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Unpacking the Mathematical Work of Teaching

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Abstract: In the context of the increased mathematical demands of the Common Core State Standards and data showing that many elementary school teachers lack strong mathematical knowledge for teaching, there is an urgent need to grow teachers' MKT. With this goal in mind, it is crucial to have research and assessment tools that are able to measure and track aspects of teachers' MKT at scale. Building on the concept of "mathematical tasks of teaching" (Ball et al., 2008), we report on a new framework that unpacks the mathematical work of teaching that could serve as a scaffold for item writers who are developing assessments of MKT. We argue that this framework supports a focus on the *mathematical* work of teaching that moves beyond common content knowledge but without moving into a space of pedagogical choice. We also illustrate how the framework was constructed to highlight connections within and across the mathematical content of elementary school. The mathematical work of teaching framework has implications for assessment development at scale, and could be useful as an organizing tool in mathematics teacher education efforts to grow teachers' MKT.

Keywords: mathematical knowledge for teaching, teacher knowledge, assessment development

Introduction

Broad consensus exists about the importance of teachers' mathematical knowledge (Adler & Venkat, 2014; Ball, Lubienski, & Mewborn, 2001; Baumert et al. 2010; Döhrmann, Kaiser, & Blömeke, 2014). Studies have linked mathematical knowledge for teaching to the quality of teachers' mathematics instruction (Eisenhart, Borko, Underhill, Brown, Jones, & Agard, 1993; Hill et al., 2008). Mathematical knowledge for teaching has also been linked to

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student achievement gains in the elementary grades (Hill, Rowan, and Ball, 2005). However, many U.S. teachers lack the deep, nuanced, and specialized mathematical knowledge needed for responsible teaching. This finding is persistent over time, grade levels, and both national and international contexts (e.g., Hill & Ball, 2004; Ma, 1999; Tatto et al., 2008). Simultaneously, the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices, 2010), which have been adopted by 47 states and territories, have set out rigorous standards for K-12 mathematics learning that consequently increase the mathematical demands of teaching. To ensure that teachers are well-positioned to help students meet these more challenging learning goals, it is now — more than ever — critically important to focus on developing their mathematical knowledge for teaching.

To investigate and grow teachers' mathematical knowledge for teaching (MKT) it is crucial to be able to measure and track the development and uses of MKT. Most work to develop measures of MKT has typically been done by groups of experts in relevant fields, such as mathematicians, mathematics educators, and teachers who have worked together to draft and revise assessment items (Hill, Schilling, & Ball, 2004). The early work in this area was focused on developing and refining the construct of mathematical knowledge for teaching while simultaneously and iteratively developing measures of the construct. The process of item development was therefore often time consuming and challenging. Because of the promising results of these earlier efforts, there is now a broad need for assessments of MKT. Building tests at scale means, however, that people who are not deeply immersed in research on MKT will have to be able to write valid MKT items. This will require detailed supports to help test developers understand the nuances of the construct of MKT and ways to assess it. In this paper, we present a framework that identifies the different ways that teachers make use of mathematical knowledge as they go about the work of teaching and provides support to assessment developers. We begin by articulating and specifying what we mean by mathematical knowledge for teaching and its relationship with the mathematical work of teaching that arises in everyday practice.

Theoretical Framing

Conceptualizing Mathematical Knowledge for Teaching

Building supports for assessment development of mathematical knowledge for teaching (MKT) rests on a clear conceptualization of what we mean by MKT, how MKT is drawn upon in practice, and the specific areas of the work of teaching that we seek to assess. Scholars of mathematical knowledge have examined such knowledge in action as it is used in the practice of teaching (Ball & Bass, 2002; Rowland, Huckstep, & Thwaites, 2005).

Our work builds on a particular practice-based perspective on mathematical knowledge for teaching that begins with the premise that, to understand the specific knowledge of mathematics needed in teaching, one must first examine the mathematical work that arises in the context of teachers' instruction in classrooms, a form of job analysis (Ball & Bass, 2002). Through detailed analysis of instruction in a 3rd grade classroom over an entire year, Ball and her colleagues identified mathematical problems that teachers regularly encounter and must solve while teaching, such as "interpreting and evaluating students' non-standard mathematical ideas" (Ball & Bass, 2002, p. 9). These analyses reveal that teaching entails significant mathematical work on the part of the teacher. To highlight the complexity and variety of ways that teachers engage in mathematical work, Ball and her colleagues (2008) present a list of 16 "mathematical tasks of teaching" that may occur within every day teaching practice that involve mathematical work on the part of the teacher. This list includes tasks such as "responding to students' 'why' questions", "finding an example to make a specific mathematical point", "evaluating the plausibility of students' claims (often quickly)", "choosing and developing useable definitions", or "recognizing what is involved in using a particular representation" (p. 10). These mathematical tasks of teaching provide the contexts in which teachers must draw on mathematical knowledge for teaching, and therefore offer a window into the mathematical knowledge entailed by teaching.

Based on their analyses of these ubiquitous tasks of teaching mathematics, Ball and her colleagues (2008) identified a provisional map of domains of mathematical understanding and skill. They argued that teaching requires both "pure" subject matter knowledge and pedagogical content knowledge (Shulman, 1986, 1987; Wilson, Shulman, and Richert, 1987). Pedagogical content knowledge comprises blends of mathematical knowledge together with other kinds of knowledge, such as knowledge of students' thinking in a particular content domain, or knowledge of likely effective approaches to or materials for teaching specific content ideas. For example, in teaching integers, teachers need to appreciate that notions of "debt," "assets" and "net worth" are unfamiliar to elementary age learners and that therefore financial contexts are not likely to be useful as a representation of integer arithmetic. Knowing ways to use number line models as a context for integer arithmetic is another example of pedagogical content knowledge — knowledge of teaching approaches and models combined with a particular topic. But knowing integers for teaching also involves content knowledge. "Common" content knowledge is the term Ball and her colleagues use to describe the knowledge that 0 is neither negative or positive or that (-3) – (-7) = 4. By this they denote knowledge that is also relevant to people who do not teach — that is, known *in common* with others. They argue that teaching also requires "specialized" content knowledge — for example, being able to explain the meaning of subtraction of a negative number and connect it to moves on the number line in ways that make conceptual sense, or being able to represent the difference – even though they might produce the same result — between subtracting -4 from 10 and adding 4 to 10. Horizon knowledge is the perspective needed to understand connections among topics or to see where ideas are headed, or to notice when students are onto a sophisticated mathematical point (Ball & Bass, 2009). In our assessment development work, we focus on specialized content knowledge, as a form of subject matter knowledge that is particularly needed in the work of teaching.

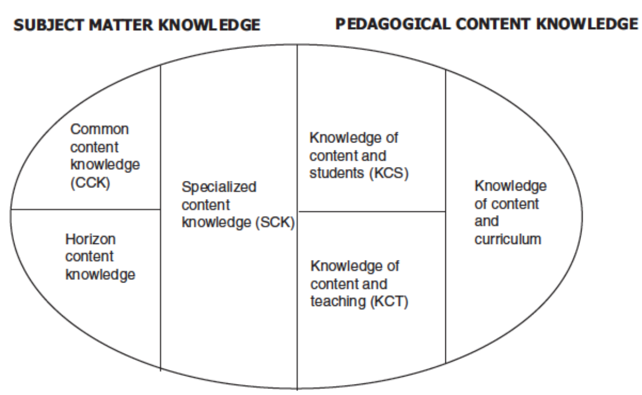

Figure 1. Domains of Content Knowledge for Teaching (Ball, Thames, & Phelps, 2008)

Research on specialized content knowledge has acknowledged that the line between specialized and common content knowledge might not be well-defined, and that particular mathematical tasks of teaching may elicit different types of knowledge by teachers or others asked to engage in these tasks (Delaney et al., 2005; Hill, Dean, & Goffney, 2007). In our work, we are less concerned with classifying assessment items as eliciting only specialized or common content knowledge; instead, we have chosen to focus on the mathematical work of teaching demanded by teaching practice and the knowledge that teachers would need to do that work, acknowledging that some mathematical work of teaching may elicit different domains of subject matter knowledge or even knowledge from multiple domains.

Building Assessments of Content Knowledge for Teaching: Challenges and Supports

Existing assessments of teacher knowledge at scale, often licensure tests, tend to focus on common content knowledge (i.e., the mathematics content that teachers teach) or horizon knowledge (i.e., perspective on how what the students are working on now connects with other mathematics). Few assessments have attempted to assess specialized content knowledge at scale. The Learning Mathematics for Teaching project (Hill & Ball, 2004; Hill, Schilling & Ball, 2004) has developed elementary and middle school level measures for research purposes that have been widely adopted and implemented. However, these measures are not intended as assessments of individual teachers. This leaves unaddressed how to develop assessments of SCK at scale and how to support item writing by test designers. Our investigation of this question has been situated in a project in which we collaborated with others to build items to measure teachers' specialized content knowledge at scale. Our goal was to develop tools that could be used to guide the development of assessments of SCK with item writers who have different expertise than the groups who have in the past worked to develop items like those in the Learning Mathematics for Teaching project.

To understand what tools and supports might be needed to accomplish this, we considered what might be challenging for assessment developers when constructing measures of specialized content knowledge. First, we hypothesized that item writers might have difficulty developing measures of more than just common content knowledge, as this is the typical focus for assessments of teacher knowledge. In particular, we anticipated that there would be challenges in understanding the differences between CCK and SCK. A second related challenge concerns the possibility that in attempting to shift from writing items focused on common content knowledge, item writers might end up going too far and focusing items on pedagogical tasks of teaching that involve more than mathematical work, such as making instructional decisions about the best ways to teach a topic. In other words, we were concerned that writers might develop items focused on pedagogical content knowledge or even pedagogical choices, both which were beyond the scope of a subject matter knowledge for teaching assessment. A third challenge might arise if item writers are not familiar with the work of teaching that draws on teachers' specialized mathematical knowledge, such as the tasks of teaching set out by Ball and colleagues (2008). Finally, we hypothesized that it might also be difficult for item writers to understand how specialized content knowledge might be used across the K-6 curriculum, and how those uses might vary. Based on these four hypothesized areas of difficulty, we developed a framework that identifies the mathematical work of teaching and is strategically designed to address each of these challenges. We highlight below how the framework supports a focus on the *mathematical* work of teaching that moves beyond common content knowledge but without moving into a space of pedagogical choice. We also illustrate how the framework was constructed to highlight connections within and across the mathematical content of elementary school.

Unpacking the Mathematical Work of Teaching Framework

The mathematical work of teaching framework expands on the mathematical tasks of teaching (Ball et al., 2008) to produce a tool that can support development of assessments of mathematical knowledge for teaching at the elementary level. The framework addresses three main goals. First, the framework supports a focus on the *mathematical* work of teaching, the mathematics that a teacher engages with while teaching content to students, as opposed to the pedagogical task of making choices about instructional strategies. Second, the framework highlights connections between the mathematical work of teaching and the mathematics content at the elementary level. Finally, the framework is usable by item writers to construct written measures of MKT, specifically measures of subject matter knowledge with a focus on specialized content knowledge. In the following sections, we unpack the mathematical work of teaching framework with respect these three goals, referencing an excerpt from the framework shown below in Table 1.

Table 1: *Mathematical work of teaching framework organized by (1) mathematical objects, (2) actions with and on those objects in teaching, and (3) specific examples.*

	MWT: Actions with and on objects	**Examples**
Explanations (includes justifications & reasoning)	**Comparing explanations** to determine which is more/most valid, generalizable, or complete explanation	Given two explanations, choose which is more complete Given multiple student explanations, determine which is most valid Given several explanations, choose the best explanation. Given conflicting explanations, determine which is valid and why. Select an explanation that best captures an underlying idea.
	Critiquing explanations to improve them with respect to completeness, validity, or generalizability.	Given an incomplete but valid explanation, determine what, if anything, is missing or needs to be added to be more complete.
	Critiquing explanations with respect to validity, generalizability, or explanatory power.	Given an explanation, determine if it is mathematically valid. Given several explanations, determine which ones are valid. Given a text, determine what may be misleading about an explanation.
	Writing mathematically valid explanations for a process, conjecture, relationship, etc.	Write a mathematically valid explanation for a process or concept. Write a mathematically valid explanation for a conjecture. Given student strategies, determine properties that could be used to justify the strategy's validity.
Mathematical structure	**Determining, analyzing, or posing** problems with the same (or different) mathematical **structure**	Given a set of problems, determine which have the same structure. Given a set of problems, choose the description of the structure type. Given a set of problems, determine which does NOT have the same structure. Write a problem that has the same structure as given problems. Given a description of a structure, determine which problems fit that structure.
	Analyzing structure in student work by determining which strategies or ideas are most closely connected with respect to mathematical structure	Given a set of student strategies, determine which have similar mathematical structure. Given a set of student strategies most of which use the same core idea but slightly differently, determine which one does not fit. Given a set of strategies and a structure, determine which strategies fit the structure.
	Matching word problems and structure	Given a structure, choose a word problem with that structure. Given a word problem, choose another problem with the same structure.

Representations	**Connecting or matching representations**	Match a representation to a given interpretation of an operation. Determine how different representations are connected. Given two claims about a representation, determine which is correct and why.
	Analyzing representations by identifying correct or misleading representations in a text, talk, or written work.	Given a written representation (e.g., number line, table, diagram), determine what may be misleading. Given a set of representations, choose which does or does not show a particular idea (table?)
	Selecting, creating, or evaluating representations for a mathematical purpose	Create a representation for a given number or operation. Select a representation that highlights a particular mathematical idea.
	Talking a representation (i.e., using words to talk through the meaning of a representation and connecting it to the key ideas)	Given a suggested way to talk about a representation in a text, evaluate whether the talk clearly connects the representation and the ideas. Given a colleague's request for feedback, determine how their talking about a representation could be improved to highlight mathematical meaning

Organization around "Mathematical Objects"

Our first task in developing the mathematical work of teaching framework was to organize the mathematical work that arises in the context of teaching in a way that would maintain a focus on the mathematics. The mathematical tasks of teaching, as set out by Ball and colleagues (2008), include a list of 16 illustrative tasks that arise in everyday practice and that entail mathematical work for the teacher. This list includes tasks such as "responding to students' 'why' questions", "finding an example to make a specific mathematical point" or "recognizing what is involved in using a particular representation" (p. 10). These provide useful examples of the mathematical work of teaching; however, they were not intended to be an exhaustive list. Therefore, we built on and expanded this list. As the list of teachers' mathematical work grew, we needed to create an organizational structure that would make the list more orderly, systematic, and useful for item writers.

The mathematical work of teaching framework is organized around a set of what we call "mathematical objects" that teachers encounter and with which they work while teaching. Examples include explanations, representations, mathematical errors, and definitions. We called these "mathematical objects" because they are the mathematical instructional objects that teachers encounter and with which they interact while teaching. For example, teachers regularly give, use, and encounter mathematical explanations; in this case, the mathematical explanation is the "object". Teachers give mathematical explanations themselves, but they also make sense of student explanations, compare different explanations in textbooks, determine if a student's explanation is valid, or critique written explanations for the purpose of improving them. We recognize that we define "mathematical objects" here in a way that is different from the way "objects" is typically used in mathematics to refer to objects such as numbers, functions, and polygons. Table 2 provides the set of mathematical objects around which we built the framework. Although this list is by no means exhaustive, we hoped to describe the diverse sets of mathematical objects that teachers typically interact with in teaching.

Table 2. *Organizing mathematical objects for the mathematical work of teaching.*

Explanations (including justification and reasoning)	Errors and incorrect thinking
Conjectures	Representations
Mathematical structure	Manipulatives
Examples, non-examples, and counter-examples	Language and definitions
Mathematical problems	Mathematical goals and topics
Strategies	

Establishing the framework around a set of mathematical objects focuses attention on the *mathematical* work of teaching. By basing the framework in these mathematical objects, the mathematics in teachers' work is foregrounded. Another way to organize such a framework would be to organize the mathematical work of teaching into pedagogical tasks or domains (e.g., the mathematical work that arises when leading a discussion). However, there are two main limitations in organizing the framework in this way. First, many of the mathematical tasks of teaching arise in the context of enacting multiple different instructional practices, but require the same mathematical work on the part of the teacher, regardless of context. For example, interpreting student mathematical errors, a key mathematical task of teaching, could arise in the context of a class discussion, but it might also arise while teachers are interpreting written work on assessments or during teachers' interactions with individuals or small groups of students. Representing the mathematical work of teaching in each of these instructional practices would result in a lengthy list of work with much repetition. A second reason to avoid organizing the framework around pedagogical domains or tasks is to keep the focus on the *mathematical* work of teaching to help item writers avoid developing items that simply assessed teachers' instructional choices. Organization around instructional practices emphasizes the teaching practice rather than the mathematics necessary to engage in that practice. Consider the instructional practice of giving oral or written feedback to students. This practice requires teachers to engage in mathematical work such as determining how a student's explanation could be improved to be more complete. Organizing the framework around mathematical objects, as opposed to instructional practices or other pedagogically focused categories, supports a focus on the mathematics in the work of teaching.

Organization around Mathematical Work of Teaching with Respect to these Objects

The framework is organized around a diverse set of mathematical objects to illuminate the varied mathematical terrain of teachers' work, ranging from the mathematics that arises in interacting with explanations and strategies, to the mathematics involved in using language and definitions carefully, to the mathematical work of choosing or constructing mathematical examples. Explicitly naming and building the framework around this set of objects highlights the diversity of mathematical work of teaching.

Each domain of mathematical objects is further defined by a set of mathematical work of teaching, or actions on that particular object. For example, the mathematical object of "representations" includes five categories of work, i.e., "connecting or matching representations", "analyzing representations", "choosing or creating representations", and "talking a representation". The categories of work for each object answer the question, "what mathematical work do teachers do with these objects while teaching"? The grain-size of these categories manages the tension between (1) defining a useful set of categories that adequately captures the nuance and variability of the ways in which teachers interact with these objects, and (2) constructing a list that is manageable to use rather than a long list of specific verbs and scenarios. These categories comprise a larger domain of mathematical work that could be defined by introducing different mathematical criteria. For example, explanations, as objects, could be critiqued by teachers with respect to the validity, generalizability, or completeness of the explanations. Rather than define a separate category for each type of critique, the framework groups them as one category of mathematical work of teaching around the mathematical object of explanations. Each category might also refer to a range of contexts in which teachers might engage in this particular mathematical work. For example, as shown in Table 1 the category of analyzing representations includes reference to a set of contexts in which teachers might encounter representations, such as written work, talk, or texts. This organization of the framework keeps the set of categories concise while also mapping the dimensions of variation in teachers' mathematical work. This could support item writers in sampling across the varied terrain of teacher's mathematical work. Although the framework is detailed, it is not fully intended to represent all of the work of teaching, but to focus on common ways that teachers interact with particular objects.

It is important to note that the high-level categories built around mathematical objects are not necessarily disjoint. We utilize the focus of the mathematical work to determine where particular tasks of teaching fit in the framework. For example, consider the mathematical work of teaching that involves analyzing student strategies in written work for evidence of use of a particular mathematical structure (e.g., examining whether strategies reveal evidence of a comparison or take-away interpretation of subtraction). This work involves both student strategies and mathematical structure. However, the framework classifies the work of teaching by the main mathematical focus or goal, so looking for structure in student strategies would be classified as belonging to mathematical structure because looking for mathematical structure is the primary mathematical work, while the student strategy was the context in which it arose.

The third level of the framework further illustrates each category of work with examples that could serve as "shells" for items or "item starters." These examples do not include all of the necessarily details that might exist in a finished item, but could serve as a starter for beginning item developer to write items in this category. The examples are specific enough to provide help in beginning to write an item in that category. Consider the category of critiquing explanations. One example in that category is "Given several explanations, determine which ones are valid." This includes information about key elements that would need to be specified in the item (i.e. several explanations) and provides the desired action on the part of the test taker (i.e. determining validity). Each category includes several examples to help item writers attend to the different contexts and ways the work might play out, but the framework makes clear that the set of examples is not exhaustive and there are other ways to construct items and scenarios in each category.

Interactions with Mathematical Content

The framework also provides support in understanding the mapping between particular K-6 mathematical content and the mathematical work of teaching. For example, the framework helps answer questions such as "in which mathematical content areas do teachers most likely interact with mathematical definitions"? Although the mathematical work of teaching can be mapped to all K-6 mathematics content, the framework focuses the supports on the most critical, or high-leverage K-6 topics (Ball & Foranzi, 2011). The mathematical work of teaching interacts with this content in a number of ways. First, there are some categories of work with respect to particular mathematical objects that are likely to emerge in instruction across all content areas. For example, interpreting students' mathematical errors is part of the mathematical work of teaching all mathematical topics (e.g., number and operations, measurement, fractions). In contrast, there are other parts of the mathematical work of teaching that are more likely to arise when teaching particular mathematical content. Consider "critiquing strategies", a category of teacher's work. While certainly possible that teachers might engage in this work across all content areas, there are some content areas in which teacher might need to do this work more frequently and with a set of common strategies. For example, when teaching multi-digit subtraction, teachers are likely to have to analyze and critique students' non-standard strategies. Similarly, making sense of student strategies is also likely to be part of the work when students are learning to compare fractions when there are many common strategies for doing so (e.g., common numerators, benchmarking). In contrast, this type of work is less likely to emerge when teaching aspects of geometry. The framework serves as a scaffold for item writers to think about in which mathematical content particular work is most likely to happen.

Annotations are included in the fourth column of the framework to foreground these connections and to highlight other considerations for writing items. These annotations address multiple areas of concern for item writers, often suggesting mathematical topics that are a good fit (or less good fit) with that category of work. In the case of the critiquing strategies example described above, the framework includes a note that "ordering numbers, operations with numbers" are fruitful areas for writing items in this category. Other times, the framework indicates that all content areas are a good fit. This column also includes annotations about the challenges of writing items in certain categories and with particular content. For example, in the topic of comparing fractions, there are a number of strategies that will result in the correct answer in some but not all cases, which makes this a productive terrain for writing items that assess candidates' ability to critique the validity and generalizability of strategies. In contrast, with whole number operations, it is much more difficult to find examples of strategies that either only work for a subset of whole numbers or strategies that result in a correct answer but are not valid. Developing items in this space is therefore quite challenging and requires very careful and strategic section of numbers. To help item writers understand this interaction between the MWT category and this content, the fourth column includes a note to indicate this difficulty.

This fourth column provides additional varied supports for understanding the MWT framework, including identifying potential item types and interactions with content. For example, some categories of the framework are areas in which others have developed items that serve as model items noted in the annotations, whereas others are novel in the sense that very few (or even no) examples of items in that space exist. The framework includes these annotations to describe the range of work teachers do and to inspire the development of new types of items that assess this range. However, items in this space are likely to be more difficult to write without examples from which to build. Therefore, notes in the fourth column alert item writers to the fact that the category was new and potentially challenging to write to. An example of this is the category of "Critiquing the use of a representation", meant to capture the work that teachers might need to do when making sense of the use of representations in particular ways by students, other teachers, or curricular materials. The final type of annotation in the framework consists of notes to support item writers in maintaining the focus on the mathematical work and refraining from building items that focus on pedagogical choices. Early observations of item writing indicate that certain categories of work (e.g., manipulatives, errors) are more likely to lead item writers into pedagogical terrain, such as presenting a student error and asking the candidate to decide the next step that would best help that student or choosing the best manipulative to help a child see his or her mistake. Annotations are included to alert item writers when a particular category of work provides challenge for retaining the focus on the mathematics, along with common mistakes made in writing items too focused on pedagogy.

Utilizing the Framework for Item Development

To further specify the use of the mathematical work of teaching framework, we will examine how the table can be used to develop items, with a focus on explanations and mathematical structure. We begin by selecting the mathematical object of explanations and one key piece of the mathematical work of teaching with that object, "critiquing explanations with respect to validity, generalizability, or explanatory power." For this example, we will focus on the criteria of generalizability. To develop an item focused on the mathematical work a teacher does when critiquing explanations for generalizability, we must consider what mathematics content makes available a variety of explanations that may or may not be generalizable. One area of mathematics that is ripe with both explanations and methods that may or may not be generalizable is numbers and operations. For this example, we focus on operations with decimals, specifically decimal multiplication. The item shown in Table 3 requires a teacher to determine for each given explanation, whether or not the explanation represents generalizable methods for multiplying any two decimals. This is mathematical work that teachers do on a regular basis in a variety of contexts.

The second sample item is focused on the mathematical work of teaching involved in creating problems with a particular mathematical structure. In this case, the selected object for the item is "mathematical structure" and is combined with the work of "determining, analyzing, or posing problems with the same (or different) mathematical structure." Again, we must consider the mathematics content that teachers are most likely to encounter the need to determining or highlighting the mathematical structure of the work. Division is one area of elementary mathematics where particular interpretations of the operation require teachers to attend to mathematical structure. The problem shown in Table 3 requires one to apply a measurement (or quotitive) interpretation of division to develop a word problem. This involves careful attention to the structure of measurement division problems with attention to the meaning of each of the parts of the problems and then transferring this meaning to a particular context.

Table 3. *Examples of items written at the intersection of the mathematical work of teaching and content.*

Explanations Sample Item
Mr. Reinke is working with his students on decimal multiplication. He asked them to solve the problem 1.2 x 0.3 and explain their process. Which of the following student explanations for multiplying 1.2 x 0.3 represent methods that are generalizable for multiplying any two decimal numbers? Select <u>all</u> that apply. (A) "I just multiplied 3x12 and got 36. I counted the total numbers behind the decimal points. That was two, so I need to have two numbers behind the decimal point in my answer." (B) " I split the problem into two problems to make it easier. So I did .2 x .3 and that got me 0.06. Then I added 0.3 to that and got 0.36." (C) "I like to change the problem so that I can use a whole number. I changed this problem to 3 x 0.12 because I can just multiply the 0.3 by ten, but I have to divide the other number by ten so I don't change the answer." (D) "I just multiply like they are fractions. So it's like multiplying 12/10 and 3/10. I multiply the 12x3 and the 10x10 and get 36/100. That's 0.36 when I write it as a decimal. (E) " I need to make the length of the numbers the same, so I can line them up. My new problem is 1.2x.30. I multiply them like regular numbers, then I just bring down the decimal point from the .30, so there are two numbers behind the decimal point."
Mathematical Structure Sample Item
Ms. Fischer is working with her students on fraction division using a measurement (or quotitive) interpretation, meaning that the quotient specifies the number of equal groups. She wants to give them word problems that use this interpretation of division so that students can practice giving explanations of fraction division. Write a word problem that uses a measurement interpretation of division and could be solved using the problem 13 ÷ ½.

Reflecting on the Framework: Affordances and Constraints

In constructing this framework for the mathematical work of teaching, we sought to develop a tool that could support a focus on the mathematical work that teachers do in the context of every day practice. In reflecting on the framework in its current version, we believe that it does provide a mapping of a practice-based view of contexts in which teachers need to draw on specialized content knowledge. Furthermore, by focusing on a wide array of mathematical objects with which teachers interact, the framework provides insight into the diverse terrain of teachers' knowledge use in teaching. By focusing on the *mathematical* work of teaching, this framework attempts to push the envelope on the types of items that could be written in ways that may not emerge when approaching item writing by starting with the knowledge to be assessed. This framework also provides key insights into the interactions between the mathematical work of teaching and the mathematical content of elementary school in ways that could support item writers to develop assessment tasks within and across different mathematical topics. Despite the potential affordances of the mathematical work of teaching framework, we also recognize that the framework, as a tool for item writing, does not necessarily provide all of the support that assessment developers might need to write items to measure teachers' specialized content knowledge. In the following section, we describe a set of additional supports that we hypothesize might be needed for item development.

The focus of the framework on the mathematical work of teaching with the additional supports of highlighting interactions with mathematical content may not provide sufficient support for item development if the item writers do not have well-developed mathematical knowledge for teaching across mathematical topics and with respect to actions on different mathematical objects. For example, to write productive items about mathematical structure, item writers would need to know common mathematical structures relevant to K-6 mathematics, such as different interpretations of subtraction or division (i.e., take-away vs. comparison, partitive vs. measurement) or common different problem structures in early addition and subtraction tasks (i.e. result unknown, change unknown). Similarly, in order to write items about the validity and generalizability of student strategies, item developers would need to know common strategies used by children in that content area, such as knowing different valid (or invalid) strategies for comparing fractions (e.g., McNamara & Shaughnessy, 2010). For items about representations as objects, item writers would need to know relevant representations for a particular content area (e.g., area models, number lines, sets, and fraction bars for fractions concepts) and how the key ideas of that content are highlighted or not in different representations, such as knowing how different mathematical ideas of decimal multiplication and place value play out in an area model (Ball, Lubienski, & Mewborn, 2001). To develop items about interpreting student errors, item writers would need to know what are likely errors within particular content areas and what the reasoning behind the errors might be, such as knowing what key mathematical ideas of place value are violated when students incorrectly regroup across zero in multi-digit subtraction (Ball et al., 2008; Fuson, 1990). All of these examples highlight the need for item writers themselves to have well-developed mathematical knowledge for teaching or the tools to access and learn this knowledge themselves in order to develop assessment tasks.

Support for item writers with respect to mathematical knowledge for teaching may be particularly needed for developing items at the lower elementary grades. Much of the key mathematics of those grades is so tacit for adults that they are likely to struggle to determine what ideas could be addressed in items, such as knowing the different mathematical ideas that must be coordinated by children when counting an ill-structured set of objects to determine "how many", such as one-to-one correspondence, verbal counting, cardinality principle, and strategies for keeping track of what has been counted (Clements & Sarama, 2014; Richardson, 2012). Similarly, to develop items about early place value, item writers would need to know the key, but often tacit, ideas of place value that are not related to operations, such as the role of zero as a place holder or that quantities are represented symbolically left to right. Item writers might also need support in knowing how to construct item scenarios with reasonable approximations of student work at the relevant grade. For example, what would a student's drawing look like when trying to represent fractions with area models? What are reasonable student explanations of their thinking around particular content? Adults with less experience in K-6 classrooms struggle to construct student talk and written work that is reasonably authentic, as adult's own ways of thinking, talking, and representing mathematics are likely to be much more sophisticated than those of children, especially at the lower elementary grades.

Another area in which the mathematical work of teaching, as written, might not be sufficient to support item development around specialized content knowledge is related to the distinction between what counts as common content knowledge and specialized content knowledge for teaching. In our initial work with this framework, we have found that some sets of actions on mathematical objects seem to sit more clearly sit in the space of specialized content knowledge, such as analyzing the validity and generalizability of student non-standard strategies, which is in alignment with what was found by Hill, Schilling, and Ball (2004). Other sets of actions emerge at times closer to the line between common and specialized content knowledge, such as making a conjecture. One could argue that this is work that students also often do in mathematics classrooms, but it is important work that is not done in other fields. The line between SCK and CCK maybe particularly challenging to distinguish in the context of the Common Core, since the Standards for Mathematical Practice now ask students (and teachers) to engage with content through mathematical practices such as constructing arguments and critiquing the thinking of others, actions which in some ways align with some of the mathematical work of teaching, such as analyzing the strategies used by others. The intent and nuance of the work may be different when students critique the explanations of peers in a classroom and when teachers are making sense of those strategies but there are some interesting similarities. This points to the possibility that item developers might need further support in how to write items that focus more squarely on specialized content knowledge.

These hypothesized supports serve as an initial set that address some key areas of concern when supporting the development of assessments. There may be other additional challenges that would arise when using the framework for this and other purposes that could require different types of support.

Discussion and Conclusion

In this paper, we have presented a framework to support the development of assessments of mathematical knowledge for teaching (Ball et al., 2008). To consider the theoretical and practical implications of this framework, we first acknowledge that this framework offers one decomposition of the mathematical work of teaching; there may be other useful ways to parse teachers' mathematical work that would foreground different aspects of practice and knowledge use. Furthermore, this framework was developed based on the concept of mathematical tasks, or the mathematical work of teaching (Ball & Bass, 2002; Ball et al., 2008) which were conceptualized based on work in elementary mathematics and with the intended goal of supporting assessment development around elementary MKT. This raises the question of whether the mathematical work of teaching framework proposed here would apply equally well for the work of teaching secondary mathematics or whether there may need to be revisions or additions. For example, at the elementary level, we chose to group explanations and justifications together as an object given the nature of mathematical arguments typically constructed at the elementary level. At the secondary level, it might be more appropriate to include "justification and proof" as a separate mathematical object with which secondary mathematics teachers interact. As part of our future work, we will be pursuing this line of inquiry as we work to support assessment development around secondary MKT.

Despite these potential limitations, the mathematical work of teaching framework offers a contribution that has both theoretical and practical implications. First, this framework builds on and expands upon the mathematical tasks of teaching (Ball & Bass, 2002; Ball et al., 2008) to provide a comprehensive and nuanced identification of the mathematical work that teachers do in the context of teaching. This provides a detailed and practice-based lens (Ball & Bass, 2002) for the contexts when teachers must draw on mathematical knowledge for teaching in their practice.

A novel contribution of this framework is the idea of organizing the work around "mathematical objects" that teachers encounter and interact with in practice. This organization highlights the central role of mathematics in the framework and also affords seeing the diverse ways that teachers interact with different types of mathematical objects (e.g., representations, explanations, mathematical structure). Drawing on a practice-based perspective on mathematical knowledge for teaching, the framework offers a systematic way to identify and examine MKT by focusing first on teaching practice and the nature of teachers' mathematical work and then including the knowledge needed to manage that work. This perspective is different from starting with teachers' mathematical knowledge. Our approach explicitly highlights the use of knowledge in practice.

The MWT framework could also serve a number of practical purposes in both assessment development and teacher education. First, the framework was designed with the purpose of supporting the development of assessments of mathematical knowledge for teaching at the elementary level. This tool, along with additional supports described in the previous section, can provide item writers ways to develop assessments of MKT at a larger scale than has previously been possible, when items such as those developed by the Learning Mathematics for Teaching project (Hill, Schilling, & Ball, 2004) have been crafted by groups of experts involving mathematicians, mathematics educators, and teachers. Specifically, this framework could serve as a tool for developing items that appraise mathematical knowledge for teaching in the context of how the knowledge might be used in practice. Furthermore, the MWT framework could provide support in maintaining the focus on the *mathematics* in the work of teaching in ways that help item writers avoid developing items that assess teachers' pedagogical choices and decisions. Similarly, the framework could help illuminate the specialized knowledge that teachers draw on in their work to support item writers in writing items that assess more than common content knowledge of particular topics. Finally, the MWT framework offers systematic ways for assessment developers to manage the connections between the mathematical work of teaching and mathematical content in ways that would allow for building assessments that tap into the diverse specialized content knowledge needed to teach the K-6 curriculum. As we work with item writers from various backgrounds to develop a content knowledge for teaching assessment for elementary mathematics, we are able to examine the utility and limitations of the MWT framework for supporting assessment development at scale.

Although the mathematical work of teaching framework was developed for the purposes of building assessments, we believe that the framework has the potential to be used for other purposes related to teacher education and professional development, because it offers a systematic identification of the mathematical work of teaching. For example, this framework could be used as an organizing principle for designing mathematics content courses for pre-service teachers to support a focus on the ways that mathematical knowledge is used in practice. Similarly, the framework could be used for as a tool for curricular mapping in mathematics teacher education so that programs could systematically design learning experiences for complementary parts of the framework in different courses (e.g., content vs. methods courses, content courses in different topics such as number or algebra). This framework could also be a useful tool for increasing the MKT of teachers and faculty, including supporting professors and instructors of mathematics content for teachers courses (who likely did not teach elementary school themselves) in better understanding the ways elementary teachers need to use mathematics in practice and the nature of this knowledge.

In the context of the increased mathematical demands of the Common Core State Standards for Mathematics and data showing that many U.S. elementary school teachers lack strong MKT (e.g., Hill & Ball, 2004; Ma, 1999; Tatto et al., 2008), there is an urgent need to develop elementary teachers' mathematical knowledge for teaching. These mathematical demands on teachers are not new (Ball, Hill, & Bass, 2005) and are likely to continue with goal of preparing skillful and responsive practitioners (Ball & Forzani, 2011) Along with ways to support teacher knowledge development, the field needs assessment tools that will allow us to measure and track teachers' growth. The mathematical work of teaching framework contributes a tool to aid in these efforts, especially in working at scale.

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