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## **Aesthetics in School Mathematics: A Potential Model and A Possible Lesson**

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**Abstract:** Earlier studies on improving classroom practice in mathematics have suggested a closer attention to nurturing an aesthetic appreciation for mathematics in students' learning experiences. Recent evidence nonetheless reveals little indication of its presence. This article offers a potential model of the case for aesthetics in school mathematics. Central to this model is the harmonious hierarchy of necessity, existence, and uniqueness without any of which the case for aesthetics in student learning might be suboptimal, if not untenable. This article offers an example of the proposed model using a possible lesson designed to engage students aesthetically in the learning of mathematics. Pedagogical implications are discussed to reflect and revisit an interpretation of learning mathematics through problem solving.

**Keywords:** mathematical aesthetics, mathematics problem-solving, teaching and learning in mathematics

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## **Introduction**

Earlier studies on improving classroom practice in mathematics have suggested a closer attention to nurturing an aesthetic appreciation for mathematics in students' learning experiences (Krutetskii, 1976; Papert, 1980; Silver & Metzger, 1989; Smith, 1927; Sriraman, 2009). Recent evidence nonetheless reveals little indication of its presence (Dreyfus & Eisenberg, 1986; Tjoe, 2015). We discuss in this article how current considerations of aesthetics in school mathematics, if any, might have inadvertently emphasized perfunctory precision over creative process. Given its current state, we argue how aesthetics can evolve into a compelling case in school mathematics.

We begin with a survey of the notions of mathematical aesthetics and its interpretations. We present a typical contemporary classroom episode of a first grade mathematics lesson in one- and two-digit addition. We explain how exposing students to such a lesson might overlook the opportunity to reveal and foster an aesthetic appreciation for mathematics. We then offer a potential model of the case for aesthetics in school mathematics. Central to this model is the harmonious hierarchy of necessity, existence, and uniqueness without any of which the case for aesthetics in student learning might be suboptimal, if not untenable. We exemplify our model with a possible lesson designed to engage students aesthetically in the learning of mathematics. Pedagogical implications are discussed to reflect and revisit an interpretation of learning mathematics through problem solving.

## **Mathematical Aesthetics**

Aesthetics has been one of the driving forces behind the activities that gave life to the advancements in mathematics as a discipline (Davis & Hersh, 1981). Its subtlety creates guidelines that many research mathematicians follow as one of the foremost principles in their

professions. It is in the search of mathematical beauty that research mathematicians often seek approvals that lead to the crowning achievement in their mathematical experience (Hardy, 1940).

Sinclair (2004) analyzes the role of aesthetic values from several conceptual insights. She draws examples from existing empirical findings such as those by Dreyfus and Eisenberg (1986) and Silver and Metzger (1989). In one of her interpretations of their work, she suggests that “mathematicians’ aesthetic choices might be at least partially learned from their community as they interact with other mathematicians and seek their approval” (Sinclair, 2004, p. 276). Furthermore, she indicates that mathematical beauty is only feasible in the process “when young mathematicians are having to join the community of professional mathematicians—and when aesthetic considerations are recognized (unlike at high school and undergraduate levels)” (p. 276).

Related to Sinclair’s (2004) interpretations of mathematical aesthetics, Karp (2008) conducts a comparative study on the aesthetic aspect of mathematical problem solving. Karp’s comparative study involved middle and high school mathematics teachers from the U.S. and Russia. In his study, teachers are asked to provide examples and explanations of “beautiful” mathematics problems and approaches in solving those problems. Karp’s (2008) findings confirm that the curricular system of education has a tremendous impact on students’ aesthetic preference in mathematics problem solving. Each group of teachers shows different perspectives on what count as mathematical “beauty.” In particular, these differences stand out from their selections of mathematics topics. American teachers put extra weight on mathematics topics as prescribed by the American curriculum, which is typically associated with real-life situations and applications. Russian teachers do likewise as recommended by Russian curriculum with its traditionally heavy emphasis on algebra, number theory, and geometry. Evidently, these Russian

problems tend to require longer approaches and are more algebraically demanding than their American counterparts. In their explanations, American teachers describe “usefulness in the teaching process,” “useful[ness] in practical life or comes the real world,” “non-standard and cannot be solved using ordinary methods that are regularly discussed in school,” “unexpectedness of the solution,” “openness of the problem,” and “a combination of methods and knowledge from different fields of mathematics” (Karp, 2008, p. 40). Russian teachers reveal in their choices of problems and solutions the sense of “overcoming of chaos,” “non-standard nature,” and “traditional fields” in their origins (p. 40). In his conclusion, Karp indicates a relative character of aesthetic preference in mathematics problem solving.

Apart from the curricular system of education, the context of cultural differences has also been observed in understanding mathematical aesthetics through classroom learning or professional experiences. Tjoe (2015) examines the hypothesis as to whether there exists a gap between different problem solvers in the criteria that might be attributed to the interpretation of mathematical “beauty.” Tjoe’s study involves expert mathematicians at the university level and mathematically gifted students at the high school level. In his study, research mathematicians and mathematically gifted students are asked to choose their most preferred approach as they are presented with a collection of many different problem solving approaches. Tjoe’s (2015) findings reveal that whereas expert mathematicians associate “beautiful” approaches with their simplicity and originality in the search for geometric reasoning or explanation, mathematically gifted students strive for the economic attribute of mathematical “beauty” in those approaches that involve fewer steps and shorter solving time. When both groups’ divergent choices of preferred approaches as well as their dissimilar interpretations of mathematical “beauty” are made known to each other, it is evident that these opposing views are not construed from their

mathematical content knowledge, but rather from their appeals of “beauty” based on their mathematics experiences (Tjoe, 2015). Tjoe suggests that “there appeared to be a profound lacuna in the understanding of mathematical aesthetics that might inadvertently subdivide the state of mathematically gifted into two groups: one group of professional research mathematicians and another group of those whose affects might be waiting to be nurtured” (Tjoe, 2015, p. 173). Given mathematical aesthetics is not a characteristic that problem solvers are born with, it is possible that one must learn to instill such habit in their everyday life as professional mathematicians. This possibility might further indicate that mathematical aesthetics is a socially constructed concept, or that mathematical aesthetics has found its decline in its inclusion in the teaching and learning of mathematics at the elementary and secondary levels.

### **A Lesson in One- and Two-digit Addition**

The Common Core State Standards Initiative describes perseverance in problem solving as one of the most important standards for mathematical practice. Students are encouraged to “check their answers to problems using a different method,” “understand the approaches of others,” and “identify correspondences between different approaches” (Common Core State Standards Initiative, 2010, p. 6). With this in mind, first grade students are expected to be proficient in varieties of ways of solving one- and two-digit addition problems (CCSS.MATH.CONTENT.1.OA.C.5-6, 1.NBT.C.4). We include in this section some of the ways that one- or two-digit addition problems (e.g.,  $5 + 6$  and  $46 + 38$ ) are typically presented in the common core aligned mathematics textbooks at the first grade level.

The most elementary way of solving one-digit addition problems is the counting via direct modeling of objects or fingers: 1) counting all (e.g.,  $5 + 6$  is solved by counting aloud one, two, three, four, five (five), ..., six, seven, eight, nine, ten, eleven (six)), 2) counting on from the

first addend (e.g.,  $5 + 6$  is solved by counting aloud five (five), ..., six, seven, eight, nine, ten, eleven (six)), and 3) counting on from the larger addend (e.g.,  $5 + 6$  is solved by counting aloud six (six), ..., seven, eight, nine, ten, eleven (five)). The next method of solving one-digit addition problems after counting via objects or fingers is to count abstractly without objects or fingers. As one moves away from relating counting to addition toward the idea of addition through reasoning, one learns to derive addition by recalling known addition facts (e.g.,  $5 + 6 = 11$  because knowing  $5 + 5 = 10$  and  $6 = 5 + 1$ , one obtains  $5 + 6 = 5 + 5 + 1 = 11$ ) or by using commutative property of addition (e.g.,  $5 + 6 = 11$  because  $6 + 5 = 11$  and  $5 + 6 = 6 + 5$  so  $5 + 6 = 11$ ). Finally, students are to perform mental math addition via retrieval from long-term memory so that one can instantly solve  $5 + 6 = 11$  on the spot.

One way to solve two-digit addition problems is to add the tens and the ones separately, and combine them (e.g.,  $46 + 38 = 84$  because  $40 + 30 = 70$  and  $6 + 8 = 14$  so  $70 + 14 = 84$ ). Another way to solve two-digit addition problems is to add on the tens followed by adding on the ones (e.g.,  $46 + 38 = 84$  because  $46 + 10 + 10 + 10 = 76$  and  $76 + 4 + 4 = 84$ ). One can also decompose an addend to make the tens (e.g.,  $46 + 38 = 84$  because  $46 + 38 = (44 + 2) + 38 = 44 + (2 + 38) = 44 + 40 = 84$ ) or compensate another addend to make the tens (e.g.,  $46 + 38 = 84$  because  $46 + 38 = 46 + (40 - 2) = (46 + 40) - 2 = 86 - 2 = 84$ ). A culminating point of addition lesson concludes with a traditional vertical algorithm for addition that sometimes commits carrying over to rote memorization (e.g.,  $46 + 38 = 84$  because  $46 + 38 = (40 + 6) + (30 + 8) = (40 + 30) + (6 + 8) = (40 + 30) + 14 = (40 + 30) + (10 + 4) = (40 + 30 + 10) + 4 = 80 + 4 = 84$ ).

These different ways of solving one- and two-digit addition problems often wear out many teachers who eventually overemphasize to their students the memory retrieval for one-digit addition problems or the vertical addition algorithm for two-digit addition problems. A popular

explanation to the overemphasis of either addition method is its *convenience*. Indeed, a number of studies demonstrate that not all of these addition strategies receive equally extensive utilization outside the teaching and learning settings (Geary & Brown, 1991; Geary & Wiley, 1991; Siegler & Shrager, 1984). Attributing their strategy choice to the most economical way of arriving at the answers, more experienced problem solvers, for instance, depend heavily on reasoning (e.g., memory retrieval) as opposed to counting to solve addition problems.

Unfortunately, such a *convenient* way of teaching the most economical approach is to a certain degree one cause of an *inconvenient* way of learning to appreciate mathematical aesthetics. Without communicating to students to help them reveal the power, usefulness, and beauty of the addition methods one after and over the other, learning mathematics might become a mere tool of pedantic precision, instead of a creative journey of problem solving. In this manner, young problem solvers might mistakenly perceive the kind of mathematics activities that research mathematicians conduct in their profession life as an impetuous act of conduct, instead of an inculcative habit of mind. It is thus essential for teachers to inform their students of the fact that the vertical addition algorithm does not simply materialize from the work of research mathematicians in the form that may be found in current mathematics textbooks. Students should also be acquainted with the aesthetics principle that directly guides the process in which research mathematicians compose, frame, and identify the vertical addition algorithm, among many other algorithms, to be what may now be considered the standard algorithm for any addition problems.

### **A Model of Aesthetics in School Mathematics**

One way to help students to understand the real process that transform the vertical addition algorithm into the standard algorithm is to engage them in a similar experience that help

define and refine the criteria of a standard algorithm from the point of view of mathematical aesthetics (Silver & Metzger, 1989). In this section, we propose a possible model of the case for aesthetics in school mathematics that integrates necessity, existence, and uniqueness. We describe the accounts of this model using backward explanations.

We maintain that our end goal is to nurture the feelings of mathematical aesthetics among our students. In the context of problem solving process, mathematical aesthetics is often identified through the uniqueness in which a problem has been approached. At one point in the problem solving process, in particular, after a problem has been successfully solved, solvers are to be able to recognize that there exists a unique problem solving approach that is preferred using a certain criteria.

On the one hand, such criteria of preference depend greatly on solvers' mathematical experience and knowledge. On the other hand, solvers can only accumulate a series of meaningful mathematical experience and knowledge, and therefore prefer a unique problem solving approach, when there is more than one problem solving approach to choose from. In other words, the existence of many different problem solving approaches is a necessary condition for the uniqueness of such problem solving approach, and perhaps more crucially, the uniqueness of the criteria which help guide solvers to prefer one problem solving approach to another. If there were only one solution method to approach a problem, then there would be no other solution method to compare with, and there would not be a need to prefer one solution method to the others. In this case, it would be a difficult effort, if not a futile one, to convince students who learn only one solution method to solve the problem that such a solution method in fact entails a great deal of aesthetics values.

Although the existence of many different solution methods can be viewed as a means to facilitate aesthetics appreciations toward the most “beautiful” problem solving approach, there needs to be a more pragmatic function (in addition to an affective one) that serves to explain why students need to learn more than one solution method. It is at this point that different numerical characteristics of problems with similar surface structures can be a determining factor. If mathematics instructors can demonstrate to their students that a certain solution method would work more effectively when applied to solve problems with a particular numerical characteristic, while other solution methods for problems with different numerical characteristics, then students may be better able to acknowledge that there is a need to study more than one solution method. In fact, it would be in the interest of the students to further recognize this utility to the extent that it will help them primarily to solve problems more competently and adaptively, and secondarily to gain exposure to and to practice satisfying their desire in their quest for the most “beautiful” solution methods.

Nonetheless, the order of presenting a series of different numerical characteristics of problems necessitates a careful deliberation of cognitive workload. In order for students to discover the power of certain solution methods, numerical characteristics of problems need to be reflected upon in a manner that unfolds the necessity of those solution methods. Correspondingly, the order of presenting the many different solution methods should be connected with the amount of cognitive workload demanded in each numerical characteristic of the problems in an increasing manner. As students grow their mathematical confidence in solving problems with numerical characteristics of lower cognitive workload, they may be introduced to problems with numerical characteristics of higher cognitive workload. With constant exposure to having to deal with problems with numerical characteristics of higher

cognitive workload, students may come to realize that they need some other solution methods that are more effective than the existing solution methods that normally work just fine with problems with numerical characteristics of lower cognitive workload.

In many respects, this model to align itself closely with the kind of preferential considerations involved in the problem solving experience that professional mathematicians encounter in their research work when deciding which approaches to pursue in solving a theorem, which existing theorems to prove, or which new theorems to conjecture. Likewise, such considerations can be related in the review process in which mathematics textbook authors conduct in determining which proofs, among existing ones, to include for each theorem in the textbooks. Indeed, a survey of research in the field of mathematics reveals that there are more proofs than there are theorems (Thurston, 1994). This indicates that more and more mathematicians are working on to use different approaches and perspectives to revisit and refine many theorems that have already been proved.

At the heart of the present model for aesthetics in school mathematics is the dynamic cycle of the need for more effective solution methods through different numerical characteristics of problems with similar surface structures, the existence of a collection of different solution methods through students' constructivism or teachers' presentation, and the uniqueness in the selection of preferred solution methods. It is the element of the gradual progression in time and difficulty that helps run the engine of this model. Students who respond to the need to learn problem solving using many different approaches may grow to become aware of "beautiful" solution methods. Students who experience a huge and sudden jump in the level of cognitive workload may in turn feel unmotivated to appreciate the "beautiful" solution methods. By creating the necessary condition for students to explore multiple solution methods, we facilitate a

learning environment where they can engage in classroom discourse to compare and contrast those solution methods, and eventually instill the feelings for aesthetics in their mathematics learning experience.

### **Aesthetics for a Lesson in One- and Two-digit Addition**

This section offers a concrete model of how a lesson in one- and two-digit addition might look like. We begin our lesson in addition as a transition from a lesson in counting. As students become acquainted with counting from one, two, three, and so on, our first examples in one-digit addition involve adding by ones:  $0 + 1 = 1$ ,  $1 + 1 = 2$ ,  $2 + 1 = 3$ ,  $3 + 1 = 4$ ,  $4 + 1 = 5$ ,  $5 + 1 = 6$ ,  $6 + 1 = 7$ ,  $7 + 1 = 8$ ,  $8 + 1 = 9$ , and  $9 + 1 = 10$  (see Figure 1). This set of problems is introduced first because it serves as a reminder to students that adding by ones is tantamount to listing or naming numerals sequentially. Different methods such as using manipulatives, counting via fingers, or talking aloud can also be utilized.

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

*Figure 1.* Adding whole numbers by ones.

Following adding by ones problems is an introduction to commutative property of addition. This includes problems such as  $1 + 0 = 1$ ,  $1 + 1 = 2$ ,  $1 + 2 = 3$ ,  $1 + 3 = 4$ ,  $1 + 4 = 5$ ,  $1 + 5 = 6$ ,  $1 + 6 = 7$ ,  $1 + 7 = 8$ ,  $1 + 8 = 9$ , and  $1 + 9 = 10$  (see Figure 2). This set of problems

establishes the foundation for future addition problems that students will encounter to the extent that it demonstrates the economy aspect of solution methods. Students will recognize that the order in which an addition is performed does not matter when adding two whole numbers: adding an addend to an augend is equivalent to adding an augend to an addend.

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Figure 2. Adding ones to whole numbers.

Given the first two sets of problems, we can now reintroduce to students that whole numbers can be represented through decomposition as (at least) two other (not necessarily distinguishable) whole numbers. Students can be engaged to examine the question of which whole numbers can make up different whole numbers. For example, the number zero can be represented as  $0 = 0 + 0$ ; the number one can be represented as  $1 = 0 + 1$  and  $1 = 1 + 0$ ; the number two can be represented as  $2 = 0 + 2$ ,  $2 = 1 + 1$ , and  $2 = 2 + 0$ ; the number three can be represented as  $3 = 0 + 3$ ,  $3 = 1 + 2$ ,  $3 = 2 + 1$ , and  $3 = 3 + 0$ ; the number four can be represented as  $4 = 0 + 4$ ,  $4 = 1 + 3$ ,  $4 = 2 + 2$ ,  $4 = 3 + 1$ , and  $4 = 4 + 0$ ; the number five can be represented as  $5 = 0 + 5$ ,  $5 = 1 + 4$ ,  $5 = 2 + 3$ ,  $5 = 3 + 2$ ,  $5 = 4 + 1$ , and  $5 = 5 + 0$ ; the number six can be represented as  $6 = 0 + 6$ ,  $6 = 1 + 5$ ,  $6 = 2 + 4$ ,  $6 = 3 + 3$ ,  $6 = 4 + 2$ ,  $6 = 5 + 1$ , and  $6 = 6 + 0$ ; the number seven can be represented as  $7 = 0 + 7$ ,  $7 = 1 + 6$ ,  $7 = 2 + 5$ ,  $7 = 3 + 4$ ,  $7 = 4 + 3$ ,  $7 = 5 +$

2,  $7 = 6 + 1$ , and  $7 = 7 + 0$ ; the number eight can be represented as  $8 = 0 + 8$ ,  $8 = 1 + 7$ ,  $8 = 2 + 6$ ,  $8 = 3 + 5$ ,  $8 = 4 + 4$ ,  $8 = 5 + 3$ ,  $8 = 6 + 2$ ,  $8 = 7 + 1$ , and  $8 = 8 + 0$ ; and the number nine can be represented as  $9 = 0 + 9$ ,  $9 = 1 + 8$ ,  $9 = 2 + 7$ ,  $9 = 3 + 6$ ,  $9 = 4 + 5$ ,  $9 = 5 + 4$ ,  $9 = 6 + 3$ ,  $9 = 7 + 2$ ,  $9 = 8 + 1$ , and  $9 = 9 + 0$  (see Figure 3).

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Figure 3. Adding through decomposition.

When faced with addition problems that result in one-digit whole numbers, students can apply their commutative property of addition. After some experience of using the counting on technique that starts either from the first addend or from the greater of the two addends, students can choose the latter as their preferred method because of its economical consideration.

The next set of problems involves whole numbers up to 18 that are the results of additional one-digit addends. For example, the number 10 can be represented as  $10 = 1 + 9$ ,  $10 = 2 + 8$ ,  $10 = 3 + 7$ ,  $10 = 4 + 6$ ,  $10 = 5 + 5$ ,  $10 = 6 + 4$ ,  $10 = 7 + 3$ ,  $10 = 8 + 2$ , and  $10 = 9 + 1$ ; the number 11 can be represented as  $11 = 2 + 9$ ,  $11 = 3 + 8$ ,  $11 = 4 + 7$ ,  $11 = 5 + 6$ ,  $11 = 6 + 5$ ,  $11 = 7 + 4$ ,  $11 = 8 + 3$ , and  $11 = 9 + 2$ ; the number 12 can be represented as  $12 = 3 + 9$ ,  $12 = 4 + 8$ ,  $12 = 5 + 7$ ,  $12 = 6 + 6$ ,  $12 = 7 + 5$ ,  $12 = 8 + 4$ , and  $12 = 9 + 3$ ; the number 13 can be represented as  $13 = 4 + 9$ ,  $13 = 5 + 8$ ,  $13 = 6 + 7$ ,  $13 = 7 + 6$ ,  $13 = 8 + 5$ , and  $13 = 9 + 4$ ; the number 14 can be

represented as  $14 = 5 + 9$ ,  $14 = 6 + 8$ ,  $14 = 7 + 7$ ,  $14 = 8 + 6$ , and  $14 = 9 + 5$ ; the number 15 can be represented as  $15 = 6 + 9$ ,  $15 = 7 + 8$ ,  $15 = 8 + 7$ , and  $15 = 9 + 6$ ; the number 16 can be represented as  $16 = 7 + 9$ ,  $16 = 8 + 8$ , and  $16 = 9 + 7$ ; the number 17 can be represented as  $17 = 8 + 9$ , and  $17 = 9 + 8$ ; and the number 18 can be represented as  $18 = 9 + 9$  (see Figure 4).

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Figure 4. Additional additions through decomposition.

Using the knowledge of commutative property of addition, students need to be prompted to realize that from the last two sets of the problems, many decompositions are symmetric to each other. Moreover, some students who may recognize that the number of decompositions of a whole number is one more than that whole number can be guided to think about whether this observation holds for the cases for 10 to 18 shown previously. Students need to see that one-digit addition problems will result between zero and 18. More specifically, they should also notice that the maximum sum of two one-digit whole numbers is 18, a useful fact that will later be employed in the traditional vertical algorithm as the standard algorithm.

At this point, we expand our addition problems using two-digit whole numbers as addends. For example, in addition to the decompositions shown previously, the number 10 can be represented as  $10 = 0 + 10$ , and  $10 = 10 + 0$ ; the number 11 can be represented as  $11 = 0 + 11$ ,

$11 = 1 + 10$ ,  $11 = 10 + 1$ , and  $11 = 11 + 0$ ; the number 12 can be represented as  $12 = 0 + 12$ ,  $12 = 1 + 11$ ,  $12 = 2 + 10$ ,  $12 = 10 + 2$ ,  $12 = 11 + 1$ , and  $12 = 12 + 0$ ; the number 13 can be represented as  $13 = 0 + 13$ ,  $13 = 1 + 12$ ,  $13 = 2 + 11$ ,  $13 = 3 + 10$ ,  $13 = 10 + 3$ ,  $13 = 11 + 2$ ,  $13 = 12 + 1$ , and  $13 = 13 + 0$ ; the number 14 can be represented as  $14 = 0 + 14$ ,  $14 = 1 + 13$ ,  $14 = 2 + 12$ ,  $14 = 3 + 11$ ,  $14 = 4 + 10$ ,  $14 = 10 + 4$ ,  $14 = 11 + 3$ ,  $14 = 12 + 2$ ,  $14 = 13 + 1$ , and  $14 = 14 + 0$ ; the number 15 can be represented as  $15 = 0 + 15$ ,  $15 = 1 + 14$ ,  $15 = 2 + 13$ ,  $15 = 3 + 12$ ,  $15 = 4 + 11$ ,  $15 = 5 + 10$ ,  $15 = 10 + 5$ ,  $15 = 11 + 4$ ,  $15 = 12 + 3$ ,  $15 = 13 + 2$ ,  $15 = 14 + 1$ , and  $15 = 15 + 0$ ; the number 16 can be represented as  $16 = 0 + 16$ ,  $16 = 1 + 15$ ,  $16 = 2 + 14$ ,  $16 = 3 + 13$ ,  $16 = 4 + 12$ ,  $16 = 5 + 11$ ,  $16 = 6 + 10$ ,  $16 = 10 + 6$ ,  $16 = 11 + 5$ ,  $16 = 12 + 4$ ,  $16 = 13 + 3$ ,  $16 = 14 + 2$ ,  $16 = 15 + 1$ , and  $16 = 16 + 0$ ; the number 17 can be represented as  $17 = 0 + 17$ ,  $17 = 1 + 16$ ,  $17 = 2 + 15$ ,  $17 = 3 + 14$ ,  $17 = 4 + 13$ ,  $17 = 5 + 12$ ,  $17 = 6 + 11$ ,  $17 = 7 + 10$ ,  $17 = 10 + 7$ ,  $17 = 11 + 6$ ,  $17 = 12 + 5$ ,  $17 = 13 + 4$ ,  $17 = 14 + 3$ ,  $17 = 15 + 2$ ,  $17 = 16 + 1$ , and  $17 = 17 + 0$ ; and the number 18 can be represented as  $18 = 0 + 18$ ,  $18 = 1 + 17$ ,  $18 = 2 + 16$ ,  $18 = 3 + 15$ ,  $18 = 4 + 14$ ,  $18 = 5 + 13$ ,  $18 = 6 + 12$ ,  $18 = 7 + 11$ ,  $18 = 8 + 10$ ,  $18 = 10 + 8$ ,  $18 = 11 + 7$ ,  $18 = 12 + 6$ ,  $18 = 13 + 5$ ,  $18 = 14 + 4$ ,  $18 = 15 + 3$ ,  $18 = 16 + 2$ ,  $18 = 17 + 1$ , and  $18 = 18 + 0$  (see Figure 5).

+	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
2	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
4	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
5	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
6	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
7	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
8	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
9	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
10	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
11	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
12	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
13	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
14	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
15	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
16	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
17	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
18	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36

Figure 5. Additional additions through decomposition of two-digit whole numbers.

After a systematic introduction to addition problems involving whole numbers between zero and 18, students can be further exposed to more complex two-digit whole numbers. For problems involving two-digit whole numbers with zero in their ones place values, we begin with the strategy of counting on by tens. For example,  $10 + 10 = 20$ ,  $20 + 10 = 30$ ,  $30 + 10 = 40$ ,  $40 + 10 = 50$ ,  $50 + 10 = 60$ ,  $60 + 10 = 70$ ,  $70 + 10 = 80$ ,  $80 + 10 = 90$ , and  $90 + 10 = 100$ . This set of problems should remind students of that involving counting on by ones technique. Students can

further extrapolate adding two-digit whole numbers with zero in their ones place values has a similar pattern that one can find when adding one-digit whole numbers.

Moreover, combining the two techniques will enable students to further generalize the pattern in adding any two-digit whole numbers: when adding two-digit whole numbers, one can add the digits in the tens place values, add the digits in the ones place values, and combine the two summands. For example,  $34 + 52 = (30 + 4) + (50 + 2) = (30 + 50) + (4 + 2) = 80 + 6 = 86$ , and  $46 + 38 = (40 + 6) + (30 + 8) = (40 + 30) + (6 + 8) = 70 + 14 = 70 + (10 + 4) = (70 + 10) + 4 = 80 + 4 = 84$ . Students need to notice that the latter example involves a higher level of cognitive workload than the former one because of the resulting two-digit summand from adding the digits in the ones place values. As such, we need to be more cognizant of presenting addition problems of two-digit whole numbers so that they will progress from those involving a one-digit summand to those involving a two-digit summand as a result of adding the digits in the ones place values.

We make a note at this point that although the previous technique of adding any two-digit whole numbers involves associative property of addition, the introduction to this property is rather informal to our first grade students. In fact, students may choose not to write formal number sentences to express any of the solution methods that are described above. Formality or standardization will nevertheless become necessary for students to adapt to as they are posed with more complex addition problems such as those involving more than two two-digit whole numbers (or even those involving two more-than-two-digit whole numbers as they will perform in later grade levels).

Consequently, students can be introduced to the traditional vertical algorithm, which essentially reverse the order of the previous technique in a vertical form. For example,  $46 + 38 = (40 + 6) + (30 + 8) = (40 + 30) + (6 + 8) = (40 + 30) + (14) = (40 + 30) + (10 + 4) = (40 + 30 +$

$10) + 4 = 80 + 4 = 84$ . We point out to students that the step that involves adding 10 to  $40 + 30$  is referred to as carrying over, and that 10 is the only possible value to carry over because of the observation made earlier that the maximum sum of two one-digit whole numbers is 18. With this traditional vertical algorithm in their tool bag, students are hoped to recognize its advantage over other solution methods in terms of generalizability: this algorithm, unlike the others, will work for addition problems involving not only two one- or two-digit whole numbers, but also more than two one- or two-digit whole numbers and two more-than-two-digit whole numbers.

Some other students might call attention to its advantage in terms of efficiency in the time that it takes to solve the addition problems. To these students, we propose a situation for addition problems: Is there a more efficient way of solving the addition problem such as  $46 + 99$  than using the traditional vertical algorithm? Students might realize that decomposing 46 into 45 and that 1 will make up 100 from 1 and 99, which is quicker to solve than the decomposition method:  $46 + 99 = (45 + 1) + 99 = 45 + (1 + 99) = 45 + 100 = 145$ . Alternatively, students might recognize that compensating 99 into 100 and that 1 will need to be taken away from 46 and 100, which is again quicker to solve than the decomposition method:  $46 + 99 = 46 + (100 - 1) = (46 + 100) - 1 = 146 - 1 = 145$ . Clearly, there exists a situation where the traditional vertical algorithm may not always offer a faster solution than the decomposition method or the compensation method. And for students to understand the need for flexibility in applying more effective solution methods to appropriate addition problems, extraordinary situations as described earlier may be indispensable.

Perhaps even generalizing the decomposition and compensation methods a little further to addition problems involving more than two two-digit whole numbers, students can recognize that some digits (whole numbers) are more compatible to other digits (whole numbers) by taking

advantage of the simplicity of multiple of tens. For example, as students recognize that the digits 6 and 4 make up 10, they can associate the corresponding addends using the commutative and associative properties of addition:  $46 + 38 + 54 = 46 + (38 + 54) = 46 + (54 + 38) = (46 + 54) + 38 = 100 + 38 = 138$ .

Although first graders are only expected to perform addition to problems involving two whole numbers up to 100 (Common Core State Standards Initiative, 2010), it is perhaps worth contemplating to what extent the power and beauty of the traditional vertical algorithm as the standard algorithm, as well as some other solution methods such as decomposition or compensation method, may become unnoticed, if not optimally appreciated, in the absence of problems requiring a higher level cognitive workload. It is through this continuous process of experiencing such problems with a variety of numerical characteristics that students can become aware of the need for more effective solution methods. And it is through this need for additional solution methods that student can not only engage in the creative praxis of constructing and inventing their own solution methods, but also learn to acknowledge the simultaneous existence of multiple solution methods. It is then through this existence of many different solution methods that students can grow to analyze those solution methods more critically toward the aesthetic goals of learning in mathematics.

### **Conclusion and Discussion**

This article presents a case of aesthetics for school mathematics. In an expository approach, it aims to illustrate the possibility that aesthetics may find its presence in school mathematics through problem solving. We recognize that mathematical aesthetics may not be necessarily interpreted by the sole means of the process in which mathematicians go through in their professional career (Poincare, 1946; Hadamard, 1945). Our approach to interpret

mathematical aesthetics through problem solving is mostly influenced by the eminent call to teach mathematics through problem solving (NCTM, 2000). Using the model proposed, we consider the necessity, existence, and uniqueness of mathematically “beautiful” solution methods in student learning. It is hoped that this proposed model might create discussions in advancing research in teaching and learning mathematics.

On the one hand, the proposed model approximates the role of generating mathematical understanding from the point of view of many of the existing teaching and learning theories in mathematics education (Cobb, 2007; Hiebert & Carpenter, 1992; Lampert, 1990; Silver & Herbst, 2007; Simon, 1995). On the other hand, the proposed model operates mathematical aesthetics around the classroom settings where mathematics is conducted in the lens of abstract problem solving process as observed in the studies by Hadamard (1945) and Poincare (1946), instead of through the concrete and static appearance of problem solving solutions as observed in the studies by Krutetskii (1976) and Sinclair (2001). At the same time, our approach might be likened to constructivism to the extent that our aesthetic engagement process promotes creativity and analytical thinking through a series of solution methods and their corresponding numerical characteristics of problems with a similar surface structure. In contrast to the current curriculum, not only does this model support students’ accumulation of a generally more substantial problem solving experience, but it also incorporates acquisition of new knowledge through the creation, presence, and evaluation of many different strategies.

The present article also demonstrates one possibility of engaging first grade students aesthetically in mathematics learning of the one- and two-digit addition. A lesson specifically considers a greater amount of depth and breath of the treatment of numerical characteristics of the addends involved in the addition problems in manner that incorporates systematically and

progressively increasing cognitive workload. Accordingly, this model has a number of implications that are worth of some reflections. First, the exposition of a substantially greater series of addition problems, although valuable in creating the need for searching different problem solving approaches, should be ensured not to exhaust the mathematical excitement of the students. Teachers should be cognizant of how much exertion in the planning of a series of problems in the addition lesson may be sufficient over a given period of time. This circumstance may also create a glimpse of constriction among other topics that can be taken account of under a particular grade level. It is conceivable that certain methods of algebra are capable of solving typical calculus problems (Tjoe, 2015). Teachers need to strike a balance in maintaining how far students can or should invoke some of the most powerful approaches in algebra class to solve problems involving a lower level of cognitive workload such as those that one can find in an elementary class, or involving a higher level of cognitive workload such as those that one can find in an AP Calculus or an advanced geometry class. Finally, future studies are called for to attend to empirical findings that support the proposed model of aesthetics engagement in school mathematics (Tjoe, 2014). Such studies may particularly relate to an experiment that compares and contrasts the efficacy of such variables as numerical characteristics of problems and order of presentation of different solution methods.

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