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Understanding Risk Through Board Games

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Abstract: In this article, I describe a potential avenue for investigating individual's understanding of and reactions to risk using the medium of board games. I first discuss some challenges that researchers face in studying risk situations. Connecting to the existing probabilistic reasoning literature, I then present a rationale for using board games to model these situations. Following this, I draw upon intuition and dual-process theory to outline an integrated theoretical perspective for such investigations. The article concludes with two vignettes demonstrating how this perspective might be used to analyze thinking about risk in a board game setting.

Keywords: risk, probabilistic reasoning, intuition, dual-process theory, board games.

Within our information-driven society, understanding and making decisions about risk has become important for personal and professional success. Each day individuals are bombarded by information from a host of organizations seeking to inform, influence, and persuade choices through the use of hypothetical situations involving risk, which I refer to as risk-laden (RL) situations. Here RL situations are defined as any situation that presents a perceived risk directly (e.g., men over the age of 25 have increased risk for gestational diabetes) or presents a risk indirectly by using probabilistic language (e.g., the odds of dying from heart disease are 1 in 7). Adults, in particular, are targeted by a host of groups (e.g., news organizations, advertisers, health professionals, financial planners, insurance companies) each of whom seeks to use RL situations to inform, influence, and motivate choices.

RL situations can be differentiated using two criteria, which I will call (and later define) density and complexity. Density refers to the extent to which probabilistic information is incorporated into a RL situation. Low-density RL situations present simple probabilities whereas high-density situations use sophisticated probabilistic language that may be embedded or indirectly expressed. Complexity refers to the context of an RL situation. Everyday contexts have a relatively low complexity whereas specialized contexts have high complexity. These criteria are useful in comparing RL situations directly. For example, consider the following two situations: (a) A meteorologist reports that there is a 60% chance of rain this weekend; and, (b) Doctor John shows Kim the following finding from a medical study, "When used in primary prevention settings, aspirin has been shown to reduce serious vascular events among individuals at average/low risk by 12% (0.51% versus 0.57%/year, P = 0.0001)" (Cuzick et al., 2014, p. 5). The first situation has a low degree of density and complexity because it presents a simple probability within an everyday context. The second has a higher degree of complexity and density because it uses sophisticated mathematical language set within a more specific context. Thus, RL situations with high degrees of density and complexity embed sophisticated probabilistic information within specialized contexts.

Although different in terms of density and complexity, both of the previously mentioned situations present similar issues in decoding. For example, what does a 60% chance of rain mean? Does each day during the weekend have a 60% chance of rain? If it rains the first day, will the next day still have the same chance of rain? Similarly, what does it mean for an individual to be at average/low risk? What information factors into this classification? What does a 12% reduction in serious vascular events

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amount to in terms of overall risk for these events? Additionally, the interpretation of each statement might change based upon the person making it. What if the meteorologist in the first situation worked for the National Weather Service? What if they worked for an insurance company? What if Dr. John were Kim's family doctor? What if he worked as a consultant for a retailer? These examples illustrate that, regardless of density and complexity, RL situations present similar challenges to individuals in terms of unpacking and understanding.

Although mathematics education researchers have investigated a range of probabilistic concepts, RL situations present different challenges for several reasons. First, RL situations often require one to make sense of several different hypothetical outcomes and weigh competing possibilities against one another in order to make sense of the situation and determine the most beneficial decision. In some instances the best decision might yield a beneficial outcome, but in others the best decision might be the outcome with the least potential consequences. Second, in contrast to familiar contexts where the probability of an event does not change (e.g., rolling a six on a fair die), RL situations can be dynamic with the likelihood of outcomes changing as events unfold. For example, weather events can change the potential risk for flooding, shifts in economic conditions influence the risk related to financial instruments, and a recent health issue can affect the risk of a medical procedure. Third, the range of contexts for RL situations can present an obstacle since individuals may have limited background knowledge about a context and must instead rely on probabilistic knowledge, experience, and intuition to make a decision.

Taken together, these points suggest that if mathematics educators wish to pursue research on mathematical thinking about risk, the setting of these investigations should be flexible with the capability of addressing both the density and complexity of RL situations. This article describes one possible avenue using the dynamic and diverse medium of board games. In what follows, I first discuss mathematics education research literature on probabilistic thinking and draw connections to RL situations. I then define what is meant by the phrase board game and present a rationale for using the medium of board games to model RL situations. Following this, a theoretical perspective is presented that may offer assistance in analyzing RL situations. The article concludes with sample vignettes modeling how the perspective might be used within a board game setting to understand an individual's thinking about risk.

Research on Probabilistic Reasoning

Probabilistic reasoning has been a focus of researchers in both mathematics education and psychology (Chernoff & Sriraman, 2014) and presents a rich foundation from which to build investigations of individual's thinking about risk. Central in probabilistic reasoning is an individual's understanding of randomness. As noted by Batanero, Green, and Serrano (1998) randomness resides at the heart of probabilistic reasoning because it serves as a string that binds together a collection of different mathematical concepts. Although there are many interpretations of randomness, the present work uses a definition presented by Moore (1990),

> Phenomena having uncertain individual outcomes but a regular pattern of outcomes in many repetitions are called *random.* "Random" is not a synonym for "haphazard" but a description of a kind of order different from the deterministic one that is popularly associated with science and mathematics. Probability is the branch of mathematics that describes randomness. (p. 98)

Based upon this definition, although the outcome of a specific event itself may be uncertain and unpredictable, if the same event is repeated a large number of times, patterns emerge and yield frequencies that, in turn, make prediction possible (Metz, 1997). Randomness connects a cluster of related ideas including uncertainty, likelihood, and chance. Therefore, an individual's probabilistic reasoning is built upon a foundation of knowledge of randomness and this collection of related ideas.

Research has documented a variety different non-normative (i.e. not held by experts) ideas that individuals may hold about randomness (Ayton & Fischer, 2004; Gold, 1997; Shaughnessy, 1992). One well-known example is the gambler's fallacy. Roughly speaking, this is the belief that a series of one outcome will create a tendency for another, opposite outcome. People who hold the gambler's fallacy often believe that they can predict the next outcome of a random process based on prior observations. For example, a coin is flipped six times resulting in the sequence THTTTT. What will be the outcome of the next flip? A person holding the gambler's fallacy would likely predict that the next outcome would be heads. This is because they believe that the observed sequence of tails would need to be balanced by a sequence of heads. However, from a normative viewpoint both heads and tails are equally likely.

Furthermore, the idea that random processes have a "bookkeeping" ability (i.e. the process remembers and reacts to previous results) is not isolated to typical probability situations. This is illustrated by the following Dear Abby letter:

> Dear Abby: My husband and I just had our eighth child. Another girl, and I am really one disappointed woman. I suppose I should thank God that she was healthy, but, Abby, this one was supposed to have been a boy. Even the doctor told me the law of averages were in our favor 100 to 1. (Dawes, 1988, p. 84)

Although the probability that a particular couple will have eight girls is quite small (roughly 0.0039 if we assume that a boy and girl are equally likely), the fact that a couple has already had seven girls does not change the probability that the next child will be a girl (it is still .5). However, as is evident in the excerpt, both the woman and her doctor believed that the next child would have to be a boy by the "law of averages." This statement is an application of the gambler's fallacy since both the woman and her doctor have assumed that the random process will "even out."

The previous Dear Abby example illustrates an issue concerning probabilistic knowledge. Unlike many other types of mathematical knowledge, which are encountered almost entirely within classroom settings, probability situations are encountered frequently as individuals go about daily activities outside of school. This is because the stream of RL situations within our modern society has become a constant part of communication. At the same time, the role of probability within the pK–12 curriculum has remained relatively minor within the United States. This is reflected in the limited emphasis of probability within current content standards (Mooney, Langrall, & Hertel, 2014). Thus, probabilistic knowledge is learned via in-school and out-of-school experiences; however, as I will discuss, out-of-school experiences do not always offer information that supports decision-making in RL situations.

One issue with out-of-school experiences is that individuals can easily be led astray by culturally accepted ideas that persist in a variety of different formats (e.g., maxims, epigrams, anecdotes, fables, proverbial sayings). Many of these ideas are encoded with probabilistic information, which can influence an individual's reasoning. For example, consider the adage "lightning never strikes twice." If a person truly believes that lightning will never strike the same place twice, then they are more likely to stand beneath an object that has previously been struck by lightening during a thunderstorm. In reality this belief is unfounded since lightning can, and does, strike the same location multiple times. Uman (1986) noted that "the Empire State Building is struck by lightning an average of about 23 times per year. As many as 48 strikes have been recorded in one year, and during one thunderstorm, eight strikes occurred within 24 minutes" (p. 47).

Another issue with out-of-school experiences is that they may teach individuals to focus on less important aspects of a RL situation. For example, a focus solely on the probability of lighting striking at the same location obfuscates other important information about the context. Since risk is intimately tied to context, the context should be considered when reasoning through RL situations (e.g., one should consider the physical landscape when making a decision about where to seek shelter in a lightening storm). Additionally, out-of-school experiences may promote a context-free application of ideas that can lead individuals astray. For example, the previous adage can be applied in relevant contexts (e.g., lightening storms) as well as other situations that an individual deems appropriate (e.g., winning the lottery), thereby obstructing and derailing decision making about unrelated events.

As mathematics education researchers, our primary goal is to investigate the learning and teaching of mathematical ideas. Although the focus of the field has tended to be on the teaching and learning of mathematics of children and young adults within school settings, there are several reasons that these conditions are constraining for studying probabilistic reasoning and its influence on decisions in RL situations. First, as previously noted, probability has a relatively weak position within most current curricula. This means that the majority of students have little in-school experience with probabilistic situations. Second, individuals spend only a short span of years in school but are subjected to the constant presence of RL information throughout their lifetimes. Although out-of-school experiences may provide individuals with some additional probabilistic knowledge, these experiences may also hinder individuals by reinforcing ideas that lack the details and specifics needed to assist them in making rational decisions about RL situations (Konold, Pollatsek, Well, Lohmeier, & Lipson, 1993). Thus, these out-of-school experiences have the potential to hinder or impede reasoning.

Third, the majority of existing research has used contrived situations crafted in contexts outside of participants' real life experience (e.g., Piaget's tip box, the hospital problem). Although this work provides insight into probabilistic reasoning, it offers little information how this reasoning is applied in more familiar contexts. Missing are studies that investigate participants' probabilistic reasoning within a context that is more closely related to an individual's normal routine. These are the RL situations that individuals are continually faced with and must make decisions about.

Fourth, viewed as a whole, the primary goal of past research has been to identify and study particular concepts or misconceptions, but little is known about how these ideas impact more complicated decisions. Studies on development of probability concepts have primarily sought to establish when particular concepts develop and if it is possible to remediate misconceptions. Judgment and decision-making studies have investigated the reasons behind the choices that people make and sought to understand commonly held misconceptions (e.g. the gambler's fallacy, the hot hand belief). Likewise, research investigating probabilistic intuitions directly has yielded mixed results with some intuitively based probabilistic misconceptions found to weaken with age, others found to grow stronger, and still others found to stabilize (Fischbein & Schnarch, 1997). Overall, missing from the literature is research that examines how these factors combine to influence decision making *in action*.

Finally, RL situations present challenges for interpretation that are different from classic probability problems. This is because RL situations are often presented as facts to be consumed rather than predictions based upon the observed frequencies of a random phenomenon. As a result, the assumptions and limitations of a prediction, which tend to be evident in more traditional probability problems, can be easily lost in communication about RL situations. The context-specific nature of risk means this poor communication can lead to misinformation and misunderstanding. Taking a previous example, it is impossible to predict with absolute certainty if it will rain today, but given that it has rained 60 out of the last 100 days with similar environmental conditions we can predict a 60% chance of rain. Thus, the risk of rain that is reported on a particular day is based upon a specific geographic location and, as experienced individuals can attest, a change in geography can invalidate this prediction. However, the RL information, which is shared and consumed, omits these specifics. This is quite different from more traditional educational settings where the context and assumptions are known or specifically focused on.

The Medium of the Board Game

For the purposes of this study, the phrase *board game* refers to a type of game that has the following characteristics: (a) a board, play matt, or clearly defined play area on a table or similar surface on which game pieces are placed and interacted with; (b) pieces, cards, or markers that are used for a variety of different purposes; (c) an external process or device that incorporates uncertainty into game play (e.g., spinner, dice, random card draw); and, (d) the absence of gambling with real-world currency (although wagering of in-game currency may be a component of gameplay). Popular games that fall under this definition include *Monopoly, Risk, Trivial Pursuit, Clue, Life,* and *Chutes and Ladders*. This definition excludes deterministic games such as chess or checkers, which do not have external processes that incorporate uncertainty into gameplay. Likewise, gambling-focused card games such as poker or blackjack are excluded from this definition.

Because board games contain external processes that incorporate uncertainty, players engage in probabilistic reasoning as part of normal gameplay. Moreover, the incorporation of uncertainty results in situations where players must make decisions based upon perceived risks and rewards. To illustrate a simple example, choices in the popular board game *Monopoly* focus on acquiring or selling properties. As the game progresses, players must make decisions about purchasing, selling, or improving properties. Since the number of spaces a player moves each turn is dependent on a dice roll, the number of turns it will take to travel the entire board, which is a primary means of collecting income, is uncertain. Consequently, the number of opportunities that a particular player will have to land on and purchase a given property is unknown. As the game progresses, opponents may benefit from acquiring properties, drawing random cards, or rent-free trips around the game board. On the other hand, opponents may be disadvantaged by paying the luxury tax, being sent to jail, or having to mortgage properties. Thus, the system of the board game, which includes the monetary assets of players as well as their property ownership, is dynamic. Players must continually reassess their options and adjust their actions as they play the game. Consequently, gameplay requires balancing the risks of running out of money with the rewards of acquiring property.

Monopoly has a relatively simplistic design when compared to many other board games because the driving force behind in-game events is the result of a dice roll. The decisions made by players about purchasing, selling, improving, or mortgaging properties have little influence on the consequences resulting from the dice roll. Instead, these decisions are mostly in reaction to random events. In contrast, many other board games provide players with a variety of choices during gameplay that can be made in anticipation of in-game events. This makes it possible for players to reduce the effect of random events or change the consequences of a particular event. By providing players with more choices, these more complex board games also provide opportunities to develop short-term and long-term strategies for managing risk.

The board game *Settlers of Catan* serves as one example highlighting these ideas. The game, which was developed in the 1990s, has grown greatly in popularity over the last decade. The objective of the game is to be the first person to accumulate 10 points. These points are gained by building structures, purchasing cards, or being awarded one of two special cards. The game board is made up of 19 hexagon tiles, which are arranged together in a specific pattern. One tile denotes the desert and the other 18 are one of five land types (hills, pasture, mountains, fields, forest). A special token (the robber) is placed on the desert and each of the other tiles is assigned one of the following numbers: 2, 3, 3, 4, 4, 5, 5, 6, 6, 8, 8, 9, 9, 10, 10, 11, 11, 12 (note that the number 7 is excluded from this list). Each of these numbers corresponds to a sum that can be obtained by rolling two six-sided dice. Figure 1 shows a portion of the board arranged in the recommended starting setup for beginners.

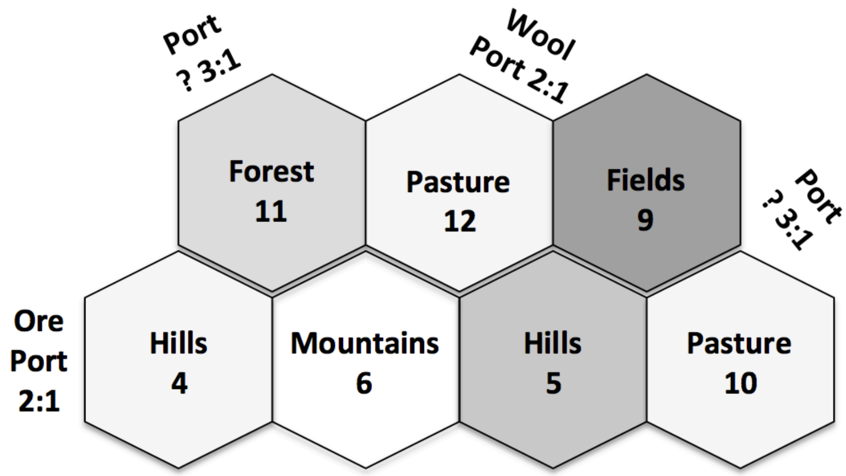

Figure 1. A portion of the *Settlers of Catan* game board

Like *Monopoly,* in-game events in *Settlers of Catan* are driven by the roll of two six-sided dice. Each turn the dice are rolled and any tile with the corresponding sum produces a resource (e.g., brick, wool, ore, grain, lumber). The most likely sum, 7, causes the robber to move to a different tile, which prevents the tile from producing resources and triggers other in-game events. Players collect resources by building structures (towns or cities) at the vertex of a tile. Thus, a player might build a town at a vertex shared by multiple tiles or choose to build at a vertex on the edge of the game board that is on a single tile. As shown in Figure 1, there are also ports along the edge of the board that, if built upon, allow a player to trade resources. For example, the port labeled ?3:1 allows players to trade three of one resource for one of another. Players also have the option to trade resources with each other or with the bank.

In contrast to *Monopoly, Settlers of Catan* offers a number of in-game choices that can change the potential risk (or reward) of a dice roll. Since gameplay is driven by collecting and using resources to acquire points, the most influential of these choices is arguably the initial placement of towns during the setup of the game. Initial placement limits future building because players must connect new structures to existing structures they control. During the game setup, players are given the initial choice to build two towns anywhere on the game board following specific in-game rules (e.g., each town is at least two edges from another town). There are a variety of strategies that players may adopt. For example, a player may choose to build at a specific vertex to collect a single type of resource. In Figure 1 a player using this strategy might build at a vertex of each hill tile to collect brick. This is likely to provide the player with a large quantity of bricks over the course of the game, which they can trade with the bank or with other players. On the other hand, a player may choose to build on tiles that have specific numbers. In Figure 1, the six on the mountain tile is the most likely sum to be rolled, followed by the five and nine. A player using this strategy may choose to build on the vertex shared by the Mountains, Pasture, and Hills thinking that two of these tiles have a greater likelihood of producing resources. These strategies, which are just two of many potential strategies, illustrate how the decisions made in playing the game are based upon both probabilistic reasoning and expectations about risk.

As the game progresses, players continue to build towns. However, these decisions must take into account the locations of opponents and the resources that a player has on hand. As areas of the game board fill with towns, the risk of not acquiring specific resources increases. At the same time, players are subject to the uncertainty of the dice roll, which can reward or penalize players in unforeseen ways. Likewise, players may choose to work together with opponents for mutual benefit or to prevent another person from acquiring points. Thus, the decisions that players make during a game offer insight into their in-action reasoning.

Although the proceeding discussion has focused on a seemingly complicated game, *Settlers of Catan*, the game has less complexity than other board games. In fact, within many board gaming circles *Settlers of Catan* is seen as a gateway game that is good for introducing new players to the medium of the board game. However, *Settlers of Catan* is often criticized for the relatively simple choices that players are allowed to make and the heavy reliance on the roll of dice.

Monopoly and *Settlers of Catan* can be seen as representing two initial points along a continuum of board games. This continuum differentiates games based upon two criteria, which I will call fortuity and intricacy. Fortuity refers to the extent to which probabilistic elements have been incorporated into a game. Board games with a low degree of fortuity incorporate a single random process or device (e.g., a dice roll each turn) whereas games with greater fortuity integrate several random processes. Intricacy is a measure of the decisions available to a player at any given point in the game. Board games that provide the player with a few simple choices have low intricacy while those that provide a large number of complicated choices have a greater degree of intricacy. Although fortuity and intricacy are independent of each other (e.g., it is possible to have a high fortuity and low intricacy), they are often linked. In particular, as board games increase in fortuity and more random processes are incorporated into gameplay, there is a tendency to increase intricacy thereby providing players with more choices to consider when making decisions. Both *Monopoly* and *Settlers of Catan* have relatively low fortuity because they incorporate simplistic random processes (e.g., dice rolls, random card draws) and present these processes clearly to players rather than embedding them into game mechanics. On the other hand, *Settlers of Catan* has a greater degree of intricacy than *Monopoly* because it offers players a variety of decisions during the game.

In addition to providing a method to categorize board games, the preceding criteria also furnish a means to connect board games to RL situations. The extent to which a board game can effectively model a given RL situation is related to the game's fortuity and intricacy. Games with a low degree of intricacy and fortuity are more suited to modeling RL situations with low density and complexity. These games have relatively simple rule sets, can be played quickly (often under an hour), and require minimal background knowledge. Consequently, they are well suited to modeling RL situations that present simple probabilistic information (low density) within a general context (low complexity).

On the other hand, board games with a high degree of fortuity and intricacy are able to model RL situations with a high degree of density and complexity. These games feature complicated rule sets, take longer amounts of time to play (four to six hours is typical), and require more extensive background knowledge. This allows the games to model RL situations with sophisticated probabilistic language (high density) that has been embedded into a specialized context (high complexity). Thus, these criteria provide a means to carefully select board games so that they closely model RL situations and provide a medium in which to investigate individual's thinking about risk.

Outlining a Theoretical Perspective

This article now offers one potential theoretical perspective for investigating RL situations using the medium of board games. This perspective is formed from the integration of two different theories on learning: intuition and dual-process theory. In what follows, I draw on Efraim Fischbein's work on the role of intuition in mathematical learning (Fischbein, 1987; Fischbein, & Gazit, 1984; Fischbein, Nello, & Marino, 1991; Fischbein, & Schnarch, 1997) and dual-process theory as describe by Kahneman (2002) and Leron and Hazzan (2006, 2009). Following a discussion of each theory, I then consider how they can be combined to form an integrated theoretical perspective for future studies.

Intuition

Fischbein's (1987) theory of intuition posits that intuitions have a cognitive-behavioral function. Fischbein defined an intuition to be "an *idea* which possesses the two fundamental properties of a concrete, objectively-given reality; *immediacy*—that is to say intrinsic *evidence*—and *certitude* (not formal conventional certitude, but practically meaningful, immanent certitude)" (Fischbein, 1987, p. 21). In other words, an intuition, or intuitive belief, is one that is immediately available to a person in a given situation and is held by the person (at least while they are expressing it) as being true. From this perspective, intuitions serve a similar behavioral function as perceptions in helping to guide our mental or practical activity. Fischbein argued that when faced with uncertainty or lack of information individuals naturally extrapolate beyond what they are given in order to make a decision. This is not, he contended, a unique event in one's existence. Rather, intuitions are part of our every-day experience. Fischbein noted,

> Promptly adjusted, well adapted reactions of a person to given circumstances are possible only if the perception of the respective reality appears to him, automatically, as coinciding with reality itself. Doubts, hesitations are useful only when referring to aims which are not directly involved in the current flow of behavior. When crossing the street, you have to believe absolutely in what you see—the approaching cars, the various distances etc.—otherwise your reactions will be discontinuous and maladjusted. Analogically, during a reasoning process, you have to believe—at least temporarily (but absolutely)—in your representations, interpretations or momentary solutions, otherwise your flow of thoughts would be paralyzed. It is this type of belief that we call an intuition. Cognitive beliefs, elaborated and confirmed repeatedly by practice, may acquire an axiomatic character." (p. 28)

Thus, Fischbein argued that intuitions reveal themselves when individuals are faced with making decisions. This is relevant to the setting of a board game, which routinely places players in situations where they must make choices in order to allow play to progress. These decisions rest in part upon their intuitions. Moreover, as Fischbein noted, intuitive beliefs can become axiomatic if they are reinforced by repeated practice. Repeated practice is common in board games where players must make the same or related decisions many times throughout the course of the game.

Fischbein saw intuitions as an essential aspect of human cognition and stressed the difference between a perception and an intuition. Perceptions are based upon one's senses and are typically correct. Intuitions, on the other hand, are "mental representations, ideas, hypothetical solutions [that] may be biased, distorted, incomplete, vague or totally wrong" (Fischbein, 1997, p. 28). Thus, it is possible to differentiate *perception of* a situation from *intuition about* the situation. Connecting back to the context of a board game, perceptions of a situation include the location of a player's pieces on the board, the amount of resources that other players have acquired, and the number of victory points one has acquired.

Intuitions, on the other hand, might include the likelihood that a player's current strategy will be successful, the anticipated moves of an opponent, or the chances that the outcome of a random process will be beneficial.

Another distinction made by Fischbein was between an intuition and a skill. An individual can be highly skilled at something without having any particular intuitions for that activity. For example, one can be very skillful at calculating probabilities for the sum of two six-sided dice without having any intuitive ideas about the roll. Intuition, on the other hand, "*is more than a system of automatized reactions, more than a skill or system of skills; it is a theory, it is a system of beliefs, of apparently autonomous expectations"* (Fischbein, 1987, p. 88). To this end, experience plays a critical role in shaping intuitions because it has the potential to stabilize expectations. Fischbein noted that expectations could become "so stable, so firmly attached to certain circumstances, that their empirical origin may, apparently, vanish from the subject's awareness" (p. 88). Whereas skills are learned through intentional practice, intuitions can be learned unintentionally through repeated experience. Consequently, the origin of an intuition may be unclear to the person holding it.

In his 1987 book, *Intuition in Science and Mathematics,* Fischbein reviewed the presence of intuition in a variety of literature including mathematics, science, philosophy, and art. He noted that the definition of intuition was varied across different disciplines and included descriptions of artistic clairvoyance, religious revelations, and scientific discovery. Drawing across all of these different examples, Fischbein identified nine properties of intuition that were shared across contexts. The first of these properties is that intuitive knowledge is immediate and self-evident. A summary of the properties is provided in Table 1.

Property	**Description**
Self-evidence and Immediacy	An intuitive cognition appears subjectively to the individual as directly acceptable. The individual does not see the need for extrinsic justification either in the form of a formal proof or empirical support.
Intrinsic Certainty	Intuitive cognitions are accepted as certain by the individual. Self-evidence and certainty are highly correlated but they are not the same thing. Certainty does not imply self-evidence nor does self-evidence imply certainty.
Perseverance	Intuitive cognitions are robust. Formal instruction aimed at providing conceptual knowledge can have little impact on an individual's intuitive background knowledge. It is possible for an individual to simultaneously hold erroneous intuitions and conceptual interpretations.
Coerciveness	Intuitions strongly affect an individual's reasoning by appearing to be absolute, unique representations or interpretations. Alternative interpretations are typically excluded or resisted.
Theory Status	An intuition is held by the individual as a theory and expressed in a particular representation using a model (paradigm, analogy, diagram, etc.). It is not a skill or perception.

Extrapolativeness	An intuition exceeds the available data. It is an extrapolation beyond the information at hand. Extrapolation alone is not enough to define an intuition, there must also be a feeling of certainty.
Globality	An intuition offers a global, synthetic view to the individual. It is concerned with the whole not the parts.
Implicitness	The activity of intuition is generally unconscious and the individual is only aware of the final product (i.e., the self-evident, intrinsically consistent cognitions).
Cognitive-Behavioral Function	Intuitions have the same behavioral function as perceptions; however, they are at a symbolic level. Consequently, intuitive cognitions prepare and guide both mental and practical activity.

Table 1. Nine Properties of Intuitions (Adapted from Fischbein, 1987)

In addition to these general properties, Fischbein also created two classification systems for intuitions. The first system categorizes intuitions by considering the relationship between an intuition and a solution to a problem. Using this scheme, an intuition falls into one of four categories: affirmatory, conceptual, anticipatory, or conclusive. The second classification system considers whether or not a particular intuition developed within the context of systematic instruction. This article focuses on Fischbein's second categorization (a more detailed description of the first categorization can be found in Fischbein, 1987).

Fischbein's second categorization system classifies intuitions developed independently of any systematic instruction as primary intuitions. These intuitions, he explained, were a result of one's personal experience.

> Our term 'primary intuitions' does not imply that these intuitions are innate, or *a priori.* Intuitions, both primary and secondary, are in fact learned cognitive capacities in the sense that they are always the product of an ample and lasting practice in some field of activity. (Fischbein, 1987, p. 69)

Thus, primary intuitions are those that individuals develop in out-of-school experiences. As noted, in order for these intuitions to develop there must be "ample and lasting practice." Consequently, primary intuitions should not be regarded as developing from one encounter with a particular idea, but rather as developing over the course of many such encounters.

Fischbein described secondary intuitions as cognitive beliefs that resulted from systematic instruction. This instruction must actively involve the learner in order for an intuition to develop. He noted,

> Such a process implies, in our view, the personal involvement of the learner in an activity in which the respective cognition play the role of necessary, anticipatory and, afterwards, confirmed representations. One may learn about irrational numbers without getting a deep intuitive insight of what the concept of irrational number represents. Only through a practical activity of measuring may one discover the meaning of incommensurability and the role and meaning of irrational numbers (Fischbein, 1987, p. 202)

As is evident in the passage, Fischbein stressed that secondary intuitions are formed only when an individual is involved in an activity that requires serious consideration of a particular idea. Thus, instruction that fails to actively engage the learner in meaningful analysis of a particular concept will not result in the development of secondary intuitions. Following from Fischbein's classification, both primary and secondary intuitions are *learned* through experience and develop throughout one's lifetime.

Dual-Process Theory

Dual-process theory has its roots in cognitive psychology (Kahneman, 2002) and has only recently been applied to mathematics education (Leron & Hazzan, 2006, 2009). The theory, which has grown from Amos Tversky and Daniel Kahneman's (1974) work concerning heuristics and biases, characterizes the mind as having two distinct systems, System 1 (S1) and System 2 (S2).

Like Fischbein's theory, dual-process theory concerns intuitions, but it takes a different perspective on their role in cognition. Dual-process theory considers the relationship between intuitive (immediate) and analytical modes of thinking and behavior. The central principle of dual-process theory is that cognition and behavior "operate in parallel in two quite different modes...roughly corresponding to our common sense notions of intuitive and analytical thinking. These modes operate in different ways, are activated by different parts of the brain, and have different evolutionary origins" (Leron & Hazzan, 2006, p. 108). The main difference between S1 and S2 is related to their accessibility. S1 is regarded as being "halfway between perception and (analytical) cognition" (p. 108). Its processes are fast, automatic, effortless, unconscious, and difficult to change. Additionally, S1 tends to contextualize and personalize problems. Decisions made by S1 are closely tied to the context of a problem.

S2 processes, on the other hand, are slow, conscious, flexible, and require effort. Unlike S1, S2 removes context and depersonalizes problems. It is more capable of creating rule-based representations and identifying underlying principles. Additionally, S2 can consider problems outside of a context.

The two systems are not isolated from each other. It is possible for skills, such as playing a particular board game, to migrate from S2 to S1. Initially, playing a board game requires a great deal of effort for novices because they must attend to the rules, understand how in-game actions are influencing gameplay, etc. However, as individuals repeatedly play a game and internalize the boundaries of the system, this skill may migrate from S2 to S1. As the context becomes familiar, in-game decisions, which were initially complicated and required a great deal of consideration, can transition from S2 to S1. Likewise it is possible for skills to migrate from S1 to S2. For example, walking along a straight line is normally handled by S1 in adults. However, if an adult is very tired this skill can require a great deal of effort and transition from S1 to S2.

Although the two systems are viewed independently, they often work together. S1 provides quick, automatic responses to appropriate situations while S2 serves as a monitor and critic of S1. However, this coordination between the two systems does not always operate well. S2 requires more effort and energy than S1. This means that, from a conservation of resources standpoint, S2 should only be used when there is a clear need. Research has documented problems in which S2 fails to engage for the majority of people (Kahneman, 2002; Leron & Hazzan, 2006). In these problems, an issue arises when S1 produces a quick (and often incorrect) response and S2 does not serve as an effective monitor. The Bat Problem, reported by Kahneman (2002) is one example,

> A baseball bat and ball cost together one dollar and 10 cents. The bat costs one dollar more than the ball. How much does the ball cost?
>
> Almost everyone reports an initial tendency to answer '10 cents' because the sum $1.10 separates naturally into $1 and 10 cents, and 10 cents is about the right magnitude. Frederick found that many intelligent people yield to this immediate impulse: 50% (47/93) of Princeton students, and

56% (164/293) of students at the University of Michigan gave the wrong answer. (p. 451)

According to dual-process theory, the specifics of this problem (i.e., the total cost of $1.10 and the bat costing $1 more than the ball) cause it to be solved quickly and erroneously by S1. The context appears straightforward and, for many people, S2 is not critical of the answer supplied by S1 because the problem does not present itself as needing such oversight. Thus, most individuals provide an incorrect answer of ten cents. Those who answer the problem correctly, in contrast, likely do so because of the involvement of S2 as either a critic of S1 or as the primary reasoning system. Research has shown that increasing the difficulty of the problem by changing the numbers yields more correct responses. It has been argued that this change in difficulty triggers the involvement of S2. Leron and Hazzan (2006) noted that many of the problems explored in probability research can be explained in terms of dual-process theory. They suggest that non-normative (i.e., different from an expert) responses to some problems may be the result of a failure by S2 to monitor S1.

The Integrated Perspective

Although the theory of intuition and dual-process theory are distinct, they complement each other. Fischbein's theory describes the characteristics of intuitions as well as their formation; however, it does not provide a detailed description of how these intuitions impact decisions. Dual-process theory, on the other hand, is concerned with the decision-making process rather than the specifics of components. These two theories can be combined to form an integrated perspective, which is modeled in Figure 2.

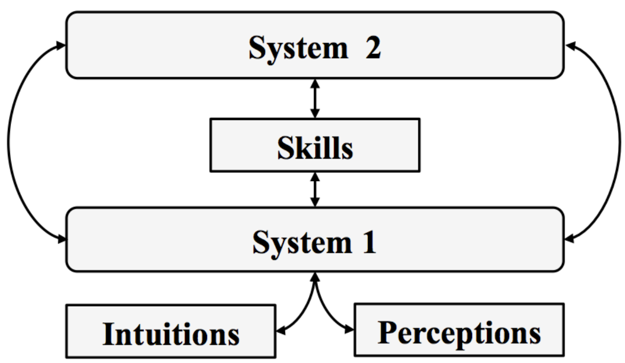

Figure 2. The integrated perspective

In this model, S2 is the higher order reasoning system with the capability to generalize and reason abstractly. However, this comes at a cost of effort and energy. S2 also serves as a monitor of S1. S1 is the immediate, context-based decision system. It's decisions come at a lower cost in terms of energy and effort but can be misguided by intuitions, perceptions, or skills. Intuitions are regarded as cognitions that result from either experience or systematic instruction and have the nine properties outlined by Fischbein (see Table 1). They are immediately available to S1 in a given situation. Perceptions are cognitions that are formed based upon available sensory information and immediately available to S1. Both intuitions and perceptions may be considered by S2, but they do not prompt direct action. Skills are effortful actions that are initially controlled by S2. If a skill is sufficiently practiced, it can become an automatic, internalized reaction that is available to S1.

There are a number of reasons this perspective is appropriate for use in studying an individual's thinking as they play board games. First, the perspective allows for the examination of immediate reactions (controlled by S1) as well as long-term strategies (controlled by S2). For example, in *Settlers of Catan* a roll of 7 triggers the movement of the robber. The player who rolled the 7 must move the robber to a new tile on the board and is then able to take resources from an opponent that controls a building on the tile. For some players, this action of moving the robber is immediate and based upon intuitions and perceptions (S1). These players focus on immediate risks or rewards. However, for others moving the robber is a much more analytical process (S2) that draws on information from a variety of sources (e.g., current game conditions, history with opponents). These players consider how moving the robber will influence their overall strategy. Consequently, their actions may defer immediate rewards in order to increase the likelihood of long-term success. Such a player may choose to use the robber as a tool to ally himself or herself with another player even though this may have negative short-term consequences.

Second, the integrated perspective provides an avenue for examining how the intuitive ideas that individuals hold about probabilistic phenomena as well as their perceptions influence in-game decisions. For example, an intuitive understanding of the probabilities of different six-sided dice roles may be an advantage in a game that incorporates quick decisions about dice outcomes (e.g., selecting tiles that are more likely to be activated). Perceptions may reinforce this intuition or appear to contradict it. For example, if a player observes a series of sums totaling two, they may ignore their intuitive understanding about randomness and adopt a misconception (e.g., the dice are not behaving normally).

Third, the perspective allows for opportunities to observe the migration of skills from S2 to S1. As noted, board game RL situations initially require a great deal of effort to understand because players must coordinate allowable actions with the likelihood of success. Consequently, these decisions are made by S2. However, as individuals repeatedly play a particular board game the in-game situations become familiar. The allowable actions become internalized and the players shift from novices towards experts. Thus, these repeated experiences with probabilistic phenomena may change from analytic cognitions made by S2 to intuitive cognitions made by S1.

Finally, the perspective provides a lens on the formation of primary intuitions. Playing a game requires individuals to repeatedly anticipate the likelihood of events and then react to the outcome. These decisions are not optional but a necessary part of gameplay. Moreover, variations of a particular event are typically repeated many times over the course of a game. Thus, players may begin to develop primary intuitions as the result of repeated play. These intuitions are considered primary because the experiences with board games are not within a formalized instructional setting.

Vignettes for Illustration

The following vignettes have been created to illustrate how this integrated perspective might be used as a lens to examine reasoning about risk within a board game setting. Although the vignettes are fictitious, they have been created to model actions and conversations that typically occur in playing the game *Settlers of Catan*.

Vignette 1: A Discussion During Game Setup

Erin, Kai, and Henry have decided to play *Settlers of Catan*. Erin and Kai are very familiar with the game having played it frequently over the last several months. Henry has only played the game once.

> *Henry:* I think I'm going to start with one town at the three-for-one port that is at the vertex of the fields and pasture tiles. I will be able to collect a lot of one resource and trade for another resource more easily.
>
> *Kai:* It's not worth your time to build on that port.
>
> *Henry*: Why do you say that?

Kai: Because you can only get resources from two tiles instead of the usual three. I don't think it is worth the risk just to gain the three-for-one trade ability.

Henry: You don't know that the town won't produce. The numbers, 9 and 10, aren't terrible.

Erin: Yeah, just because the tiles aren't a 6 or 8 doesn't mean they won't activate. I've seen a lot of games where 6 and 8 are hardly ever rolled.

Kai: I'm not saying that it won't produce. I'm saying that three-for-one isn't a really great advantage when you have to limit yourself to two tiles. You can trade four-for-one with the bank at any time or get a better exchange with other players.

The vignette begins with Henry articulating his strategy. He is attempting to develop a generalized strategy, which is focused on having a better trade ratio, that might benefit him throughout the game. Kai points out a risk of the strategy, i.e., building on the port means the town will only be on two tiles. Henry and Erin respond to Kai's comments with a focus on the likelihood of rolling the numbers on the tiles. This allows some information concerning their primary intuitions about probability. Henry's response that "the numbers, 9 and 10, aren't terrible" suggests he knows that 9 and 10 are less likely than sums of 6 or 8, but more likely than a sum of 2 or 12. Thus, although his original statement appeared to be focused on a trade deal, there is some evidence that he also considered the likelihood of the tiles producing resources. Erin's response appears to outline a primary intuition about the frequency of sums in a game that she has learned through repeated play. A question that might be asked is whether her response shows this experience has negatively influenced her probabilistic reasoning by suggesting that theoretical probability is not connected to real-world practice or positively influenced her reasoning by reinforcing that individual outcomes of a random event are uncertain. In his follow-up comments Kai indicates that he believes the risk prompted by not having a third tile is not worth the reward of a three-for-one trade ability. His focus is on the likelihood of a town generating resources when it is on three tiles versus when it is on two tiles. Overall, this excerpt demonstrates S2 at work in trying to generalize and abstract a RL situation. The comments reveal how each player's perceptions of the board and probabilistic intuitions are being used by S2 to form a general strategy.

Vignette 2: A Midgame Discussion

As play progresses, Henry develops an early lead with seven points. Erin is close behind with six points (recall that the first player to score 10 points is the winner). On his turn, Kai rolls the dice and gets a sum of seven. This triggers movement of the robber. He decides to move the robber onto a tile that is bordered by towns from both Henry and Erin. As part of the robber action, Kai must take a card from one of the players and he decides to take a card from Erin.

Erin: Why are you taking a card from me? Henry is in the lead.

Kai: I don't think he is going to win.

Erin: He's only four points from victory!

Henry: And I've been able to use my port effectively to get goods.

Kai: Your towns are mostly on tiles with unlikely numbers. So I doubt you're going to keep benefiting from dice rolls. Besides you need to upgrade some of your towns to cities in order to win and even if your tiles produce they wouldn't provide the right resources.

Henry: Well I disagree. The dice have been in my favor today.

Erin: It's like I said before, you really can't predict the roll. In some games a 6 and 8 are hardly ever rolled. That's just how it goes.

Henry: Yeah, I've been doing a pretty good job of rolling what I need. It's all technique.

Kai: (Shaking his head). Look, Erin is just about to claim the longest road card giving her two more points. She is also close to having the largest army card, which is another two points. She can end this game in two rounds.

Henry: We'll see about that. (Rolls a seven). Ha, see that? Told you I've got the skills. (He quickly moves the robber to a tile bordered by Kai and takes a card).

Kai: Why would you do that? Erin is clearly in the lead here.

Henry: I disagree. Besides you have a lot of cards.

Kai: But only one of my cards is a resource that you need for upgrading. Most of Erin's cards are resources that you need. That move doesn't make sense.

This vignette demonstrates probabilistic decisions being made *in-action* by players. Kai's comments indicate that he is assessing the overall progress of the game and weighing the states of each opponent (S2). His assessment that Henry is not likely to win is based on information about the likelihood that Henry's towns will produce combined with knowledge that even if the towns did produce the resources would not be immediately helpful. Henry's comments indicate that he is focused on more immediate conditions (S1). He also appears to have intuitive beliefs about controlling the outcome of a dice role indicating probabilistic misconceptions. Moreover, his reaction to the roll of a 7, which focuses on taking a card from Kai, is a quick judgment that appears to be in response to Kai's comments rather than based on in-game conditions. For her part, Erin's comments suggest that she is aware that the individual outcome of a random event is uncertain. Overall, this excerpt demonstrates how players might be using S1 and S2 to make decisions about a given situation. Kai's decisions indicate that he is still trying to generalize and predict future outcomes. Henry, on the other hand, appears to be making quick, in-the-moment decisions without focusing on the overall risks presented by Erin.

Concluding Thoughts

This article has sought to illuminate one possible avenue for studying individual's thinking about risk. As with any research agenda, there are some initial challenges that must be addressed. In particular, researchers must decide which individuals to include, which board games to use, and the specific RL situations they will investigate. For example, it seems likely that such investigations would occur both within and outside of traditional classrooms and draw on a small set of games with a limited range of fortuity and intricacy. Additionally, it is probable this research would involve individuals from a variety of age ranges some of whom are likely well passed school age. The study of such environments will require frameworks that have the flexibility to examine how various cognitions (intuitions, perceptions, skills, etc.) influence both long-term and short-term decisions. The integrated theoretical perspective outlined previously may offer this flexibility; however, it should be seen as an initial framework in need of refinement and extension.

Challenges not withstanding, I believe that the possibilities of this medium are exciting. Work within the field of mathematics education has already laid the groundwork for meaningful investigations and the popularity of board games has grown greatly in the last decade. If the field wishes to investigate thinking about this topic, it seems reasonable that we do so in ways that can model the prevalence of probability in modern life and the diversity of contexts in which individuals encounter RL situations. Moreover, the experiences should have some authenticity for participants and be dynamic. Board games offer one potential avenue with a great diversity in design and components. This medium has the potential to engage individuals in authentic RL situations while at the same time providing researchers with some control over the system. Thus, the atmosphere appears right for mathematics education researchers to take advantage of the medium of board games for studying RL situations. As a community, it is time to take a seat at the table, roll the dice, and make the next move.

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