The Mathematics Enthusiast

Volume 11
Number 3 *Number 3*

Article 3

12-2014

Generalizing Cantor-Schroeder-Bernstein: Counterexamples in Standard Settings

Tien Chih

DOI: https://doi.org/10.54870/1551-3440.1327

Counterexamples to Cantor-Schroeder-Bernstein

Tien Chih
Newberry College

ABSTRACT: The Cantor-Schroeder-Bernstein theorem [Can97][Sch98][Ber05] states that any two sets that have injections into each other have the same cardinality, i.e. there is a bijection between them. Another way to phrase this is if two sets A, B have monomorphisms from A to B and B to A, then they are isomorphic in the setting of sets. One naturally wonders if this may be extended to other commonly studied systems of sets with structure and functions which preserve that structure. Given two objects with injective structure preserving maps between them are the structures of these objects the same? In other words, would these two objects be isomorphic in their respective setting? We see that in vector spaces, which are determined completely by their bases sets, this is true. However, when the objects are graphs, groups, rings or topological spaces, one may find counterexamples to such an extension. This is interesting, as it contradicts the naive intuition that two objects which are "subobjects" of each other must be the same. In this paper we provide some of these counterexamples.

Keywords: Cantor-Schroeder-Bernsetin Theorem, Counterexamples, Object Isomorphisms, Set Theory

1 Introduction

The Cantor-Shroeder-Bernstein theorem was first proposed by Cantor in 1895 [Can97]. It was proved erroneously by Schroeder in 1896 [Sch98]. Then finally proved by Bernstein in 1897 [Ber05]. It states that if there were two sets with injective functions into each other, then they were isomorphic as sets i.e. there was a bijection between them. This provides a convenient way to establish the existence of bijections between certain (non-countable) sets, in particular the real numbers $\mathbb{R}$ and the set of 0 and 1 valued sequences $\{0,1\}^{\mathbb{N}}$, which demonstrates the uncountablity of $\mathbb{R}$ [Mun75].

Naturally, one wonders if this result could be extended to sets with structure, together with injective maps that preserve said structure. If such an extension held, this would provide an extremely powerful way to establish isomorphisms between different structures. However, we will see that this is generally not true. We begin with a proof of the Cantor-Schroeder-Bernstein Theorem. We then discuss why it may seem plausible to believe that this result may be extended. We then provide counterexamples to such an extension for Topological Spaces, Rings both with and without unity, Groups and Graphs. We use standard definitions and notation which may be found in [BM08], [DF04], [Mun75].

2 Proof of the Cantor-Schroeder-Bernstein Theorem

We first begin with the author's proof of the Cantor-Schroeder-Bernstein Theorem:

Theorem 2.1 (Cantor-Schroeder-Bernstein)**.** *Let A, B be sets, and let $f : A \hookrightarrow B, g : B \hookrightarrow A$, then $A \cong B$*

This proof is similar to the one found in [Hal60].

Proof. (Note: *we let $\bar{X}$ denote the complement of X*).

Let $S := \bigcup_{i=0}^{\infty} (fg)^i(\overline{f(A)}) \subseteq B$, where $(fg)^0$ is the identity by convention. We then define

$$h : A \to B, x \mapsto \begin{cases} g^{-1}(x) & \text{if } x \in g(S) \\ f(x) & \text{otherwise} \end{cases}$$

and propose that h is a bijection.

It is clear that $g^{-1}|_{g(S)}$ serves as a bijection from $g(S)$ to S. It is also clear that since f is a injection, that it is a bijection onto whatever its image is. Thus it suffices to show that the image of $\overline{g(S)}$ under f is exactly the complement to the image of $g^{-1}|_{g(S)}$, that is $\bar{S}$

In other words, I claim: $f(\overline{g(S)}) = \bar{S}$, and showing this completes the proof.

$\subseteq$. Suppose, that the left hand side is not contained in the right hand side, then $\exists\ x \in f(\overline{g(S)}) \cap S$. Since $x \in S, x = (fg)^m(z), z \in \overline{f(A)}, m \in \mathbb{N}$. But if $m > 0$, then $x \in f(g(S))$, which is a contradiction to $x \in (\overline{g(S)})$. Thus $x = z \in \overline{f(A)}$, but we see that $f^{-1}(x)$ is defined, and $x \in \overline{f(A)}$ are precisely the elements where f^{-1} is not defined. Thus this is also a contradiction, and therefore no such x exists

Note that this also says that h is injective, that the map defined as f does not overlap with the map defined as g^{-1}.

$\supseteq$. Let $x \in \bar{S}$, and notice that $\bar{S} = \overline{\bigcup_{i=0}^{\infty}(fg)^i(\overline{f(A)})} = \bigcap_{i=0}^{\infty}\overline{(fg)^i(\overline{f(A)})}$. Then since $(fg)^0 = id_B$, we may say that $x \in f(A)$. Thus we may write $x = f(a), a \in A$, and we want to show that $a \in \overline{g(S)}$, but this must be true, otherwise $a \in g(S)$ implies $x = f(a) \in S$, contradictory to our original choice of x.

Note that this also says that h is surjective, that the map defined as f is onto the subset of B, that is not mapped to by g^{-1}.

$\square$

We can see here that it possible to construct a map via the two given injections. It also clear, that this map, defined in such a complicated and fractured way, would be very unlikely to preserve any structure A, B might have had, even if f, g originally were structure preserving maps of some sort. However, it does not rule out the possibility that there is some other bijection between A, B, such that the structure of the sets are preserved. Moreover, perhaps sets with particularly strong structures would require f, g to be defined in such a way that would only allow the existence of such embeddings if they were the same object.

3 Vector Spaces

An extension of Cantor-Schroeder-Bernstein does in fact hold for vector spaces over a division ring D. In fact it does so via an application of the Cantor-Schroeder-Bernstein Theorem.

Theorem 3.1 (Dimension Theorem [Axl97]). *Given a division ring D and a D (left) vector-spaceV, then given any bases for V, $\mathcal{B}_1, \mathcal{B}_2$, then the cardinalities $|\mathcal{B}_1| = |\mathcal{B}_2|$. Thus dimension is an invariant.*

Proof. Suppose $|\mathcal{B}_2| > |\mathcal{B}_1|$. Notice that given any element $v \in V$, we may write $v = \sum_{b_i \in E_j} d_i \cdot b_i$, where $d_i \in D$ and $E_j \subseteq \mathcal{B}_1, |E_j| < \infty$. Note that the collection of all E_j (all finite subsets of $\mathcal{B}_i$)

has the same cardinality as $\mathcal{B}_1$. Thus there is a $b \in \mathcal{B}_2$ which is not the linear combination of any of the elements of $\mathcal{B}_1$. Thus $\mathcal{B}_1 \cup \{b\}$ is a linearly independent set, contradicting $\mathcal{B}_1$ a basis.

$\square$

Then, by Cantor-Schroeder-Bernstein, we have the following result:

Theorem 3.2. *Let D be a division ring, and let V_1, V_2 be (left) vector-spaces over D, such that there are linear monomorphisms $\varphi_1 : V_1 \to V_2$ and $\varphi_2 : V_2 \to V_1$. Then $V_1 \cong V_2$.*

Proof. Let $\mathcal{B}_i$ be a basis for V_i. Then consider that $\varphi_1(\mathcal{B}_1)$ is a linearly independent set in V_2, and so extends to a basis $\mathcal{C}_2$ in V_2. Similarly, $\varphi_2(\mathcal{B}_2)$ extends to a basis $\mathcal{C}_1$ of V_1.

So by the dimension theorem, we have bijections: $\psi_i : \mathcal{B}_i \to \mathcal{C}_i$. Then $\psi_2^{-1} \circ \varphi_1 : \mathcal{B}_1 \to \mathcal{B}_2$ is an injection. Similarly, $\psi_1^{-1} \circ \varphi_2 : \mathcal{B}_2 \to \mathcal{B}_1$ is also an injection. So by the Cantor-Schroeder-Bernstein Theorem, there is a bijection $\tau : \mathcal{B}_1 \to \mathcal{B}_2$.

This bijection between bases τ, extends uniquely to a linear isomorphism T, and thus $V_1 \cong V_2$.

$\square$

4 Topological Spaces

When one considers extensions of the Cantor-Schroeder-Bernstein Theorem, a setting to consider would be topological spaces. However in the setting of topological spaces we find a very simple counterexample.

Proposition 4.1. *Let $A := (-1, 1)$ with the subspace topology of $\mathbb{R}$, and let $B := [-1, 1]$ also with the subspace topology. Then there are embeddings from A to B and from B to A, but A and B are not homeomorphic.*

Proof. A clearly embeds into B via the inclusion $\varphi : A \hookrightarrow B, x \mapsto x$, and $\psi : B \hookrightarrow A, y \mapsto y/2$, since both maps are non-zero linear maps when extended to $\mathbb{R}$, they are continuous and injective.

To show that A and B are not homeomorphic, suppose that $\chi : A \to B$ were a homeomorphism, then let $z = \chi^{-1}(-1)$. z would then be a limit point of the sets $(-1, z), (z, 1)$, and thus -1 would be a limit point of the image of both sets under χ, and since they are non intersecting, so would their images. This is not possible and thus a contradiction. $\square$

We notice that there are many ways to show that $(-1, 1)$ and $[-1, 1]$ are not homeomorphic. For example, one may consider that by the Heine-Borel Theorem [Mun75], $[-1, 1]$ is compact while $(-1, 1)$ is not. Most of these arguments are topological in nature, and do not extend to other settings. In the following sections, we will see an argument for non-isomorphism which extends nicely to other settings.

5 Rings

In this section, we begin with a counterexample to an extension of Cantor-Schroeder-Bernstein for rings *without* unity. We then modify these rings to exhibit a counterexample for rings *with* unity.

First, some preliminary facts are stated:

Lemma 5.1 ([DF04])**.** *If D is a division ring, then for $n \in \mathbb{Z}_+$, the matrix ring $M_n(D)$ is a simple ring.*

Proposition 5.2. *Let $R := \bigoplus_{i=1}^{\infty} M_{2i}(\mathbb{Z}_2), S := \bigoplus_{i=1}^{\infty} M_{2i+1}(\mathbb{Z}_2)$, then there exist embeddings from R to S and from S to R, but these are non isomorphic rings.*

Proof. We first show that there exists injections from $R \hookrightarrow S$ and $S \hookrightarrow R$. Notice that for a given ring T, there there is a very natural embedding $\iota : M_k(T) \hookrightarrow M_{k+1}(T)$ for any value k, where $\iota(M)_{(a,b)} = M_{(a,b)}$ if $0 \le a, b \le k$ and $\iota(M)_{(a,b)} = 0$ otherwise. Thus there are natural embeddings φ, ψ:

$$\begin{tikzcd}
R: & M_2(\mathbb{Z}_2) \arrow[d, hook, "\varphi"'] & \oplus & M_4(\mathbb{Z}_2) \arrow[d, hook, "\varphi"'] & \oplus & M_6(\mathbb{Z}_2) \arrow[d, hook, "\varphi"'] & \oplus & \cdots \arrow[d, hook, "\varphi"'] \\
S: & M_3(\mathbb{Z}_2) \arrow[urr, hook, "\psi"] & \oplus & M_5(\mathbb{Z}_2) \arrow[urr, hook, "\psi"] & \oplus & M_7(\mathbb{Z}_2) \arrow[urr, hook, "\psi"] & \oplus & \cdots
\end{tikzcd}$$

Where $\varphi|_{M_k(\mathbb{Z}_2)} : M_k(\mathbb{Z}_2) \hookrightarrow M_{k+1}(\mathbb{Z}_2)$ and similarly for ψ.

However, these rings are not isomorphic, and to show this, we first need to state some facts:

We then show that there are no onto homomorphisms, and thus no isomorphism, from S to R.

Let $\chi : S \to R$ be a ring homomorphism, and let $\rho_{2n} : R \to M_{2n}(\mathbb{Z}_2)$ be the projection homomorphism. Notice that $\chi|_{M_{2n+1}(\mathbb{Z}_2)}$ is also a ring homomorphism for any n. Since $M_{2n+1}(\mathbb{Z}_2)$ is simple, it's image is either 0 or isomorphic to itself. Consider the composition $\rho_2 \circ \chi|_{M_{2n+1}(\mathbb{Z}_2)}$. Since each $M_{2n+1}(\mathbb{Z}_2)$ has vector space dimension greater than four over $\mathbb{Z}_2$, their image under $\rho_2 \circ \chi|_{M_{2n+1}(\mathbb{Z}_2)}$ must be 0. Thus, no homomorphism from $S \to R$ is onto the first summand of R, and thus there are no onto homomorphisms, or isomorphisms from S to R. $\square$

However, this example is limited, since we are taking the infinite direct sums of rings, neither ring contains a unity element. We have not shown that this extension would not hold when R, S

are unital rings. However, we may use a standard construction to construct two rings with unity from these rings.

Theorem 5.3. *Let S be a ring without unity, such that S is an T-algebra, and T is a ring with unity. Then define S' on the set $T \oplus S$, with coordinate wise addition, and multiplication $(a, B) * (c, D) = (ac, aD + Bc + BD)$, where aD, Bc are algebra actions, and BD is product in S. Then S' is a ring with unity: $(1_T, 0)$, with S as a subring.*

This is often called the *standard (universal) unitization* of the algebra S.

Proof. It is elementary, albeit tedious and unenlightening to check the axioms of associativity, distributivity, and closure. It is similarly simple to check that $\{0\} \oplus S$ is a subring isomorphic to S. Thus, here only the fact that $(1_T, 0) = 1_{S'}$ is checked.

Let $(a, b) \in S'$, $(1_T, 0)(a, B) = (1_T a, 1_T B + 0a + 0b) = (a, B + 0 + 0)$. The proof is the same for the product in reverse order. □

Corollary 5.4. *Then $R' := (\mathbb{Z}_2, R), S' := (\mathbb{Z}_2, S)$ as defined above are unital rings.*

Proof. Since R, S are direct sums of matrix rings over $\mathbb{Z}_2$, they are clearly $\mathbb{Z}_2$ vector spaces, and thus $\mathbb{Z}_2$ algebras. □

Again, there is a clear embedding from R' into S' and vice versa, and again, these are not isomorphic rings.

Proof. Let $\chi : S' \to R'$ be a ring homomorphism, and consider $\chi((a, B)) = \chi((a, 0)) + \chi((0, B))$. Notice that $a \in \mathbb{Z}_2$, thus $(a, 0)$ is either the zero or the unity of S'. Ring homomorphisms clearly preserve the zero, and if χ does not preserve the unity, then it is not an isomorphism, and we are done. Thus, without loss of generality, we may assume $\chi((a, 0)) = (a, 0) \in R'$.

We then consider $\chi|_S$, and we have already seen that this map is never onto $M_2(\mathbb{Z}_2) \subset R'$, and thus not onto R', and so S', R' are unital rings that are non-isomorphic. □

6 Groups

The construction for groups is essentially a repurposing of the arguments for rings. By taking infinite direct sums of simple groups with natural injections into one another, the same arguments hold to show that they are non-isomorphic.

Definition 6.1. Given a positive integer n, A_n denotes the group of even permutations on $[n]$. We call A_n the *alternating group* of degree n.

Theorem 6.2 ([Sco87])**.** *For positive integer $n, n \geq 5$, A_n is a simple group.*

Proposition 6.3. *Let $G := \bigoplus_{n=1}^{\infty} A_{2n+3}, H := \bigoplus_{n=1}^{\infty} A_{2n+4}$, where A_i is the alternating group on i elements. Then there are injections from G into H and vice versa, but these groups are non-isomorphic.*

Proof. To see that these groups embed into one another, we observe that there is a natural embedding $\iota : A_i \hookrightarrow A_{i+1}$ for any choice of i, since the even permutations on the an $i+1$ element set include the even permutations that do not act on the $i+1$st element. Thus we have embeddings $\varphi : G \to H$ and $\psi : H \to G$:

$$\begin{tikzcd}
G: & A_5 \arrow[d, hook, "\varphi"'] & \oplus & A_7 \arrow[d, hook, "\varphi"'] & \oplus & A_9 \arrow[d, hook, "\varphi"'] & \oplus & \cdots \arrow[d, hook, "\varphi"'] \\
H: & A_6 \arrow[urr, hook, "\psi"] & \oplus & A_8 \arrow[urr, hook, "\psi"] & \oplus & A_{10} \arrow[urr, hook, "\psi"] & \oplus & \cdots
\end{tikzcd}$$

where $\varphi|_{A_k} : A_k \hookrightarrow A_{k+1}$ is the natural embedding, and similarly for ψ.

However, to show that they are not isomorphic, let $\chi : H \to G$ be a group homomorphism and let $\rho_5 : G \to A_5$ be the projection homomorphism. Recall that each A_i is simple for $i \geq 5$ and each $A_{2n+4}, n \geq 1$ contains strictly greater than $|A_5|$ elements. Because A_{2n+4} is simple, $\rho_5 \circ \chi|_{A_{2n+4}}$ must be isomorphic to either A_{2n+4} or $\{e\}$ and so $\rho_5 \circ \chi|_{A_{2n+4}}$ must be the trivial map. Thus χ is not onto, and since this is true for any χ, G and H are not isomorphic groups. $\square$

7 Graphs

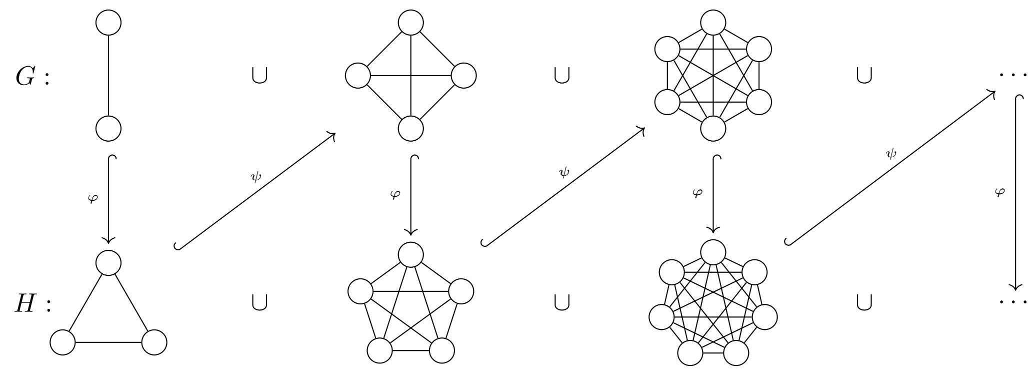
The same idea may also be used for graphs. Although there is no equivalent concept to a non-trivial simple graph, we may use vertex degree arguments to show non-isomorphism instead.

Definition 7.1. Let K_n denote the *complete graph* on n vertices, that is a graph with n vertices where each vertex is incident to each other distinct vertex.

Proposition 7.2. *Let $G := \bigcup_{i=1}^{n} K_{2i}, H := \bigcup_{j=1}^{n} K_{2j+1}$, the disjoint union of non-trivial even and odd degree complete graphs, respectively. Then G, H embed into one another but are non-isomorphic graphs.*

Proof. We first show the existence of embeddings $G \hookrightarrow H$, and $H \hookrightarrow G$. Notice that given any positive integer n, there is a natural embedding $\iota : K_n \to K_{n+1}$, by mapping the n vertices of K_n

to any n vertices of K_{n+1}. Thus we may define φ, ψ:



where $\varphi|_{K_n}$ is the natural embedding $K_n \hookrightarrow K_{n+1}$ and similarly for ψ.

However, every vertex of G has odd degree and every vertex of H has even degree. Thus G and H cannot be isomorphic.

$\square$

8 Conclusions

We find that there are in fact very few structures where mutual embeddings between objects implies isomorphism or equivalence between these objects. In particular the examples that we found for both unital and non-unital rings, groups and graphs are all variations of the same "graded-objects" theme: where we have some sort of structure indexed by $n \in \mathbb{Z}^+$ where the nth structure may be embedded into the $n+1$st, but are fundamentally non-isomorphic. These examples are reminiscent of Hilbert's hotel-type arguments. We conclude that although it is tempting to naively believe that two objects which are "sub-objects" of each other should be the same, that in reality this notion of "sub-object" is far to complex for any structure more robust than sets.

Acknowledgements

I would like to thank Eric Cheesebro and Demitri Plessas for introducing me to the Cantor-Schroeder-Bernstein Theorem and it's implications. I would like to thank Jennifer Halfpap Kacmarcik and Rachael Chaphalkar for encouraging me to take these stray thoughts and collect them into a paper. I would like to thank my advisor George McRae for his input and advice. Finally, I would especially like to thank Nikolaus Vonnesson, for providing the key insight which allowed me to construct the appropriate counterexamples.

References

[Axl97] Sheldon Axler. *Linear algebra done right*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1997.

[Ber05] Felix Bernstein. Untersuchungen aus der Mengenlehre. *Math. Ann.*, 61(1):117–155, 1905.

[BM08] J. A. Bondy and U. S. R. Murty. *Graph theory*, volume 244 of *Graduate Texts in Mathematics*. Springer, New York, 2008.

[Can97] Georg Cantor. Beiträge zur Begründung der transfiniten Mengenlehre. *Math. Ann.*, 49(2):207–246, 1897.

[DF04] David Steven Dummit and Richard M. Foote. *Abstract algebra*. John Wiley & sons, Hoboken, NJ, 2004.

[Hal60] P.R. Halmos. *Naive Set Theory*. Undergraduate Texts in Mathematics. Springer, 1960.

[Mun75] James R. Munkres. *Topology: a first course*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1975.

[Sch98] E. Schröder. *Ueber zwei Definitionen der Endlichkeit und G. Cantor'sche Sätze*. Nova acta Academiae Caesareae Leopoldino-Carolinae Germanicae Naturae Curiosorum. 1898.

[Sco87] W. R. Scott. *Group theory*. Dover Publications, Inc., New York, second edition, 1987.