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Making Progress on Mathematical Knowledge for Teaching

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Abstract: Although the field lacks a theoretically grounded, well-defined, and shared conception of mathematical knowledge required for teaching, there appears to be broad agreement that a specialized body of knowledge is vital to improvement. Further, such a construct serves as the foundation for different kinds of studies with different agendas. This article reviews what is known and needs to be known to advance research on mathematical knowledge for teaching. It argues for three priorities: (i) finding common ground for engaging in complementary studies that together advance the field; (ii) innovating and reflecting on method; and (iii) addressing the relationship of such knowledge to mathematical fluency in teaching and to issues of equity and diversity in teaching. It concludes by situating the articles in this special issue within this emerging picture.

Keywords: mathematical knowledge for teaching, MKT, specialized knowledge, pedagogical content knowledge, PCK, mathematics teacher education, method, mathematical fluency, equity, diversity.

Introduction

A century ago, a central focus of teacher education in the United States was on developing a thorough understanding of subject matter, but the mid-twentieth century witnessed a steady shift to an emphasis on pedagogy generalized to be largely independent of subject matter. By the 1980s, an absence of content focus was so prevalent that Shulman (1986) referred to this as a "missing paradigm" in teacher education. A similar tendency can be seen in other countries. For example, a few decades ago, it was possible to become qualified for teaching mathematics in grades 1–9 in Norway with no more mathematics than a short course in didactics. A widespread

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assumption seemed to be that prospective teachers already knew the content they needed, from their experiences as students, and they only required directions in how to teach this content. Shulman's call for increased attention to subject matter reoriented research and practice. However, the connection between the formal education of mathematics teachers and the content understanding important for their work is not straightforward. Teachers' formal mathematics education is not highly correlated with their students' achievement (Begle, 1979) or with the depth of understanding they seem to have of the mathematical issues that arise in teaching (Ma, 1999).

One of Shulman's (1986) most important contributions was the suggestion that the work of teaching requires professional knowledge that is distinctive for the teaching profession. He proposed different categories of professional knowledge for teaching. One of these categories was distinctive content knowledge, which Shulman described as including a deep knowledge of the structures of the subject (e.g., Schwab, 1978), beyond procedural and factual knowledge. Another category of knowledge was what Shulman termed "pedagogical content knowledge," which is aspects of the content most germane to its teaching (1986, p. 9). The idea about an amalgam of subject matter knowledge and pedagogical knowledge has continued to appeal to researchers working in different subject areas, and Shulman's foundational publications are among the most cited references in the field of education. (Google Scholar identifies over 13000 publications that cite his 1986 article.)

In the last two decades, researchers and mathematics educators have increasingly emphasized the significance of mathematical knowledge that is teaching-specific. Such knowledge is seen as different from the mathematics typically taught in most collegiate mathematics courses and from the mathematics needed by professionals other than teachers. Although it includes knowing the mathematics taught to students, the kind of understanding of the material needed by teachers is different than that needed by the students. Even though the literature suggests a general consensus that mathematics teaching requires special kinds of mathematical knowledge, agreement is lacking about definitions, language, and basic concepts. Many scholars draw on Shulman's notion of pedagogical content knowledge (or PCK) and view this knowledge as being either a kind of "combined" knowledge or a kind of "transformed" knowledge. Grounded in Shulman's proposals, the phrases "for teaching" and "practice-based" have been emphasized to indicate the relationship of the knowledge to specific work of teaching (e.g., Ball, Thames, & Phelps, 2008). For this article, we adopt these phrases but maintain an ecumenical view of a more extended literature.

With growing interest in ideas about specialized professional content knowledge, the early 2000s saw a spate of large-scale efforts to develop measures of such knowledge and the use of such a construct as the basis for a wide range of research studies, such as evaluating professional development (e.g., Bell, Wilson, Higgins, & McCoach, 2010), examining the impact of structural differences on the mathematical education of teachers (e.g., Kleickmann et al., 2013), arguing for policies and programs (e.g., Hill, 2011), and investigating the role of professional content knowledge on mathematics teaching practice (e.g., Speer & Wagner, 2009). Instruments for measuring such knowledge represent a crucial tool for making meaningful progress in a field. They operationalize emerging thinking, invite scrutiny, and support the investigation of underlying models.

This special issue on developing measures and measuring development of mathematical knowledge for teaching continues this focus on instruments, along with a concomitant regard for broader purposes and potential ways to advance the field. In an effort to situate this special issue, this introductory article provides some selected highlights from the field — focusing on what is being studied, how, and to what ends. To accomplish this, we draw on both a detailed review of articles sampled from 2006 to 2013 and our wider reading of the literature. We then nominate some key areas for making progress on research and development of the specialized mathematical knowledge teachers need and we use this framing to characterize the agendas and contributions of the collection of articles assembled in this special issue. The article consists of three major sections.

1. Lessons from Empirical Research
2. Next Steps for the Development of Mathematical Knowledge for Teaching
3. Articles that Develop Measures and Measure Development

The first describes a review we conducted and discusses three broad arenas of work suggested by this review. The second discusses three proposals for future research. The last briefly situates the articles in this issue within the lessons and directions discussed.

Lessons from Empirical Research

In our reading of empirical literature concerned with the distinctive mathematical knowledge requirements for teaching, several broad strands of research stand out. We begin by describing a formal review we conducted of empirical research that began appearing in about 2006, in the wake of a number of conceptual proposals (beyond PCK), and that began using these proposals as a conceptual basis for empirical study. This review informs our overall reading of the field. Combining this review with our wider reading in the field, we then identify and discuss three major arenas of work.

Reviewing the literature. In the course of other research we were conducting, we reviewed international empirical literature published in peer-reviewed journals in English between 2006 and 2013.[1] Wanting to survey the topic across theoretical perspectives, we developed and tested inclusive search terms:

- Mathematics
    - o math* (the asterisk is a placeholder for derived terms)
- Content knowledge
    - o know* AND (content OR special* OR pedagog* OR didact* OR math* OR teach* OR professional OR disciplin* OR domain) OR “math for teaching” OR “mathematics for teaching” OR “math-for-teaching” OR “mathematics-for-teaching

[1] This more formal review, which we use to inform our wider reading of the literature, was funded by the National Science Foundation under grant DRL-1008317 and conducted in collaboration with Arne Jakobsen, Yeon Kim, Minsung Kwon, Lindsey Mann, and Rohen Shah, who we wish to thank for their assistance with searching, conceptualizing codes, coding, and analysis. The opinions reported here are those of the authors and do not necessarily reflect the views of the National Science Foundation or our colleagues.

- Teaching
    - teaching OR pedagog* OR didact* OR instruction*

These search terms initially yielded over 3000 articles from the following six databases:

- PsycInfo
- Eric
- Francis
- ZentralBlatt
- Web of Science
- Dissertation Abstracts

Broadened search terms, additional databases, and inclusion of earlier publication years yielded none to negligible additional articles.

Based on a reading of abstracts, 349 articles were identified as potential empirical articles (as characterized by the American Educational Research Association, 2006) in which some concept of distinctive mathematics needed for teaching was used as a conceptual tool to formulate research questions or structure analysis. Our goal was not to reach high standards of reliability, but rather to use a systematic process to collect a corpus of relevant studies representing the literature from this period. In coding the articles, we sought to be descriptive rather than evaluative and iteratively worked between an inductive examination of a sample of articles and initial conceptualizations of empirical research combined with a basic model of educational change. After reading full articles, 190 of the 349 remained in the final set. A set of core codes were developed for the following categories:

1. Genre of the study
2. Research problem used to motivate the study
3. Variables used
4. Whether or not and how causality was addressed
5. Findings

Additional codes included sample size, instruments used for measuring mathematical knowledge for teaching, school level or setting, professional experience of the teachers, geographic region, and mathematical area addressed. Each article was read and coded by two team members, with a decision as to whether it satisfied our inclusion criteria, and if so, codes were reconciled. (For a more detailed description of the methods used, see Kim, Mosvold, and Hoover (2015).)

In table 1, we present some patterns that emerged from some of the additional, descriptive codes.

Table 1. *Selected descriptive codes for sample size, instrument used, level of schooling, and geographic context.*

Categories and codes	*Number of papers*
Sample size	
Small scale (<10)	60
Medium 1 (10–29)	51
Medium 2 (30–70)	34
Large scale (>70)	43
None	2
Instrument	
COACTIV	4
CVA	3
DTAMS	3
LMT (including adaptations)	31
TEDS-M	2
Non-standardized	56
None	91
Level of teachers	
Primary (K–8)	81
Middle (5–9)	45
Secondary (7–13)	41
Tertiary	3
Across levels	20
Regions	
Africa	7
Asia	27
Europe	22
Latin America	3
North America	112
Oceania	15
Across regions	4

We observe that many studies are small-scale, and a large number of the studies apply non-standardized instruments or no instruments. In the studies where standardized instruments were used to measure teachers' knowledge, the instruments developed to measure mathematical knowledge for teaching in the Learning Mathematics for Teaching (LMT) project were most common. An abundance of studies focuses on primary teachers, and most studies were carried out in North America.

Table 2 provides the fourteen categories developed for coding the research problem. We have grouped these into three domains and use these groups to discuss the literature in the following sections.

Table 2. *Research problems addressed.*

Problems	*Number of papers*	*%*
Nature and composition of SM	55	28.9
What is SM?	34	
What relationships exist among aspects of SM or with other variables?	21	
Improvement of SM	81	42.6
What professional development improves teachers' SM?	28	
What teacher education improves teachers' SM?	28	
What curriculum/tasks improve teachers' SM?	10	
What teaching practice improves teachers' SM?	0	
How SM develops?	15	
How to scale up the teaching and learning of SM?	0	
Contribution of SM	33	17.4
Does SM contribute to teaching practice?	6	
What does SM contribute to teaching practice?	12	
Does SM contribute to student learning?	15	
What does SM contribute to student learning?	0	
Other		
What SM do teachers know?	21	11.1
How policy influences teachers' SM?	0	0
Total	*190*	*100*

In order to make table 2 more readable, we use the abbreviation SM to signify any of the variety of ways in which mathematical knowledge for teaching might be conceptualized and named. The intention is not to introduce yet another term or acronym for such knowledge. In this article, we have adopted more generic language to express an inclusive notion of such knowledge and we avoid the use of any specific acronym label.

For the purpose of this introductory article, we used patterns evident in the review described above to inform our extended reading of the field. Together, these efforts led us to identify three broad themes. First, a number of studies investigate the nature and composition of teacher content knowledge. Given that foundational research into teaching-specific mathematical knowledge pointed to its elusiveness and complexity, it is not surprising that scholars continue to investigate what it is — its components, measurement, features, and related constructs. A second group of studies, which constitutes the majority of published articles, investigates approaches to increasing teacher knowledge, in both the context of pre-service teacher education and the professional education of practicing teachers. A third group of studies, fewer in number, investigates effects of teachers' knowledge on both teaching and student learning. In the following sections, we use these three broad themes to organize our comments on selected highlights from the literature. Following these, we provide suggestions about possible next steps for further research in this field.

Nature and composition of mathematical knowledge for teaching. Current studies continue to probe ideas about the nature and composition of teaching-specific knowledge of mathematics. Some studies consider the construct in broad terms. They may identify or elaborate aspects or frameworks, characterize or critique the construct, compare different representations or sub-domains, or compare such knowledge with other kinds of mathematical knowledge. Others examine a constrained area of knowledge: some in relation to specific mathematical topics; some in relation to specific practices of teaching, or at specific levels (such as interpreting and responding to student thinking, curriculum use, or proving in high school geometry); and some in relation to specific qualities (such as connectedness). However, these studies do not build on each other in obvious ways and clear lessons are hard to identify. The one avenue of work that represents progress for the field is the development of instruments, and we focus our discussion there.

Instruments provide a crucial tool for investigating the nature and composition of mathematical knowledge needed for teaching. They serve to operationalize ideas about mathematical knowledge for teaching and test assumed models of the role it plays. They are used to investigate the teaching and learning of such knowledge, relationships with other variables, and other questions important for practice and policy. On the one hand, rigorous instrument development is expensive relative to budgets available for most studies and many instruments are used in a single study and limited in the extent to which they meet psychometric standards and establish validity. On the other hand, several larger efforts have invested in building instruments for large-scale studies and wider use in the field. The Learning Mathematics for Teaching (LMT) instruments for practicing elementary and middle school teachers (Hill, Schilling, & Ball, 2004) include nearly 1000 items on over a dozen different instruments and have been used in numerous program evaluations and studies of relationships and effects. They have been extensively validated (Schilling & Hill, 2007) and adapted internationally (Blömeke & Delaney, 2012). The Diagnostic Teacher Assessment in Mathematics and Science (DTAMS) instruments for practicing middle school teachers (Saderholm, Ronau, Brown, & Collins, 2010) include 24 forms in four content areas, have been administered and rigorously analyzed with a sample of several thousand teachers, and are currently being expanded. The Teacher Education and Development Study in Mathematics (TEDS-M) instruments for pre-service primary and lower secondary teachers (Tatto et al., 2008; Senk et al. 2012) include over 100 items and were originally administered to 23,000 pre-service teachers in 17 countries.

These instruments represent an important contribution to the field. Extensive cross-professional-community review and the building of agreed-on formulations of important content knowledge have played a major role in the development of these measures. The synthesis of ideas and the integration of expertise from multiple professional communities have helped to clarify and improve ideas about mathematical knowledge for teaching. In addition, the availability of common instruments has enabled meaningful comparison and interpretation across programs, countries, and studies in ways that contribute to the maturity of research on mathematical knowledge for teaching.

Several other efforts have developed instruments with less focus on broad consensus or widespread use. The COACTIV instrument for practicing secondary teachers (Kunter, Klusmann, Baumert, Voss, & Hacfeld, 2013) produced items of a genre similar to those described above and used these to investigate relationships to other variables and to understand issues of practice and policy related to the mathematical education of teachers. Some instruments have been developed to focus on mathematical knowledge related to a specific topic, such as fractions (Izsak, Jacobson, de Araujo, & Orrill, 2012), geometry (Herbst & Kosko, 2012), algebra (McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012), and continuous variation and covariation (Thompson, 2015). Others have focused on specific aspects of teaching, and the mathematical knowledge required in these specific teaching practices, such as choosing examples (Chick, 2009; Zodik & Zaslavsky, 2008) and scaffolding whole-class discussions to address mathematical goals (Speer & Wagner, 2009). Many instruments have been developed in relation to specific lines of research and often in response to perceived issues with more established instruments. A number of researchers are concerned about a potentially narrow interpretation of knowledge as declarative or about a possible discrepancy between knowledge and knowledge use. These concerns have led some scholars to explore different conceptualizations of the mathematics teachers need and to look for alternative formats for measuring it (e.g., Kersting, Givvin, Thompson, Santagata, & Stigler, 2012; McCray & Chen, 2012; Thompson, 2015).

Although the development of instruments is an important step toward building a robust conception of teaching-specific knowledge of mathematics, these efforts also reveal a lack of shared language and meaning of foundational concepts. Differences in meaning for the construct PCK have been noted in the past (Ball, Thames, & Phelps, 2008; Depaepe, Verschaffel, & Kelchtermans, 2013; Graeber & Tirosh, 2008). These differences persist, yet they are often overlooked with regard to instruments.

For example, Kaarstein (2014) examined whether the LMT, TEDS-M, and COACTIV instruments, each referencing Shulman and stating that the respective instrument measures PCK, measure the same thing. To study this issue, she constructed a taxonomy of the different levels of categories in Shulman's initial framework as well as the frameworks that were used to develop the three instruments. She then selected three items — one from each instrument — and categorized them according to each of the three frameworks. Her main argument is that content knowledge and pedagogical content knowledge are supposed to be distinct categories, and therefore three projects that use the same basic categories should categorize items in the same way. However, from her analysis the items would be placed in different basic categories using the criteria reported by the projects. As an example, an item that was categorized as a specialized content knowledge item (measuring content knowledge) in the LMT project would probably have been categorized as a PCK item in TEDS-M and COACTIV. Kaarstein's argument does not necessarily threaten the validity of the measures from each of the three projects, but her observation deserves further attention.

Similarly, a study by Copur-Gencturk and Lubienski (2013) echoes this concern. In order to investigate growth in pre-service teacher knowledge, they used two different instruments: LMT and DTAMS. When comparing groups of teachers who had participated in different kinds of courses, they concluded that the LMT and DTAMS instruments measure aspects of mathematical knowledge for teaching that are substantially different. Teachers who participated in a hybrid mathematics content/methods course had the most significant increase in their LMT score, and this score remained stable although they took an additional content course. Teachers' DTAMS score also increased during the hybrid course; during the content knowledge course, only the content knowledge part of their DTAMS increased. This study thus supports the idea that there is specialized mathematical content knowledge not influenced by general mathematics content courses. That different instruments measure different aspects of knowledge is not necessarily surprising, but it is worrying if instruments ostensibly designed to capture the same construct in fact measure significantly different facets of that knowledge, with little clarity about these differences.

The concerns raised by Kaarstein (2014) and Copur-Gencturk and Lubienski (2013) suggests that the limited specification of the construct and the different ways of operationalizing it makes it difficult to interpret results. This limits the extent to which results from these instruments, taken individually or together, can inform the conceptualization of mathematical knowledge for teaching or practical decisions needed to design learning opportunities.

Developing teachers' mathematical knowledge for teaching. With a growing sense of the mathematics important for improving teaching and learning, practitioners have turned their attention to increasing teachers' knowledge of professionally relevant mathematics and scholarly work has followed suit. A large number of studies make it clear that the design and evaluation of teacher education and professional development programs in developing teachers' mathematical knowledge for teaching are top priorities. From several decades of research, we propose what we see as a few related emerging lessons:

- Teaching teachers additional standard disciplinary mathematics beyond a basic threshold does not increase their knowledge in ways that impact teaching and learning.

- Providing teachers with opportunities to learn mathematics that is intertwined with teaching increases their mathematical knowledge for teaching.
- The focus of the content, tasks, and pedagogy for teaching such knowledge requires thoughtful attention to ways of maintaining a coordination of content and teaching without slipping exclusively into one domain or the other.

These lessons are rooted in early efforts to document effects of teachers' mathematical knowledge on student learning and are reinforced by current research on the design and implementation of teacher education and professional development. We begin by briefly reflecting on that early work and then tracing these lessons into current research.

Much of the impetus for the surge in research on teaching-specific knowledge began with reviews of several decades of large-scale research that found surprisingly little to no effect of teachers' mathematical knowledge on their students' learning (Ball, Lubienski, & Mewborn, 2001). The studies reviewed were often conducted with large datasets but very coarse measures. Taking Shulman's (1986) suggestion that the content knowledge needed by teachers was characteristically different from that needed by other professionals, researchers began to look more closely at the measures used in those studies and at the findings. The clearest finding that emerged was that methods courses consistently showed positive effects while content courses did not (e.g., Begle, 1979; Ferguson & Womack, 1993; Guyton & Farokhi, 1987; Monk, 1994). The second was that positive effects were more likely when the content taught to teachers was more closely related to the content they subsequently taught. For instance, several scholars found effects when using student exams to measure teachers' knowledge (Harbison & Hanushek, 1992; Mullens, Murnane, & Willett, 1996). Reinforcing these results, Monk (1994) found that coursework in calculus influenced the achievement of secondary teachers' students in algebra classes, but not in their geometry classes. In general, when the mathematics taught or measured is meaningfully connected to classroom materials or interactions, it is modestly associated with improved teaching and learning.

For some practitioners and policy-makers, the implication of these empirical studies, combined with logical arguments for teaching-specific professional knowledge, has been enough to lead to prioritizing mathematical knowledge for teaching in the mathematical education of teachers. Nevertheless, many policies continue to press for increases in the number of mathematics courses required of teachers, regardless of their connection to teaching, despite abundant evidence that such policies are unlikely to improve teaching and learning (e.g., Youngs & Qian, 2013). Such policies have probably been less the result of lingering doubt about empirical results and more the result of overextending the notion that knowing content well is key to good teaching, even in the face of disconfirming evidence. Of course, a certain threshold level of knowledge of the subject is essential, but preparing teachers by requiring mathematics courses that are not directly connected to the content being taught or to the work involved in teaching that content is misguided.

More recent studies continue to reinforce these established lessons. One recent line of inquiry is the investigation of features of innovative, well-received professional development programs. To us, the most compelling result emerging from these studies is that professional development requires designing pedagogically relevant movement between mathematical and pedagogical concern both to motivate teachers' investment in mathematical issues and to keep the mathematical attention on mathematics that matters for the work of teaching. To elaborate, we offer several examples that contribute to this claim.

With deep regard for the limited effects of decades of substantial national investment in professional development, several research groups have organically developed approaches informed by thoughtful reflection and attention to disciplined observation of teachers' engagement with and actual uptake of ideas and practices. One important insight emerging from these decades-long investments is that cycling through mathematical considerations, pedagogical considerations, and reflective enactment is vital to the design of professional development. For instance, Silver, Clark, Ghousseini, Charalambous, and Sealy (2007) set out to provide evidence for whether and how teachers might enhance their mathematical knowledge for teaching through monthly practice-based professional development workshops designed to cycle from activities of doing mathematics, to examining case-based pedagogical and student-related issues, to planning, teaching and debriefing lessons collaboratively (all related to a common mathematics task or set of tasks). Examining the interactions of one teacher, they document ways these activities provided opportunities for teachers to build connections among mathematical ideas and to consider these ideas in relation to student thinking and teaching. They do not measure teacher learning. Nor do they disentangle effects of what they refer to as a professional-learning-task cycle from a number of other important features of their professional development program. However, they document dynamics in which the teacher, from an initial experience solving a nontrivial mathematics problem, supported by mathematically sensitive facilitation, successively engages in mathematical issues and pedagogical issues in ways that visibly build connections among mathematical ideas, pedagogical practice, and growing mathematical knowledge for teaching. In addition, they argue that their cyclic design increased teachers' motivation for learning mathematics, both in the workshops and in their daily practice.

> Through successive opportunities to consider mathematical ideas in relation to the activities of classroom practice, our participants came to see their pedagogical work as permeated by mathematical considerations. (p. 276)

Similarly, in working to close the gap between a reform vision and the actual practice of mathematics teaching and learning, Koellner et al. (2007) implemented a model of professional development designed to help teachers deepen their mathematical knowledge for teaching through a cycle of solving a mathematics problem, teaching the problem, and analyzing first teacher questioning and then student thinking in videos of their teaching. In order to understand the learning opportunities afforded by what they refer to as a problem-solving-cycle design, they analyzed artifacts from two years of a series of monthly, full-day workshops with ten middle school mathematics teachers, including workshop videos and interviews with facilitators. The researchers used the knowledge domains identified in Ball, Thames, and Phelps (2008) to analyze several teacher interactions. They found that different learning opportunities were afforded by different activities: specialized content knowledge was developed by comparing, reasoning, and making connections between the various solution strategies; knowledge of content and teaching was developed by analyzing teacher questioning in the video clips from the teachers' lessons; and knowledge of content and students was developed by analyzing students' solution methods (interpreting them and considering their implications for instruction). More importantly, the researchers found that reflecting on and discussing the nature of student thinking and teacher questioning of students evident in videos of their own teaching led teachers to extend their mathematical knowledge for teaching as they re-engaged with the mathematics problem and reconsidered how they might teach the problem in light of their new regard for how students might approach the problem. Throughout the analysis, the authors found that specialized content knowledge interacts with pedagogical content knowledge in interpreting student thinking and planning lessons. The authors argue that the workshops developed teachers' mathematical knowledge for teaching by supporting teachers' current knowledge, while gradually challenging them to gain new understanding for the purpose of their work as teachers.

The lessons from studies such as these are subtle. The movement between mathematical study and pedagogical practice is central, but attention needs to be given to dynamics regarding teachers' motivation, the timing of different activities, and specific mathematical opportunities arising from specific pedagogical activities. In reading these reports, one gets the sense that really smart enactment of the professional development was key to success and that replicating effects might be challenging. From this work, it would seem important to discern the essential design features and elaborate the necessary character of facilitation.

One effort along these lines is a study by Elliott, Kazemi, Lesseig, Mumme, and Kelley-Petersen (2009). In the context of supporting facilitators' enactment of mathematically focused professional education, they analyzed facilitators' learning and the use of two frameworks provided as conceptual tools: (i) sociomathematical norms for cultivating mathematically productive discussion in professional development, adapted from Yackel and Cobb (1996) and (ii) practices for orchestrating productive mathematical discussions, adapted from Stein, Engle, Smith, and Hughes (2008). In their study, Elliot and colleagues collected extensive documentation and analyzed the learning of 5 of the 36 facilitators trained at two sites in 6 two-day seminars across an academic year. They found that although facilitators responded positively to the frameworks, they experienced tensions in using the frameworks to ask questions about colleagues' mathematical thinking and they struggled with the fact that teachers positioned themselves and others as better or worse in mathematics. These dynamics got in the way of productive mathematical discussions and frustrated facilitators. The analysis revealed that one way to mitigate these tensions was by helping facilitators to identify mathematical ideas that teachers would readily see as worth developing. This led the researchers to see a need for developing more nuanced and detailed purposes for doing mathematics in professional development in ways that teachers would see as relevant to their work.

This then led the researchers to realize that they needed a way to focus the purpose and work of professional development on connections between mathematics and the work of teaching. To accomplish this, they added a third framework to their design. The authors argue that the mathematical-knowledge-for-teaching framework engaged facilitators in understanding the ways in which specialized content knowledge (SCK) connects mathematics to teaching and that the framework provided a meaningful articulation of the purpose of the professional development and a helpful focus for the mathematical tasks and discussions that took place.

> By understanding how a SCK-oriented purpose for PD is tied to classroom teaching and being able to articulate that understanding to teachers in accessible ways, leaders will be able to begin to address the pressure they felt to assure relevance in their PD. (Elliott et al., 2009, p. 376)

Again, the dynamics between mathematics and the motivation and use of that mathematics is key to effective teacher learning of professionally relevant mathematics.

The field is also beginning to see evidence that these insights have measurable yield. For instance, Bell, Wilson, Higgins, and McCoach (2010) argue that it is the practice-based character of the nationally disseminated Developing Mathematical Ideas (DMI) mathematics professional development program that best explains participating teachers' learning of mathematical knowledge for teaching. The researchers examined pre and post teacher content knowledge for 308 treatment and comparison teachers across 10 well-established sites. They found significantly larger gains for treatment teachers' scores and that these gains were related to breadth of opportunity to learn provided by facilitators. Methodically considering a number of alternative explanations for treatment teachers' improvement, the researchers emphasize the classroom-practice feature of the professional development, where teachers move back and forth between seminars and their own classrooms, receiving written feedback from regularly observing facilitators. Referring to Ball and Cohen's (1999) argument that teacher learning needs to be embedded in practice, they point out that connecting to practice can leverage teacher learning in and from their daily work, greatly expanding overall capacity for teacher learning and improvement. They argue that the practice-based nature of their design contrasts with professional development that takes place apart from teachers' practice.

> DMI is quite different in this regard, for it encourages teachers to take their nascent SCK, KCS, and KCT into their classrooms and try things out. Repeatedly, teachers told us of their revelations — both in seminars and in their own schools — as they drew on their growing knowledge of and enthusiasm for mathematics and teaching mathematics in their classrooms. This anecdotal evidence aligns with results from S. Cohen's (2004) yearlong study of changes in teachers' thinking and practices over the course of their participation in DMI seminars. (Bell et al., 2010, p. 505)

These different studies compellingly add to the arguments that teachers need mathematical knowledge that is connected to the work they do and that situating the learning of mathematical knowledge in teachers' practice supports the learning of mathematical knowledge for teaching. Bell et al.'s (2010) large-scale study of the effect of professional development on teacher learning corroborates the qualitative, small-scale findings of the other studies. The professional development models highlighted set teachers up to learn in and from their practice. Together, the studies discussed above point to the coordinated nature of mathematical knowledge for teaching and the ways in which the coordination between mathematics and pedagogy is essential to teaching and learning mathematical knowledge for teaching.

Impact of mathematical knowledge for teaching. Whereas more studies have investigated the nature and composition of mathematical knowledge for teaching and developing teachers' knowledge, fewer studies have investigated the impact such knowledge has on teaching and learning. As mentioned earlier, several studies report positive effects of mathematical knowledge for teaching on student learning. Crucial to this research has been the development of robust instruments assessing mathematical knowledge for teaching. The field has found evidence linking mathematical knowledge for teaching to student achievement using the LMT instrument (e.g., Hill, Rowan, & Ball, 2005; Rockoff, Jacob, Kane, & Staiger, 2011), the COACTIV instrument (e.g., Baumert et al., 2010; Kunter et al., 2013), and the Classroom Video Analysis (CVA) instrument (e.g., Kersting et al., 2010, Kersting et al., 2012). A fewer number of studies have investigated links between teaching practice and mathematical knowledge for teaching and/or student achievement (e.g., Hill, Kapitula, & Umland, 2011). In these studies, student learning is mostly measured by standardized test scores, and the studies vary in how they measure teaching quality. These studies indicate that, generally speaking, mathematical knowledge for teaching impacts teaching and learning.

We acknowledge the importance of studies that identify an influence of teachers' mathematical knowledge on teaching and learning, but are particularly excited about studies that unpack the dynamics of how mathematical knowledge for teaching impacts teaching and learning. In their study of 34 teachers, Hill, Umland, Litke, and Kapitula (2012) demonstrated that the connection between mathematical knowledge for teaching (measured with the LMT instrument) and the quality of instruction is complex. While weaker mathematical knowledge for teaching seemed to predict poorer quality of instruction, and stronger mathematical knowledge for teaching seemed to predict higher quality of instruction, teachers who performed in the midrange on the LMT measure varied widely in the quality of their instruction. Student achievement also varied widely for teachers with mid-range mathematical knowledge for teaching. Furthermore, Hill et al.'s (2008) study of 10 teachers found that although use of supplemental curriculum materials, teacher beliefs, and professional development are factors of potential influence, these factors might all cut both ways depending on the teachers' mathematical knowledge for teaching. These two studies underscore that simply establishing impact of knowledge on teaching is not enough to make decisions about teacher education or policy.

To frame a fuller consideration of impact, we reflect briefly on the nature of teaching and learning. Teaching mathematics involves managing instructional interactions, including everything teachers say and do together with students focused on content, where teacher knowledge is a resource for the work (Cohen, Raudenbush, & Ball, 2003). This observation suggests that in addition to general effect studies on teaching and learning, it would be helpful to know more about which specific aspects of teaching and learning are influenced by teacher content knowledge, which specific aspects of teacher content knowledge are influential, and how the influences impact interactions among teacher and students around content. In other words, we propose that Cohen et al.'s conceptualization of teacher content knowledge as a resource impacting instructional interactions is important for framing an investigation of mathematical knowledge for teaching suited to informing the improvement of teaching and learning.

A promising direction in recent work has been initial investigation of the specific influence that mathematical knowledge for teaching has on teaching. One example of this kind is Speer and Wagner's (2009) case study of one undergraduate instructor's scaffolding of classroom discussions. Using Williams and Baxter's (1996) constructs of social and analytic scaffolding as a frame, Speer and Wagner argue that aspects of pedagogical content knowledge are important for helping students find productive ways of solving particular problems and for understanding which student contributions — correct or incorrect — are important to emphasize in a discussion. They trace ways in which particular knowledge of students' understanding aids teachers in assuring that the lesson reaches intended mathematical goals and in understanding the role of particular mathematical ideas in students' development.

In a similar vein, an exploratory study by Charalambous (2010) investigated teachers' knowledge in relation to selection and use of mathematical tasks. He investigated the teaching of two primary mathematics teachers with different levels of mathematical knowledge for teaching and found notable differences in the quality of their teaching. He used Stein and colleagues' mathematical tasks framework to examine the cognitive level of enacted tasks, and he formulated three tentative hypotheses about mechanisms of how mathematical knowledge for teaching impacts teachers' selection and use of mathematical tasks. First, he hypothesizes that strong mathematical knowledge for teaching may contribute to a use of representations that supports students in solving problems, whereas weaker mathematical knowledge for teaching may limit instruction to memorizing rules. Second, he proposes that mathematical knowledge for teaching appears to support teachers' ability to provide explanations that give meaning to mathematical procedures. Third, he proposes that teachers' mathematical knowledge for teaching may be related to their ability to follow students' thinking and responsively support development of understanding.

These two studies exemplify potential analyses of mathematical knowledge for teaching in relation to frameworks of teaching and learning. They leverage findings about teaching to probe the contributions of mathematical knowledge for teaching in ways that begin to unpack the specific role such knowledge plays. They are not the only studies to do so, but to date studies in this realm are rare. Building on these ideas, further conceptualization of distinctly mathematical tasks of teaching might provide even more focused contexts for studying mathematical knowledge for teaching as a resource for teaching. Establishing agreed-upon conceptualizations of mathematical knowledge for teaching related to well-studied components of the work of teaching and using these as a common ground for instrument development would provide a solid foundation for advancing the field.

From this brief review of recent progress on identifying, developing, and understanding the impact of mathematical knowledge for teaching, we now turn our attention to proposing directions for future work.

Next Steps for the Development of Mathematical Knowledge for Teaching

As described above, compelling examples of mathematical knowledge for teaching and evidence associating it with improved teaching and learning have sparked interest in making it a central goal in the mathematical education of teachers. However, various impediments exist. The lack of rigorous, shared definitions and the incomplete elaboration of a robust body of knowledge create problems for meaningful measures and curricula development. Underlying these challenges are competing ideas about how to conceptualize the knowledge, questions about the relationship among knowledge, knowledge use, and outcome, and the need for ways to decide claims about whether or not something constitutes professional knowledge.

We suggest three priorities for research and development of mathematical knowledge for teaching: (1) focused studies that together begin to compose a more coherent, comprehensive, and shared understanding of what it is, how it is learned, and what it does; (2) innovation and reflection on method for investigating it; and (3) studies of mathematical fluency in teaching and the nature of mathematical knowledge for equitable teaching. Below, we argue that each of these is vital to long-term progress in improving the mathematical education of teachers and the mathematics teaching and learning that depends on it.

Investigating focused issues while contributing to a larger research program. Scores of articles in the previous decade have argued for particular ways of distinguishing and conceptualizing important knowledge, and many others have sought to establish its presence and overall impact. With a sense of the importance of mathematical knowledge for teaching, additional studies explored the teaching of such knowledge. However, on the whole, conceptual work has been exploratory, measures have been general, and studies of the mathematical education of teachers have been limited by under-specification of the body of knowledge. We suggest that the field would benefit from focused studies that build on each other in ways that begin to put in place the machinery needed to develop an overall system for educating teachers mathematically. Such a system would include clear content-knowledge standards for professional competence, comprehensive content-knowledge course and program curricula, robust exit or professional content-knowledge exams, and rationale for what is to be taught in pre-service programs and what is better addressed in early career professional development or later on. To get there, we propose collectively pursuing several focal areas of study.

First, mathematical knowledge for teaching needs to be elaborated — for specific mathematical topics and tasks of teaching, across educational levels. Some of this work is underway, but we suggest that more needs to be done in ways that research studies, taken together, define a body of professional knowledge and provide a basis for curricula, standards, and assessments. One area of need that stands out is the investigation of the mathematical knowledge demands associated with particular domains of the work of teaching, such as leading a discussion, launching students to do mathematical work, or deciding the instructional implications of particular student work. This is a particularly challenging area of study because the field lacks comprehensive, robust specifications of the work of teaching. It is also a potentially promising area of study. Where initial decompositions of teaching are available, such as for orchestrating discussions, awareness of the mathematical knowledge entailed in the teaching can position teachers to learn both the domain of teaching and the mathematical knowledge more productively (Boerst, Sleep, Ball, & Bass, 2011; Elliott et al., 2009). Nonetheless, domains of teaching need additional parsing before they can be fully leveraged.

A second proposed area of study is determining meaningful "chunking" of mathematical knowledge for teaching and practical progressions for teaching and learning it. In considering the mathematics that students need to learn, topics are typically decomposed into a sequence of small-sized learning goals. In contrast, teachers' mathematical knowledge for teaching is not simply a mirror image of student curriculum. Teachers need knowledge that is different in important ways from the knowledge students need to learn. Mathematical knowledge for teaching is related to student curriculum, but it is not clear what this relationship implies for how it is best organized. In contrast to the mathematics that students need to learn, the specialized mathematics that teachers need to learn appears to be constituted in ways that span blocks of the student curriculum.

For instance, a teacher who learns how to model the steps of the standard addition algorithm using base ten blocks might still need to think through modeling subtraction, but as a minor extension of what is already learned, not as a new topic, requiring a new program of instruction. The question deserves more careful examination, but our experience is that teachers who participate in professional development related to a particular strand of work on place value exhibit significantly increased mathematical knowledge for teaching more generally across whole number computation, but with little to no impact on their mathematical knowledge for teaching topics related to geometry, data analysis, or even rational number computation. This is just a conjecture, but we offer it as a way to indicate an area of study that would contribute to improved approaches to the mathematical education of teachers. How big are these chunks? What are possibilities for structuring the chunks? Which have the greatest impact for beginning teachers? Some of these questions could be investigated as part of the elaboration research described above. Our point is that beyond the important goal of identifying knowledge for specific mathematical topics and tasks of teaching, across educational levels, research on how best to organize that knowledge might usefully inform the mathematical education of teachers.

This discussion leads to a third proposed line of investigation, one that explores mathematical knowledge for teaching along a professional trajectory from before teachers enter teacher preparation, through their training and novice practice, and into their maturation as professionals. This would require navigation among questions about what teachers know, what might be learned when, what is essential to responsible practice, and what can be sensibly coordinated with growing professional expertise. For this, the field would need to know more about the mathematical knowledge for teaching that prospective teachers bring to teacher education and whether there are things that might more readily be learned in the program and others that might be more productively required before admission. The field would need to know more about mathematical knowledge for teaching that is readily acquired from experience, as well as the supports needed to do so. Researchers would need to investigate how to distinguish between the mathematical knowledge for teaching that is essential to know before assuming sole responsibility for classroom instruction and the knowledge that can be safely left to later professional development. We suggest that such studies would contribute to developing coherence, efficiency, and responsibility in an overarching picture of the mathematical education of teachers.

Another proposed area of study would extend work that examines effects of specific mathematical knowledge on specific teaching and learning in ways that identify underlying mechanisms and informs views of when and how mathematical knowledge is used in teaching. We noted above a need for more studies that unpack relationships among mathematical knowledge for teaching, teaching practice, and student learning. Such studies might examine the nature of student learning gains resulting from specific teacher knowledge or they might investigate the mechanisms by which teachers' mathematical knowledge for teaching has an impact. They would provide a better understanding of the nature and role of mathematical knowledge in teaching, informing both its conceptualization and validating underlying assumptions about its significance.

Finally, we suggest that the field would benefit from more studies of effects at a mid-range level, above that of idiosyncratic, individual programs and courses and below that of large-scale, international studies. In their 2004 International Congress on Mathematics Education plenary, Adler, Ball, Krainer, Lin, and Novotna (2008) observed that the majority of studies in teacher education are small-scale qualitative studies conducted by educators studying the teachers with whom they are working within individual programs or courses. The TEDS-M study and the development of some of the instruments described above have supported an increase in large-scale and cross-case studies, but as Adler and her colleagues point out, the study of courses, programs, and teachers by researchers who are also the designers and educators of those programs and teachers creates both opportunities and risks. From our review, our sense is that many small studies are driven by convenience and reduced cost, but at the expense of rigorous design and skeptical stance. Mid-sized studies would be enhanced by efforts such as developing collaborative investigations across remote sites with either similar or contrasting interventions. This is consistent with arguments about research on professional development made by Borko (2004).

Next, we argue that the agenda sketched above will require explicit development of methods for conducting such research efficiently and effectively.

Innovating and reflecting on method.[2] We propose that a central problem for progress in the field is a lack of clearly understood and practicable methodology for the study and development of mathematical knowledge for teaching. First, many researchers, including graduate students, seem eager to conduct studies in this arena, but choices about research design and approaches to analysis are uncertain. In our review of the literature, we found that methods vary widely, are relatively idiosyncratic, and are in general weak — in some cases attempting to make causal claims from research designs poorly suited for such claims and in others providing thoughtful claims but from unclear

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[2] Material in this section is based on work supported by the National Science Foundation under grant number 1502778. Opinions are those of the authors and do not necessarily reflect the views of the National Science Foundation.

processes and underdeveloped logical rationale. We suspect that a lack of clarity and rigor of methods, including in our own work, are a result of several factors: unresolved and underdeveloped conceptualization of the terrain; competing purposes of research (often within a single study); and uncertain grounds for making claims about whether something does or does not constitute professional knowledge. Struggles to design robust studies and to articulate methods used suggest a need for increased attention to method.

This should not be surprising. The vitality of research in areas still in early stages of theory development requires a concomitant consideration of method. Importing method from other arenas is appropriate, but regard for the theoretical foundations of the object of study and their implications for all aspects of method is also important. We propose that reflective innovation of method, grounded in emerging theory of teaching, can better account for confounding variables that are relevant to teaching and can inform the alignment among research questions, design, analysis of data, claims, and interpretations. To ground this proposal, we reflect on two approaches that have been evident in efforts to study the nature and composition of mathematical knowledge for teaching (interview studies and observational studies) and then suggest directions for potential innovations.

Early investigations of teacher content knowledge were mostly limited to correlational studies (e.g., Begle, 1979). Correlational studies remain prominent in the field (e.g., Baumert et al., 2010 Hill, Rowan, & Ball, 2005; Kersting et al., 2010), but in the 1980s and 1990s studies began using teacher interviews to investigate teacher knowledge (see Ball, Lubienski, & Mewborn, 2001). This early work was limited in two ways. First, it tended to focus on identifying deficits in teachers' mathematical knowledge instead of clarifying the mathematical knowledge requirements of teaching. Second, although some good interview prompts emerged and supported a surfeit of studies, generating additional high-quality prompts has not been easy. The strength of these early interview prompts was that they were focused, specific, and offered compelling examples of specialized mathematical knowledge that would be important for teachers to know. The weaknesses were that they focused the conversation on teachers' lack of knowledge, while providing little insight into how to rectify these lacks, and they left the difficult work of generating good prompts invisible.

Similarly, methods for observational studies have often been weakly specified and hard to use by other scholars as the basis for complementary study. For example, because of the shortcomings of teacher interviews, the research group at the University of Michigan developed a practice-based approach to the study of video records of instruction (Ball & Bass, 2003; Thames, 2009). This approach requires simultaneously conceptualizing the work of teaching together with the mathematical demands of that work. It is empirical, interdisciplinary, analytical-conceptual research that involves developing concepts and conceptual framing by parsing the phenomenon and systematically testing proposals for consistency with data and with relevant theoretical and practice-based perspectives. The approach is time-intensive and expensive, requires skillful use of distributed expertise, and is sufficiently underspecified to make broader use challenging. For instance, early on, these researchers wrote about ways in which inter-disciplinary perspectives were central to their analyses (e.g., Ball, 1999; Ball &

Bass, 2003), but this characterization, although it captures an important feature of the approach, is inadequate as a characterization of their approach and as a method for others to use. It is underspecified, relies more on experienced judgment than on independently usable criteria or techniques, and leaves key foundational issues in doubt (Thames, 2009).

Reflecting on our own use of these approaches, we offer several, somewhat ad hoc, observations.

- Teaching is purposeful work and, as such, imposes logical demands on the activity, and these logical demands play a role in warranting claims about the work of teaching and mathematical knowledge needed for teaching.
- Mathematical knowledge for teaching is professional knowledge, and central to its development and articulation is professional vetting or consensus building based on cross-community professional judgment.
- The pedagogical context provided in crafted items and prompts entails engagement in the work of teaching and in the use of mathematical knowledge and, as such, provides crafted instances for the study of specialized knowledge for teaching.
- There is an iterative process among the development of instructional tasks, assessment items, and interview prompts and our increasing capacity for eliciting and engaging mathematical knowledge for teaching.
- Analysis of mathematical knowledge as professional knowledge for teaching, whether in situ or in constrained instances, is fundamentally an empirical, conceptual-analytic, normatively informed process, not a strictly descriptive one.

We believe that the first two observations have important methodological implications that are as of yet unrealized. Key to understanding teaching and its knowledge demands is understanding its contextual rationality. In other words, meaningful study of teaching must account for the directed and contextual nature of the work. We suspect that such study will require the development and use of methods fit to the work of teaching and that this means greater reliance of underlying theory of teaching in designing studies and choosing methods of analysis. As Gherardi (2012, p. 209) succinctly summarizes in her writing about conducting practice-based studies, “Hence, empirical study of organizing as knowing-in-practice requires analysis of how, in working practices, resources are collectively activated and aligned with competence.” She argues for thoughtful consideration of how methodological approaches are positioned in relation to the nature of practice and its constant reconstitution in the context of professional work. We agree with this position and suggest that it is exactly these issues that need to be taken up in an investigation of method for studying mathematical knowledge for teaching.

The second observation in our list above raises additional concerns for the development of new methods. Mathematical knowledge for teaching is professional knowledge, in the sense that it is shared, technical knowledge determined by professional judgment (Lortie, 1975; Abbott, 1988), but it is distinctive as a body of knowledge in that it requires the coordination of mathematics with teaching, which are different areas of expertise resident in different professional communities (Thames, 2009). Thus, the study of such knowledge requires coordination across different professional communities with different disciplinary foundations. In other words, the study of mathematical knowledge for teaching requires, or is at least enhanced by, collective work across distinct professional communities with different expertise and different professional norms and practices, and such work requires special consideration and support (Star & Griesemer, 1989).

With the call for cross-professional coordination, the study of mathematical knowledge for teaching involves much more than an assembly line model where different professional constituents inject their specific expertise into a product handed down the line. It calls for specification of processes for collectively considering whether a proposed claim of professional knowledge is warranted. It is about establishing protocols for merging and melding different expertise in the midst of improvement work that attends to overall coherence and practical merit. It requires specific ways of working together, tools for organizing the scholarly work, and boundary objects that provide meaningfully bridges among communities (Akkermann & Bakker, 2011). Each of these adds to the need for new methodology.

Innovation and reflection on method can be carried out in numerous ways. Researchers can simply attend more closely to decisions of method and explicit reporting of method. Alternatively, they can deliberatively develop, implement, and study methods. In order to provide a sense of the kind of innovation and reflection that might be done, we discuss some of the ways we have begun to explore methodological approaches for the study of mathematical knowledge for teaching.

An emerging approach we find promising is to use sites where professional deliberation about teaching are taking place as sites where we might productively research the work of teaching and its mathematical demands. In recent studies, we have designed interview protocols as a tool for generating data useful for studying the mathematical work of teaching. For instance, to investigate the work involved in providing students with written feedback, Kim (this issue) provides a strategic piece of student work and asks interviewees to provide written feedback and to explain the rationale for the feedback. Here, instead of videotaping classroom instruction and analyzing the mathematical demands of teaching, Kim analyzes those demands as they play out in a constrained slice of the mathematical work of teaching as evidenced in responding to a teaching scenario provided.

We see this approach as an instance of a more general phenomenon, one of using sites of professional deliberation about teaching as research sites for studying teaching. For example, a group of mathematics teachers and mathematics educators in a professional development setting might discuss responses to a particular pedagogical situation in ways akin to the pedagogical deliberations of a teacher engaged in teaching. Thus, this professional development event can be useful for studying professional practice. It may even have the advantage that professional action and reasoning are more explicitly expressed, yet of course, with certain caveats in place as well, such as recognizing that real-time demands of teaching are suspended. Similar opportunities can arise in other settings where pedagogical deliberations take place, such as teacher education or the development of curriculum or assessment. For instance, recent investigation of the design process for producing tasks to measure mathematical knowledge for teaching suggests that writing and reviewing such tasks can provide insight into teaching and its mathematical demands, even to the point of serving as a site for investigating mathematical knowledge for teaching (Jacobson, Remillard, Hoover, & Aaron, in press; Herbst & Kosko, 2012).

We propose that such an approach is distinctively different from general interview techniques that have teachers reflect on their teaching. Crucial to this difference is that the prompts are designed to provide authentic pedagogical contexts with essential, yet minimal, constraints for directing targeted pedagogical work (such as a crucial instructional goal, a key excerpt from a textbook, or strategically selected student work). Good pedagogical context needs to be based on initial conceptions of key aspects of the work, and constraints need to be designed to engage initial ideas about the nature and demands of the work. Otherwise, the pedagogical context of the tasks is unlikely to engage people in authentic pedagogical work.

Our recent experience with interview prompts of this kind has convinced us of their potential for studying teaching and teacher knowledge. Several advantages are evident: constraints provided can be manipulated; different professional communities can be engaged; and bounded instances of work examined. The development of this approach would support new lines of research that specify teaching and its professional knowledge demands in ways that can better inform professional education and evaluation. They are also easy to use and require only modest time and expense.

Such innovations begin to suggest the development of a "laboratory science" approach for studying mathematical knowledge for teaching that takes advantage of the tools of constrained prompts, the generative analytic techniques of instructional analysis, and the multiple sites available for such study. By a "lab science" approach we mean direct interaction with the world of instruction or slices of instruction using tools, data collection techniques, and models and theories of teaching. Analogous to the ways in which experimental psychologists isolate phenomena under controlled conditions in a laboratory setting or biochemists manipulate protein processes at the bench, we propose that the study of specialized teacher content knowledge can isolate activities of teaching and the use of resources, examine those activities and resources in detail, and systematically manipulate constraints to better understand phenomena. This work can be done deductively, to test specific hypotheses, inductively, to discern functional relationships, or abductively, to refine current understanding. Such an investigation of method should be intimately tied to underlying foundational issues, both shaped by theoretical commitments and giving precise definition and form to underlying theory.

In conclusion, we suggest that the development of usable, practical, and defensible method, whether along the lines we have sketched here or along other lines, will be critical to carrying out the extensive agenda described earlier for building a understanding of mathematical knowledge for teaching adequate for sustainable improvement of the mathematical education of teachers. We now sketch two areas of study largely missing from the literature on knowledge distinctive for teaching mathematics and argue that both of these are essential to viable progress on building a theory and practice of mathematical knowledge for teaching.

Addressing Two Key Issues: Mathematical Knowledge for Fluent and Equitable Teaching. Although there has been substantial progress in conceptualizing and understanding the mathematical understanding needed for the practice of teaching, significant issues remain. We focus here on two aspects that seem to us to be particularly critical to progress on mathematical knowledge for teaching. One centers on the communicative demands of teaching, the other on what is involved in teaching to disrupt the historical privileging of particular forms of mathematical competence and engagement, resulting in persistent inequity in access and opportunity. We argue that both of these are key to the long-term viability of efforts to improve the mathematical education of teachers.

Teaching is inherently a communication-intensive practice. Teachers listen to their students, explain ideas, and pose questions. They read their students' written work and drawings, and provide written feedback. Throughout these communications, they use mathematics in a range of specialized ways. They must hear what their students say, even though students talk and use mathematical and everyday language in ways that reflect their emergent understanding. Similarly, they must interpret students' writing and drawings. When they talk, teachers must attune their language to students' current understanding, and yet do so in ways that are intellectually honest and do not distort mathematical ideas to which they are responsible for giving their students access.

What is involved in this sort of mathematical communication in the context of teaching? Because teaching is fast-paced and interactive, the demands are intense. Talk and listening cannot be fully scripted or anticipated. A special kind of mathematical fluency is required, tuned to the work of teaching. Asking a question in the moment; explaining in response to a student's puzzlement; listening to, interpreting, and responding to a child's explanation — each of these involves hearing and making sense of others' mathematical ideas in the moment, speaking on one's feet while seeking to connect with others. Although much of the work on mathematical knowledge needed by teachers is situated in relation to what teachers do, including using representations and interpreting students' thinking, as yet little of it has focused on the mathematical fluency needed for the work teachers do in classrooms, live, in communicating with students. As compellingly argued by Sfard (2008) and others (e.g., Resnick, Asterhan, & Clarke, 2015), it is this communicative work that is central to the practice of education. Failing to investigate and squarely address communicative mathematical demands of teaching may result in an impoverished theory of mathematical knowledge for teaching in ways that sorely limit its utility and impact.

Another major area of work centers on the need to address the persistent inequities in mathematics learning both produced and reproduced in school. Goffney and her colleagues have begun to identify a set of practices of equitable mathematics teaching (Goffney, 2010; Goffney & Gonzalez, 2015; Goffney, 2015), and several of the articles in this volume explore the measurement of mathematical knowledge for equitable teaching. The driving question is what do teachers need to appreciate and understand about mathematics in order to be able to create access for groups that have been historically marginalized? Part of this has to do with a flexible understanding of the mathematics that enables teachers to build bridges between mathematics and their students. One aspect of this is to represent mathematics in ways that connect with their students' experience. Another is to be able to recognize mathematical capability and insight in their students' out-of-school practices. Each of these entails a flexibility of mathematical understanding, particularly of mathematical structure and practice. But it also involves the ability to recognize as mathematical a range of specific activities, reasoning processes, and ways of representing. Being able to do this can enable teachers to broaden both what it means to be "good at math" as well as what can be legitimated as "mathematics."

Equity is not a new focus in mathematics education (e.g., Schoenfeld, 2002), and there have been studies on the effect of gender and language on mathematics teachers' knowledge (Blömeke, Suhl, & Kaiser, 2011) as well as the distribution of teacher knowledge in different populations of teachers (Hill, 2007). In our review of the literature, we observed that most studies on equity were focused on aspirations and imperatives (i.e., arguments for teaching for equity). Few studies focused directly on specific practices of equitable mathematics teaching or knowledge for equitable mathematics teaching. We argue that increased focus in this area is crucial for three reasons. First is the underlying principle that extant inequity in mathematics teaching and learning is morally reprehensible in a civilized society (Perry, Moses, Cortez, Delpit, & Wynne, 2010). Second is our contention that, while certainly not in itself a solution, teacher content knowledge is both an indispensible and an untapped resource for disrupting the historical privileging of particular forms of mathematical competence and engagement. Third, as with nearly all achievement measures in early stages of development, current instruments are significantly biased because of the contextual features of where, as well as for and by whom, they are developed. The field needs good instrumentation, for research and for practice. Overly delaying the development of unbiased instruments may well undermine the political viability of well-meaning efforts to improve the mathematical education of teachers. Such development will require solid research in this difficult yet important arena.

In proposing these two areas of study, we acknowledge the conceptual and methodological challenges each presents. We suspect that research in these areas has been underdeveloped in large part because these foci involve subtle social dynamics less readily captured in print and in more conventional measures. These challenges simply add to our concern that concerted attention be given them. Our argument here is that these two areas of study are not merely our favored topics, but that that they are essential to long-term success.

Articles that Develop Measures and Measure Development

The agenda sketched above is both a reflection of emerging work in the field and a proposal for future work. In many ways, the articles in this special issue, though specifically addressing measurement, resonate with themes above. For instance, the discussion about focused studies that contribute to a larger research program suggests some benefits of creating a common framework for describing mathematics teaching. In their article, Selling, Garcia and Ball (this issue) present a framework for unpacking the mathematical work of teaching that is promising in this respect. Whereas other frameworks often start with what teachers do, they focus first on the mathematical objects involved in the work of teaching and then follow up by describing actions that teachers do on these objects. This idea builds upon and extends the notion of mathematical tasks of teaching that has been highlighted in previous publications on the practice-based theory of mathematical knowledge for teaching (e.g., Ball et al., 2008; Hoover, Mosvold, & Fauskanger, 2014), as well as in previous efforts to conceptualize the work of teaching (e.g., Ball & Forzani, 2009). A main aim with this framework is to inform and assist future development of items and instruments for measuring mathematical knowledge for teaching.

Phelps and Howell (this issue) discuss the role of teaching contexts in items developed to measure mathematical knowledge for teaching. Given that mathematical knowledge for teaching is understood as knowledge applied in the work of teaching, a teaching context that illustrates a certain component of this work is typically included in items. Phelps and Howell discuss different ways in which context can be critical to assessing mathematical knowledge for teaching. They argue that attention to the role of context might provide better understanding of the knowledge assessed in particular items and might also inform further development of a theory in which teaching context is used to define knowledge.

Whereas both of these first articles point to core issues regarding the conceptualization of mathematical knowledge for teaching — in the context of item and instrument development — the next two articles deal more directly with measurement. Kim (this issue) focuses on designing interview prompts for assessing mathematical knowledge for teaching. In particular, her discussion focuses on the task of providing written feedback to students. To model this task, she combines a decomposition of the task with aspects of the pedagogical context involved and sub-domains of mathematical knowledge for teaching.

Where Kim's study is more qualitative and conceptual in nature, Orrill and Cohen (this issue) draw on psychometric models in their work. Their study hinges on the issue of defining the construct measured, and they use a mixture Rasch model to analyze different subsets of items to support an argument that the domain definition has strong implications on the claims one tries to make about teachers' performance. In light of our observations about the lack of consensus about how to define the constructs that are being measured and discussed across studies, a focus on careful construct definition and implications is particularly relevant.

In the international literature on teaching and learning, a focus on equity is prevalent. In research on mathematical knowledge for teaching, the discussion of knowledge for teaching equitable mathematics also receives some attention — although issues of equity and diversity have not been emphasized in frameworks of mathematical knowledge for teaching. In this connection, Wilson's (this issue) and Turkan's studies of mathematical knowledge for teaching English language learners draw attention to this missing area of research. Both involve design and application of measures. Wilson proposes a new aspect of pedagogical content knowledge that is connected specifically to the work of teaching mathematics to English language learners. Turkan addresses practicing teachers' reasoning about teaching mathematics to ELLs. Based on analysis of data from cognitive interviews, she argues that there is a unique domain of knowledge necessary for teaching ELLs — thus supporting Wilson's argument — and she calls for further investigations to identify and assess this knowledge.

Finally, this special issue includes two articles that investigate teachers' views. Koponen, Asikainen, Viholainen and Hirvonen investigate the views of teachers as well as teacher educators about the content of mathematics teacher education. Results from their survey indicate that teachers as well as teacher educators in the Finnish context emphasize the need for courses in content knowledge that is distinctive for teaching — not just more advanced. They argue that the mathematical content of teacher education needs to be tightly connected to the mathematics being taught, and even pedagogical courses need to include knowledge connected with mathematics, in particular focusing on knowledge of teaching and learning of mathematics. In the last article of the special issue, Kazima, Jakobsen and Kasoka investigate Malawian teachers' views about mathematical tasks of teaching and the potential usefulness of adapted measures of mathematical knowledge for teaching among Malawian pre-service mathematics teachers. The measures as well as the applied framework of mathematical knowledge for teaching were developed in the United States. Despite the significant cultural differences between Malawi and the United States, the authors argue that the framework as well as most of the items function well in the Malawian context.

Together, this collection of articles on the development and use of measures lies at a transition from the lessons of past studies of mathematical knowledge for teaching into vital arenas of research needed for systemic improvement on the mathematical education of teachers.

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