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Mathematical Creativity: The Unexpected Links

Amine El-Sahili

Nour Al-Sharif

Sahar Khanafer

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Amine El-Sahili[1], Nour Al-Sharif[2], Sahar Khanafer[3]

Lebanese University, Beirut, Lebanon

Abstract

Creativity in mathematics is identified in many forms or we can say is made up of many components. One of these components is *The Unexpected Links* where one tries to solve a mathematical problem in a nontraditional manner that requires the formation of hidden bridges between distinct mathematical domains or even between seemingly far ideas within the same domain. In this article, we design problems that express unexpected links in mathematics and suit students of intermediate and secondary levels. We prove their feasibility through teachers' testimonies and through introducing them in classrooms and collecting students' attitudes with respect to understanding and interest. Results confirm that students can sense such component and that designed problems had caught teachers' and students' interest.

Keywords: *Creativity, Mathematical creativity, Unexpected Links, Classroom problems.*

1 Introduction

In 2004, Sriraman conducted a qualitative study in which he interviewed five creative mathematicians to get an insight on some characteristics of mathematical creativity. He ended his study with an inspiring conjecture that captured our interest:

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[1] *Email: sahili@ul.edu.lb*

[2] *Email: nour.alsharif.87@gmail.com*

[3] *Email: saharkhanafer90@gmail.com*

"It is my conjecture that in order for mathematical creativity to manifest itself in the classroom, students should be given the opportunity to tackle non-routine problems with complexity and structure - problems which require not only motivation and persistence but also considerable reflection." (Sriraman, 2004)

We believe that problems demonstrating unexpected links between two or more domains in mathematics represent one type of the non-routine problems addressed by Sriraman in his conjecture. Through well designed problems of this kind, students could be trained to find unusual connections between seemingly far domains in mathematics in order to reach a solution.

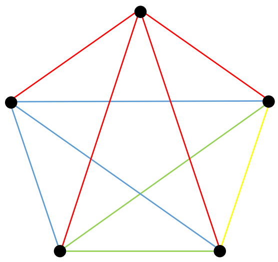
This approach in solving a mathematical problem is viewed in the work of many creative mathematicians. In 1896, Hadamard (Hadamard, 1893; Hadamard, 1896) and De la Vallée Poussin (De la Vallée Poussin, 1896) established and proved (independently) the famous *Prime Number Theorem* using complex analysis. This theorem describes the general distribution of prime numbers among positive integers. It states: *If* $\pi(x)$ *is the number of primes less than or equal to* x, *then* $\lim_{x\to\infty}\frac{\pi(x)\ln(x)}{x}=1$*; that is,* $\pi(x)$ *is asymptotically equal to* $\frac{x}{\ln(x)}$ *as* $x\to\infty$. Although there is no clear connection between complex analysis and the distribution of prime numbers, the proof depends greatly on Riemann's zeta function from complex analysis.

Another example is related to the infinity of primes. Fürstenberg defined a topology on the set of integers $\mathbb{Z}$ and linked it with elementary properties on numbers to prove that the set of prime numbers is infinite (p:5) (Aigner & Ziegler, 2010).

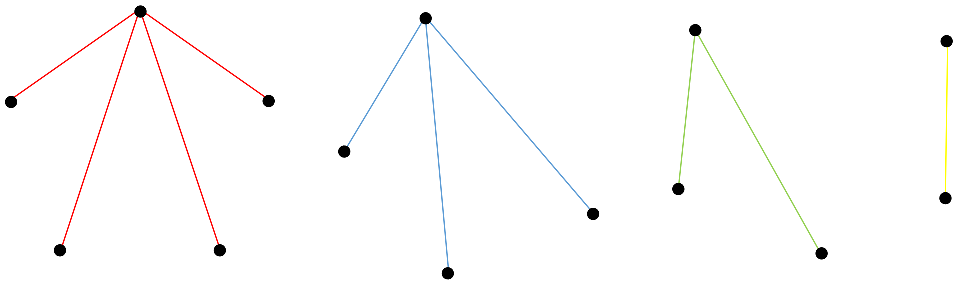
Moreover, in graph theory, Erdös and Rényi introduced for the first time the probabilistic methods to prove the existence of some graphs that are usually difficult to find (Aigner & Ziegler, 2010). The usage of this method represents the unexpected link between graph theory and probability.

In addition, the friendship theorem is a real situation problem that was translated into a graph theoretical problem and then solved using both graph theory and linear algebra techniques. It states: *"Suppose in a group of people we* *have the situation that any pair of persons has precisely one common friend. Then there is always a person who is everybody's friend."*

Also, in graph theory, Tverberg (1982) established a proof of a theorem about the decomposition of a complete graph into complete bipartite graphs. This proof makes use of a system of linear equations to show that the minimum number of stars necessary to cover a complete graph is $n-1$. The following example illustrates this theorem for the complete graph K_5:



The decomposition of K_5 into the four stars $K_{1,4}, K_{1,3}, K_{1,2}$ and $K_{1,1}$ respectively:



After reviewing the above examples, one might think that such approach to solve mathematical problems can only be used by professional mathematicians or postgraduate math students since they possess deep knowledge in mathematics, yet we claim that such approach can be inserted in the educational curriculum of intermediate and secondary students. In this way students will be able to experience such type of creativity in solving mathematical problems by constructing links among distinct domains. This is clearly shown in the content of this paper where examples are given and tested to prove the credibility of this claim.

2 Literature Review

The examples mentioned above, and others, provided us with the very first flame that ignited the idea that there is a correspondence between solving mathematical problems using unexpected links and mathematical creativity.

2.1 Mathematical Creativity

For a long period of time, the dominant view was that creativity in mathematics is limited to “genius” individuals (p.148) (Weisberg, 1988). This view has shifted and, today, mathematical creativity is seen as a skill that can either be fostered and encouraged or suppressed and deprived (Silver, 1997; Leikin, 2009). Due to this new view, many contemporary research in mathematics education aim to define mathematical creativity, identify its characteristics, search for tools that assess it, or even tools that implement it in the general school population. A common point in all of these researches is that there is no universally accepted definition of mathematical creativity or even creativity in general (Sriraman, Havold, and Lee, 2013; Treffinger et al., 2002; Mann, 2005; Haylock, 1997).

In this paper, we adopt the definition that was suggested by Sriraman during a conversation with Liljedahl (Liljedahl & Sriraman, 2006). He differentiated between mathematical creativity at two levels:

- At the professional level, mathematical creativity can be defined as the:
    a) Ability to produce original work that significantly extends the body of knowledge (which could also include significant synthesis and extension of ideas).
    b) Ability to open avenues of new questions for other mathematicians.
- At the classroom level, mathematical creativity is defined as the process that results in:
    a) Novel and/or insightful solutions.
    b) The formulation of new questions and/or possibilities that allow an old problem to be regarded from a new point of view.

2.2 Unexpected Links

Throughout the literature, many are the ideas that indirectly support our claim. Sriraman, in his conversation with Liljedahl (Liljedahl & Sriraman, 2006), gave an example to clarify what he meant by saying: "original work…could also include significant synthesis and extension of ideas" while defining mathematical creativity at the professional level. The example was Witt's proof (1931) of Wedderburn's theorem; *a finite division ring is a field*, which uses algebra, complex analysis and number theory.

It is well known that a commutative division ring is a field, so it is brilliant how the finite condition on the division ring forces the multiplication in this ring to become commutative and thus transforms it into a field. In this subject, Herstein writes *"It is so unexpectedly interrelating two seemingly unrelated things, the number of elements in a certain algebraic system and the multiplication of that system"* (as cited in Aigner & Ziegler (2010), p.31). Also Aigner and Ziegler, in their book *Proofs from the Book* (p.31), comment that this proof *"combines two elementary ideas towards a glorious finish"*. Thus the combination of these different domains in mathematics to formulate a proof of the theorem is considered to be mathematically creative.

Booden states that "sometimes creativity is the *combination of familiar ideas in unfamiliar ways*" (p: xi) (Booden, 2004). Chamberlin and Moon note that mathematical creativity is realized when one *creates a non-standard solution of a problem* that can be solved using standard methods (Chamberlin & Moon, 2005). Nadjafikhah et al. conclude their study with three points that could consist a creative act in mathematics: creating a new fruitful mathematical concept, *discovering an unknown relation*, and recognizing the structure of a mathematical theory (Nadjafikhaha, Yaftianb, & Bakhshalizadehc, 2012).

Moreover, in almost all contemporary research about fostering or assessing mathematical creativity, experiments were evaluated depending on the three components of creativity that were originally defined by Torrance (1966, 1974) and then redefined to suite the assessment of creativity in mathematics. One of them is flexibility: "flexibility refers to apparent *shifts in approaches* taken when generating responses to a prompt" (Silver, 1997). This is consistent with the notion of *unexpected links* where both tend to solve a problem in an unusual approach that relates several distinct domains in mathematics.

In an article about aesthetics and creativity, Brinkmann and Sriraman sent a questionnaire to some mathematicians (Brinkmann & Sriraman, 2009). Some of the answers assert that mathematicians sense elegance and beauty in proofs that demonstrate unexpected links between two or more domains in mathematics:

- "A particularly intensive appeal comes from the suddenly (and sometimes unexpected) pure discovery, the clear understanding of a mathematical phenomenon, often from a completely new perspective, a *new harmonic interplay of different fields, first appearing not to be related to each other*."
- "Beauty within mathematics manifests itself on the one hand by typical mathematical-logical arguments, especially if these arguments show *unexpected and important connections, in an (at first) surprising manner and then mostly also in a surprising simple manner*."

They conclude that aesthetic appeal plays a crucial role in the creative work of contemporary mathematicians. And since there is a governing call to view school students as budding mathematicians, *it is ironic that aesthetics has not received much attention by the community of mathematics educators* (Brinkmann and Sriraman, 2009); Especially that in order to motivate students towards getting engaged in creative mathematical thinking or even mathematical thinking, they must sense the beauty of mathematics (Sinclair, 2009). For as Hardy (1940) said: "*There is no permanent place in the world for ugly mathematics*" (Hardy, 1940). Thus unexpected links is one method that can motivate students toward beautiful creative mathematical thinking.

As reinforcement to our claim, we testify by Poincaré's statement:
"Elegance may result from the feeling of surprise caused by the unlooked-for occurrence of objects not habitually associated. In this, again, it is fruitful, since it discloses thus relations that were until then unrecognized. ***Mathematics is the art of giving the same names to different things****"* (Verhulst, 2012).

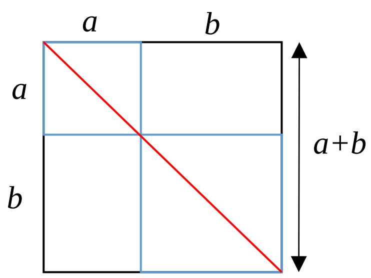
2.3 Calls to Foster Mathematical Creativity

While reviewing the general literature about mathematical creativity, we stood upon many recommendations to foster it in classrooms. Sriraman emphasized that: “it's in the best interest of the field of mathematics education that we identify and nurture creative talent in the mathematics classroom” (Sriraman, 2004). In fact, limiting the use of mathematical creativity in classrooms transforms mathematics into a set of skills to master and rules to memorize (Mann, 2005), whereas “the wellsprings of mathematics are not utility and relevance, but creativity, imagination and appreciation of the beauty of the subject” (Whitcombe, 1988).

Furthermore, researchers recommend mathematics educators to identify and develop mathematical creativity (Nadjafikhaha, Yaftianb, & Bakhshalizadehc, 2012). “All students, especially those with potential talent in mathematics, need academic rigor and challenge as well as creative opportunities to explore the nature of mathematics and to employ the skills they have developed” (Mann, 2005).

3 Unexpected Links: Among and Within Branches of Mathematics

First algebraists established some algebraic formulas through geometric demonstrations. Euclid, in his book *Elements*, uses geometric notions to express what we today consider as algebraic formulas. For example, he states and illustrates the following proposition: *“If a straight-line is cut at random, then the square on the whole (straight-line), is equal to the (sum of the) squares on the pieces (of the straight line), and twice the rectangle contained by the pieces”* (p.52-53) (Fitzpatrick, 2007).



The algebraic translation of this proposition presents the identity: $(a+b)^2 = a^2 + 2ab + b^2$ which is gulped into students' memory at the intermediate levels with just a simple proof by expansion. Whereas if demonstrated as it was originally developed by Euclid, it will certainly capture their attention.

Moreover, Al-Khawarizmi, in a printed translated copy of his book (Rosen, 1831), provided the geometric illustration of the equation $x^2 + 10x = 39$ through which he found one of its solutions, $x = 3$. His method can be generalized to find one solution, $x = \frac{-p + 4\sqrt{4q + p^2}}{2}$, of any quadratic equation of the form $x^2 + px + q = 0$.

In addition, a simple demonstration of unexpected links, in algebra, grows upon understanding the nature of "equal". At early stages, students might enclose on the idea that equality is a trivial notion, for example: $2 = 2$ or $1 \neq 2$. Whereas, while ascending to higher levels of education, they encounter a much more sophisticated understanding of equality. For example, after being introduced to fractions, students will view the equality of $\frac{8}{4} = \frac{6}{3} = 2$ or $\frac{513}{456} = \frac{9}{8}$. In intermediate levels, they get familiarized with more complex equalities such as $\sqrt{2} + \sqrt{5} = \sqrt{7 + 2\sqrt{10}}$ which is an expression of equality among unequal objects at the first glance. At higher levels, students will counter an advanced expression of equality and inequality as $\sqrt{2} \neq \frac{p}{q}$ with $p, q \in \mathrm{N}$. Another example is that concerning the representation of a real number as a combination of imaginary numbers, such as $3 = \left(1 + \sqrt{-5}\right) + \left(2 - \sqrt{-5}\right)$ or $5 = \left(2 + \sqrt{-1}\right)\left(2 - \sqrt{-1}\right)$.

Moving to the domain of geometry, a notable observation in Euclid's Elements is the interrelated study of two different geometric structures, circles and triangles. For example, Euclid made use of circles "to construct an equilateral triangle on a given finite straight-line" (Euclid's st1proposition, book I), (Fitzpatrick, 2007) (p.8).

Furthermore, constructing precise structures (shapes) out of random ones is one of the beams of hidden links within plane geometry. One of these shapes is Euler line that joins the orthocenter, centroid, and circumcenter of any triangle. Another shape is the *Nine-Point Circle;* the feet of the three altitudes of any triangle, the midpoints of the three sides, and the midpoints of the segments from the three vertices to the orthocenter, all lie on this circle. Surprisingly, the center of this circle lies on Euler line.

On the other hand, it is remarkable and interesting to shed the light on a theorem that links algebra and projective geometry. It states: *in a finite projective plane, Desargues' theorem implies Pappus' theorem follows from the algebraic result that a finite division ring is a field*. *"The idea to pursue here is the assignment of number-like objects to the elements (points, lines) of various geometries"* (p.254) (Kleiner, 2012). An algebraic is system is constructed and studying the geometric properties is replaced by studying the associated algebraic system.

Moreover, Kleiner emphasizes the importance of building bridges between distinct domains of mathematics and how such bridges that were built between algebra and geometry resulted in the creation of a new field which is analytic geometry. "*Building bridges between different, seemingly unrelated, areas of mathematics is an important and powerful idea, for it brings to bear the tools of one field in the service of the other"* (p.12) (Kleiner, 2012).

Also, Pythagoras theorem represents the bedrock that resulted in linking arithmetic and geometry and thus in the creation of a new mathematical branch which is analytic geometry. "*The Pythagorean theorem was the first hint of a hidden, deeper relationship between arithmetic and geometry, and it has continued to hold a key position between these two realms throughout the history of mathematics"* (p.2) (Stillwell, 2010).

Another good example shows the link between analysis and graph theory, where Cauchy-Schwarz inequality: $\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$ *for real numbers*

$a_i, b_i,\ 1 \le i \le n$, is applied to get, surprisingly, the following result: *a graph on* n *vertices and without triangles, has at most* $n^2/4$ *edges*.

As a conclusion, we choose to end this part with the following quotation:

"Where things get really interesting is when unexpected bridges emerge between parts of the mathematical world that were previously believed to be very far remote from each other in the natural mental picture that a generation had elaborated. At that point one gets the feeling that a sudden wind has blown out the fog that was hiding parts of a beautiful landscape" (p.3) (Connes).

4 Embedding Unexpected Links through Classroom Examples

In a regular mathematics school curriculum, students are deprived the opportunity of realizing hidden links among different mathematical domains and thus they grow up to believe that such links do not exist.

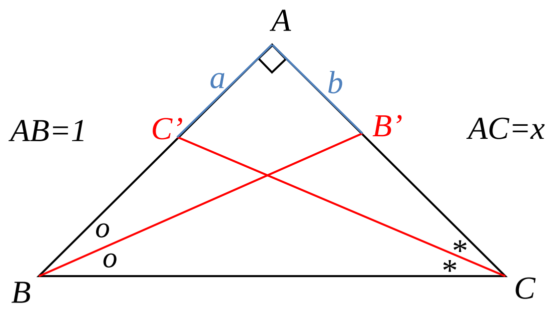
In this section, we propose examples that open the eyes of school students towards the existence of links between mathematical domains that seem very far apart. Note that as we descend from higher levels to lower ones, creating and demonstrating examples of this kind becomes more and more difficult. This is due to the accompanied limitation in students' acquired mathematical knowledge.

4.1 Real Functions and Geometry

This is an example for a third-secondary class in which a geometric problem is solved by the aid of an exponential function:

Problem: Let ABC be a right triangle at A, with (BB') and (CC') the bisectors of $A\hat{B}C$ and $A\hat{C}B$ respectively, such that $\mathrm{BB'} = \mathrm{CC'}$. Show that if the function $f(t) = \dfrac{e^t - e^{-3t}}{2}$ is strictly increasing, then ABC is isosceles.

A. Suppose $\mathrm{AB}=1$ and $\mathrm{AC}=x>0$. Let $\mathrm{AC'}=a$ and let $\mathrm{AB'}=\mathrm{b}$.



1. Find a and b in terms of x. (Hint: apply the *Angle bisector Theorem*: consider a triangle ABC, if AE is the angle bisector of $B\hat{A}C$ then $\dfrac{EB}{EC}=\dfrac{AB}{AC}$).

2. Using the previous part, show that:

$$x+\frac{x}{\left(x+\sqrt{1+x^2}\right)^2}=\frac{1}{x}+\frac{\frac{1}{x}}{\left(\frac{1}{x}+\sqrt{1+\frac{1}{x^2}}\right)^2}$$

(Hint: Show that $a^2+x^2=b^2+1$)

3. Find a function g such that $g(x)=g\left(\dfrac{1}{x}\right)$.

B. Consider the hyperbolic sine function $\sinh(t)=\dfrac{e^t-e^{-t}}{2}$ and the hyperbolic cosine function $\cosh(t)=\dfrac{e^t+e^{-t}}{2}$.

1. Define the domain of definition of $\sinh$ and $\cosh$.
2. Show that $(\sinh(t))'=\cosh(t)$ and $(\cosh(t))'=\sinh(t)$.
3. Study the variations of $\sinh$ and $\cosh$.
4. Deduce that for $x>0$ there exists $t>0$ such that $x=\sinh(t)$.
5. Show that $\cosh^2(t)-\sinh^2(t)=1$.

C. Let $g(x)=x+\dfrac{x}{\left(x+\sqrt{1+x^2}\right)^2}$ and $x=\sinh(t)$ with $t>0$.

1. Find the exponential function $f(t)$ such that $f(t) = g(x)$.

2. Show that f is strictly increasing.

3. Deduce that g is strictly increasing.

D. Using the previous parts, deduce that ABC is isosceles. (In other words, deduce that $x = 1$)

As much as this problem can catch the attention of the students, it will before, catch the attention of their teachers. Even at the level of math teachers, which most of them are expected to be holders of at least a bachelor degree in pure mathematics, the main statement of the problem will be surprising and finding the link will not be an easy task. They will be in need of some steps similar to the ones mentioned above to formulate the big picture of the solution.

4.2 Inequalities and Functions

Here is an algebraic problem that can be introduced to first and second secondary students. It reflects a link between inequalities and variation of functions.

Problem. Compare $a = \sqrt[3]{5 - \sqrt[3]{2}}$ and $b = \sqrt[3]{2 + \sqrt[3]{4}}$.

1. Compare $\sqrt[3]{2}$ and $\frac{-1+\sqrt{13}}{2}$.

2. Let $f(x) = x^2 + x - 3$. Show that $f(x) < 0$.

3. Compute $b^3 - a^3$.

4. Deduce that $a > b$.

4.3 1 in Disguise

The following problem demonstrates what we have pointed out previously that unexpected links can appear within the understanding of simple mathematical objects, such as "=". The problem will show that “1” can take a complex surprising disguise: $\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}$. The following problem is suitable for second secondary students:

Problem: Show that $\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}$ is rational.

1. Show that $\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}$ is a root of x^3+3x-4.
2. Prove that $x^3+3x-4=(x-1)(x^2+x+4)$.
3. Show that x^3+3x-4 has one real root and two imaginary roots.
4. Deduce that $\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}=1$; that is, $\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}$ is rational.

4.4 Triangles and Prime Numbers

For students of grade 10, we suggest this example that links triangles and prime numbers:

Problem: Let ABC be a right triangle at A. Let $A\hat{B}C=x$ and $A\hat{C}B=y$ such that $x>5$ and $x,y\in \mathbb{N}$ (measure in degrees). Show that if x is prime, then y is prime, provided that y is not divisible by 7.

1. Suppose that y is not prime number. Show that y has a divisor less than 10.
2. Deduce that y has a prime divisor p less than 10.
3. What is the set of possibilities of p?
4. Show that if $p\in\{2,3,5\}$, then x is no more prime.
5. Deduce that, according to the given, y is prime.

This solution provides the student with a new way of thinking and solving where s/he doesn't just view x and y as angles that sum up to 90. S/he manipulates these integers in a simple logical manner to reach the result.

4.5 Linked properties in a triangle: Heights, Perpendicular Bisectors and Bisectors

In the 8th grade, students learn that in any triangle, the three heights, the three bisectors, as well as the three perpendicular bisectors, meet in one point. These three concepts are apparently unrelated due to the different nature underlying their definitions. Surprisingly, we can find out how tightly they are linked to each other in the following manner:

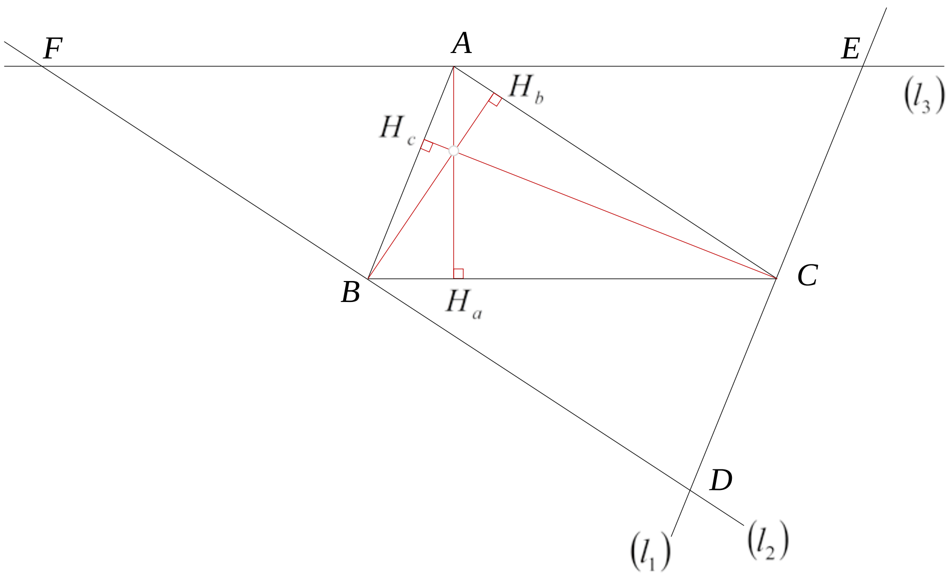
4.5.1 Perpendicular Bisectors and Heights

In the first part, we link a property of the perpendicular bisectors in a triangle to a property related to its heights:

Knowing that the perpendicular bisectors of any triangle meet in one point prove that the three heights also meet in one point?

Consider any triangle ABC. Let AH_a, BH_b and CH_c be the heights relative to the bases BC, AC and AB respectively. Since we don't know about the nature of their intersection, if it is one point or it forms a triangle, we will hide this intersection.

1. Let (l_1) be the line passing in C and parallel to (AB), (l_2) be the line passing in B and parallel to (AC), and (l_3) be the line passing in A and parallel to (BC). Let D, E and F be the points of intersection of (l_1) and (l_2), (l_1) and (l_3), (l_2) and (l_3) respectively.



2. Prove that $ABCE$, $ACBF$ and $ACDB$ are parallelograms.

3. Prove that A, B and C are the midpoints of $[EF]$, $[FD]$ and $[DE]$ respectively.

4. Deduce that (AH_a), (BH_b) and (CH_c) are the perpendicular bisectors of $[EF]$, $[FD]$ and $[DE]$ respectively.

5. Deduce that the 3 heights of triangle ABC meet in a common point.

4.5.2 Angle Bisectors and Heights

In this section, another interesting hidden link between bisectors and heights is illustrated.

Knowing that the angle bisectors of any triangle meet in one point prove that the three heights also meet in one point?

Let ABC be any triangle. Let H_a, H_b and H_c be the feet of the heights drawn from A, B and C, respectively, to the relative bases.

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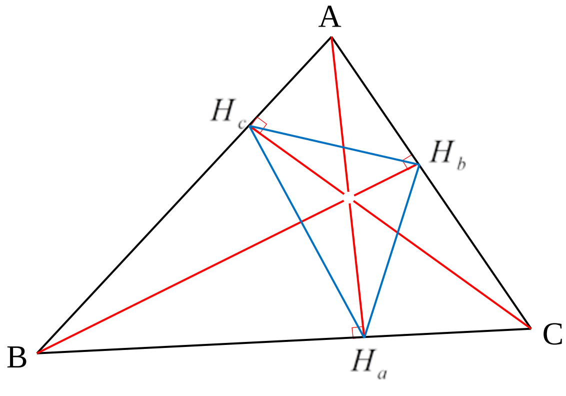


1. Prove that BCH_bH_c is an inscribed/cyclic quadrilateral.

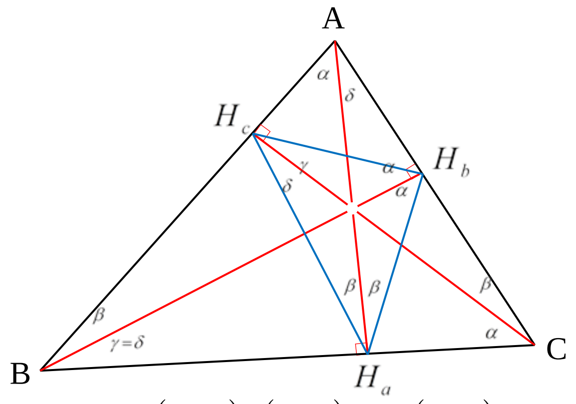
2. Deduce that $H_c\hat{H}_bB = H_c\hat{C}B = \alpha$, $H_b\hat{H}_cC = H_b\hat{B}C = \beta$ and $H_b\hat{C}H_c = H_b\hat{B}H_c = \gamma$.

3. Show that ABH_aH_b is an inscribed quadrilateral.

4. Deduce that $B\hat{A}H_a = B\hat{H}_bH_a = \gamma$, $A\hat{H}_aH_b = \delta$ and $H_a\hat{A}H_b = \beta$.

5. Prove that ACH_aH_c is an inscribed quadrilateral.

6. Deduce that $C\hat{H}_cH_a = \beta$, $H_c\hat{H}_aA = \delta$ and $\alpha = \gamma$.



7. Deduce that the heights (AH_a), (BH_b) and (CH_c) meet in one point.

The teacher can end up this problem by posing some *open questions*:

- How can we show that if the three heights of any triangle meet in one point, then:
    1. The three angle bisectors also meet in one point?
    2. The three perpendicular bisectors also meet in one point?
- Is it possible to find equivalence between both properties?

4.6 Attractive Induced Formula

Now we move into a lower level. We propose the following example for grade 7 (second intermediate). It is based on a well-known proposition in these grades:

$$\text{If } \frac{a}{b} = \frac{c}{d} \text{ then } \frac{a}{b} = \frac{a+c}{b+d}.$$

We can represent this formula in a more beautiful and attractive way:

$$\frac{12}{35} = \frac{1212}{3535} = \frac{121212}{353535} = \cdots$$

$$\text{Or:} \quad \frac{431}{205} = \frac{431431}{205205} = \frac{431431431}{205205205} = \cdots$$

Introducing the above property with the symphonic numerical equality will not reflect any link at the first glance. However, certain demonstrations will clarify this hidden link and can allow students in lower intermediate levels to understand and be interested in such attractive formula.

1. Using your calculator find the ratios $\frac{13}{75}$, $\frac{1313}{7575}$, $\frac{131313}{757575}$.
2. What can you conclude? $\left(\frac{13}{75}=\frac{13131313}{75757575}=\cdots\right)$
3. Can you give a 3-digit example with the same previous feature?
4. Show that: “If $\frac{a}{b}=\frac{c}{d}$ then $\frac{a}{b}=\frac{a+c}{b+d}$ ” .
5. Use this property to give the reason behind the above equalities?
6. Can we deduce that $\frac{25}{471}=\frac{2525}{471471}$? Why? (try to use the formula)
7. Apply the strategy on the 3-digit-number example.

We claim that this example will lighten an aesthetic numerical form of the main property. It will attract students, stay in their minds and develop their way of thinking to discover similar properties.

5 Research Methodology

Our notation: *The Unexpected links*, between distinct mathematical domains or even between seemingly far ideas within the same domain, is a component of mathematical creativity that is sensed by students of pure mathematics in postgraduate levels. However, students of lower levels, especially school students, are deprived from the opportunity to experience such beautiful and creative links in mathematics.

In addressing this problem, we hypothesize that *problems that hold unexpected links can be designed to suit intermediate and secondary scholastic levels, where they can be understood by students and can catch their interest*.

With this hypothesis the following questions rise up:

1. Is *The Unexpected Links* a form of mathematical creativity?
2. Are problems that demonstrate the unexpected links applicable in regular mathematics classrooms?
3. Can students understand such problems?

4. Are these problems interesting for students?

The first question was answered through the illustrations we provided from the literature review and history of mathematics. The second question is answered through teachers' interviews and questionnaires. Whereas for the last two questions, their answers are provided through the observations noted while demonstrating the problems in classrooms and through the data collected from post questionnaires distributed to the students.

5.1 Design

Our experiment is divided into two parts, a teachers' experiment and a student’s one. For the teachers' experiment, we were obliged to use two procedures according to the availability of teachers and their time. Sometimes we had the opportunity to group a number of teachers together and introduce the problems to them; while other times, when teachers had no common free time, we had to introduce the problems to each teacher separately.

For the same reasons, we were able to conduct semi-structured interviews on a group of teachers while on others we had to pass a questionnaire. Note that both interviews and questionnaires held the same contents but interviews gave us a better detailed insight on teachers' opinions. For the students' experiment, we introduced each single problem into a classroom. All classrooms were subjected to post questionnaires.

5.2 Participants

We applied our experiment on teachers from two schools and on students from one school. We had the chance to conduct the experiment on 10 mathematics teachers: 4 teachers of only secondary levels, 2 teachers of only intermediate levels, and 4 teachers of both levels. All have been mathematics teachers for more than 7 years except for one intermediate level math teacher whom has been teaching mathematics for only 3 years. Thus all are well experienced in teaching mathematics. As for students, we introduced our problems in 7 classrooms: 4 secondary levels and 3 intermediate. Each classroom contained at least 16 students.

5.3 Data Analysis

We will present our data in 3 sections:

- Teachers' interview: for each problem, we provide teachers' opinions and then we draw a conclusion.
- Students' experiment: for each problem, we provide the procedure, step by step, by which it was introduced and we provide students' interaction in each step, when remarked. Also we provide some observations noted throughout the whole demonstration.
- Questionnaires' analysis: a post questionnaire was distributed to each student. This questionnaire investigates students' impressions towards the presented problem.

5.3.1 Teachers' Interviews

We denoted the questions by Q1, Q2, Q3…etc. and we named the teachers by T1, T2, T3…etc. The statements provided in square brackets are the researchers' intervention in order to rectify or explain some points to the interviewed teacher. If a teachers' opinion is not provided, then this means that the answer is missing. The questions of the teachers' interview are provided in appendix B.

1) Real Functions and Geometry:

This is a problem we proposed for 3^{rd} secondary students. 7 teachers' opinions are provided here.

Q1 :

T1 You can explain all concepts and make students understand them, but with respect to them this is a very high level problem. They won't understand anything; maybe 1 or 2 students will be able to proceed with you. Now, you can tell them the rules of $\sinh$ and $\sinh$ if they know exponential functions; you can make them understand all concepts as you like. But I think students can't relate all parts together and notice how the question is proceeding… Even if 1 or 2 students proceed with you, they will get lost and will need to review the problem at home.

T2 No, they know all.

T3 No.

T4 No.

T5 No.

T6 Yes, rules, calculation, and geometric properties that are rarely used.

T7 No.

Q2 :

T1 No, a part alone no. But you will need to work with clever students that can solve. Sometimes, even those things that we find easy (cross multiplication ...etc.) make many students lost. For example, if you ask life science section students to find a in terms of x, only 2 or 3 students can solve it. Now for general science sections, yes maybe more than half of the class can solve such things. If you are addressing general sciences students, yes every part alone is good.

T2 If for general sciences students, then of course all parts are not hard. For life science students, some parts are hard.

T3 No.

T4 No.

T5 No.

T6 Yes, the linkage of geometric exercises to tough irrational functions and composite functions.

T7 No.

Q3 :

T1 Yes, it is hard. If the problem is for general sciences students and the teacher explains every part, then yes students can understand. [This is what I meant, if the teacher explains every part, can they understand it?] Yes they can understand but I am telling you that not all students of general sciences can proceed with you, the clever ones only. [That is the weak students cannot understand?] No, the weak ones cannot understand.

T3 No.

T4 No.

T5 No.

T6 Yes, the final linkage and the use of bijective functions are hard for students to understand. The meaning of $g(x) = g\left(\frac{1}{x}\right)$.

T7 No.

Q4 All teachers responded with *yes*. T1 added *not only students also teachers. I am sure of that, I saw such kinds of examples but this is a very nice one.* Also, T2 added *students will enjoy*.

Q5 All teachers responded with *yes*.

Q6 :

T1 One remark only, that this is a hard question for secondary levels. But as an idea it is very nice and it is very beautiful to link a function in this manner and show that math is interrelated. It is a very nice idea. Maybe if simplified more, the same ideas but in a low dosage, in a way that if introduced into a general sciences section, the whole class will be enjoying it, in my opinion it would be more beautiful.

T2 Really, it is a nice idea.

T6 You can make the hidden link with what we call "verbal questions"; the questions are exactly those of linkage… I was shocked by this function, $f(t) = \frac{e^t - e^{-3t}}{2}$ when I read it at first in the main statement of the problem. [One of the goals is to surprise you and students by the statement of the problem that holds within the linkage of two seemingly far and distinct concepts, and then we go through the proof step by step]. There are other simpler examples on such links; this is my basic note.

T7 Relating the ideas together is hard.

Most teachers agree that the problem makes no use of mathematical concepts that students are not aware of, except for one geometric property, *The Angle-Bisector Theorem*; it is rarely or never used in schools. For the hardness, teachers agree that each part separately is generally not hard, especially for general sciences students, yet the final linkage of the whole problem is hard. They commented that general sciences students can comprehend the solutions, if explained thoroughly, while life science students will face difficulties. The problem was interesting for teachers, specifically the appearance of an exponential function in a geometric problem.

In conclusion, by considering teachers' opinions, the problem is suitable for general sciences sections, provided that all hints we proposed are introduced. In addition, the problem conveys our notation and has the potentiality to fulfil our aim.

2) Inequalities and Variation of Functions:

We proposed this problem for 2^{nd} secondary students. 8 teachers' opinions are provided here.

Q1 All teachers responded with *no*.

Q2 All teachers responded with *no*, except for T8 who responded with *yes*, *the linkage between the function and comparison in part 3 is hard for students to know and deduce alone*.

Q3 All teachers responded with *no*, except for T2 whose answer is missing. Also, T6 added the following comment *it just needs a rearrangement so that they will understand it better*.

Q4 All teachers responded with *yes*, except for T2 whose answer is missing.

Q5 All teachers responded with *yes*, except for T2 whose answer is missing.

Q6 :

T1 It is very nice.

T6 Rearrange the parts to make the concept closer to the level of the class. In rearranging this problem, ask about the study of the function before the comparison of the numbers in part 1. When I firstly read part 1 after the statement of the main problem, I said "why this comparison is put here? What is its role?" This part transmits a sense of weirdness and foreignness and, due to that, I was no more encouraged to solve it. However, when you start by the function, I will feel that it is true and more beautiful. Also the most beautiful is to ask: Compare $\sqrt[3]{2}$ and

$\frac{-1+\sqrt{13}}{2}$ and then put $\sqrt[3]{2}$ on the table. Where is its position? It is between the roots. [That is right, it becomes easier] Aha, you said the word I am looking for! There is a big difficulty in the way you proposed…This linkage is very helpful when it is hard to treat a problem by our familiar methods; it makes the problem easier and useful. But all my remarks don't affect the objective of this research; the objective of the research is sacred and doesn't hold any problem.

T7 The problem is very nice. It is nice how in the problem we move and relate the table of signs with the comparison.

T8 The linkage between the function and comparison in part 3 is hard for students to know and deduce alone. [If I solved with the students to reach this part, is it hard to see that $b^3 - a^3 = f\left(\sqrt[3]{2}\right)$?] Look, this type of questions is for special persons, not for all students. [No, my aim is that all students understand such questions.] Aha, you mean that such problem is given directly to the student? Or he should have been trained on similar patterns? [No, there is no previous training at this level. What is important to me is that the student knows all the concepts in the problem, as studying a function with its table of variations…, and that he understands all the parts to reach the link. I will explain the link for him; could he then understand it all?] Yes, of course. But if you bring the question with its parts and ask a student to solve it alone, there will be parts he can surely solve, but linking the ideas together with such given, having cubic roots, will be a bit hard for him. I am saying this according to what we teach and the examples of functions that we solve in classrooms; they almost contain a square root in a square root. Usually, solving with cubic roots is less familiar than solving with square roots. For this, seeing a square root in the question and dealing with it is very normal, it is “nothing” with respect to students. However, when a cubic root appears, they take more time thinking of it.

As a problem, it is very beautiful. Some people may not face any problem in solving it and they know all its concepts… Students are familiar with such patterns, while in your problem they may feel awkward with the cubic root. But when you explain it and give them the indications, they can understand, but not all students can solve it, alone, from the first time. With respect to the concepts, nothing is new; it is all clear and understood.

All teachers gave positive attitudes towards this problem with respect to used concepts, hardness, and understanding. Some teachers found that linking the ideas together might be hard for students to reach, if they solved alone. So they recommended a simple rearrangement in the structure of the problem. In all, according to teachers, the problem is suitable to fulfill our aim.

3) 1 in Disguise:

This is a problem we proposed for 2^{nd} secondary students. 8 teachers' opinions are provided here.

Q1 All teachers responded with *no*, except for T2 whose answer is missing. Also T1 added the following comment: *But I repeat that the students must be of good level*.

Q2 All teachers responded with *no*, except for T6 who responded with *yes, calculation in part 1, students hate to write it* and T2 whose answer is missing.

Q3 All teachers responded with *no*, except for T2 whose answer is missing.

Q4 All teachers responded with *yes*, except for T2 whose answer is missing.

Q5 All teachers responded with *yes*, except for T2 whose answer is missing.

Q6 :

T2 This is an ordinary question. Students can solve.

T6 It is true that when one looks at this number with cubic root, he can't believe it is equal to 1!... Nice, nice, nice! It is the example that impressed me the most. Really it is a very excellent question. I answered this questions with total conviction; it is beneficial.

T7 If the calculation of part one is simplified, then the question will be nicer.

T8 Nice question. It is nice and easier than the previous one.

This problem scored total acceptance among teachers. It was very attractive; they only remarked that the calculations of part one might be tiresome for students.

4) Triangles and Prime Numbers:

This is a problem we proposed for 1st secondary students. 3 teachers' opinions are provided here. 1 of the teachers responded to all questions while the others gave a general remark.

Q1 T6 responded with *yes, in every step*.

Q2 T6 responded with *yes, they can't solve it alone*.

Q3 T6 responded with *no, after it is explained, it will be understood*.

Q4 T6 responded with *yes, for sure*.

Q5 T6 responded with *yes*.

Q6 :

T6 All the contents of the problem are not acquired in the Lebanese program. This question is useful for grade 10, but the arithmetic is not given in the curriculum. It is a presumptive imaginary question. This kind of mathematics does not exist in our curriculum. It is all new. Parts 1, 2, 3 are understandable but this concept, “suppose not” to reach a contradiction, does not exist in our curriculum. It can be understood. The level of this problem is much higher than the level of 10th grade and it is even not found in the Lebanese curricula.

T7 It is hard. We do not teach such things in school (arithmetic). Even the contradiction is given at the final levels of secondary stages and not in this strong form.

T8 The idea of supposing the contrary rarely appears in the curriculum… Such examples appear in the 10th grade at the end of the year with space. But we are not explaining them anymore. Maybe it is given once through explaining the lesson, but not through the students' applications. This is logic; that is it needs a lot of application and work to be understood by students and to be well-established in their minds. If you explain it, the student will understand, but after a week you will see that he forgot it. It is hard for grade 10; you are overburdening the problems!!

Teachers agree that this is a very hard problem for 10^{th} grade students to solve, particularly alone. Yet they don't deny that students can understand the solution if it is well explained.

5) Perpendicular Bisectors, Heights, and Angle Bisectors:

This title holds within two problems that were both proposed for students of grades 8 and 9. The opinions of two teachers are provided here.

Q1 :

T8 Yes, the concept of “cyclic”. We don't use this term, we ask students to “prove that the four points are on the same circle”. The other concepts are all known.

T9 No.

Q2 Both teachers responded with *no*.

Q3 Both teachers responded with *no*.

Q4 Both teachers responded with *yes*.

Q5 Both teachers responded with *yes*.

Q6 :

T8 The student can prove that four points are on the same circle, but we don't use the terms “cyclic” or “inscribed”. In the 8^{th} grade, they learn that four points are on the same circle if they form two right angles facing same diameter, but we don't say it is cyclic since “cyclic quadrilaterals” are not required from them. Also, 8^{th} grade students don't know that *if we have four points forming two equal angles facing the same chord, then the four points are on the same circle*.

The problem of perpendicular bisector and height is suitable for grade 8 ... It is not hard and the student can solve, deduce and be happy and interested with the question. However, the problem of the bisector and height is not suitable for grade 8 since it holds within properties in the circle and the concept of four points on same circle… the circle is given at the end of the year for 8^{th} grade. Thus, it is better to give this problem to students of grade 9 than those of grade 8.

T9 The problem of the perpendicular bisector and height is suitable for all students of grade 8. But that of the bisector and height just suits the special students in grade 8, whereas it can be given for all students of grade 9.

Teachers agree that some terms should be exchanged with more familiar terms in these grades. Also, the perpendicular bisector and heights problem is suitable for 8^{th} and 9^{th} grade students while the bisectors and heights is only suitable for 9^{th} grade students. With such recommendations and modifications, teachers find the problems not hard and understandable.

6) Attractive Induced Formula:

This exercise was proposed for students of grade 7. The opinions of two teachers are provided here.

Q1 Both teachers answered with *yes, the formula: If* $\frac{a}{b}=\frac{c}{d}$ *,then* $\frac{a}{b}=\frac{a+c}{b+d}$.

Q2 Both teachers responded with *yes, it is hard to prove the formula*.

Q3 :

T9 No, if they take the lesson of proportion and variables.

T10 No, if it is all explained in details.

Q4 T9 answered with a *yes*, and T10's response is missing.

Q5 T9 answered with a *yes*, and T10's response is also missing.

Q6 :

T9 A 7^{th} grade special student can understand the formula. Usually we don't give the proportions in grade 7, which is related to this question.…The idea is beautiful and nice to be given to all students.

[In general, if I give this exercise to a 7^{th} grade class and explained it?] If I want to talk in general, I can say that at the age of grade 7 the student can understand this formula if he took the lesson.

T10 Students can understand this formula if it is explained in details. It needs time to be well understood. [Of course it will take a complete period to be presented and clarified]. That is true; it is not easy. And not all the students are going to understand it. If you designed this question for all students, the intelligent one will catch the idea directly, the student who is fair or good enough may also understand it; however it is impossible for a weak student to understand it. I don't expect he can work on this formula; the weak student can study this formula or apply it directly, but he cannot understand the rule and how we moved from the necessary condition $\frac{a}{b}=\frac{c}{d}$ to the sufficient one $\frac{a}{b}=\frac{a+c}{b+d}$. The link is somehow hard for him. The intelligent and the fair students can understand it because in the seventh class they are exposed to similar ideas or even harder ones. Nothing is hard in this question and they can get it part by part.

Teachers agree that students will not be able to comprehend the proof of the formula and some students might find the structure of the formula hard to digest. Yet they agreed that almost all students can understand the formula and the whole problem if explained well.

5.3.2 Students' Experiment

1) Real Functions and Geometry:

We introduced this problem to a 3^{rd} secondary general sciences classroom consisting of 18 students.

1. We stated and provided the drawing of the main problem prompt. While writing the exponential function on board, one student directly commented with astonishment that what brought this function to this problem!
2. We gave students some time to think it out and try to propose how the ideas are related. After a while, they started to comment that what does the

function have to do with the problem? When nobody had an answer, we wanted to proceed, but a student commanded more time to think. Students started to suggest some scenarios, none were correct. Unfortunately, due to lack of sufficient time, we couldn't provide more time to hear more suggestions.

3. We stated part A. We asked students that, with those notations, what are they required to prove? They directly answered that they should prove $x = 1$.
4. We stated part A(1). Since we already knew from teachers that students do not know the Angle-Bisector Theorem, we directly provided and demonstrated it. Students were able to comprehend the theorem and apply it to triangle ABC.
5. We asked students to use this theorem and solve part A(1). One student commented that he didn't understand the whole problem, so we re-illustrated. One student said that he found b, his answer was wrong. Due to lack of time, we had to move on and demonstrate the solution. We provided how to find a, then a student directly provided the answer of b. All students understood the steps.
6. As a reminder to be used in later parts, we wrote a and b in terms of x on one corner of the board.
7. We stated part A(2). We told students that to find the solution, they should notice some given from the figure and combine it with the results of the previous part. A student directly pointed out the given which makes use of Pythagoras theorem. Then we demonstrated the solution with the participation of some students.
8. On the same corner of the board, we added the obtained equation.
9. We stated part A(3). Students acknowledged that the function g has the form of the previous equation and thus the answer was illustrated.
10. We commented that, starting from this point, we are relating the geometric problem to functions. We added to the same corner of the board this new result.
11. We asked students if they were able to find the variation of this function g. They responded that they should find g', so we told them to try. When

nobody commented, we told them that finding the variation of g in this manner is very hard and they should follow another path.

12. We illustrated the hyperbolic functions. Directly students commented that they didn't take them in their mathematics classroom. We emphasized that these functions are in terms of exponential functions that they already know.
13. We hastily went through part B, since we had too little time. We asked for the domain of definition of $\sinh$ and $\cosh$, students directly answered correctly. We asked for their derivatives, students were able to suggest how to derive them and gave the correct answers. Then we asked them to deduce the variation of $\sinh$ only, since the next parts depend just on it.
14. We asked: what do we deduce by having $\sinh$ strictly increasing and continuous? Students directly answered that $\sinh$ becomes bijective. Then we remarked that this leads to the following conclusion, part B(3), that: for $x > 0$ there exists $t > 0$ such that $x = \sinh(t)$. We emphasized on this point. Students acknowledged this deduction.
15. We stated part B(5). Students provided the proof of this part.
16. We commented that due to part B(4), we can now apply a change of variable on g in order to find its variation in an indirect way, thus justifying the given in part C.
17. We stated part C(1). Students directly said that they should substitute $x = \sinh(t)$ in $g(x)$. We gave students some time to solve. Due to lack of time, we had to demonstrate before any of the students was able to give an answer. We started solving and students were helping.
18. We stated part C(2). Students were directly able to provide the solution.
19. We stated part C(3). Then, we demonstrated the solution with the assistance of students. On the same corner of the board, we added the following: g is strictly increasing. We asked students to relate our last findings and reach the final result. Students started to suggest, none were correct, until one student said that this gives us that $x = \frac{1}{x}$. Afterwards, we demonstrated how this result is derived and how it leads to $x = 1$.
20. We demonstrated the steps of the whole problem together.

21. We commented that solving the main geometrical problem in geometric methods is really hard, so we went through another simpler path were we linked 2 domains in mathematics to reach the solution.

Most students were participating and anticipating. Some were curious and others were confused.

2) Inequalities and Variation of Functions:

We introduced this problem to a 2^{nd} secondary sciences classroom consisting of 18 students.

1. We stated the main problem. We provided the expansion: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ and we gave students time to solve. None gave an answer.
2. We asked students to find the solution through the calculator. All were able to realize that $a > b$ with a minor difference. We commented that although these two numbers are very close to each other, comparing them together in a traditional manner is hard and thus we need to go through another path.
3. We stated part 1. With our assistance, students were able to provide an illustration of the solution.
4. We stated part 2. Students suggested the solution. It was a trivial part for them. We emphasized on the result found here and we asked students whether they can speculate from this point a scenario of the rest of the solution. Some students suggested, but only one had suggested a relatively close scenario where he said to subtract a and b from each other and apply a table of sign.
5. We stated part 3. Students were able to participate in the computation. We asked students if they can link the parts together and find the final solution. Students tried to relate x of part 2 to the answer, but none gave an explicit and clear answer. We had to intervene and help them out.
6. We asked students to try to suggest how the solution will appear throughout the derived results. A student started to give a correct solution. Then, with our aid, students were able to demonstrate the solution.
7. We commented on the situation, that although those two numbers are relatively close to each other, traditional methods couldn't give a

demonstration of the comparison. So, we had to go through another path that involves the usage of a function and its sign.

Students were participating and group discussions were active. Some students expressed interest and anticipation in viewing results while others were not focused.

3) 1 in disguise:

We introduced this problem to a 2^{nd} secondary sciences classroom consisting of 21 students.

1. First, we cleared out what does a rational number mean. Then we stated the main problem. We gave students time to think. Some students used the calculator and found that this number is equal to 1. None of the students was able to find a demonstration of the solution. So we commented that through traditional methods that involve powers of 3, the demonstration might be impossible. Hence we need another untraditional method.
2. We stated part 1. All students suggested that they should substitute the number in the equation and prove it to be zero. Due to lack of time, we had to demonstrate the solution quickly. Students were participating.
3. We stated part 2. Students directly gave a demonstration.
4. We stated part 3. Students didn't know what imaginary numbers were, so we had to modify the question, we asked them to show that x^3+3x-4 has only one real root. Directly students used the previous part and answered that $x=1$ is the real root.
5. We started to connect the parts together by emphasizing on $\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}$ and 1 being real roots of the equation that has only one real root. Then we asked students what does this situation lead to? All answered that they should be equal.
6. We stated part 4. We re-illustrated the whole steps.
7. We commented that in order to solve this problem we had to go through such untraditional path.

Participation and suggesting was a major observation that went through all the problem. While group discussions took place among some students, confusion was noted among others.

4) Triangles and prime numbers:

We performed this experiment in a first secondary class consisting of 24 students.

1. The problem was put on the board in front of students, pointing out on the conditions. At the first glance, all students seemed to be astonished with nothing to say.
2. We illustrated a numerical example of x and y, some students gave other examples, then we showed how it is possible to set all the possibilities of x and y to prove the result, knowing that $1 \leq x, y < 90$ and they are integers.
3. We moved to the algebraic proof and started by the supposition "y not prime". All students were surprised and paying attention to this idea. Though they were not familiar with this method of proof, we clarified it in a simple manner and continued.
4. Stating part 1: Students needed a detailed orientation towards this question; we explained the proof and repeated each step to be clear for all.
5. After stating part 2, we gave a numerical example to make the question more understandable. Then, we recalled that every number has prime factors that are found by making its prime factorization. With more explanation, every point was clear till we reached the required result.
6. We stated part 3. Most of the students answered that they are all primes between 1 and 10, and then a student stated the possibilities $\{2,3,5,7\}$.
7. We stated part 4. This is the heaviest part of the problem since it requires deeper thinking and relating the given, definitely by pursuing the path of proving by contradiction! The solution was demonstrated step by step and students shared by agreeing on each step.
8. The final part was to deduce, according to the given, that y is prime. At this level, we rewrote the previous results together till we reach that the possibilities of $p \in \{2,3,5\}$ are rejected. Here, a student knew that the remaining possibility of p is 7. Then, we explained the reason behind rejecting $p = 7$. Afterwards, a student said: "then it means y should be

prime" and another one added: "you tried all possibilities to reach the required result".

9. We summarized the whole parts and students understood that the main supposition was wrong and so y is prime.

Students revealed some participation when each part was stated and they tried to give suggestions of the solutions. This didn't hide a sense of distraction by some students, and dullness at certain levels, especially when we started "supposing the contrary". Sarcastic comments were also heard through the demonstration.

5) Heights and Angle Bisectors:

This problem was demonstrated for 25 students of grade 9.

1. We started by recalling the definitions of the height and the angle bisector in a triangle, and asking if there is something common between these definitions. All students said "no". We asked about a property of the three angle bisectors of any triangle, a student answered that they meet in one point.
2. We showed the proof of this property in details. Students shared in the proof and gave suggestions; many seemed interested, and some were excited to solve by themselves.
3. We stated the main prompt of problem on the board. A student seemed curious to start solving and asked us to draw the figure with heights on the board.
4. Part 1 was introduced. It was clear that students didn't know the meaning of "inscribed". So, we clarified what it meant and recalled that if two angles facing same chord are equal then the 4 points forming these angles are on the same circle; then, we showed the proof on the figure.
5. We moved to part 2 and stated that, in a circle, two angles facing the same chord are equal. We pointed on the naming and choice of equal angles; that is, to choose equal angles according to their common extremities on the circle. Then we deduced the corresponding equal angles.
6. Students read the next part 3; they were a bit confused. We proceeded by the same previous pattern, providing the students with some orientations to

reach the result. Directly, we moved to finding the inscribed angles required in part 4. Together we shared with the students and proved the equal angles as explained previously; most of them got the method we are working with.

7. At the time we were moving to part 5, a student directly solved it; s/he named the equal right angles facing the same chord, and deduced that the 4 points, A, C, H_a and H_c are on the same circle. Then, students started naming the equal inscribed angles with some help and orientation. Proving $\alpha = \gamma$ made some confusion in determining the names of the corresponding angles; however, it was noted that a student was naming these angles and all was clear.
8. At the end, we stated part 7 which is the deduction of the main statement of the problem. First we emphasized on the given: *knowing that the three bisectors meet in one point*. Following our directions, students noticed that the heights of ABC are the bisectors of $H_aH_bH_c$ and reached the result.

While explaining the problem, a student showed some fear; he was afraid if this was required for tests or their math program. We calmed him and explained our solely goal to solve together a problem, whose concepts are known for them, and see a certain link. Besides, the names of angles, α, β, δ and γ, made some disturbance, as well as the notations of feet of the heights, H_a, H_b and H_c. These notations were uncomfortable for some students and caused confusion.

In summary, students were participating, speculating, suggesting and showing some interest during the whole period. In addition, some signs of distraction, straying and dullness were recorded. Some students were also playing, not concentrating, and not interested at all.

Finally, it is worthy to state the comment of a student who asked whether the problem is applicable when “the heights are outside the triangle”. It was a remarkable question.

6) Heights and Perpendicular Bisectors:

This problem was explained to 25 students of grade 8.

1. At the start, we recalled the definitions of the height and the perpendicular bisector in a triangle, and then we asked about the common between these two concepts. Many answers were detected: both make $90°$, both are straight lines, and both are parallel (if drawn on the same base).
2. We asked about the property of the three perpendicular bisectors of any triangle; they remembered that they meet in one point. Then, we explained the proof and students shared along with our orientation.
3. The main problem was given. We sketched the figure on the board and started with the steps. ABC is an arbitrary triangle with its three corresponding heights whose intersection is erased. We drew the required constructions in the given.
4. We stated part 1. Many students shared in the answer. They showed that, in each one of these quadrilaterals, the opposite sides are parallel.
5. We proceeded with part 2. With some aid and orientation, a student showed that A is the midpoint of $[EF]$. Then, students recognized the used method and some of them participated in the proof of the other midpoints.
6. We stated part 3. Starting with AH_a, a student knew that it is perpendicular to $[EF]$ by noticing alternating angles between parallel lines. After giving the complete demonstration of AH_a, students continued the proof of the other perpendicular bisectors following our directions.
7. Before reaching the end, we ran out of time and the class was out of control! We were obliged to state the result and explain it without giving them time to think of it. We showed the link by which the heights of triangle ABC became the perpendicular bisectors of triangle DEF, and thus they meet in one point, by using the main given.

There was participation in the class, as well as speculation and suggestions with some group discussions. Some distractions were also detected, and many students were not participating; some were playing and eating as if they had a recess time! Unfortunately, some students were naughty and added sarcastic comments.

7) Attractive Induced Formula:

We proposed this problem to 16 students of grade 7.

1. We started by the first part. Students calculated each fraction and found them equal. One of the students said that “they will be equal since we are repeating the same number up and down”. We asked about the reason, then we gave a numerical example that proves to this student that his claim is wrong. A remarkable answer was given by another student: “they are not equal because ‘5’ is one digit, not two”. He noticed the remark and almost understood the procedure. We agreed with him/her, and moved to continue the exercise.
2. We stated part 2. After some orientation, a student answered: “every time we repeat a number more times we get the same result”. We asked him/her for an example and S/he gave: $\frac{13131313}{75757575}$. Students checked that $\frac{13}{75}=\frac{13131313}{75757575}$.
3. Then, we stated part 3. A student found an example: $\frac{143}{734}=\frac{143143}{734734}$ and found them equal. Another student gave another example: $\frac{125}{137}=\frac{125125}{137137}$.
4. We moved to part 4. Students were confused, they didn't know the meaning of this proposition and weren't that much familiar with variables. There was confusion in making manipulations of variables and changing the signs when moving to different sides of the equation. We demonstrated the parts of the proof and repeated it. But when we kept sensing their confusion, we skipped it and moved to its application.
5. We stated part 5. Due to students confusion, we showed the link directly by putting $a=13$, $b=75$, $c=1300$ and $d=7500$.We applied the formula: $\frac{13}{75}=\frac{1300}{7500}$, then $\frac{13}{75}=\frac{1300+13}{7500+75}=\frac{1313}{7575}$. They participated in reaching it and some said that they “now understand it”. We told them that the aim of this property is to apply it on any number we want. Then, we asked them to continue the next step. They did it: $\frac{1313}{7575}=\frac{131300+13}{757500+75}=\frac{131313}{757575}$.

6. Finally, we asked about part 7. We gave an example and showed how to add 3 zeros: $\frac{132}{564}=\frac{132000+132}{564000+564}=\frac{132132}{564564}$. Some students shared and gave correct ideas, and others commented and deduced that "we add zeros according to the number of digits".

Note that although we didn't have enough time to demonstrate that "the equality of such fractions appears only if the two numbers in the numerator and the denominator have the same number of digits", students caught this idea from the beginning when we asked them to check it through a numerical example, but we hadn't the opportunity to clarify it by using the formula.

Students of this class recorded a good participation in each step; they were interested, curious, anticipating, speculating, suggesting and showed persistence in finding the solutions. These bright-sided observations were penetrated by some distraction and dullness while explaining the proof of the formula.

5.3.3 Questionnaires' Analysis

The structure of the post questionnaire is provided in appendix B.

Real Functions and Geometry:

61.1% of the students understood the whole problem and 38.9% understood parts of the problem. 88.9% of the students found the problem interesting.

Inequalities and Functions:

77.8% of the students understood the whole problem and 22.2% understood parts of the problem. 88.9% of the students found the problem interesting.

1 in disguise:

94.7% of the students understood the whole problem and 5.3% understood nothing in the problem (two answers were recorded as missing). 89.5% of the students found the problem interesting (two answers were recorded as missing).

Triangles and Prime Numbers:

45.8% of the students understood the whole problem, 8.3% understood parts of the problem and 45.8% understood nothing in the problem. 62.5% of the students found the problem interesting.

Heights and Angle Bisectors:

43.5% of the students understood the whole problem, 30.4% understood parts of the problem and 26.1% understood nothing in the problem (two answers were recorded as missing). 80% of the students found the problem interesting.

Heights and Perpendicular Bisectors:

56% of the students understood the whole problem, 32% understood parts of the problem and 12% understood nothing in the problem. 80% of the students found the problem interesting.

Attractive Induced Formula:

18.8% of the students understood the whole problem, 62.5% understood parts of the problem and 18.8% understood nothing in the problem. 93.8% of the students found the problem interesting.

6 Conclusions, Limitations and Implications

6.1 Conclusion

According to the analyzed data, "The Unexpected Links" is a component of mathematical creativity that is not patent of professional mathematicians or postgraduate mathematics students. It can be made available to intermediate and secondary school students through well-structured and suitable mathematics problems that meet their acquired mathematical knowledge.

The problems we proposed fulfilled their role, they were understandable for almost all students and they caught students' interest.

According to teachers' testimonies, classrooms' observations and students' interactions, some modifications are required to some problems. We suggest the following:

- A retest of the problem of triangles and prime numbers in higher levels (grades 11 and 12).
- The elimination of the proof of the formula in the attractive induced formula problem and the re-introduction of it in grade 8 where students are more familiar with variables and their manipulations.

- The rearrangement of the parts of the problem of inequalities and functions in accordance with a teacher's recommendation.
- The elimination of the proofs of the intersection of angle bisectors and perpendicular bisectors, since they are distracting the students from the main problem.
- Changing the letters used to denote the angles in the problems concerning heights, angle bisectors and perpendicular bisectors.
- The modification of part 3 of the 1 in disguise problem in a way that doesn't include imaginary numbers.

Appendix A provides the modified structures of the problems.

6.2 Limitations

We can divide the limitations into two kinds: the first in the phase of preparing problems and the second in the experimentation phase. With respect to problem design, as we move to lower level, such kinds of problems are hard to create. This is due to students' limited acquired knowledge. With respect to the experimentation phase:

- We faced some difficulties in finding schools that agree to undergo such experiment.
- We had a difficult time in finding teachers; some barely gave us 5 minute to introduce the problems and interview or apply a questionnaire on them.
- School's administration wouldn't allow us to take more than one period (50-55 minutes) in one classroom.

6.3 Implications

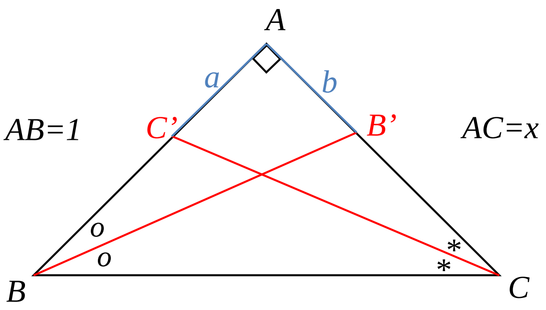
This paper provides evidence that problems demonstrating the unexpected links can be introduced in regular mathematics classrooms; students can comprehend such problems that contain deep links among and within mathematical domains. Now, a single problem has no power to impose a striking impact on students' methods and strategies in solving mathematical problems. Whereas, we believe, even we conjecture that a long term exposition of students to such problems may train them to use this creative method in solving mathematics.

Appendix A

1. Real Functions and Geometry

Let ABC be a right triangle at A, with (BB') and (CC') the bisectors of $A\hat{B}C$ and $A\hat{C}B$ respectively, such that $\mathrm{BB'} = \mathrm{CC'}$. Show that if the function $f(t) = \dfrac{e^t - e^{-3t}}{2}$ is strictly increasing, then ABC is isosceles.

A. Suppose $\mathrm{AB} = 1$ and $\mathrm{AC} = x > 0$. Let $\mathrm{AC'} = a$ and let $\mathrm{AB'} = \mathrm{b}$.



1. Find a and b in terms of x. (Use *the Angle-Bisector Theorem*: consider a triangle ABC, if AE is the angle bisector of $B\hat{A}C$ then $\dfrac{EB}{EC} = \dfrac{AB}{AC}$).
2. Using the previous part, show that:

$$x + \frac{x}{\left(x + \sqrt{1+x^2}\right)^2} = \frac{1}{x} + \frac{\frac{1}{x}}{\left(\frac{1}{x} + \sqrt{1 + \frac{1}{x^2}}\right)^2}.$$

(Hint: Show that $a^2 + x^2 = b^2 + 1$)

3. Find a function g such that $g(x) = g\left(\frac{1}{x}\right)$.

B. Consider the **hyperbolic sine function** $\sinh(t) = \dfrac{e^t - e^{-t}}{2}$ and the **hyperbolic cosine function** $\cosh(t) = \dfrac{e^t + e^{-t}}{2}$.

1. Define the domain of definition of $\sinh$ and of $\cosh$.
2. Show that $(\sinh(t))' = \cosh(t)$ and $(\cosh(t))' = \sinh(t)$.

3. Study the variations of $\sinh$ and of $\cosh$.
4. Deduce that for $x > 0$ there exists $t > 0$ such that $x = \sinh(t)$.
5. Show that $\cosh^2(t) - \sinh^2(t) = 1$.

C. Let $g(x) = x + \dfrac{x}{\left(x + \sqrt{1 + x^2}\right)^2}$ and $x = \sinh(t)$ with $t > 0$.

1. Find the **exponential function** $f(t)$ such that $f(t) = g(x)$.
2. Show that f is strictly increasing.
3. Deduce that g is strictly increasing.

D. Using the previous parts, deduce that ABC is isosceles. (In other words, deduce that $x = 1$)

2. Inequalities and Functions

Compare $a = \sqrt[3]{5 - \sqrt[3]{2}}$ and $b = \sqrt[3]{2 + \sqrt[3]{4}}$.

1. Let $f(x) = x^2 + x - 3$. For what values of x, $f(x) < 0$?
2. Compare $\sqrt[3]{2}$ and $\dfrac{-1 + \sqrt{13}}{2}$.
3. Compute $b^3 - a^3$.
4. Deduce that $a > b$.

3. 1 in Disguise

Show that $\sqrt[3]{2 + \sqrt{5}} + \sqrt[3]{2 - \sqrt{5}}$ is a rational.

1. Show that $\sqrt[3]{2 + \sqrt{5}} + \sqrt[3]{2 - \sqrt{5}}$ is a root of $x^3 + 3x - 4$.
2. Prove that $x^3 + 3x - 4 = (x - 1)(x^2 + x + 4)$.
3. Show that $x^3 + 3x - 4$ has only one real root.
4. Deduce that $\sqrt[3]{2 + \sqrt{5}} + \sqrt[3]{2 - \sqrt{5}} = 1$; that is, $\sqrt[3]{2 + \sqrt{5}} + \sqrt[3]{2 - \sqrt{5}}$ is rational.

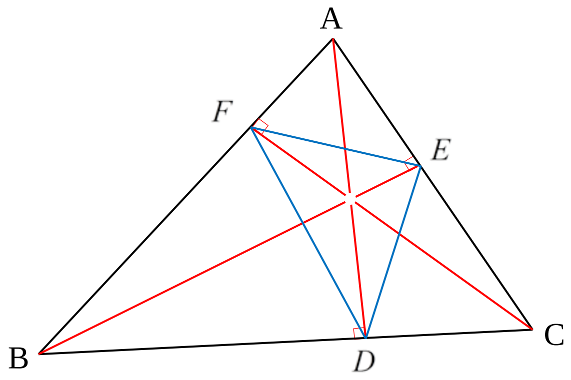
4. Triangles and Prime Numbers

Let ABC be a right triangle at A. Let $A\hat{B}C = x$ and $A\hat{C}B = y$ such that $x > 5$ and $x, y \in \mathrm{N}$ (measure in degrees). Show that if x is prime, then y is prime, provided that y is not divisible by 7.

1. Suppose that y is not prime. Show that y has a divisor less than 10.
2. Deduce that y has a prime divisor p less than 10.
3. What is the set of possibilities of p?
4. Show that if $p \in \{2,3,5\}$, then x is no more prime.
5. Deduce that, according to the given, y is prime.

5. Heights and Angle Bisectors

Let ABC be any triangle. Let D, E and F be the feet of the heights drawn from A, B and C, respectively, to the relative bases. We will hide their intersection. Consider the triangle DEF.



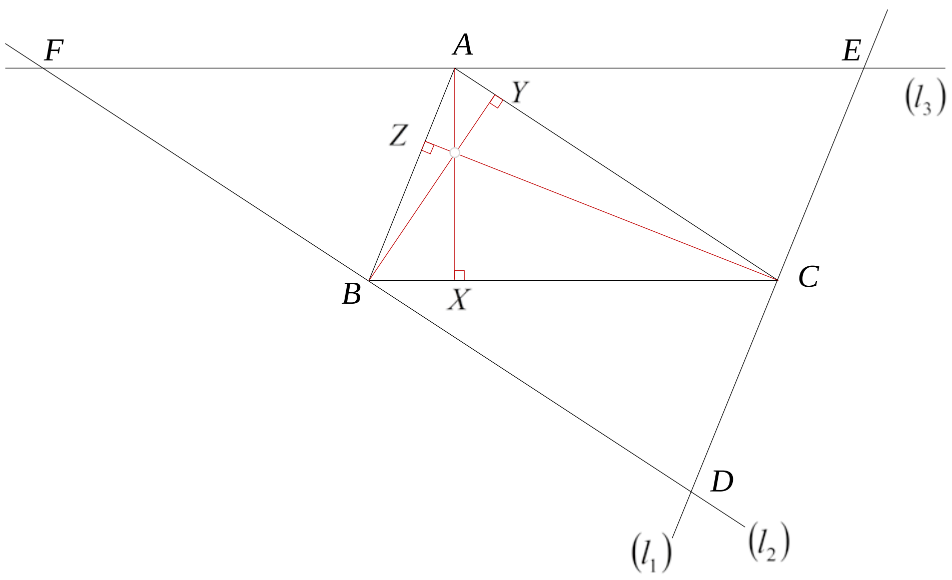
1. Prove that B, C, E and F are on the same circle.
2. Deduce that $F\hat{E}B = F\hat{C}B = x$, $E\hat{F}C = E\hat{B}C = y$ and $E\hat{C}F = E\hat{B}F = z$.
3. Prove that A, B, D and E are on the same circle.
4. Deduce that $B\hat{A}D = B\hat{E}D = t$, $A\hat{D}E = z$ and $D\hat{A}E = y$.
5. Prove that A, C, D and F are on the same circle.
6. Deduce that $C\hat{F}D = y$, $F\hat{D}A = z$ and $x = t$.
7. Deduce that the heights (AD), (BE) and (CF) meet in one point.

6. Heights and Perpendicular bisectors

Consider any triangle ABC. Let AX, BY and CZ be the heights relative to the bases BC, AC and AB respectively. We will hide their intersection.

Let (l_1) be the line passing in C and parallel to (AB), (l_2) be the line passing in B and parallel to (AC), and (l_3) be the line passing in A and parallel to (BC). Let D, E and F be the points of intersection of (l_1) and (l_2), (l_1) and (l_3), (l_2) and (l_3) respectively.

1. Prove that $ABCE$, $ACBF$ and $ACDB$ are parallelograms.
2. Prove that A, B and C are the midpoints of $[EF]$, $[FD]$ and $[DE]$ respectively.
3. Deduce that (AX), (BY) and (CZ) are the perpendicular bisectors of $[EF]$, $[FD]$ and $[DE]$ respectively.
4. Deduce that the 3 heights of triangle ABC meet in a common point.



7. Attractive Induced Formula

1. Using your calculator find the values of: $\frac{13}{75}$, $\frac{1313}{7575}$, $\frac{131313}{757575}$.
2. What can you conclude?

3. Can you give a 3-digit example with the same previous feature?
4. Show that: "If $\frac{a}{b}=\frac{c}{d}$ then $\frac{a}{b}=\frac{a+c}{b+d}$" . [Note that this part is asked for grade 8; if the problem is given to grade 7, it is skipped.]
5. Use the following property to give the reason behind the above equalities?

$$\text{“If } \frac{a}{b}=\frac{c}{d} \text{ then } \frac{a}{b}=\frac{a+c}{b+d}\text{”}$$

6. Can we deduce that $\frac{25}{471}=\frac{2525}{471471}$? Why? (try to use the formula)
7. Apply the strategy on the 3-digit-number example.

Appendix B

Q1 According to your teaching experience, does this problem make use of any mathematical concepts that students of this grade do not know or are not aware of?

Q2 According to your teaching experience, do you find any part of this problem hard for students to solve?

Q3 According to your teaching experience, do you find the solution of any part of this problem hard for students to understand?

Q4 In your opinion, do you think that this problem triggers students' curiosity and help them see that mathematical properties and formulas are linked together?

Q5 In your opinion, does this problem represent a good example of our notation "the unexpected links"?

Q6 Any remarks?

Appendix C

Questionnaire

Choose <u>one</u> answer only:

1. I understood:
   a. The whole problem.
   b. Nothing in the problem.
   c. Other (some parts).

2. The problem was interesting:
    a. Yes.
    b. No.

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