

The Mathematics Enthusiast

Volume 12
Number 1 *Numbers 1, 2, & 3*

Article 17

6-2015

Teaching Risk in School

Andreas Eichler

Markus Vogel

Follow this and additional works at: <https://scholarworks.umt.edu/tme>



Part of the [Mathematics Commons](#)

Let us know how access to this document benefits you.

Recommended Citation

Eichler, Andreas and Vogel, Markus (2015) "Teaching Risk in School," *The Mathematics Enthusiast*. Vol. 12 : No. 1 , Article 17.

Available at: <https://scholarworks.umt.edu/tme/vol12/iss1/17>

This Article is brought to you for free and open access by ScholarWorks at University of Montana. It has been accepted for inclusion in *The Mathematics Enthusiast* by an authorized editor of ScholarWorks at University of Montana. For more information, please contact scholarworks@mso.umt.edu.

Teaching Risk in School

Andreas Eichler

University Kassel, Germany

Markus Vogel

University of Education Heidelberg, Germany

Abstract: Although risk is an important topic for society it is seldom addressed when teaching statistics and probability. In this paper we refer to this discrepancy identifying three obstacles for teaching risk in school regarding the mathematical and the situational aspect of risk. Based on two educational constructs, i.e. probability literacy and modelling, we discuss existing approaches for teaching risk in school and propose two strategies for promoting risk as a valuable issue for students based again on the distinction of the mathematical and situational aspect of risk.

Keywords: risk, risk literacy, probability literacy, modeling.

Obstacles for Teaching Risk in Schools

Although risk is understood as a “part of our everyday lives” (Erickson, 2010) and risk is “a hot topic” (Spiegelhalter, 2014) that should be taught in school (cf. Gigerenzer, 2013), this hot topic seems to play at most a minor role in school, particularly if mathematics teaching is regarded. We approach this only seemingly paradox with the following situation (Latten, Martignon, Monti, & Multmeier, 2011, p. 21) that was developed for mathematics teaching:

Is it more risky to possess an Alsatian dog (German shepherd) than a Chihuahua? It is known that four out of nine Chihuahuas and three out of ten Alsatian dogs bite their owner at most once.

The specific task was to compare the “risk” of possessing an Alsatian dog or a Chihuahua. The intended answer was that it is more risky to possess an Alsatian dog, because the result of a bite from this dog breed is considerably worse than a bite from a Chihuahua, although the probability of a Chihuahua’s bite is bigger than that for a bite from an Alsatian dog. This task includes a first obstacle for teaching risk in school. The obstacle is not the dog-task itself, but the amount of different answers. Following different authors who work on risk issues, we found mainly three possible different answers:

1. As mentioned above, the answer of the task developers is that the bite of the Alsatian dog represents the bigger risk. This answer is based on the definition of risk as the arithmetical product of a measure of the dis-utility represented by a random event and the probability p of this event, i.e. $risk = p \times \text{measure of the dis-utility of an event}$ (Latten et al., 2011). A disadvantage of this definition is the problem of measuring a dis-utility (c.f. Kent, Pratt, Levinson, Yogui, & Kapadia, 2010). Of course a bite from a dog represents a dis-utility. However it is not clear how to measure the loss of healthiness based on a dog’s bite. Further, the mentioned arithmetical product does not exist in certain cases. For example a bite from the white killer shark (in Steven Spielberg’s film Jaws) results in a person’s death. Thus, if we define the dis-utility of a person’s death as infinite the risk of a bite from the killer shark is infinity. However, the definition of risk

discussed above is often used in the literature and is close to the theory of the Subjective Expected Utility, a theory of human decisions (Savage, 1954).

2. In contrast, Gigerenzer (2013, pp. 39-40) defines risk as a probability of an event that is measurable. The event could represent both good luck and bad luck or rather “nice or nasty” outcomes (Spiegelhalter, 2011). A disadvantage of this definition is the unclear distinction between risk and probability. Although Gigerenzer (2013) discusses a lot of examples that intuitively could be assigned to the construct of risk (e.g. financial risks, diseases, games of chance), nearly every probability of a random event could be understood as a risk since nearly every random event represents in a certain sense a positive or a negative outcome. For example, both the probability of a bite from an Alsatian dog and the probability of a 6 when throwing a die represent a risk. The definition of Gigerenzer (2013) is close to the (economical) risk definition of Keynes (1921), who defines the Laplacean probability and the frequentistic probability as risk, but a subjective probability as (a person’s) uncertainty.

3. On a first view, the third “definition” includes the demand not to define the term risk: “Risk is a strange concept. Different disciplines have tried to define it precisely, but perhaps it is better to be informal” (Spiegelhalter, 2011, p. 17). On a second view, risk is implicitly defined as a situation’s characteristic consisting of a pair of a probability and the related random event that mostly could be understood as a bad event. Also this definition has the difficulty that there is no clear distinction between a situation with risk and a situation of uncertainty (without risk). For example, a pair of a dog’s bite and the related probability could be a characteristic of a risky situation as well as the pair of a 6 when throwing a die and the probability of getting a 6. However, we follow in this paper this third definition that is as a specific situation’s characteristic also used by the International Risk Governance Council (IRGC, 2005; Boeckelmann & Mildner, 2011).

Besides these three definitions, there are further aspects of the construct of risk that we mostly neglect in this paper. For example, many researchers refer to the human influence on risk (Slovic, 2010) or rather the human inability to cope with risks (e.g. Kahneman, 2011). Further, another aspect of risk is the distinction between calculable situations and not calculable situations. The latter form of situations is discussed as unknown unknowns (e.g. Boeckelmann & Mildner, 2011; Gigerenzer, 2013) or black swans (Taleb, 2007).

The plurality of definitions of the concept of risk and the fact that not all definitions are appropriate to confine the concept of risk to already existing concepts, e.g. probability, is a first obstacle for teaching risk in school, particularly, for teaching risk as a part of mathematics. Thus, the plurality of definitions and the absence of clear differentiation from other existing and well defined concepts seem to impede a clear conception of a risk curriculum. E.g. Kent et al. (2010) state that “risk is a difficult topic for teachers and the detailed questions about what and how to teach remain (for mathematics especially) largely unanswered”.

A further obstacle for teaching risk in school could be caused by the situations that are connected with risk in the related literature and, particularly, in daily media. Thus, risky situations are often nasty, complex and refer to data for which the process of data generation is hard to understand. For example, a lot of risky situations include nasty outcomes, i.e. include a heavy loss of healthiness (diseases, accidents) or a heavy loss of financial resources (economics, games of chance). However, it should be scrutinized whether situations referring to cancer or death by accident are suitable for teaching mathematics in school. Further, risky situations that are adequate for teaching mathematics in school

must be graspable for students in a mathematical sense. Finally, the demand of a data driven curriculum (Moore, 1997) includes that data and data generation are crucial issues for statistics and probability teaching. By contrast, for example, it seems to be a big challenge to unfold the generation of data that are necessary to judge the risk of using different forms of transportation (walking, cycling, motorbike, car; Spiegelhalter, 2011).

Although there are possible obstacles for teaching risk in schools regarding the *mathematical aspect* of risk in terms of an appropriate definition as well as referring to the *situational aspect* of risk, the demand to improve students' risk literacy as a crucial skill of "informed decision making" (Till, 2014) in situations of uncertainty is, nevertheless, important for teaching mathematics in school. For this reason, we integrate firstly the mathematical aspect and the situational aspect of risk into overarching models of teaching mathematics or teaching statistics and probability. Afterwards we allocate existing approaches for teaching risk or risk literacy to these educational models. Finally, we propose different examples aiming to reduce the obstacles of teaching risk in school.

Probability Literacy and Risk Literacy

Gal and Garfield (1997, p. 3) give a meaty fundament to reflect what is important to be reached as an overarching goal at the end of the curriculum: students should become "[...] informed citizens who are able to [...] comprehend and deal with uncertainty, variability, and statistical information in the world around them, and participate effectively in an information-laden society". Referring to this fundament, the issue of decision-making under uncertainty is close to the aim of risk literacy as mentioned above. As with this brief summary of the overarching goals, other well-known conceptions for overarching goals of teaching statistics and probability seem to integrate risk – partly without mentioning the term risk, or risk literacy. For example, Franklin et al. (2005, p. 1) emphasize the ability to deal with "decisions in our personal lives". Of these conceptions we particularly focus on Gal's definition of statistical literacy (Gal, 2002) and probability literacy (Gal, 2012) since both conceptions include explicitly both the mathematical aspect of dealing with decisions of uncertainty and the situational aspect. For example, Gal (2002) refers to a statistical and mathematical knowledge base (mathematical aspect) and to world knowledge and critical skills (situational aspect) as main aspects of statistical literacy. Similarly, Gal (2012) describes "probability literacy" as a part of the broader construct of statistical literacy. He defines probability literacy as follows: "The ability to access, use, interpret, and communicate probability-related information and ideas, in order to engage and effectively manage the demands of real-world roles and tasks involving uncertainty and risk." (Gal, 2012, p. 4). For the sake of emphasizing the risk aspect Till (2014) speaks of "risk literacy" when he refers explicitly to this definition of Gal (2012). For probability literacy Gal (2012) defines two points of view as being constitutive:

- The internal view addresses the students' learning within the pure mathematical world by understanding and applying logic, abstract concepts, proofs, formulas, graphs, etc. In general, it is about dealing with mathematical objects and mathematical representations or computing probabilities in specific probability models. The internal view could represent the mathematical aspect of risk.
- The external view means basically that teaching probability should (beyond the internal view) comprise the role of probability-related ideas when for example talking with students about sampling or testing hypotheses. At its broadest level the role of probability-related ideas should

be seen within their potential to contribute to the educational goal that the students can become informed citizens. In this broad sense, the external view represents the situational aspect of risk.

Both views are intertwined. Thus, when focusing on teaching risk within this perspective of Gal's definition of probability literacy those problem situations, which are feasible for the students with regard to both the situational context and the situation's underlying mathematical structure, become interesting.

Probability literacy as well as risk literacy fit also the main aspects of modelling as an overarching conception of mathematics education (Blum, Galbraith, Henn, & Niss, 2007). Thus, risk in its mutual relationship (cf. Eichler & Vogel, 2012) between real-world affairs comprising real situations and empirical data, and mathematical-world affairs comprising probabilities could be seen as a specific form of modelling (fig. 1) referring to the following four modelling phases:

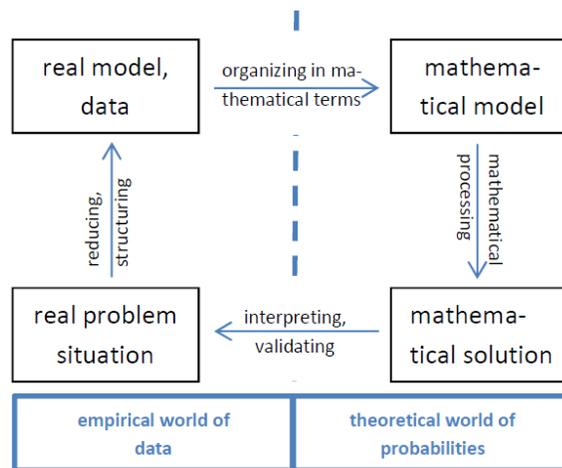


Figure 1. A model of modelling

- A real problem situation concerning risk has firstly to be structured and reduced to the relevant information.
- The so-called “real model” (Blum et al., 2007) is the basis for applying mathematics originating from the theoretical world of probabilities, for example relating two random events and the concept of probability to each other by asking for a certain conditional probability within a risky situation.
- Based on these mathematical models some mathematical results can be derived like, for example, fixing a concrete probability as a risk.
- These mathematical results have to be translated in the real-world, e.g. for a decision in a risky situation.

The underlying educational idea of modelling also refers to teaching risk based on probabilistic concepts and methods. Firstly, it is about students' ability to access real-world problems with their available mathematical concepts and methods: “Our mathematical concepts, structures, ideas have been invented as tools to organize the phenomena of the physical, social and mental world.” (Freudenthal, 1983, ix). Secondly, it is about students' ability to deepen their understanding of mathematics and to develop new mathematic concepts and methods (c. f. Klieme, Neubrand, & Lüdtke, 2001, 143).

Both ideas can be interpreted in the light of Gal's (2012) points of views mentioned above: The internal view of educational goals of probability education can be seen in relation to the second idea, in

concrete, the deepening of understanding of probabilistic concepts and methods; the external view can be seen in relation to the first idea, in concrete, enhancing intellectual accessibility to risky problem situations or situations comprising uncertainty applying available knowledge about probabilistic concepts and methods.

Existing Conceptions of Teaching Risk

Independently from the obstacles that we identified above, there exist several educational approaches for teaching risk in schools. We report some of these ideas referring to our theoretical framework including

- the differentiation of teaching risk and teaching statistics (or probability), and
- ideas for dealing with the internal view (mathematical aspect) and with the external view (situational aspect) of risk.

Not surprisingly, researchers that address the question of risk literacy, predominately rank teaching risk among teaching statistics and probability (e.g. Latten et al., 2011; Martignon & Monti, 2010). In addition, Kent et al. (2010) explicitly mention risk also as a part of science teaching in UK.

Regarding the internal view, an explicit risk curriculum is missing. However, Till (2014) following Martignon and Krauss (2009) mention proportions, conditional probabilities and expected values as a basis for risk literacy. Implicitly, Gigerenzer and his team at the Harding Center for risk literacy (<https://www.harding-center.mpg.de/en/>) give notes for possible subjects of a risk curriculum when listing technical terms referring to the rubric of health information that are a basis for risk literacy. These technical terms encompass concepts that are common for a probability curriculum except of the concept of distribution, e.g. probabilities (and frequencies), conditional probabilities, Bayes' formula, independence. A substantial extension of the probability curriculum is represented by the crucial distinction of absolute risk and relative risk (e.g. Gigerenzer, 2013).

A main focus of the approaches to teaching risk literacy, which we have quoted in this paper, is on the interface between the internal view (mathematical aspect) and the external view (situational aspect) of risk in terms of fostering understanding and interpreting risk situations by using specific representations of the situations. The tree diagram with natural frequencies is a main part of these representations (e.g. Gigerenzer, 2013) since research yielded its potential for solving tasks in risky situations (Sedlmeier & Gigerenzer, 2001). Other representations that are often discussed are for example fast and frugal trees (e.g. Laskey & Martignon, 2014) or colored cubes in a hands-on activity (e.g. Kurz-Milcke & Martignon, 2006). Following Spiegelhalter and Gage (2014), who on their part took the concept from Gage (2012) the use of physical models (the colored cubes) and multiple representations is a core for teaching probability (and risk).

Although there are situations of risk with nasty outcomes in the literature (e.g. Kent et al., 2010, referring to a "Painometer" or Eichler & Vogel, 2014 referring to HIV), there are a lot of attempts to transfer main elements of risk literacy to situations in which the risky event is far from being a threat for students. This is especially the case if risk literacy is addressed in primary schools. For example, Till (2014) uses the situation of snails biting plants in a garden (task referring to proportion) or focuses on the situation that a person has to judge the gender of a person when knowing a specific characteristic of the person like the hair length (Bayesian formula). Another example for secondary schools is given by Erickson (2010), who addresses, in a playful way, insurance problems. Although we did not find a

clearly given rationale to use situations to promote risk literacy that are not nasty it seems to be an undeclared demand to mostly avoid these nasty situations for teaching risk in schools.

The mentioned conceptions for teaching risk in school stem from educational literature. However, from a teacher's perspective, the first and foremost question in preparing a course concerning risk is the question of available material in text books. Kapadia (2008) states with regard to an analysis of mathematical text books used in England: "It is certainly true that the majority of teachers make links to real-life situations, indeed they often occur in text-books. But the lack of reference to risk is very important in the context of the current world." A corresponding screening of a small sample (n=13) of well-known mathematical text books for secondary school in Germany yields a similar result: 307 tasks were taken into account as touching different contextual themes in the field of making decisions under uncertainty and handling risk in a broad sense. For example these tasks include questions about tests concerning production quality and production errors, questions about elections and predicting probable outcomes of elections, questions about the principle of flights overbooking including calculations of possible losses, about elementary insurance problems, and about the diagnostic power of different health tests. These tasks address mathematical competencies going beyond those mathematical competencies like proportional reasoning, dealing with expected values and understanding conditional probabilities which Martignon and Krauss (2009) call "tool box for decision-making and reckoning with risk". The main mathematical topics of the investigated upper-secondary tasks were: conditional probabilities (including Bayes formula), distributions (binomial and normal), testing hypotheses and confidence intervals. With regard to the tasks' contextual background it was interesting to see that in less than 5% of the regarded 307 tasks the word "risk" was used explicitly. Although we do not raise the claim of representativeness our finding meets the statement of Kapadia (2008) quoted above.

A Proposal for Dealing with Obstacles for Teaching Risk in Schools

We identified three obstacles for teaching risk in schools. The first obstacle refer to the mathematical part of risk and, more precisely, is based on the absence of a mathematical definition of the construct of risk and on a difficult delimitation of risk to existing concepts of probability. A first proposal for promoting risk as a part of mathematics teaching includes aspects

- to understand risk as a pair including a mathematical aspect and a situational aspect,
- to identify the difference between risk and an uncertain situation that we do not understand as risk, as only being related to situational aspects but not to the mathematical aspects. Thus, risk is inextricably intertwined with interpreting the real world, but does not exist solely in the mathematical world;
- to acknowledge that referring to situational aspects or rather modelling is one of more goals of mathematics teaching. Thus, both the internal view and the external view in their broadest sense are important issues for improving probability literacy.

Secondly, another obstacle refers to emphasizing risky situations including nasty outcomes as a part of mathematics teaching. Our proposal of promoting the teaching of risk in school is to consider more explicitly the situational aspect of risk and to vary more explicitly risky situations with nasty outcomes and situations that are structurally equivalent but have no nasty outcomes.

Finally, the third obstacle, i.e. the nearly complete absence of the construct of risk in text books might be a consequence of the two obstacles summarized above. For this reason, we neglect the third

obstacle in the following, but discuss an example that illustrates our proposal for promoting teaching risk in school.

The Contextual Side: Varying Situations

We differentiate three kinds of problem situations (Eichler & Vogel, 2009; Eichler & Vogel, 2014):

- Problem situations which contain all necessary information and represent a concept of probability (or statistics) like a conditional probability in a strongly reduced context or virtual context we call *virtual problem situations*.
- Problem situations which demand analyzing a situation's context that is more "authentic" and provide a "narrative anchor" (Cognition and Technology Group at Vanderbilt, 1990), but still provide demands that do not aim to reconstruct real world problems we call *virtual real world problem situations*.
- Problem situations that include the aim to reproduce real problems of a society we call *real world problem situations*.

These problem situations differ with regard to their level of the situation's reality. Varying available material along these different levels of reality is based on the idea of reflecting on a certain probabilistic concept and/or method in a continuum between the empirical world of society, nature and techniques which comprises the whole complexity of reality including risks and the theoretical world of pure mathematics, in specifically the world of probabilistic concepts and methods. We give an example of the three worlds referring to the same mathematical structure.

A *real-world problem situation* that is both strongly related to the term of risk (e.g. Gigerenzer, 2013) and connected to a very nasty outcome is an HIV-test. The interpretation of the result of an HIV-test is further doubtless a highly relevant topic of public health debates all over the world. The core problem can be captured by tasks like the following which we found in different variations in our screening of available materials in German text books (cf. as well Eichler & Vogel, 2010):

How reliable is an HIV-test?

In Germany the prevalence of HIV infection is estimated to be 0.1%, according to official sources. An HIV test which is usually conducted as a multi-stage process can provide a high level of certainty regarding the presence of the infection.

At the first stage such an HIV test has the following characteristics: sensitivity and specificity. The sensitivity of an HIV test gives a correct indication of infection in 99.9% of all cases tested. The specificity of the test means that 0.5% of cases will be wrongly identified as having HIV, although infection is not present.

If a person's first test yields a positive result, how certain can this person be that he or she is actually infected?

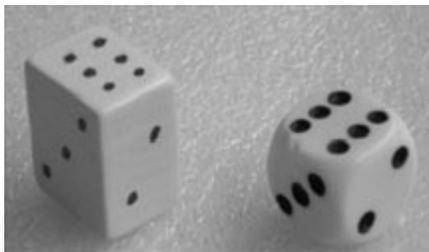
Research results of the working group of Gigerenzer (2004) yielded that even for professional physicians it is not easy to grasp such conditional probability problems concerning appropriate interpretation of medical test results. In terms of the modelling cycle presented above, a possible modelling structure could be as follows:

- The real problem situation here is the question off the reliability of an HIV test given the situation as described in the task.
- Building the so-called real model would mean extracting the relevant information given in the text (i.e. base-rate; sensitivity and specificity of the test), relating them to each other and setting them in relation to what is asked for in the task, a measurement for the certainty of really being infected given a positive test result.
- Transforming the information of the real-world into the mathematical world can be described as three-stepped: Firstly, the relevant information is to be captured within terms of (conditional) probabilities, here it would be (with T+ standing for a positive test result and HIV- standing for not being infected): base-rate $P(HIV) = 0.001$ (which implies $P(HIV-) = 0.999$), it is given here proactively only because of its need in the Bayes formula), test sensitivity $P(T+|HIV) = 0.999$ and test specificity $P(T+| HIV-)=0.005$. In a second step the requested information, i.e. the probability of being really infected has also to be expressed mathematically by using conditional probabilities: $P(HIV|T+)$. Third, these probabilities have to be applied to the Bayes-formula:

$$P(HIV|T+) = \frac{(P(T+|HIV) \cdot P(HIV))}{(P(T+|HIV) \cdot P(HIV) + (P(T+|HIV-) \cdot P(HIV-))} = \frac{0.999 \cdot 0.001}{0.999 \cdot 0.001 + 0.005 \cdot 0.999} = 0.17$$

- This mathematical result, i.e. 17%, has to be transformed back into the real-world and interpreted in the light of the given information: Following the findings of Gigerenzer's research most of the people having heard from a positive HIV-test result would probably not think that the risk to be really infected is that low.

Of course, maybe there are students who know this topic very well because of being touched with the underlying problem by knowing people being HIV-infected within their family or their circle of friends. In this case, it is questionable from a pedagogical point of view whether one should tackle such an example in this class if the teacher decides to tackle it he or she has to be very careful. However, it is possible to integrate the same mathematical structure in an (uncritical) *virtual real world problem* that is shown below (c.f. Eichler & Vogel, 2014).



Students are playing a game along the following rules (c.f. Riemer, 1991): There are small groups of students each with one referee. The referee has two dice, an ordinary unbiased die representing an equal distribution of probability and a cuboid shaped die with the distribution: 1–0.05; 2–0.10; 3–0.35; 4–0.35; 5–0.10; 6–0.05. The dice are covered for the other members of the referee's group. The referee has covertly chosen one of the dice and throws it repeatedly. The other students try to find out which die the referee has chosen by processing information about the outcomes of each throw and by estimating step-by-step the probabilities for the die being covertly thrown.

From a mathematical perspective, the die task is structurally equivalent to the HIV test (fig. 2), particularly, if one result of throwing the die is given.

HIV-task	Die-task
available information prevalence: $P(HIV+) = 0.001$ thus, $P(HIV-) = 0.999$ sensitivity of the test: $P(T+ HIV+) = 0.999$ specificity of the test: $P(T+ HIV-) = 0.005$	available information prevalence: $P(N) = 1/2$ (normal die) thus, $P(C) = 1/2$ (cuboid die) sensitivity of the test: $P(6 N) = 1/6$ specificity of the test: $P(6 C) = 0.05$
calculation: $P(HIV+ T+) = \frac{P(T+ HIV+) \cdot P(HIV+)}{P(T+ HIV+) \cdot P(HIV+) + P(T+ HIV-) \cdot P(HIV-)}$ $= \frac{0.999 \cdot 0.001}{0.999 \cdot 0.001 + 0.001 \cdot 0.005} = 0.17$	calculation: $P(N 6) = \frac{P(6 N) \cdot P(N)}{P(6 N) \cdot P(N) + P(6 C) \cdot P(C)}$ $= \frac{1/6 \cdot 1/2}{1/6 \cdot 1/2 + 1/20 \cdot 1/2} = 0.77$

Figure 2. Structural equivalence between the HIV-task and the die-task

The risk regarding this virtual reality could be defined. Consider for example the situation where the correct conclusion regarding which die was thrown when showing a certain number e.g. 6 yields a win of 5 Euros, but the false assumption costs one billion Euros. This would be a very risky situation compared to the situation where a correct or incorrect answer would have no consequences outside the game.

The same mathematical structure could also be integrated in *virtual world problem situation*. As mentioned before, we suggest reducing the complexity of the situation by keeping the mathematical structure. Here in the context it could be a short comic-story like the following one:

For many years Kurt has had a rainmaker friend who is able to make it rain by singing his special rain song. Because it is very exhausting the rainmaker only sings on 10% of all days. Gradually year by year, the rainmaker loses his magic power increasingly and he now succeeds only 80% of the time in making it rain when he sings his special rain song.

One day in May, Kurt had his first rendezvous with Karla, the beauty queen of his class. Kurt chose May for his date because he knows that in May it rains only 10% of the days (without accounting for the rainmakers influence) and thus, the probability of having a bad hair day seems to be low. All went perfectly, but just right at that moment when he wanted to kiss her it started to rain, and she leaves not being kissed because of having her hairstyle totally destroyed.



Kurt is frustrated and he suspects his rainmaker friend. On what probability for being guilty is Kurt justified in suspecting the rainmaker?

With regard to the purpose of concentrating closer on the mathematical structure a virtual problem situation has the advantage of being freely constructible for the mathematical narrator. Thus, exaggerations and simplifications are possible as well as variations of story influencing parameters which can be used to emphasize structural aspects of the underlying mathematical model like the Bayesian formula. Regarding risk, a potential nasty outcome in this situation is always a virtual nasty outcome. A further advantage of virtual world problem situations is that students accepting such a simplified virtual world situation could refer more directly to the situation’s underlying mathematical structure, because understanding and processing the contextual information is not as hard. The modelling cycle turns out to be useful for describing the solutions process’s structure of a virtual problem situation of decision–making under uncertainty by transferring its terms and considering them in parallel (fig. 3).

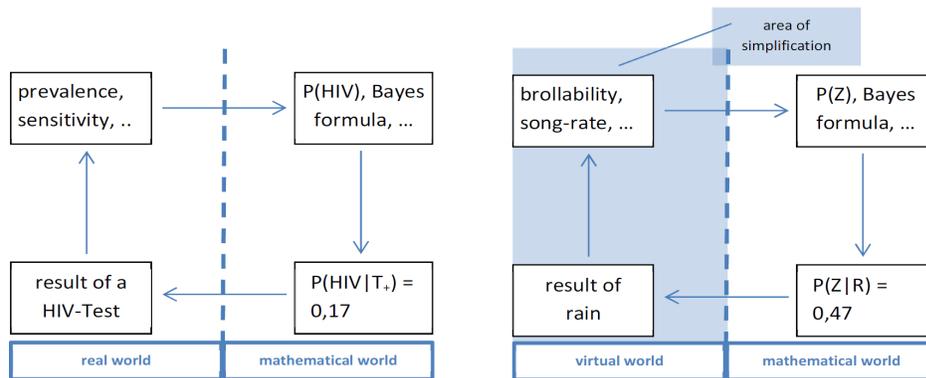


Figure 3. Analogies between solving the HIV-problem and the rainmaker-problem

In this parallel point of view, it turns out that the problem solving structure of a virtual problem situation of uncertainty like the rainmaker-problem can be seen as being similar to the modelling structure of the HIV example discussed above. A decisive difference is that a serious real-world problem situation is exchanged for a simplified virtual real-world problem situation, thus, the students have to what can be called “re-model” the virtual situation prepared by the teacher.

Again, from a mathematical perspective, the rainmaker task is structurally equivalent to the HIV test (Figure 4).

HIV-task	Rainmaker-task
<p>available information prevalence: $P(HIV+) = 0.001$ thus, $P(HIV-) = 0.999$ sensitivity of the test: $P(T+ HIV+) = 0.999$ specificity of the test: $P(T+ HIV-) = 0.005$</p> <p>calculation:</p> $P(HIV+ T+) = \frac{P(T+ HIV+) \cdot P(HIV+)}{P(T+ HIV+) \cdot P(HIV+) + P(T+ HIV-) \cdot P(HIV-)}$ $= \frac{0.999 \cdot 0.001}{0.999 \cdot 0.001 + 0.001 \cdot 0.005} = 0.17$	<p>available information prevalence: $P(Z) = 0.1$ thus, $P(Z) = 1 - 0.1 = 0.9$ sensitivity of the test: $P(R Z) = 0.8$ specificity of the test: $P(R Z-) = 0.1$</p> <p>calculation:</p> $P(Z R) = \frac{P(R Z) \cdot P(Z)}{P(R Z) \cdot P(Z) + P(R Z-) \cdot P(Z-)}$ $= \frac{0.8 \cdot 0.1}{0.8 \cdot 0.1 + 0.1 \cdot 0.9} = 0.47$

Figure 4. Structural equivalence between the HIV-task and the rainmaker-task

At the end, we claim a psychological argument for considering the variation of situations including potentially a risky situation: Referring to situated learning theory Brown, Collins, & Duguid (1989) argue that in general, learning and cognition are fundamentally situated. Following this approach

learning situations (HIV, dice, rainmaker) do not need necessarily to be real in the sense of what we call real-world problem situations (see above). It is not the level of reality but the level of the perceived authenticity and realism of a learning situation containing multiple possible perspectives on the problem (cf. Mandl, Gruber, & Renkl, 1997) which turns out decisive for coping with the situation. The internal logic structure of the situation must be consistent, traceable and comprehensible. By following the goal to prevent the so-called inert knowledge, the research group Cognition and Technology Group at Vanderbilt (1990) developed and applied in practical oriented research the theory of anchored instruction. The central characteristic of this approach is its narrative anchor by which the interest of the learners should be produced and the problem should be embedded in the context so that situated learning can take place. Thus, within a pedagogical interpretation of these theoretical approaches of situated learning and anchored instruction and a basic implication of cognitive load theory (Chandler & Sweller, 1991) the learning about structural mathematical concepts and methods like proportional reasoning or probabilistic argumentation is expected to be facilitated when we reduce the complexity of the problem situation with should students should cope. This includes virtual worlds like for example being given in tales, narratives or comic stories.

The mathematical side: Understanding structures

When concentrating on the mathematical aspects of teaching the Bayes formula the formula’s structure comes into the foreground but not only the probabilities resulting from calculations with the formula. The student will not understand what the formula $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A) \cdot P(A)}{(P(B|A) \cdot P(A) + (P(B|A-) \cdot P(A-))}$ mathematically means by “only” calculating one example after another in classroom. For teaching purposes the mathematical relationship of the formula-combined probabilities becomes important with regard to static and dynamic aspects of variation. To reach this by teaching the fundamental idea is not to focus on the single representation of the Bayes formula but, by following Spiegelhalter and Gage (2014) to transform it into multiple representations which open up different perspectives on the underlying mathematical structure. It is necessary to transform the Bayes formula into different representations.

As already mentioned above, a very common and often used representation is the tree diagram, either with probabilities or by following Gigerenzer (2013) with natural frequencies. In case of our rainmaker problem this would appear as is shown in figure 5.

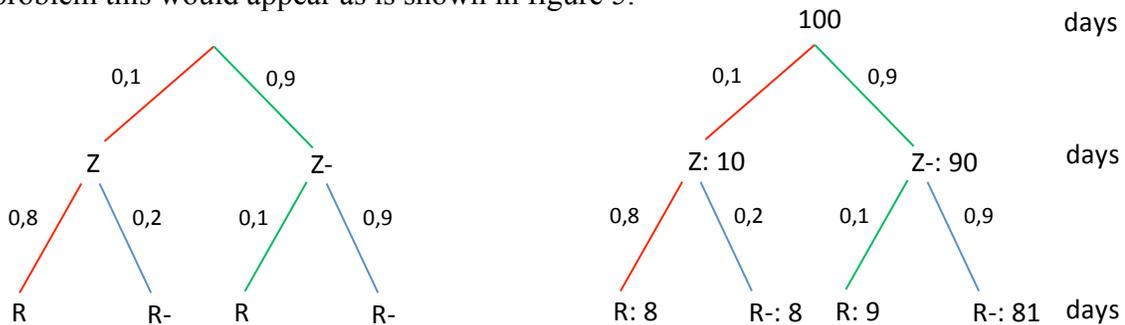


Figure 5. Tree diagrams with probabilities and natural frequencies concerning the rainmaker problem

The structural relationship to the Bayes formula could be represented by the following equation (fig. 6), the last “equation” visualizes those paths which are interesting for calculation by using colors.

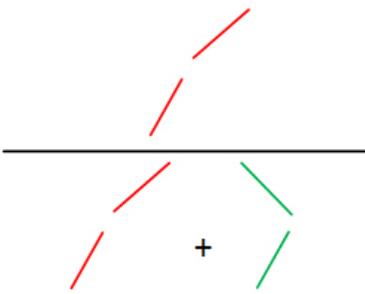
$$P(Z|R) = \frac{P(Z \cap R)}{P(Z \cap R) + P(Z \cap \bar{R})} = \frac{\text{probability for singing and rain}}{\text{probability for rain}} = \frac{\text{red path}}{\text{red path} + \text{green path}}$$


Figure 6. Relationship between Bayes formula and tree diagram

If there were natural frequencies used instead of probabilities the solution as well as the calculation of the problem becomes much easier for most of the people according to research findings of Sedlmeier & Gigerenzer (2001). However, the paths which have to be related to each other do not change, whether absolute frequencies were used instead of probabilities nor the values changed, for example for examining the changing proportions by varying the probability for singing with $P(Z)=0,1$; $P(Z)=0,5$; $P(Z)=0,9$. The tree visualization is static within this point of view (fig. 6).

For emphasizing the dynamic point of view we propose to use a unit square. A computer-based version of the unit square appears in outlines of the screenshot of figure 7: It is dynamically linked with the Bayes formula using the same colours as before for highlighting connections between corresponding elements.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	
1	Unit square																										
2																											
3	conditioning event: B																										
4	P(B) = 0,5																										
5																											
6	P(\bar{B}) = 0,5																										
7																											
8																											
9																											
10	conditioned event: A																										
11	P(A B) = 0,75																										
12																											
13	P(\bar{A} B) = 0,25																										
14																											
15	P(A \bar{B}) = 0,4																										
16																											
17	P(\bar{A} \bar{B}) = 0,6																										
18																											
19																											
20	formula and calculation																										
21	$P(B A) = \frac{P(A B) \cdot P(B)}{P(A B) \cdot P(B) + P(A \bar{B}) \cdot P(\bar{B})} = \frac{0,75 \cdot 0,5}{0,75 \cdot 0,5 + 0,4 \cdot 0,5} = 0,652$																										
22																											
23																											
24																											

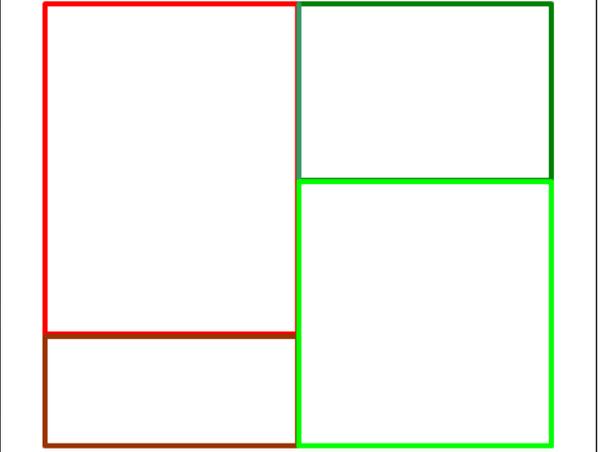


Figure 7. Computer-based learning environment concerning the Bayes formula

By using such a computer-based learning environment the students can explore the formula's structure, which is an important goal for mathematics teaching, particularly, when the mathematical aspect of probability literacy is regarded. In figure 8 the meaning of changing the base-rate is displayed within the visualisation of the unit square concerning the context of the rainmaker-problem.

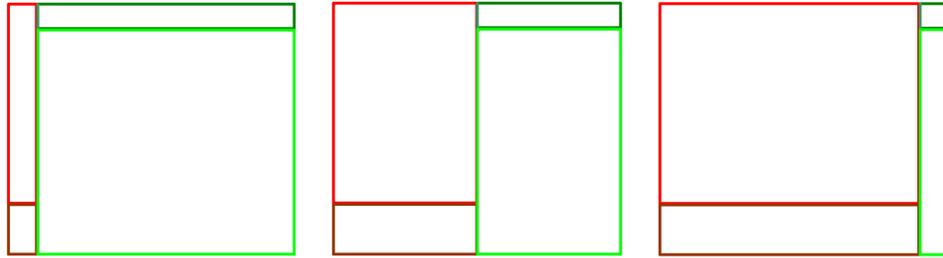


Figure 8. Influence of changing base-rate within the unit square ($P(Z)=0,1$; $P(Z)=0,5$; $P(Z)=0,9$)

The meaning of these changes with regard to a corresponding visualized version of the Bayes formula (cf. Eichler & Vogel, 2010) result in what can be seen in figure 9.

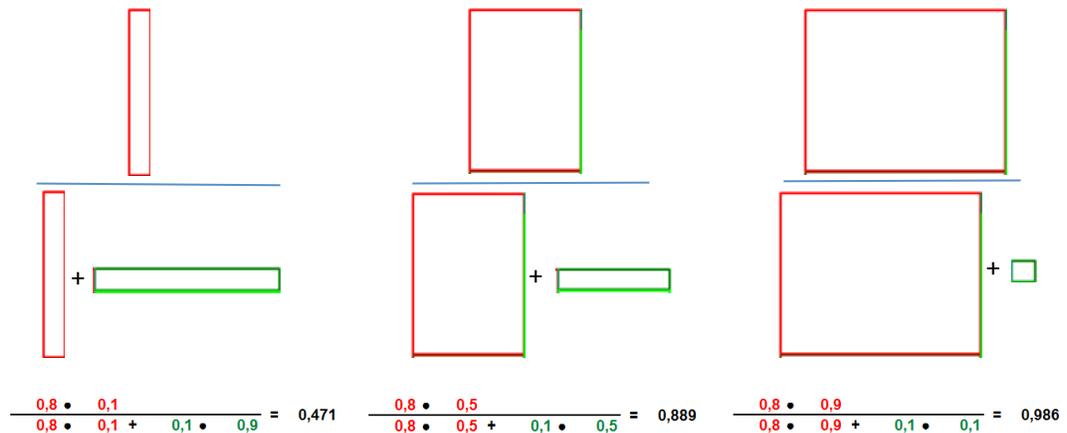


Figure 9. Bayes formula visualized by the unit square

In this point of view, the influence of varying the base-rate of rain-singing to the result could become more obvious by displaying the proportions of corresponding areas of the unit square. The unit square contains, at least partially, graphical elements of the mathematical structure of the Bayes formula but the tree diagram does not. For this reason, we prefer the unit square when focussing on teaching the mathematical understanding of the Bayes formula. For only finding a solution the tree diagram with the natural frequencies might be easier for most of the people outside the classroom.

Of course, this kind of visualization can also be applied to real-world problems like the HIV problem; here varying the base-rate would correspond to judging the test results in countries with different estimations of prevalence (cf. www.unaids.com) as well as to virtual real-world problems like the gambling situation mentioned above, in which varying the base-rate would mean changing the referee's preference for one of the two given dice, the normal one or the cuboid one. The unit square visualization focusses on mathematical structure but not the context.

Discussion and Conclusion

Why is risk a hot topic for society at large but plays a minor role in mathematics teaching in school? In this paper we claim that this fact could be provoked by obstacles for teaching risk in schools

regarding the mathematical aspect and the situational aspect of the construct of risk. For this reason, it is a main issue of this paper to propose strategies to overcome the mentioned obstacles.

The first strategy concerns reducing contextual complexity by changing from the real world to the virtual real world (die task) or even to the virtual world (rainmaker). For example, this change means reducing the significance of the situation's risk because risk could matter in the world of games of chance, but does not matter personally within the virtual world. For example, the risk of Kurt is no risk for the reader of the comic. Serious risk is an issue of the real world but not the virtual world. This is at the core of the underlying pedagogical idea of changing from the real world to the virtual world: In the virtual world the students can think about the virtual context and the underlying mathematics without being seriously personally concerned. When they have learned about the structure and the contextual meaning in a virtual problem situation or in a game of chance they can transfer their knowledge to reality and thus, think about risk in its serious meaning concerning reality, like e.g. in the HIV problem.

This variation principle of tasks and problem situations can be applied to other mathematical concepts like confidence intervals, testing hypotheses or modeling with normal or binomial distributions. By changing different mathematical concepts as well as reality levels of problem situations, the experienced students can learn about the difficulties concerning different interpretations of what is meant if someone speaks of "risk" whenever they enter the real world.

Another main aim of this paper was to make a distinction between applying a heuristic to solve a real problem (the tree diagram with natural frequencies) and applying a heuristic to understand the structure of a mathematical problem (the unit square). Whereas the tree diagram with natural frequencies might be the optimal strategy for persons outside mathematics classes at school when faced with a problem like the HIV-test, the unit square seems to be more appropriate for yielding a mathematical and conceptual understanding of the Bayesian formula and the impact of the parameters to the arithmetical results (c.f. Eichler & Vogel, 2010; Böcherer-Lindner & Eichler, 2015).

In the end, we hold the view that acknowledging the perspective of teaching mathematics considering both the mathematical as well as the situational aspects of risk could be a sound strategy for making a hot topic for the society into a substantial topic within the teaching of statistics and probability.

References

- Blum, W., Galbraith, P., Henn, H.-W., & Niss, M. (Eds.) (2007). *Modelling and applications in mathematics education. The 14th ICMI study*. New York: Springer.
- Boeckelmann, L. & Mildner, S.-A. (2011). Unsicherheit, Ungewissheit, Risiko [Uncertainty, incertitude, risk]. *SWP-Zeitschriftenschau* 2, 1-8.
- Böcherer-Lindner, K. & Eichler, A. (2015). Vergleich konkurrierender Visualisierungen zum Bereich der bedingten Wahrscheinlichkeiten [Comparison of different visualisations of conditional probabilities]. *Beiträge zum Mathematikunterricht*. Retrieved from http://www.mathematik.tu-dortmund.de/ieem/cms/de/home/bzmu_home.html.
- Brown, J. S., Collins, A. & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, 18(1), 32–42.
- Chandler, P. & Sweller, J. (1991). Cognitive load theory and the format of instruction. *Cognition and Instruction*, 8(4), 293-332.
- Cognition and Technology Group at Vanderbilt (1990). Anchored instruction and its relationship to situated cognition. *Educational Researcher*, 19(6), 2-10.

- Eichler, A., & Vogel, M. (2009). *Leitidee Daten und Zufall [Key idea of data and chance]*. Wiesbaden: Vieweg+Teubner.
- Eichler, A. & Vogel, M. (2010). Die (Bild-)Formel von Bayes [The (picture) formula of Bayes]. *PM - Praxis der Mathematik in der Schule*, 52(32), 25-30.
- Eichler, A. & Vogel, M. (2012). Stochastik – fit für die Zukunft [Stochastics – fit for the future]. *PM - Praxis der Mathematik in der Schule*, 54(48), 2-9.
- Eichler, A. & Vogel, M. (2014). Three Approaches for Modelling Situations with Randomness. In E. J. Chernoff & B. Sriraman (Eds.), *Probabilistics thinking* (pp. 75-100). Dordrecht: Springer.
- Erickson, T. (2010). Exploring risk through simulation. In C. Reading (Ed.), *Proceedings of the Eighth International Conference on Teaching Statistics*. Lubljana, Slovenia: International Statistical Institute and International Association for Statistical Education. Retrieved from http://iase-web.org/documents/papers/icots8/ICOTS8_8C3_ERICKSON.pdf.
- Franklin, C., Kader, G., Mewborn, D., Moreno, J., Peck, R., Perry, M., & Schaeffer, R. (2005). *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report*. Alexandria, VA: American Statistical Association.
- Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: Reidel.
- Gal, I. (2002). Adults' Statistical Literacy: Meanings, Components, Responsibilities. *International Statistical Review*, 70(1), 1-51.
- Gal, I. (2012). Developing Probability Literacy: Needs and Pressures Steeming from Frameworks of Adult Competencies and Mathematics Curricula. In S.-J. Cho (Ed.), *Proceedings of the 12th International Congress on Mathematical Education (8 July – 15 July, 2012), COEX, Seoul, Korea* (pp. 1-7). Retrieved from <http://www.icme12.org/upload/upfile2/tsg/2088.pdf>.
- Gal, I., & Garfield, J. (1997). Curricular Goals and Assessment Challenges in Statistics Education. In I. Gal & J. Garfield (Eds.), *The Assessment Challenge in Statistics Education* (pp. 1-13). Retrieved from <https://www.stat.auckland.ac.nz/~iase/publications/assessbk/chapter01.pdf>.
- Gigerenzer, G. (2004). *Das Einmaleins der Skepsis. Über den richtigen Umgang mit Zahlen und Risiken [Calculated risk: How to know when numbers deceive you]*. Berlin: BvT Berliner Taschenbuch Verlag.
- Gigerenzer, G. (2013). *Risk savvy, how to make good decisions*. New York: Penguin.
- International Risk Governance Council (IRGC) (2005). *Risk Governance: Towards an integrative approach*. Genf: IRGC.
- Kahneman, D. (2011). *Thinking fast and slow*. London: Allen Lane.
- Kapadia, R (2008). Chance Encounters - 20 years later. Fundamental ideas in teaching probability at school level. *Proceedings of the 11th International Congress on Mathematical Education (6 July – 13 July, 2008), Monterrey, Mexico* (pp. 1-8). Retrieved from http://iase-web.org/documents/papers/icme11/ICME11_TSG13_24P_kapadia.pdf.
- Kent, P., Pratt, D., Levinson, R., Yogui, C., & Kapadia, R. (2010). Teaching uncertainty and risk in mathematics and science. In C. Reading (Ed.), *Proceedings of the Eighth International Conference on Teaching Statistics*. Lubljana, Slovenia: International Statistical Institute and International Association for Statistical Education. Retrieved from http://iase-web.org/documents/papers/icots8/ICOTS8_8C1_KENT.pdf.
- Keynes, J. M. (1921). *A treatise on probability*. London: Macmillan.
- Klieme, E., Neubrand, M., & Lüdtke, O. (2001). Mathematische Grundbildung: Testkonzeption und Ergebnisse [Mathematical literacy: Development of the test and results]. In J. Baumert, E. Klieme, M. Neubrand, M. Prenzel, U. Schiefele, W. Schneider, P. Stanat, K.-J. Tillmann, & M. Weiß (Hrsg.), *PISA 2000. Basiskompetenzen von Schülerinnen und Schülern im internationalen*

- Vergleich [PISA 2000. Students' basic competencies in international comparison]* (pp. 139-190). Opladen: Leske+Budrich.
- Kurz-Milcke, E. & Martignon, L. (2006). Educating children in stochastic modelling: Games with stochastic urns and colored tinker cubes. In A. Rossman, & B. Chance (Eds.), *Proceedings of the Seventh International Conference on Teaching Statistics*. Salvador, Brazil: International Statistical Institute and International Association for Statistical Education. Retrieved from <http://iase-web.org/documents/papers/icots7/C443.pdf>.
- Laskey, K. & Martignon, L. (2014). Comparing fast and frugal trees and Bayesian networks for risk assessment. In K. Makar (Ed.), *Proceedings of the Ninth International Conference on Teaching Statistics*. Flagstaff, USA: International Statistical Institute and International Association for Statistical Education. Retrieved from http://iase-web.org/icots/9/proceedings/pdfs/ICOTS9_8I4_LASKEY.pdf.
- Latten, S., Martignon, L., Monti, M., & Multmeier, J. (2011). Die Förderung erster Kompetenzen für den Umgang mit Risiken bereits in der Grundschule [Promoting risk literacy already in primary school]. *Stochastik in der Schule* 31(1), 17-25.
- Mandl, H., Gruber, H. & Renkl, A. (1997). Situiertes Lernen in multimedialen Lernumgebungen [Situating learning in multimedia environment]. In L. J. Issing & P. Klimsa (Eds.), *Information und Lernen mit Multimedia [Information and learning with multimedia]* (pp. 166-178). Weinheim: Psychologie Verlags Union.
- Martignon, L. & Krauss, S. (2009). Hands-On Activities for Fourth Graders: A Tool Box for Decision-Making and Reckoning with Risk. *International Electronic Journal of Mathematics Education*, 4(3), 227-258.
- Martignon, L. & Monti, M. (2010). Conditions for risk assessment as a topic for probabilistic education. In C. Reading (Ed.), *Proceedings of the Eighth International Conference on Teaching Statistics*. Lubljana, Slovenia: International Statistical Institute and International Association for Statistical Education. Retrieved from http://iase-web.org/documents/papers/icots8/ICOTS8_8C2_MARTIGNON.pdf.
- Moore, D. (1997). New pedagogy and new content. The case of statistics. *International Statistical Review*, 65, 123-165.
- Riemer, W. (1991). *Stochastische Probleme aus elementarer Sicht [Stochastic problems from an elementary perspective]*. Mannheim: BI-Wissenschaftsverlag.
- Savage, L. J. (1954). *The Foundations of Statistics*. New York, Wiley.
- Sedlmeier, P. & Gigerenzer, G. (2001). Teaching bayesian reasoning in less than two hours. *Journal of Experimental Psychology: General*, 130(3), 380-400.
- Slovic, P. (2010). The psychology of risk. *Saúde e Sociedade*, 19(4), 731-747. Retrieved from <http://www.scielo.br/pdf/sausoc/v19n4/02.pdf>.
- Spiegelhalter, D. (2011). Quantifying uncertainty. In L. Skinns, M. Scott, & T. Cox (Eds.), *Risk* (pp 17-33). Cambridge: CUP.
- Spiegelhalter, D., & Gage, J. (2014). What can we learn from real-world communication of risk and uncertainty? In K. Makar (Ed.) *Proceedings of the Ninth International Conference on Teaching Statistics*. Flagstaff, USA: International Statistical Institute and International Association for Statistical Education. Retrieved from http://iase-web.org/icots/9/proceedings/pdfs/ICOTS9_PL2_SPIEGELHALTER.pdf.
- Taleb, N. N. (2007). *The black swan: The impact of highly improbable*. New York: Random House.
- Till, C. (2014). Fostering Risk Literacy in Elementary School. *Mathematics Education*, 9(2), 83-96.