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Interview Prompts to Uncover Mathematical Knowledge for Teaching: Focus on Providing Written Feedback

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Abstract: One area of study that has been gathering enthusiastic attention and interest is mathematical knowledge for teaching (MKT). How to research MKT, however, is still unsettled despite the plethora of unexamined areas of practice. As one of ways to unearth and measure MKT, this study uses interview prompts designed to providing written feedback, as a target area of practice. This study specifies in what ways the interview prompts are used in order to provide a comprehensive method to researching MKT. From interviews across professional communities with different kinds of mathematical expertise, the author develops a conceptual model based on the tasks of teaching, elements of pedagogical context, and domains of MKT. This model provides the fluent character of proficient MKT decision making in teaching practice and explains key features of the design of prompts for investigating and measuring MKT. From the analysis, two claims emerged: bidirectional approaches to investigating MKT and continuous and spontaneous aspects of MKT.

Keywords: mathematical knowledge for teaching, providing written feedback, pedagogical context, measurement, conceptual study.

Introduction

Mathematical knowledge that is specifically connected to the work of teaching has been investigated empirically and theoretically, leading to a significant progression of its conceptualization. In particular, Ball, Thames, and Phelps (2008) developed a practice-based theory of content knowledge for teaching and introduced *mathematical knowledge for teaching* (MKT) as the mathematical knowledge needed to carry out the work of teaching mathematics. Such knowledge has been studied by examining teaching practice, such as job analysis (Ball & Bass, 2003) and by developing its measurement (Hill, Schilling, & Ball, 2004). MKT has been identified as important in teaching mathematics (Lewis & Blunk, 2012) and in student achievement (Hill, Rowan, & Ball, 2005; Rockoff, Jacob, Kane, & Staiger, 2011).

Research on MKT has been conducted with practices of teaching that are prominent in mathematics classrooms, though not all practices. Ma (1999), for example, used four items with some exceptional tasks of teaching and mathematical demands, but did not include all topics and practices in and from teaching. Items developed by the University of Michigan include

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An earlier version of this research was presented with Mark Hoover at the 2015 KSME International Conference on Mathematics Education and at the 2015 annual meeting of the American Educational Research Association.

substantial tasks of teaching, but not a sufficiently organized approach to mathematical topics and teaching practice.¹ Ball et al. (2008) emphasized a practice-based approach to study content knowledge for teaching. While introducing their conceptualization about such special knowledge, they focused on several and seminal practices of teaching, such as presenting mathematical ideas, responding to students' "why" questions, choosing and developing useable definitions. Nonetheless, the conceptualization has not yet been broadened to explore all practices or all topics in terms of MKT. In other words, these studies do not cover the extensive terrain of mathematical demands in teaching across contexts. For sustainable and elaborated development of the study of MKT, the mathematical demands entailed in teaching now needs to be explored systemically using a clear and comprehensive map of the practice of teaching, mathematics, features of learners, and national or international curriculum or grade levels.

How MKT is studied has yet to receive sufficient attention. Ball and her colleagues studied MKT with a set of analytic tools they developed, using their wide range of experiences and disciplinary backgrounds, for coordinating mathematical and pedagogical perspectives (Ball et al., 2008; Thames, 2009). However, their experiences have not been shared in terms of researching MKT. A robust, reliable, and consistent study of MKT can be expected with an appropriate and good method. Substantial areas of mathematical demands in different practices of teaching still call for investigations from many researchers. Like the collaborative work on the Human Genome Project (International Human Genome Sequencing Consortium, 2001; Naidoo, Pawitan, Soong, Cooper, & Ku, 2011), research on MKT would need cooperative work for elaborate and systematic conceptualization. A major prerequisite for such collective work is the identification of a method to research MKT. If relevant methods of studying MKT are specified, research of MKT will be powerfully advanced. To systemically research MKT, a comprehensive method needs to be specified.

To address the problems, the current study focuses on both using interviews as a comprehensive way to study and measure MKT and researching mathematical demands in providing written feedback, which is an unexamined teaching practice. It does so by building on lessons from the interview prompts used in Ball (1988) and Ma (1999) and on items used to measure MKT (Ball, Bass, & Hill, 2004). Specifically, this paper explores the ways in which interview prompts are developed and used to provide the content and character of MKT, and what, through the use of such interview prompts, might be learned. Particularly, what is entailed mathematically and pedagogically in providing written feedback?

Measuring MKT

Based on Shulman's (1986) notion of pedagogical content knowledge, Ball et al. (2008) specified several subdomains within pedagogical content knowledge (knowledge of content and students [KCS], knowledge of content and teaching [KCT], and knowledge of content and curriculum [KCC]). Further, they identified an important subdomain of "pure" content knowledge unique to the work of teaching, specialized content knowledge [SCK]. SCK is "distinct from the common content knowledge [CCK] needed by teachers and non-teachers alike" (p. 389). They argued that, "teachers' opportunities to learn mathematics for teaching could be better tuned if we could identify those types more clearly" (p. 399).

In earlier research, Ball (1988) created problems that were used in interviews with prospective teachers to explore their proficiency in meeting the mathematical demands of teaching. Ma (1999) extended the use of these interview prompts with practicing teachers in China. Their findings and arguments were critically informed by the carefully designed

mathematical teaching problems represented in the interview prompts. Such mathematical teaching problems are, in short, tools for uncovering mathematics entailed in teaching. Based on analyses of teaching, Ball and Hill's *Learning Mathematics for Teaching* (LMT) project developed multiple-choice items to measure MKT (Ball, Hill, & Bass, 2005; Hill, Ball, & Schilling, 2008; Hill, Schilling, & Ball, 2004). Following the same approach, Herbst and Kosko (2014) developed MKT items for secondary geometry teaching.

To identify key design features of MKT items, my colleagues and I analyzed representations of teaching embedded in MKT assessment items.² We found that the reasoning of item doers was shaped by elements of pedagogical context: student background, teaching purposes, and classroom artifacts, as represented in the teaching situations described in the items. The pedagogical context given in items created situations that required doing mathematics while holding onto a specific pedagogical purpose; the context gave the tasks an MKT, rather than a disciplinary mathematics character. We also found that competent performance on the items depended on reasoning that used, in integral ways, features of the pedagogical context. This conceptualization of pedagogical context offers further ideas about how to design items in ways that measure mathematical knowledge that is fundamentally linked to teaching and not simply disciplinary knowledge that remains remote from effective teaching and learning.

The current study uses open-format prompts designed to investigate the work of teaching that is not readily evident from video of teaching (which was used to design the pedagogical contexts and mathematical tasks of teaching in LMT items). The pedagogical focus of the sub-study reported here is the task of providing written student feedback. This task of teaching requires consideration of the mathematics problem, students' responses to it, and decisions about how best to guide students toward the intended purpose of the problem. It does not mean to suggest that this task of teaching cannot be studied observationally or measured using multiple-choice items. Rather, it needs a means of uncovering more about what was involved in the task and have thus explored the use of designed interview prompts to expand the investigation of MKT. Hence, the current study investigates the following questions: What are the ways in which interview prompts can be developed and used to provide insight into the content and character of MKT? What is entailed, mathematically and pedagogically, in providing written feedback?

Providing Written Feedback

The purpose of feedback is to reduce discrepancies between what is understood and what is intended to be understood (Hattie & Timperley, 2007). Feedback aims to modify a student's thinking to improve his or her learning (Shute, 2008). Apparently, major and complex tasks of teaching involved in the providing of feedback include the sizing up of students' current understandings, directing students to a desired goal, and deciding what information will be provided and in what ways. The complexity of providing feedback is evidenced by a large body of research that encompasses many conflicting findings and no consistent pattern of results. Nevertheless, Hattie and Timperley (2007) reviewed research on feedback and clarified that

... feedback needs to be clear, purposeful, meaningful, and compatible with students' prior knowledge and to provide logical connections. It also needs to prompt active information processing on the part of learners, have low task complexity, relate to specific and clear goals, and provide little threat to the person at the self level. (p. 104)

Shute (2008) made a similar claim that feedback should be nonevaluative, supportive, timely, and specific. Based on a literature review of feedback that concentrates on specific

information to a student about a particular response to a task, Shute (2008) suggested that feedback should address specific features of the student's work with a description of the what, how, and why of a given problem and suggest improvements that the student can manage. This clear feedback would reduce uncertainty in relation to how well students are performing on a problem and what needs to be accomplished to attain the goals.

Feedback, in fact, can promote learning if it is received mindfully (Bangert-Drowns, Kulik, Kulik, & Morgan, 1991). Yeager et al. (2014) emphasized that trust is crucial for successfully delivering written feedback, adding that "mistrust can lead people to view critical feedback as a sign of the evaluator's indifference, antipathy, or bias, leading them to dismiss rather than accept it" (p. 805). Harber et al. (2012) pointed out that many teachers tend to overpraise students for mediocre performance, particularly those subject to negative stereotypes, in order to enhance student self-esteem. However, unlike the teachers' intention, this overpraising in written feedback hinders the development of trust and reinforces minority students' perceptions that they are being viewed stereotypically (Croft & Schmader, 2012; Harber et al., 2012). Feedback should be based on a reflection of a teacher's high standards, not his or her bias, and offer students both an assurance about their potential to reach such high standards and the resources with which they might do so (Yeager et al., 2014). However, written feedback is more unbiased and objective than face-to-face feedback (Kluger & DeNisi, 1996; Shute, 2008).

Critical review about feedback, as specified previously, concentrates on features of feedback that function well to improve students' learning. However, it does not show what tasks of teaching are organically entailed in providing written feedback. The work of teaching includes the activities in which teachers engage and the responsibilities they have to teach content (Ball & Forzani, 2009). The work of teaching occurs in the dynamics initiated by a teacher before, during, and after instruction so as to help students learn the content (Sleep, 2009). Furthermore, in teaching mathematics, specific mathematical goals are critical throughout instruction (Schoenfeld, 2011). Recognizing the varied understandings of the mathematics, for example, probing so as to see what students do or do not know, and responding in ways that address students' errors and that build on student understanding help move students toward goals in instruction. Evidently, providing written feedback is purposeful work: control or modification of a student's ways of thinking or answers to learn something that a teacher desires. Providing written feedback assumes that a teacher gets and synthetically and analytically evaluates signals from students' responses or reasoning, tasks or problems given, and pedagogical situations enacted and created by the teachers and the students. Then, the teacher makes a decision to put forward certain comments to the individual student or to the whole group of students. This description offers a simple glimpse of a feedback loop because it is important to understand the circumstances that result in the differential outcomes and responses of students (Hattie & Timperley, 2007). This loop could be repeated in a lesson until the teacher is satisfied with the control or with the students' revised ideas or answers. Each teacher might have different perspectives and rationales on what feedback works well and what should be highlighted in which situations. However, providing written feedback can be specified with sub-tasks of teaching that are responsive and intrinsic to teaching and entail professional norms of teaching.

Design and Analysis of the Prompts and Interviews

Design of the Interview Prompts

Prompt development started with initial descriptions of the high-leverage practices given in the professionally vetted version on the *TeachingWorks* website.³ The 19 high-leverage practices are tasks central to teaching, which are expected to increase the likelihood for students' learning across a broad range of subject areas, grade levels, and teaching contexts. These high-leverage practices are warranted by research evidence, wisdom of practice, and logic and were developed through many discussions with researchers, expert teachers, faculty members of teacher education, and education policy makers. Of the 19 practices, 16 are crucial in teaching mathematics, as shown in Figure 1. The figure illustrates both apparent interrelationships of the different practices of teaching and gives an overall picture of the practices of teaching mathematics. In other words, Figure 1 represents a comprehensive map of the practice of teaching, which the current study uses to study the mathematical demands entailed in teaching.

Figure 1 shows that providing oral and written feedback is based on assessing students' knowledge. Such an assessment includes selecting and using particular methods to check understanding and monitor student learning and composing, selecting, interpreting, and using information from methods of summative assessment. Providing feedback is influenced by appraising, choosing, and modifying tasks and texts for a specific learning goal. Furthermore, providing feedback may consist of leading a whole-class discussion and setting up and managing small group work. In other words, providing feedback is related to other practices in teaching

Figure 1. High-leverage practices in teaching mathematics

mathematics. Although providing feedback is typical work in teaching, it is crucial to approach this work from a holistic perspective about teaching practice.

Teachingworks explains the providing of oral and written feedback to students on their work as follows:

Effective feedback helps focus students' attention on specific qualities of their work; it highlights areas needing improvement; and delineates ways to improve. Good feedback is specific, not overwhelming in scope, and focused on the academic task, and supports students' perceptions of their own capability. Giving skillful feedback requires the teacher to make strategic choices about the frequency, method, and content of feedback and to communicate in ways that are understandable by students.

This explanation highlights both feedback about a task and processing of a task, which Hattie and Timperley (2007) differentiated. The explanation also emphasizes the need to attend to the student's work and to give specific comments about that work to advance it in a way that they can truly accept and understand (Shute, 2008; Yeager et al., 2014).

To access potentially tacit knowledge and reasoning about what is involved in providing feedback, situations were generated using realistic pedagogical contexts focused on written feedback. On this point, Common Core State Standards were used as a second map to select mathematical topics and determine students' backgrounds. These standards outline a clear set of mathematical skills and knowledge that students will learn in a more organized way both during the school year and across grades. The standards' coherent composition and specific statements offer clear features of mathematical topics and possible information about students' backgrounds that were necessary in creating scenarios for interview prompts. Scenarios of instruction, which would require feedback, and possible student work were sketched. Different elements were considered as well: which elements of pedagogical context are used and how they flow in the scenarios and what mathematical ideas, reasoning, and practices are addressed and how they can be unveiled and interpreted. In deciding each of these elements, the focus was on keeping the situation realistic.

Figure 2 and Figure 3 offer an example that sheds light on the transformation of one initial interview prompt into one used in interviews. In fact, the transformation took place over the course of ten revisions over several meetings with colleagues. The figures illustrate some of the challenges that can occur in developing interview prompts. Both prompts are about providing written feedback as a major practice of teaching, lines of symmetry as a mathematical topic, and fourth-grade students as student background. Nonetheless, Figure 2 does not present a teaching purpose for the instruction; different triangles are used repeatedly without being developed in the instruction; there is no mathematical foundation that interviewees or Mrs. Johnson in the scenario can use to provide feedback; specifically, it is unclear what Mrs. Johnson did pedagogically when drawing symmetry lines on the board; and, at the fourth grade level, the congruency marks offer no indication of whether lines of symmetry exist or not. Figure 3—the final version—specifies first what the teaching purpose is in the provided situation (helping fourth-grade students understand lines of symmetry and how to draw them for two-dimensional figures); it offers a concise and short explanation about the activity that Mrs. Johnson had with her students (using squares and figuring out how to find lines of symmetry); a definition of a line of symmetry is given by the teacher in the scenario, which can be used by the interviewee to provide written feedback in the interview; and student work is provided that includes errors and a

partial understanding about lines of symmetry. Phrases in the scenario have been revised repeatedly to create a succinct and unambiguous scenario that requires written feedback.

Three major questions were decided on: What does the interviewee first notice? What

In a lesson about lines of symmetry for triangles with her fourth grade students, Mrs. Johnson handed out the copies of an equilateral triangle, an isosceles triangle, and a scalene triangle, and students had time to try to fold each triangle into matching parts. Mrs. Johnson then explained lines of symmetry for triangles as lines across the triangles such that the triangles can be folded along the line into matching parts. She also drew symmetry lines on the board as shown in the following:

Then, to assess her students' understanding, Mrs. Johnson gave students time to individually practice drawing lines of symmetry for triangles. When she was monitoring, she noticed that several students drew lines on the textbook as following:

Figure 2. Example of initial interview prompt

Mrs. Johnson wanted to help her fourth-grade students understand lines of symmetry and how to draw them for two-dimensional figures. At the beginning of a lesson, she had the students cut out squares and look for ways to fold a square into two matching parts. The students identified two ways: folding it diagonally and folding it from the middle of one side to the middle of the opposite side. They used rulers to draw lines along their folds. Mrs. Johnson explained that these lines are called lines of symmetry and wrote the following definition on chart paper.

A line of symmetry is a line across a figure such that the figure can be folded along the line into matching parts.

Then, the students did a similar activity with triangles, cutting out triangles and looking for ways to fold each triangle into two matching parts. Here, they noticed that some triangles do not have a line of symmetry.

At the end of this lesson, Mrs. Johnson gave her students a page of practice exercises, where she included three figures that are not triangles and squares. For these figures, one of her students handed in the following:

Figure 3. Example of interview prompt used in actual interviews

written feedback does he/she offer? What would he/she do for a fifteen-minute lesson? These three questions aimed to grasp an interviewee's knowledge and reasoning for providing written feedback. Along with the major questions, several minor questions were added to identify the interviewee's rationale behind his or her responses. Figure 4 includes a list of major and minor questions on the left and on the right one page of possible responses for each question, used to prepare interviews for the interviews. Minor questions helped interviewers prepare for the interviews as well, deciding which minor questions to ask depending on the interviewees' responses. The initial prompts were discussed with colleagues who had teaching experience in grades K-9 and who had studied mathematics and mathematics education. Such discussions were meant to help the researcher anticipate responses and revise the prompts based on whether the prompt would activate and support the task of providing feedback. In other words, collective work was initiated with the designing of interview prompts.

For this sub-study of providing written feedback, interviews were conducted for each interview prompt with two practicing mathematicians, two prospective teachers, and three expert teachers.⁴ Interviewees were carefully selected and contacted. The mathematicians' responses in the interviews offered insight into how disciplinary thinking functions and what appropriately supporting learners means in instruction. The prospective teachers had no prior regular teaching experience and were attending the School of Education working toward their teaching certifications. They were considered a group that needed more professionalism rather than what mathematicians and expert teachers performed in order to investigate mathematical reasoning and knowledge entailed in providing feedback and identify features of MKT in such a task of teaching. To recruit expert teachers, the project team first contacted researchers who were

Grade 4	Geometry [A line of symmetry]	Providing feedback about lines of symmetry
Questions <ul style="list-style-type: none"> • [after getting a response to the first question] How did you figure that out? How do you know? • [after getting a response to the second question] Why did you do that? Why is that? • What did you do to write this feedback? What did you consider? What did you rule out? • [after getting a response to the third question] Why will you plan it? Why is that? <p>[Follow-up questions for Question 2]</p> <ul style="list-style-type: none"> • What does the student appear to know and not to know about lines of symmetry? • Which two-dimensional figures work well with this student's solution to find all lines of symmetry? • What are some other figures where the student might miss some of the lines of symmetry? • What mathematically might be the source of the student's limited understanding? • Did you use or consider this source in your feedback? [If so, ask] How did you use it? [If not, ask] Why didn't you use it? • Do you have any particular mathematical issues or intentions that you considered or emphasized in your feedback? How did you do that? Why are they important? • What do students need to know or what can students do in order to recognize a line of symmetry, identify line-symmetric figures, and draw lines of symmetry? What is important for students to understand about lines of symmetry? • Did you use or consider this knowledge or mathematical importance in your feedback? [If so, ask] How did you use it? [If not, ask] Why didn't you use it? • Would it be helpful for students to be familiar with certain properties of lines of symmetry? • Did you use or consider these properties in your feedback? [If so, ask] How did you use them? [If not, ask] why didn't you use them? • [In the feedback, if the interviewee mentioned the square that the class worked] You pointed out the square in your feedback. Why did you mention it specifically in your feedback? • [In the feedback, if the interviewee mentioned the definition that the teacher wrote down on chart paper] You pointed out the definition in your feedback. Why did you include it in your feedback? There are other possible resources in this classroom. Why didn't you use them? [After hearing the response, and then ask] What do you think about the activity with square? • [If the interviewee did not mention any activities that happened in the provided lesson, ask] How do you think about the activities that given in the lesson? Have you considered them to formulate your feedback? • What would you expect from your feedback? What would this student actually recognize or do after getting your feedback? • Would developing strategies for finding lines of symmetry be the most helpful? • Would it be best to address directly the issue of lines of symmetry passing either through vertices or midpoints of sides? <p>[The following three questions might not needed in the interview, but if the interviewee does not do these things or the interviewer decides to need to know, ask them]</p> <ul style="list-style-type: none"> • Any other more lines of symmetry in the first figure? What about the second and third one? • How do you find them? How do you find a line of symmetry? Are you sure the pairs are matching? • Is there a way to fold a trapezoid into two matching parts? • If a line of symmetry crosses a line of the figure, must it be perpendicular? • How would you show a figure does not have any lines of symmetry? • How would you show that you have all lines of symmetry for a given figure? • What numbers of lines of symmetry are possible? Is there a triangle with just two lines of symmetry? • Are there any types of lines of symmetry that cannot be found in quadrilaterals? <p>[Follow-up questions for Question 3]</p> <ul style="list-style-type: none"> • [If the interviewee did not specify the purpose of this 15-minute lesson, ask] What would be the purpose of this small lesson? • What mathematics would you like to emphasize in that lesson? • [If the interviewee would use some examples of figures, ask] Why did you plan to use these figures? • [If the interviewee does not mention any examples of figures, ask] If you are using examples of figures in this lesson, which would you like to use? Why? 		
<p><i>In reviewing this student's work, what do you notice?</i></p> <ul style="list-style-type: none"> → All the lines drawn are lines of symmetry. They all pass through vertices, not edges. Some lines of symmetry are missing. The missing ones all pass through edges, not vertices. → They seemed to think that all lines of symmetry are lines from a vertex to another vertex. → They seemed to think that lines of symmetry are diagonal. <ul style="list-style-type: none"> • How did you figure that out? How do you know? <p>→ All lines that this student drew pass through vertices, but there is no line between a vertex and a midpoint of a side.</p> <p>[It might be possible to ask additional questions here, but the main investigation in this prompt will be in the second question.]</p> <p><i>Given the lesson just taught, what written feedback might be good to offer this student? Please write your feedback.</i></p> <ul style="list-style-type: none"> → Look at lines of symmetry in your square. One method is folding it diagonally, and another is folding it vertically, or horizontally. Think about lines of symmetry that are from the middle of one side to the middle of the opposite side. → Some lines of symmetry are from the middle of one side to the middle of the opposite side. → All of the lines of symmetry you drew pass through two vertices. If you think about the square example, were there other types of lines of symmetry? Do you see any of those types on these figures? → Why are these lines of symmetry? How do you know if you have them all? <ul style="list-style-type: none"> • Why did you do that? Why is that? [It is possible to ask additional questions here.] <p>→ This student needs to draw lines of symmetry that he/she missed first time. I also hope helping him/her learn how to draw them as the proposed purposes here.</p> <ul style="list-style-type: none"> • What did you do to write this feedback? What did you consider? What did you rule out? [It is possible to ask additional questions here.] <p>→ I used my analysis of the student work. Here, I found this student did not draw a certain type of lines of symmetry. Actually, the class worked with a square in the lesson. So, I used this resource in my feedback.</p> <p><i>If several students had completed the practice exercises as above and there was an opportunity to spend an additional 15-minutes addressing students' misunderstanding, what would you recommend doing?</i></p> <ul style="list-style-type: none"> → The purposes of tomorrow lesson will be same as today's because it is a short lesson and some students did not achieve today's purposes. First, I'll briefly review a page of practice exercises that the students worked today. Here, I'll emphasize all lines of symmetry are not from a vertex to another vertex. Some lines of symmetry are from the middle of one side to the middle of the opposite side, in particular, for an isosceles trapezoid, and a regular hexagon. Second, I'll bring different types of quadrilaterals (rectangles, rhombuses, and isosceles trapezoids) and regular hexagons and octagons, and have the students do similar activities, such as cutting them and finding ways to fold them into two matching parts. It would not take long. Then, I'll bring all figures that we worked on today and tomorrow and arrange them on the board so that the students can apply the working definition that they used today and figure out which figures have lines of symmetry and which figures do not. → For a regular pentagon, a line of symmetry goes from a vertex to the midpoint of a side. → I will bring these figures again on the board and ask some students to draw lines of symmetry and talk about them if they miss some lines. If so, I will ask students to think back the square examples and emphasize lines of symmetry from one midpoint of one side to another midpoint of another side. <ul style="list-style-type: none"> • Why will you plan it? Why is that? <p>→ I would like to help them understand of bisecting an angle or a side as a critical way to draw a line of symmetry.</p> <p>[It might be possible to ask additional questions here.]</p> <p>[Follow-up questions for Question 2]</p> <ul style="list-style-type: none"> • What does the student appear to know and not to know about lines of symmetry? 		

Figure 4. Examples of questions and possible responses

leading research about teaching, teachers, or professional development in the United States. Lists of teachers were gathered based on the researchers' recommendations of expert teachers. They were knowledgeable in mathematics as well as mathematical practice in instructional situations and each had more than ten years of teaching experience. The prospective and expert teachers' responses in the interviews provided insight into what professional reasoning runs in instructional contexts and how it plays out. Observing what these three groups of interviewees did was to help the research team perceive and specify clearly both mathematical and pedagogical reasoning that is engaged in the work of teaching mathematics. Furthermore, interviews were conducted by several interviewers. Each interview was conducted with one interviewee by one interviewer. As previously specified, the current study was collaborative group work, and the collectively performed interviews helped the project team to deepen their insights about mathematical demands entailed in providing written feedback.

Vignette of One Interview and Its Analysis

Each interview was analyzed separately, and then all interviews and analyses were synthetically and analytically probed to investigate and characterize the mathematical demands of providing written feedback. The next section provides a more detailed explanation about the analysis, but this section shows a vignette of one interview and its analysis to illustrate a way of examining and researching MKT.

One interviewee had taught elementary students for approximately 25 years and, for the past 12 years, had worked as a math coach at the K-8th grade level. She also received an M.A. in teaching and learning mathematics. In the interview with the prompts shown in Figure 5, the interviewee immediately recognized that the student in the prompt used a system to make the list. The interviewee noted that the student changed the last two over and over again as well as reordered the first two letters. She was also curious about what made the student decide that the list was complete. The interviewee then wrote down as feedback: "I see you are using a system to find all of the orders. I noticed that your orders are coming in pairs: where the last two groups switch order before the first two groups switch order. How would you know you have all the

<p>To provide an opportunity for his fifth-grade students to use patterns to reason about whether their solution is complete, Mr. Rios assigned the following problem:</p> <p>With four schools participating in the County Honors Chorus, in how many different orders can the school groups be placed on the risers from left to right? The school groups are Talbot, Littlewood, Wyles, and Meadowbrook.</p> <p>Now that students have attempted the problem, he wants to give each student feedback that will support development of a complete list, followed by reasoning that the list is complete.</p> <p>He finds the following student solution:</p> <ol style="list-style-type: none"> 1. TLWM 2. TLMW 3. MWLT 4. MWTL 5. MTLW 6. MTWL 7. TMLW 8. TMWL 9. TWLM 10. TWML 11. LMTW 12. LMWT 13. LTWM 14. LTWM 15. LWTM 16. LWMT 17. WTLM 18. WTML 19. WMLT 20. WMTL 21. WLTM 22. WLMT 	<p><u>Questions by an interviewer</u></p> <ol style="list-style-type: none"> 1. In reviewing this student's work, what do you notice? 2. What feedback might be good to offer this student? Please write your feedback. <ul style="list-style-type: none"> • Why did you decide that? Why is that? [It is possible to ask additional questions here.] • What did you do to write this feedback? What did you consider? What did you rule out? [It is possible to ask additional questions here.] • What does the student appear to know and not to know about making a list systematically? • What might be the source of the student's limited understanding? • Did you use or consider this source in your feedback? [If so, ask] How did you use it? [If not, ask] Why didn't you use it? • Do you have any particular mathematical issues or intentions that you considered or emphasized in your feedback? How did you do that? Why are they important? • What do students need to know or what can students do in order to have a full list of arrangement of elements? • Did you use or consider this knowledge or this ability in your feedback? [If so, ask] How did you use it? [If not, ask] Why didn't you use it? • What would you expect from your feedback? What would this student actually recognize or do after getting your feedback? • [If the interviewee does not do these things, please ask them] I have several mathematics questions. What solutions were missed in the list? • How do you find all orders? How do you find all orders of four different letters? What about five letters? • How would you show that you have all arrangements of the four letters? 3. If you have 15 minutes to close the problem considering this student work, what would you like to do? <ul style="list-style-type: none"> • Why will you plan it? Why is that? • [If the interviewee did not specify the purpose of this 15-minute lesson, ask] What would be the purpose of this small lesson? • What mathematics would you like to emphasize in that lesson?
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Figure 5. One of interview prompt (permutation) developed in the study and questions by an interviewer

choices for arranging the classes?” The interviewee emphasized the importance of the student’s decision point about when he or she finished making a whole list. For an additional 15 minute-lesson, the interviewee planned to start the lesson by asking how many orders there were with W first and then L, for she expected the student to recognize there were six orders each with W and L, but only four orders with M. The interviewee would have asked, “Should there be six with M?” and “What makes you think there should be six?” And the interviewee would then let the student look at what he or she had gotten correct and what he or she had missed. Her reason for planning the lesson in this way was that there was a mistake in the middle of the list, but, at the bottom of the list, orders with W and L were listed very systemically. Because the student seemed to have a better sense of making a list with L and W, the interviewee expected that the student could figure out the regularity in orders with W and L and apply it to have a complete list.

The interviewee’s responses were analyzed in terms of the mathematical work of teaching engaged in providing written feedback. First, there were shown several sub-tasks of teaching related to providing written feedback. (i) Identifying a mathematical feature that needs to be emphasized in the situation—the interviewee immediately recognized the main mathematical topic of this situation was having a complete list to show all the orders of the four schools. Throughout the interview, although the interviewee used different terms, such as “organization,” “regularity,” “system,” and “efficient,” she emphasized a systematic way to make a complete list with any number of cases from patterns involved in this situation. (ii) Playing with ways in which the student’s work is produced from the given problem—the interviewee investigated the student’s work and recognized that there is a systemic way to make the list. The interviewee also acknowledged that this student had some sense about permutation, but not enough to make a whole list. The interviewee also reflected on what this student’s orders would be if there were two or three schools because the student might be able to make a generalizable list from the smaller example. (iii) Formulating critical feedback—rather than correcting the student’s work or directly specifying what to do next, the interviewee wanted to give something that made the student think more broadly and deeply. She also clarified two elements in her feedback. One was what the teacher noticed in the student’s work and the other was a question to help the student reflect on his or her work and reasoning and investigate and find a way to build a complete list.

The mathematical knowledge and reasoning entailed in providing written feedback were also analyzed. The interviewee was very confident that the most important thing in this situation was letting the student recognize whether or not he or she had all the orders and helping the student understand a generalizable way to make a full list from patterns involved in this situation. Her responses and comments were consistently geared toward the student. However, the interviewee did not ascertain what orders the student had missed until the interviewer asked about it. At the beginning of the interview, she briefly recognized that the orders starting with L and W were systematically listed, but those starting with T and M were not. Rather than just fixing the student’s work, the interviewee put more value on mathematical generalization and working with the student’s reasoning toward it. It was also interesting that the interviewee recognized completeness as a critical issue in this situation. However, the interviewee did not seem to know the total number of permutations using combinatorics ($4!$), but she could explain her reasoning to get a full list of orders by using a tree diagram.

The final thing examined in the analysis of this interview was distinctive mathematical knowledge and reasoning for teaching. The interviewee recognized that the last entries in the

student's list were mathematically well organized, but she did not use it directly in her feedback. Rather, she focused on reasoning related to mathematical completeness and mathematical generalization to approach this situation, and she seemed to believe that mathematics instruction needed to concentrate on them rather than on merely getting correct answers. This might be the reason why she did not rush to solve the provided problem. She investigated the student's work and tried to specify possible reasoning that the student might have used in the work. Her feedback was specific enough to make the student think about what he or she had done. In general, her sense about providing written feedback is to give the student a chance to reflect on his or her work based on what the teacher noticed in the work rather than correcting the work.

Analysis of the Interviews

Each interview was summarized and analyzed carefully by the interviewer using the following questions: How did the interviewee respond to each question of the prompt? What was the rationale for each response? What is the mathematical work of teaching as it relates to this teaching situation as suggested through this interview? What mathematical knowledge and reasoning are entailed in this work as suggested through this interview? What does this interview suggest about what might be distinctive about mathematical knowledge and reasoning for teaching? During this time, we continually revised the interview prompt so as to create with clear language realistic situations.

The summary notes were major resources for analyzing the data. During research meetings, the interviewers shared their experiences in the interviews and discussed major interests of the current research using the questions introduced in the previous paragraph. Figure 6 and Figure 7 show two summary notes, one from an interview conducted with an expert teacher and one from an interview conducted with a mathematician. One of the predominant features shown in the interview with the expert teacher was the teacher figuring out and using information from the pedagogical context. The mathematician, on the other hand, merely checked the correctness of the answer, neglecting to dig in and create feedback in the provided situation. In other words, the student work was mathematically analyzed by the expert teacher using pedagogical considerations of the teaching purpose; her analysis was greatly used to formulate feedback. In that situation, the bottom orders in the student's list constitute a critical clue that formulates feedback according to the given teaching purpose, "use patterns to reason about whether their solution is complete." Checking the correctness of the answer is insufficient to formulate feedback. This task of teaching requires complicated sub-tasks of teaching that are mathematically and pedagogically delicate and that also require comprehensive analyzing and decision making.

This research, again, does not aim to characterize the three different groups of interviewees in the context of providing written feedback. However, interviews with prospective teachers presented challenges to approaching the given situation and making a decision. Their reasoning, as revealed in the interviews, was inconsistent. Some prospective teachers only dug into the problem used in the given situation, just like learners would. Some prospective teachers solved the problem while others did not. Most of them recognized some information about the pedagogical context. Some of them used it to formulate feedback while others did not. It seems clear that one thing that is critical in providing feedback is seizing and analyzing mathematical and pedagogical information simultaneously, immediately and synthetically. It is a real challenge, however, to gather such information. Furthermore, interviewees in each group did not always respond to the prompts in the same way. In other words, mathematical, pedagogical, partial,

analytical, and synthetic tendencies and professionalism for figuring out the provided situation and making a decision were different across participants.

MKT Interview Summary Field Notes		
Date of Interview: [REDACTED] Researcher: [REDACTED] Interviewee Type: Expert Teacher Files: [REDACTED]		
Description of Response and Rationale		
Summarize the participant's responses to the prompt and its questions in the table below. Use the following questions to guide your responses 1. How did the interviewee respond to each question/request of the prompt? If short, record the response; otherwise summarize the response in a sentence or two. 2. What was the rationale for each response?		
Question/Request	Response	Rationale
In reviewing this student's work, what do you notice?	The participant immediately recognized that there is an organization to list letters. She specified that the student changed the last two over and over again as well as reordered the first two letters. She was also curious about what makes the student decide that the list is completed.	This student has a system to make a list, and the teacher needs to understand what the system is, and what regularity the student used.
What feedback might be good to offer this student? Please write your feedback.	I see you are using a system to find all of the orders. I noticed that your orders are coming in pairs: where the last two groups switch order before the first two groups switch order. How would you know you have all the choices for arranging the classes?	The participant identified that the feedback consists of two parts: what the teacher notices in the student's work and getting information about the whole process of solving the problem. What she first does is says what she sees in the student's work. If her observation is different from what the student intended, the student will let her know his/her original intention, or he/she could recognize what the teacher or the other sees in his/her work. Furthermore, in this situation, it is important to know the student's decision point about when he/she
Reflection on the Mathematical Work of Teaching Reflect on the interview and consider what you may be learning in relation to the work of teaching in the teaching situation and its mathematical demands. Please record your thoughts in relation to the following three questions. 3. What is the mathematical work of teaching related to this teaching situation as suggested by this interview? What did this person do in responding (conceived as work		
of teaching)? What did the person's work suggest about the mathematical work of teaching? (one paragraph) (i) Identifying a mathematical feature that needs to be emphasized in the situation: The participant immediately recognized the main mathematical topic of this situation is having a complete list to show all orders of four schools. Throughout the interview, although the participant used different terms, such as organization, regularity, system, and efficient, she emphasized a systematic way to make a complete list with any number of cases from patterns involved in this situation. (ii) Playing with ways in which the student's work is produced from the given problem: the participant investigated the student's work and recognized that there is a systemic way the student used to make a list. The participant also acknowledged that this student had some sense about permutation, but not enough to make a whole list. The participant also reflected that if there are two or three schools, what this student's orders would be because the student might be able to make a generalizable list. However, it does not seem that this participant critically use the possible sources of the student's misunderstanding. (iii) Formulating critical feedback: rather than correcting the student's work or directly specifying what to do next, the participant would like to give something that makes the student think more broadly and deeply. She also clarified the two elements as feedback. One is what the teacher noticed in the student's work and the other is a question to help the student reflect on his/her work and reasoning and investigate and find a way to build a complete list. 4. What mathematical knowledge and reasoning are entailed in this work as suggested by this interview? (one paragraph) The participant was very confident that the most important thing in this situation is letting the student recognize whether or not he/she had all orders and helping the student understand a generalizable way to make a full list from patterns involved in this situation. Her responses and comments were consistently geared toward them. However, the participant did not investigate exactly what orders the student missed until before the interviewer asked it. At the beginning of the interview, she briefly recognized that the orders starting with L and W are systematically listed, but not those starting with T and M. Rather than just fixing the student's work, the participant put more value on mathematical generalization and working with the student's reasoning toward it. It was also interesting that the participant recognized completeness as a critical issue in this situation. However, the participant did not seem to know the total number of combinations based on combinatorics (4!) very well, but she could explain her reasoning to get a full list of orders by using a tree diagram. 5. What does this interview suggest about what might be distinctive about mathematical knowledge and reasoning for teaching? (This might be about something the interviewee evidenced, or about something noticeably absent or different from what would be expected.) (one paragraph) The participant exactly recognized that the last entries in the student's list are mathematically well organized, and she did not use it directly in her feedback. Rather, she focused on reasoning related to mathematical completeness and mathematical		
If you have 15 minutes to close the problem considering this student work, what would you like to do?	The lesson would start to ask how many orders are there with W first and then L. After the student understands that there are six orders each with W and L, the student might recognize there are only four orders with M. If so, the participant would like to ask "should there be six with M?" and "what makes you think there should be six?" And the participant then lets the student look at what he/she has and what he/she missed. The participant might also use the smaller numbers of schools so that the student can see patterns easily rather than bigger numbers of cases.	finished making a whole list. There is a mistake in the middle of the list. But, at the bottom of the list, orders with W and L are listed very systematically. Because the student seemed to have a better sense at making a list with L and W, the participant expects that the student can figure out the regularity in orders with W and L and recognize regularity to have a complete list. The problem here is how many different orders can the school groups be placed, so it is important to help the student recognize regularity to build whole orders. The purpose of this instructional situation is helping students find all the different ways to be placed. But, the purpose of this small lesson would be reflecting the student's work in order to help him/her see the regularity, use it, and clarify the system to build a whole list efficiently. Regularity of the system that includes all possibilities helps the student find all of them.

Figure 6. Example of summary notes of one expert teacher

MKT Interview Summary Field Notes		
Date of Interview: [REDACTED] Researcher: [REDACTED] Interviewee Type: Mathematician Files: [REDACTED]		
Description of Response and Rationale		
Summarize the participant's responses to the prompt and its questions in the table below. Use the following questions to guide your responses 1. How did the interviewee respond to each question/request of the prompt? If short, record the response; otherwise summarize the response in a sentence or two. 2. What was the rationale for each response?		
Question/Request	Response	Rationale
1. In reviewing this student's work, what do you notice?	I would take all the Ts. And then, the others are systematically ordered. Some organized ways are tried to make orders. I like an organized way to approach the problem. ... There are Ts until 10 th . I can see all Ts.	I don't know how to handle it. ... Understanding what the question is always the hardest part. If there is an error, where does it come from? (And then, responded to this question)
2. What feedback might be good to offer this student? Please write your feedback.	Go back and check this out. ... Look systematically to work this out. ... (After responding for the question about what this student knows, the interviewer ask again this question) Try to do an organizing way. All the possibility to do with Ts at the first. That's all you have to do. Are there anything that does not fit? Are there anything that you missed?	I don't know what to do. I don't want to interrupt. This should be 24, not 22. I would not ask a formula. I don't want to say binomial. I think this student know organizing all the Ws in the first column, but not the other ones. Ls are pretty good, but not the Ts. Because of the lack of confidence, this student missed the orders. ... Let students play with the problem rather than teaching how to solve it. I expect that the student is confident to solve the problem quickly.
3. If you have 15 minutes to close the problem considering this student work, what would	I don't know. I really do not know. ... I might go through with students. But, let students	This is a so detailed question. ... I don't want to say if you do something, it
5. What does this interview suggest about what might be distinctive about mathematical knowledge and reasoning for teaching? (This might be about something the interviewee evidenced, or about something noticeably absent or different from what would be expected.) (one paragraph) Catching up the teaching purpose in the provided situation, analyzing the student work, noticing the student error and using student work to formulate feedback may be a distinctive feature of mathematical knowledge and reasoning for teaching. Furthermore, this knowledge may contain that the idea about a systemic approach is part of the content that needs to be taught to students. When asked about giving another problem, he agreed using another problems, but did not come up with appropriate problems (easy, difficult, complex, etc.). This seems to suggest that being aware of appropriate additional problems is distinctive about knowledge for teaching.		

you like to do?	do this problem and argue each other. That would be wonderful. ... I might have some conversation with students, but I don't know.	would be come up. ... I don't know because I have never been there. I don't have any strong opinion. ... I would do with similar problems. I mean examples are wonderful. (The interview asks "Do you have any particular example?") No. I don't have any now. (The interviewer asks "Would you use complex or easy one?") I don't know.
Reflection on the Mathematical Work of Teaching		
Reflect on the interview and consider what you may be learning in relation to the work of teaching in the teaching situation and its mathematical demands. Please record your thoughts in relation to the following three questions. 3. What is the mathematical work of teaching related to this teaching situation as suggested by this interview? What did this person do in responding (conceived as work of teaching)? What did the person's work suggest about the mathematical work of teaching? (one paragraph) The interviewee seems to immediately recognize 22 orders is a wrong answer. However, analyzing student work or creating specific feedback were not shown. He strongly believed the power of students' argument rather than teaching particularly how to solve the problem, but did not provide any guidance to help the student move forward in his/her thinking. The teacher must recognize the error that the student is making. They must next be able to attribute the error to the student's thinking or reasoning. They must then be able to provide feedback that addresses the error in thinking or reasoning in a way that is actionable by the student. Several of these pieces were missing in this participant's response. For example, in his feedback, this participant asked the student to go back and check again, but did not provide any guidance to help the student move forward in their thinking.		
4. What mathematical knowledge and reasoning are entailed in this work as suggested by this interview? (one paragraph) The interviewee thought that 22 orders was incorrect and the orders were not well organized in the top ones. However, he did not try to figure out which orders the student was missing. Part of the mathematical work of teaching seems to be finding the correct solution. It seems like the mathematical knowledge and reasoning involved in this work includes being able to determine the understanding underlying a mistake and connect it to the student's broader thinking and reasoning.		

Figure 7. Example of summary notes of one mathematician

Based on investigating the distinctive features of mathematical knowledge and reasoning for teaching as it relates to providing written feedback using the interviews with the three different groups, my colleagues and I were able to build up our understanding of the main features of MKT entailed in providing written feedback. Throughout the analysis and discussion

of summary notes, features related to confident and professional reasoning and performance in providing feedback were discovered. The first feature concerned whether or not interviewees had similar teaching experiences to those presented in the situation provided, showing that coherent and logical reasoning exists for approaching and examining provided instructional situations and making a decision based on the teaching purposes embedded in the situation. Confidence about such reasoning was critical as well. The teacher's confidence seemed to determine how well he or she explained the rationales of the analysis and decision making.

The second feature was the explicit recognition that providing feedback aimed to help extend students' understanding by offering specific advice rather than correcting their responses. The appropriateness of specific comments was also considered because comments that were too particular could reduce the chances that students would improve their response. Comments that were too vague offered no help to students.

The third feature was recognizing the instructional purpose in the provided situation and sticking to that purpose while investigating information, formulating feedback and specifying the rationales of such feedback. For example, the teaching purpose in the prompt shown in Figure 5 is to provide "an opportunity for his fifth-grade students to use patterns to reason about whether their solution is complete." This purpose includes pedagogical concerns (e.g., providing an opportunity for fifth-grade students) and mathematical topics (e.g., patterns and mathematical completeness). Immediate recognition of the purpose is prominent at the beginning of approaching the provided situation, and this recognition works critically throughout the providing of feedback.

The fourth feature is analytically and synthetically recognizing information from the pedagogical context. Figuring out and mathematically analyzing student work was critical. This task includes probing the reasoning behind the student work and having reasonable assumptions to explain what gives rise to students' wrong responses. Furthermore, other tasks related to recognizing information included looking for instructional resources, evaluating them to formulate feedback, and deciding which one should be used to create feedback. Identifying mathematical content areas related to the provided mathematical topic was important, too.

Based on the features discovered in the providing of feedback on a task of teaching, my colleagues and I have tried to develop a consistent and logical framework that illustrates proficiency in providing feedback in terms of sub-tasks of teaching. In the process of developing the framework, different domains of MKT used in the particular task emerged. Sub-domains of MKT are categorized by different features of "mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (Ball et al., 2008, p. 399). For example, CCK is the mathematical knowledge and skill used in settings other than teaching, such as "simply calculating an answer or, more generally, correctly solving mathematics problems" (Ball et al., 2008, p. 399). CCK seemed dominant in the mathematician's reasoning in the situation of formulating feedback while SCK, KCT and KCS seemed prevalent in the expert teachers' reasoning. More interestingly, while SCK seemed to be extensively implemented to investigate the mathematics embedded in the provided situation, KCT and KCS seemed to be at work when figuring out the pedagogical context and making a decision regarding the student in the provided situation. CCK seemed to function among SCK and KCS, but primarily garnered attention in identifying that the sub-domains of MKT overlap, through consecutive sub-tasks of teaching.

It was also discovered that different aspects of pedagogical context function across different tasks of teaching. For example, identifying the instructional purpose was closely related to teaching purpose and mathematical topic, and investigating student work was connected to mathematical topic. Student background, such as fifth-grade students, was significant to formulate feedback.

The mathematical work associated with the task of providing written feedback was iteratively analyzed using the interviews as empirical grounding, as explained in this section. The current study used an interactional conceptualization of teaching, as described in Cohen, Raudenbush, and Ball (2003) and a practice-based conceptualization of MKT by Ball and Bass (2003). In other words, rather than trying to describe the MKT held by interviewees or the particular ways in which they reasoned about the prompts, I was trying to use the ways in which they reasoned about the prompts to characterize professionally defensible knowledge and practice. The research questions for the larger study are as follows.

1. What is the mathematical work of teaching in unexamined areas of practice?
 - a. areas perhaps not readily studied using video
 - b. specific practices not previously studied
 - c. key areas with distinctive MKT demands (such as impromptu talk)
2. What mathematical knowledge and reasoning are entailed in this work?
3. What is distinctive about mathematical knowledge and reasoning for teaching?
4. What are key features of the design of prompts and interview methods for uncovering MKT?

The methods of analysis for the study reported here are consistent with the job analysis and conceptualization described by Thames (2009), but are applied to the interview data rather than to video.

Revision of the Prompts

Based on the analysis of the interview, the prompts were both revised and analyzed. The first major question—what does an interviewee notice first—was written in the document. It was decided, however, that the question should be removed and asked orally. The printed question seemed to force the interviewee to focus on noticing things within the provided situation rather than allowing the interviewee to get started in the given situation and find whatever the interviewee happened to pick up on. Moreover, conducting interviews helps the project team examine the prompts, specifically how providing written feedback is decomposed, what mathematical demands are entailed in each task of teaching, and which elements of pedagogical contexts function in each task of teaching. Although the prompts to investigate MKT were developed by the author and the project team, conducting and analyzing the interviews with the developed prompts helped extend our understanding of MKT as well. The characteristics found by the iterative process are specified in the following section.

What is Entailed in Providing Written Feedback

From the iterative and synthetic analysis of the interviews and the interview prompts, it was found that skillfully providing students with written feedback requires teachers to draw on purpose and relevant information given in the pedagogical context and flexibly use knowledge resources across different domains of MKT. MKT seems to involve an ability to recognize important and adequate information about pedagogical context in teaching to make a decision about which actions to perform. To characterize this distinctive knowledge and reasoning with

competent performance of the task, this study uses three features: tasks of teaching, pedagogical context, and domains of MKT.

Sub-Tasks of Teaching

To develop a specification of the mathematical demands of providing feedback, the sub-tasks of teaching entailed in responding to the prompt were analyzed first. Specifying the work of teaching mathematics is critical to both characterize teaching in terms of the dual foci of mathematics and instruction that Ball (1993) specified as the nature of teaching mathematics. Providing written feedback is not just one simple task, but includes several combined tasks integral to developing feedback. Characterizing the work of teaching is a critical step in identifying MKT, as is specifying what this work is and what it requires. I found competently providing feedback, expressed in general terms, involves four sub-tasks:

1. tracking on the instructional purpose of the problem and/or when in the students' learning trajectory the problem is being used, and what that implies for the mathematical territory of the problem;
2. making sense of the student work in relation to the instructional goals, the mathematical structure and territory of the problem, and the multiple ways that the problem can be approached;
3. identifying resources in the student work in relation to helping the student recognize the need for further work and have a way to make further progress;
4. deciding on a clear aim for the feedback and using the resources to design feedback that supports students in being able to work toward the instructional goal.

These sub-tasks tend to unfold in a linear fashion, but each sub-task includes a non-linear fashion of tasks. Furthermore, the sub-tasks may unfold flexibly and may cycle. These can also be expressed in more particular terms for the given scenario in each specific prompt.⁵ This general description of providing written feedback characterizes that the tasks of teaching include the providing of written feedback.

Role of Pedagogical Context

As previously identified, one component that is salient to reasoning in teaching practice is pedagogical context. My analysis suggests four elements that can support MKT reasoning about providing feedback. These elements consist of the instructional purpose given in the scenario, the mathematical topic discussed in the provided instruction, instructional resources provided to support instruction, which can be used in instruction, and student background, which can offer information for feedback. These elements of pedagogical context provide support for reasoning about providing feedback. They might also be suited for other tasks of teaching.

Another issue is that these four elements of pedagogical context do not operate in all four sub-tasks simultaneously. In other words, there are targeted or untargeted elements in each aspect of the work. In the first sub-task, teaching purpose and mathematical topic are used to identify the instructional purpose and to consider the mathematical structure. Second, the mathematical topic and instructional resources are used to draw a map between the student work and the original problem and resources used in instruction. Teaching purpose is briefly used to interpret how well the student work matches the teaching purpose. Third, mathematical topic and instructional resources are reconsidered, but their uses are different. Here, they are used so as to enable the student to recognize the need for further work. All elements are used in different

moments in the different sub-tasks, which are elaborated in the mathematical work of teaching, to formulate feedback. In particular, the instructional purpose provides direction for feedback.

Overlapping Sub-Domains of MKT

In parallel to the analysis of elements of pedagogical context, I examined each sub-task in terms of the domains of MKT. Each sub-task involves different demands in relation to MKT's sub-domains. First, to recognize the mathematical structure and identify resources, SCK and KCT play a role in identifying the instructional purpose, establishing core ideas of the provided problem, and determining which resources were mathematically used. The second sub-task entails the use of SCK, CCK, and KCS to identify relationships between the problem focused on and student work, solve the problem, identify which resources were used and how they were used in student responses, and generate possible reasons for the student work. The third sub-task requires SCK to find resources used to suggest ways to make further progress. The fourth sub-task involves SCK, KCS and KCT to encourage the student to review and develop her or his work. Providing written feedback entails SCK, CCK, KCS and KCT, though the use of these sub-domains shifts across its subtasks. SCK plays a major role, but CCK, KCS and KCT are also critical in providing feedback.

Figure 8 synthetically provides a view of the distinctive character of competent performance of the task conceptualized in relation to three basic features: the tasks of teaching, the pedagogical context, and the domains of MKT. This diagram, which is chronological from top to bottom, has a major axis with the sub-tasks of teaching for providing written feedback. Each task also entails different elements of the pedagogical context, which are represented by

Figure 8. Proficiency in providing written feedback

dark or light gray rectangles depending on the targeted or untargeted elements. Furthermore, this diagram shows which different domains of MKT functions in each task of teaching, which are represented by abbreviations in the arcs.

Discussion

This study aims to investigate the ways in which interview prompts are developed and used to provide the character of MKT. It also explores what is entailed mathematically and pedagogically in providing written feedback to demonstrate this type of method for studying MKT. From the analysis of the interview prompts and the interviews conducted with people who have different experiences related to teaching mathematics, the current study sets up the three interrelated aspects of mathematical demands: general description of sub-tasks of teaching entailed in providing written feedback, multiple and selective use of elements of pedagogical context in each sub-task, and continuous and simultaneous functions of different domains of MKT. Furthermore, this study specified how the interview prompts were used as a comprehensive way of studying MKT. Based on the analysis above, two issues have emerged related to investigating MKT and aspects of MKT.

In both generating and modifying prompts and probing interviews, prominence is given to the bidirectional approaches to investigating MKT. Developing prompts first aimed at creating situations that provided space for interviewees to organically provide written feedback. The development made certain assumptions about what providing written feedback involves, how interviewees might respond to prepared questions, and what reasoning would prevail. Because interview prompts touch on specific mathematical areas, the mathematical features embedded in the prompts were also carefully considered, including the mathematical facts, practices, and reasoning that would be related to teaching and learning in these particular instructional situations. Developing interview prompts entails analyzing the prompts themselves with such questions. Hypothesized ways of addressing the developed situations mathematically and pedagogically were set up and used to trim and elaborate the prompts. The prompts were designed for the interviews, but the process of creating the prompts also shed light on the instructional situations embedded in the prompt.

Conducting interviews aims to examine reasoning used in a given situation. Interviews with the three groups of people offered a sense of the pedagogical and mathematical reasoning at play in the given instructional situation. Although possible responses were prepared when developing the interview prompts, the interviews provided the project team with both a finer sense of how to revise the interview prompts and a better awareness of aspects of MKT entailed in providing written feedback than merely the step of developing the prompts. Conducting and analyzing interviews enabled the project team to scrutinize the interview prompts and trim them extensively. Going back and forth between creating and revising interview prompts and conducting and probing interviews ultimately unveiled the distinctive characteristics of MKT. This bidirectionality is also used as a way to design items and examine teaching (Jacobson, Remillard, Hoover, & Aaron, in press), and a common feature of MKT research. An interview is an efficient way of targeting and investigating a particular work of teaching in studying MKT. Creating the situation for an interview, however, requires intentional effort unlike the use of video clips, which show instruction clearly without requiring effort to create certain instructional situations. In this sense, the use of interviews in this study is not typical to MKT research and requires close attention to the bidirectionality between developing interview prompts and analyzing interviews.

The second major claim made by this study is that the features of MKT are continuous and simultaneous rather than separate and isolated. Ball (1993) described the nature of pedagogical deliberations and claimed to have understood more about the processes of pedagogical and mathematical deliberation in teaching mathematics. Shulman (1987) also claimed to have followed the process of pedagogical reasoning and actions that is used in teaching. He emphasized the continuity of the process of reasoning with the identification that “pedagogical reasoning and action involve a cycle through the activities of comprehension, transformation, instruction, evaluation, and reflection” (p. 14). These researchers pointed out the process in reasoning and performance engaged in teaching. The current study claims not only continuity in each of three different features but also simultaneousness among them.

To specify the complicated nature of teaching as an elaborated process in terms of tasks of teaching, domains of MKT, and pedagogical context, the current study suggests the three features as a way of describing the deliberation for professional and competent performance of providing written feedback, and to conceptually elaborate MKT. However, this does not mean that the three features operate independently. The three features are closely related to moment-to-moment teaching practice because the deliberations in teaching practice are continuous. Furthermore, elements of each feature functions continuously rather than separately or absently. Sub-tasks of teaching are continuously performed and elements of pedagogical contexts are always functioning. Different domains of MKT functions continuously and simultaneously. Different elements of the pedagogical context are continuously drawn on. This means that each feature includes continuity as well.

Teaching is intricate (Ball & Forzani, 2009). Therefore, investigating MKT requires a careful, analytic, and clear method to deepen and widen the study of nature of MKT by researchers interested in it. Also, an insightful lens is needed to scrutinize its intricacies and clarify the distinctive features of teaching mathematics in terms of MKT. Bidirectionality between the interview prompts and the interview as an approach to researching MKT offers a microscope by which we can discern the critical characters of MKT and teaching. As analysis using the bidirectional approach, the current study also claims that continuity and simultaneousness are major features among the three features used in this study to conceptualize the competent performance of providing written feedback. However, one of major limitations of the current study is that the method for studying MKT illustrated here requires use and validation by multiple groups of researchers to get universality as a robust method for studying MKT.

Conclusions

This study has focused on providing written feedback and investigating key features in the design of prompts for uncovering MKT. This study argued that three dimensions—the decomposition of the task of providing feedback, elements of the pedagogical context, and sub-domains of MKT—are useful in characterizing the distinctive MKT reasoning involved in providing feedback and in designing pedagogical scenarios that can be used as part of a wide variety of tools for engaging, studying, and measuring MKT. This study has found the conceptual tools described here helpful across other datasets but there is a need to continue the analysis across other tasks of teaching and other data sources to both test and refine these initial ideas.

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Footnotes

¹ Their items mainly concentrated on the following practices of teaching in algebra and geometry for elementary levels: evaluating understanding, choosing examples – illustrating a concept, evaluating explanations, choosing representations, evaluating difficulty, choosing examples – selecting a problem for an exercise, and figuring out non-standard work.

² I would like to acknowledge the contribution of the MTLT project, particularly, Yvonne Lai, Erik Jacobson, and Mark Hoover for outlining the foundation of this work.

³ Here is the link: <http://teachingworks.soe.umich.edu/work-of-teaching/high-leverage-practices>

⁴ I would like to thank the MTLT project, particularly, Rachel Snider, Lindsey Mann, Joy Johnson, and Mark Hoover, for conducting interviews.

⁵ Appendix shows particular description engaged in the interview prompts shown in Figure 5.

Appendix

Particular description of the work of providing written feedback in the interview prompt (permutation), shown in Figure 5

1. Tracking on the purpose of engaging students in using patterns to reason about the completeness of a solution to a permutation problem.
2. Working through the student solution, imagining how the student was likely thinking, and noticing that the student: (i) begins by swapping the two rightmost characters, then swapping the first two leftmost characters with the two rightmost and again swapping the two rightmost, then continuing to look for a new sequence on which to swap without repeating; (ii) shifts to the more systematic approach of fixing the first and recursively generating all swaps on the remaining three characters starting with the 11th ordering; and (iii) has missed two orderings beginning with “M” because the initial pattern did not have a systematic approach to determining the first two characters, but has been quite orderly and careful throughout.
3. Identifying that: (i) the solution is missing two orderings or that “M” is the start of only 4 orderings while the other letters have 6 and (ii) the second half of the list (for orders beginning with “L” and “W”) competently uses a powerful standard system for finding all orderings. In addition, recognizing that the student’s approach for the second half can be used to systematically list all solutions and reason that you have them all.
4. Deciding to have the student use the work in the second half of the list to reconsider or redo the work in the first half. Further, using the two missing orderings to get the student to realize the need for further work and drawing the student’s attention to the pattern in the second half of the list and the idea of using that pattern to make a complete list and reason that it is complete.