometrie*. Éditions Jacques Gabay.

Common Core State Standards Initiative. (2010). *Common core state standards for mathematics*. Washington DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/

De Bock, D., Van Dooren, W., Janssens, D., & Verschaffel, L. (2002). Improper use of linear reasoning: An in-depth study of the nature and the irresistibility of secondary school students'errors. *Educational Studies in Mathematics*, *50*, 311–314.

De Bock, D., Verschaffel, L., & Janssens, D. (1998). The predominance of the linear model in secondary school students' solutions of word problems involving length and area of similar plane figures. *Educational Studies in Mathematics, 35*, 65–85.

Doyle, W. (1983). Academic work. *Review of Educational Research, 53*, 159-199.

Even, R., & Lappan, G. (1994). Constructing meaningful understanding of mathematics content. In D. B. Aichele & A. F. Coxford (Eds.), *Professional development for teachers of mathematics* (pp. 128-143). Reston, VA: NCTM.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: D. Reidel Publishing Company

Gattegno, C. (1987). *The science of education: Part 1: Theoretical considerations*. New York: Educational Solutions.

Givental, A. (2007). *The Pythagorean theorem : What is it about?* Retrieved from http://math.berkeley.edu/~giventh/papers/eu.pdf

Hart, K. (1984). *Ratio: Children's strategies and errors*. Windsor: NFER-Nelson.

Heath, T.L. (Ed.). (1956). *The thirteen books of Euclid's elements*. NY: Dover Publications.

Hiebert, J. (2003). What research says about the NCTM Standards. In J. Kilpatrick, G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 5–26). Reston, VA: NCTM.

Karplus, R., Pulos, S., & Stage, E. K. (1983). Early adolescents' proportional reasoning on 'rate' problems. *Educational Studies in Mathematics, 14*, 219-233.

Lamon, S. J. (1999). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. Mahwah, New Jersey: Lawrence Erlbaum Associates.

Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In J. Hiebert & M. Behr (Eds.), *Number Concepts and operations in the middle grades* (pp. 93-118). Reston, VA: NCTM.

Lester, F. K., Masingila, J. O., Mau, S. T., Lambdin, D. V., dos Santos, V. M. P., & Raymend, A. M. (1994). Learning how to teach via problem solving. In D. B. Aichele & A. F. Coxford (Eds.), *Professional development for teachers of mathematics: 1994 yearbook* (pp. 152-166). Reston, VA: NCTM.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: NCTM.
Otte, M. (1990). Arithmetic and geometry: Some remarks on the concept of complementarity. *Studies in Philosophy and Education*, *10*, 37-62.
Radford, L. (1996). The roles of geometry and arithmetic in the development of algebra: historical remarks from a didactic perspective. In N. Bendarz, C. Kieren, & L. Lee (Eds.), *Approaches to algebra* (pp. 39-54). Dordrecht: Kluwer Academic Publishers.
Shifter, D., & Szymaszek, J. (2003). Structuring a rectangle: Teachers write to learn about their students' thinking. In D. H. Clements (Ed.), *Learning and teaching measurement: 2003 yearbook* (pp. 143-156). Reston, VA: NCTM.
Stein, M., Engle, R., Smith, M., & Hughes, E. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, *10*, 313-340.
Stein, M., Grover, B., & Henningsen, M. (1996). Building students capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33,* 455-488.
Streefland, L. (1985). Search for the roots of ratio: Some thoughts on the long term learning process, Part II-The outline of the long term learning process. *Educational Studies in Mathematics, 16*, 75- 94.
Sztajn, P., Confrey, J., Wilson, P., & Edginton, C. (2012). Learning trajectory based instruction: Toward a theory of teaching. *Educational Researcher, 41*, 147-156.
Tourniaire, F., & Pulos, S. (1985). Proportional reasoning: A review of the Literature. *Educational Studies in Mathematics*, *16*, 181-204.
Van de Walle, John A., Karp, K. S., & Bay-Williams, J. M. (2010). *Elementary and middle school mathematics: Teaching developmentally* (7th ed.). Boston, MA: Pearson.
van den Kieboom, L., & Magiera, M. (2012). Cultivating Algebraic Representations. *Mathematics Teaching in the Middle School*, *17*, 353-357.
Van Dooren, W., De Bock, D., Depaepe, F., Janssens, D., & Verschaffel, L. (2003). The illusion of linearity: Expanding the evidence towards probabilistic reasoning. *Educational Studies in Mathematics, 53.*113–138.
Vollrath, H. J. (1977). The understanding of similarity and shape in classifying tasks. *Educational Studies in Mathematics*, *8*, 211-224.