The Mathematics Enthusiast

Volume 7 | Number 1 Article 11

1-2010

When is .999... less than 1?

Karin Usadi Katz

Mikhail G. Katz

When is .999... less than 1?

Karin Usadi Katz and Mikhail G. Katz[0]

We examine alternative interpretations of the symbol described as *nought, point, nine recurring*. Is "an infinite number of 9s" merely a figure of speech? How are such alternative interpretations related to infinite cardinalities? How are they expressed in Lightstone's "semicolon" notation? Is it possible to choose a canonical alternative interpretation? Should unital evaluation of the symbol $.999\ldots$ be inculcated in a pre-limit teaching environment? The problem of the unital evaluation is hereby examined from the pre-$\mathbb{R}$, pre-lim viewpoint of the *student*.

1. Introduction

Leading education researcher and mathematician D. Tall [63] comments that a mathematician "may think of the physical line as an *approximation* to the infinity of numbers, so that the line is a practical representation of numbers[, and] of the number line as a visual representation of a precise numerical system of decimals." Tall concludes that "this still does not alter the fact that there are connections in the minds of students based on experiences with the number line that differ from the formal theory of real numbers and cause them to feel confused."

One specific experience has proved particularly confusing to the students, namely their encounter with the evaluation of the symbol $.999\ldots$ to the standard real value 1. Such an evaluation will be henceforth referred to as the *unital evaluation*.

We have argued [32] that the students are being needlessly confused by a premature emphasis on the unital evaluation, and that their persistent intuition that $.999\ldots$ can fall short of 1, can be rigorously justified. Other interpretations (than the unital evaluation) of the symbol $.999\ldots$ are possible, that are more in line with the students' naive initial intuition, persistently reported by teachers. From this viewpoint, attempts to inculcate the equality $.999\ldots = 1$ in a teaching environment prior to the introduction of limits (as proposed in [65]), appear to be premature.

[0]Supported by the Israel Science Foundation (grants no. 84/03 and 1294/06).

To be sure, certain student intuitions are clearly dysfunctional, such as a perception that $\forall\epsilon\exists\delta$ and $\exists\delta\forall\epsilon$ are basically "the same thing". Such intuitions need to be uprooted. However, a student who intuits $.999\ldots$ as a dynamic process (see R. Ely [23, 24]) that never quite reaches its final address, is grappling with a fruitful cognitive issue at the level of the world of proceptual symbolism, see Tall [63]. The student's functional intuition can be channeled toward mastering a higher level of abstraction at a later, limits/$\mathbb{R}$ stage.

In '72, A. Harold Lightstone published a text entitled *Infinitesimals* in the *American Mathematical Monthly* [38]. If $\epsilon > 0$ is infinitesimal (see Appendix A), then $1-\epsilon$ is less than 1, and Lightstone's extended decimal expansion of $1-\epsilon$ starts with more than any finite number of repeated 9s.[1] Such a phenomenon was briefly mentioned by Sad, Teixeira, and Baldino [51, p. 286].

The symbol $.999\ldots$ is often said to possess more than any finite number of 9s. However, describing the real decimal $.999\ldots$ as possessing an *infinite number of* 9*s* is only a figure of speech, as *infinity* is not a number in standard analysis.

The above comments have provoked a series of thoughtful questions from colleagues, as illustrated below in a question and answer format perfected by Imre Lakatos in [37].

2. Frequently asked questions: when is .999... less than 1?

Question 2.1. Aren't there many standard proofs that $0.999\ldots = 1$? Since we can't have that and also $0.999\ldots \neq 1$ at the same time, if mathematics is consistent, then isn't there necessarily a flaw in your proof?

Answer. The standard proofs[2] are of course correct, in the context of the standard real numbers. However, the annals of the teaching of decimal notation are full of evidence of student frustration with the unital evaluation of $.999\ldots$ This does not mean that we should tell the students an untruth. What this does mean is that it may be instructive to examine why exactly the students are frustrated.

Question 2.2. Why are the students frustrated?

Answer. The important observation here is that the students are not told about either of the following two items:

(1) the real number system;
(2) limits,

before they are exposed to the topic of decimal notation, as well as the problem of unital evaluation. What we point out is that so long as the number system has not been specified explicitly, the students' hunch that $.999\ldots$ falls infinitesimally short of 1 can be justified in a rigorous fashion, in the framework of Abraham Robinson's [47, 48] non-standard analysis.[1]

[1] By the work of J. Avigad [3], the phenomenon can already be expressed in primitive recursive arithmetic, in the context of Skolem's non-standard models of arithmetic, see answer to Questions 6.1 and 6.2 below.

[2] The proof exploiting the long division of 1 by 3, is dealt with in the answer to Question 8.1.

Question 2.3. Isn't it a problem with the proof that the definitions aren't precise? You say that 0.999... has an "unbounded number of repeated digits 9". That is not a meaningful mathematical statement; there is no such number as "unbounded". If it is to be precise, then you need to provide a formal definition of "unbounded", which you have not done.

Answer. The comment was not meant to be a precise definition. The precise definition of .999... as a real number is well known. The thrust of the argument is that *before* the number system has been explicitly specified, one can reasonably consider that the ellipsis "..." in the symbol .999... is in fact ambiguous. From this point of view, the notation .999... stands, not for a single number, but for a class of numbers,[3] all but one of which are less than 1.

Note that F. Richman [46] creates a natural semiring (in the context of decimal expansions), motivated by constructivist considerations, where certain cancellations are disallowed (as they involve infinite "carry-over"). The absence of certain cancellations (i.e. subtractions) leads to a system where a strict inequality $.999\ldots < 1$ is satisfied. The advantage of the hyperreal approach is that the number system remains a field, together with the extension principle and the transfer principle (see Appendix A).

Question 2.4. Doesn't decimal representation have the same meaning in standard analysis as non-standard analysis?

Answer. Yes and no. Lightstone [38] has developed an extended decimal notation[4] that gives more precise information about the hyperreal. In his notation, the standard real .999... would appear as

$$.999\ldots;\ldots 999\ldots$$

Question 2.5. Since non-standard analysis is a conservative extension of the standard reals, shouldn't all existing properties of the standard reals continue to hold?

Answer. Certainly, $.999\ldots;\ldots 999\ldots$ equals 1, on the nose, in the hyperreal number system, as well. An accessible account of the hyperreals can be found in chapter 6: *Ghosts of departed quantities* of Ian Stewart's popular book *From here to infinity* [55]. In his unique way, Stewart has captured the essense of the issue as follows in [56, p. 176]:

> The standard analysis answer is to take '...' as indicating passage to a limit. But in non-standard analysis there are many different interpretations.

In particular, a *terminating* infinite decimal $.999\ldots;\ldots 999$ is less than 1.

Question 2.6. Your expression "terminating infinite decimals" sounds like gibberish. How many decimal places do they have exactly? How can infinity terminate?

Answer. If you are troubled by this, you are in good company. A remarkable passage by Leibniz is a testimony to the enduring appeal of the metaphor of infinity, even in its,

[3] For a more specific choice of such a number, see the answer to Question 8.3.

[4] For details see Appendix A, item A.11 below.

paradoxically, *terminated* form. In a letter to Johann Bernoulli dating from june 1698 (as quoted in Jesseph [30, Section 5]), Leibniz speculated concerning

> lines [...] which are terminated at either end, but which nevertheless are to our ordinary lines, as an infinite to a finite.

He further speculates as to the possibility of

> a point in space which can not be reached in an assignable time by uniform motion. And it will similarly be required to conceive a time terminated on both sides, which nevertheless is infinite, and even that there can be given a certain kind of eternity [...] which is terminated.

Ultimately, Leibniz rejected any metaphysical reality of such quantities, and conceived of both infinitesimals and infinite quantities as ideal numbers, falling short of the reality of the familiar appreciable quantities.[5]

Question 2.7. You say that in Lightstone's notation, the nonstandard number represented by $.999\ldots;\ldots 99\hat{9}$ is less 1. Wouldn't he consider this as something different from $.999\ldots$, since he uses a different notation, and that he would say

$$.999\ldots;\ldots 99\hat{9} < .999\ldots = 1?$$

Answer. Certainly.

Question 2.8. Aren't you arbitrarily redefining $.999\ldots$ as equal to the non-standard number $.999\ldots;\ldots 99\hat{9}$, which would contradict the standard definition?

Answer. No, the contention is that the ellipsis notation is ambiguous, particularly as perceived by pre-lim, pre-$\mathbb{R}$ students. The notation could reasonably be applied to a class of numbers[6] infinitely close to 1.

Question 2.9. You claim that "there is an unbounded number of 9s in $.999\ldots$, but saying that it has infinitely many 9s is only a figure of speech". Now there are several problems with such a claim. First, there is no such object as an "unbounded number". Second, "infinitely many 9s" not a figure of speech, but rather quite precise. Doesn't "infinite" in this context mean the countable cardinal number, $\aleph_0$ in Cantor's notation?

Answer. One can certainly choose to call the output of a series whatever one wishes. The terminology "infinite sum" is a useful and intuitive term, when it comes to understanding standard calculus. In other ways, it can be misleading. Thus, the term contains no hint of the fact that such an "$\aleph_0$-fold sum" is only a partial operation, unlike the inductively defined n-fold sums. Namely, a series can diverge, in which case the infinite sum is undefined (to be sure, this does not happen for decimal series representing real numbers).

[5] Ely [24] presents a case study of a student who naturally developed an intuitive system of infinitesimals and infinitely large quantities, bearing a striking resemblance to Leibniz's system. Ely concludes: "By recognizing that some student conceptions that appear to be misconceptions are in fact nonstandard conceptions, we can see meaningful connections between cognitive structures and mathematical structures of the present and past that otherwise would have been overlooked."

[6] For a more specific choice of such a number, see the answer to Question 8.3.

Furthermore, the "$\aleph_0$-fold sum" intuition creates an impediment to understanding Lightstone extended decimals

$$.a_1 a_2 a_3 \ldots ; \ldots a_H \ldots$$

If one thinks of the standard real as an $\aleph_0$-fold sum of the countably many terms such as $a_1/10, a_2/100, a_3/1000$, etc., then it may appear as though Lightstone's extended decimals *add* additional positive (infinitesimal) terms to the real value one started with (which seems to be already "present" to the left of the semicolon). It then becomes difficult to understand how such an extended decimal can represent a number *less* than 1.

For this reason, it becomes necessary to analyze the *infinite sum* figure of speech, with an emphasis on the built-in *limit*.

Question 2.10. Are you trying to convince me that the expression *infinite sum*, routinely used in Calculus, is only a figure of speech?

Answer. The debate over whether or not an *infinite sum* is a figure of speech, is in a way a re-enactment of the foundational debates at the end of the 17th and the beginning of the 18th century, generally thought of as a Newton-Berkeley debate.[7] The founders of the calculus thought of

(1) the derivative as a ratio of a pair of infinitesimals, and of
(2) the integral as an infinite sum of terms $f(x)dx$.

Bishop Berkeley [8] most famously[8] criticized the former in terms of the familiar *ghosts of departed quantities* (see [55, Chapter 6]) as follows. The infinitesimal

$$dx$$

appearing in the denominator is expected, at the beginning of the calculation, to be nonzero (the *ghosts*), yet at the end of the calculation it is neglected as if it were zero (hence, *departed* quantities). The implied stripping away of an infinitesimal at the end of the calculation occurs in evaluating an integral, as well.

To summarize, the integral is not an infinite Riemann sum, but rather the standard part of the latter (see Section A, item A.12). From this viewpoint, calling it an infinite sum is merely a figure of speech, as the crucial, final step is left out.

A. Robinson solved the 300-year-old logical inconsistency of the infinitesimal definition of the integral, in terms of the standard part function.[9]

Question 2.11. Hasn't historian Bos criticized Robinson for being excessive in enlisting Leibniz for his cause?

Answer. In his essay on Leibniz, H. Bos [17, p. 13] acknowledged that Robinson's hyperreals provide

> [a] preliminary explanation of why the calculus could develop on the insecure foundation of the acceptance of infinitely small and infinitely large quantities.

[7] See footnote 28 for a historical clarification.
[8] Similar criticisms were expressed by Rolle, thirty years earlier; see Schubring [52].
[9] See Appendix A, item A.3.

F. Medvedev [41, 42] further points out that

> [n]onstandard analysis makes it possible to answer a delicate question bound up with earlier approaches to the history of classical analysis. If infinitely small and infinitely large magnitudes are regarded as inconsistent notions, how could they serve as a basis for the construction of so [magnificent] an edifice of one of the most important mathematical disciplines?

3. Is *infinite sum* a figure of speech?

Question 3.1. Perhaps the historical definition of an integral, as an infinite sum of infinitesimals, had been a figure of speech. But why is an infinite sum of a sequence of real numbers more of a figure of speech than a sum of two real numbers?

Answer. Foundationally speaking, the two issues (integral and infinite sum) are closely related. The series can be described cognitively as a proceptual encapsulation of a dynamic process suggested by a sequence of finite sums, see D. Tall [63]. Mathematically speaking, the convergence of a series relies on the completeness of the reals, a result whose difficulty is of an entirely different order compared to what is typically offered as arguments in favor of unital evaluation.

The rigorous justification of the notion of an integral is identical to the rigorous justification of the notion of a series. One can accomplish it finitistically with epsilontics, or one can accomplish it infinitesimally with standard part. In either case, one is dealing with an issue of an entirely different nature, as compared to finite n-fold sums.

Question 3.2. You have claimed that "saying that it has an infinite number of 9s is only a figure of speech". Of course "infinity" is not a number in standard analysis: this word refers to a number in the cardinal number system, i.e. the cardinality[10] of the number of digits; it does not refer to a number in the real number system.

Answer. One can certainly consider an infinite string of 9s labeled by the standard natural numbers. However, when challenged to write down a precise definition of $.999\ldots$, one invariably falls back upon the *limit* concept (and presumably the respectable epsilon, delta definition thereof). Thus, it turns out that $.999\ldots$ is really the limit of the sequence .9, .99, .999, etc.[11] Note that such a definition never uses an infinite string of 9s labeled by the standard natural numbers, but only finite fragments thereof.

Informally, when the students are confronted with the problem of the unital evaluation, they are told that the decimal in question is zero, point, followed by infinitely many 9s. Well, taken literally, this describes the hyperreal number

$$.999\ldots;\ldots 999000$$

perfectly well: we have zero, point, followed by H-infinitely many 9s. Moreover, this statement in a way is truer than the one about the standard decimal, as explained above

[10] See more on cardinals in answer to Question 6.1.

[11] The related sequence .3, .33, .333, etc. is discussed in the answer to Question 8.1.

(the infinite string is never used in the actual standard definition). The hyperreal *is* an infinite sum, on the nose. It is not a limit of finite sums.

Question 3.3. Do limits have a role in the hyperreal approach?

Answer. Certainly. Let $u_1 = .9, u_2 = .99, u_3 = .999$, etc. Then the limit, from the hyperreal viewpoint, is the standard part of u_H for any infinite hyperinteger H. The standard part strips away the (negative) infinitesimal, resulting in the standard value 1, and the students are right almost everywhere.

Question 3.4. A mathematical notation is whatever it is defined to be, no more and no less. Isn't .999 . . . defined to be equal to 1?

Answer. As far as teaching is concerned, it is not necessarily up to research mathematicians to decide what is good notation and what is not, but rather should be determined by the teaching profession and its needs, particularly when it comes to students who have not yet been introduced to $\mathbb{R}$ and lim.

Question 3.5. In its normal context, .999 . . . is defined unambiguously, shouldn't it therefore be taught as a single mathematical object?

Answer. Indeed, in the context of ZFC standard reals and the appropriate notion of limit, the definition is unambiguous. The issue here is elsewhere: what does .999 . . . *look like* to highschoolers when they are exposed to the problem of unital evaluation, *before* learning about $\mathbb{R}$ and lim.[12]

Question 3.6. Don't standard analysis texts provide a unique definition of .999 . . . that is almost universally accepted, as a certain infinite sum that (independently) happens to evaluate to 1?

Answer. More precisely, it is a limit of finite sums, whereas "infinite sum" is a figurative way of describing the limit. Note that the hyperreal sum from 1 to H, where H is an infinite hyperinteger, can also be described as an "infinite sum", or more precisely H-infinite sum, for a choice of a hypernatural number H.

Question 3.7. There are certain operations that happen to work with "formal" manipulation, such as dividing each digit by 3 to result[13] in 0.333 . . . But shouldn't such manipulation be taught as merely a convenient shortcut that happens to work but needs to be verified independently with a rigorous argument before it is accepted?

Answer. Correct. The best rigorous argument, of course, is that the sequence

$$\langle .9, .99, .999, \ldots \rangle$$

gets closer and closer to 1 (and therefore 1 is the limit by definition). The students would most likely find the remark before the parenthesis, unobjectionable. Meanwhile, the parenthetical remark is unintelligible to them, unless they have already taken calculus.

[12] A preferred choice of a hyperreal evaluation of the symbol ".999 . . ." is described in the answers to Questions 8.2, 8.3, and 8.4.

[13] The long division of 1 by 3 and its implications for unital evaluation are discussed in detail in the answer to Question 8.1.

4. Meanings, standard and non-standard

Question 4.1. Isn't it very misleading to change the standard meaning of .999..., even though it may be convenient? This is in the context of standard analysis, since non-standard analysis is not taught very often because it has its own set of issues and complexities.

Answer. In the fall of '08, a course in calculus was taught using H. Jerome Keisler's textbook *Elementary Calculus* [33]. The course was taught to a group of 25 freshmen. The TA had to be trained as well, as the material was new to the TA. The students have never been so excited about learning calculus, according to repeated reports from the TA. Two of the students happened to be highschool teachers (they were somewhat exceptional in an otherwise teenage class). They said they were so excited about the new approach that they had already started using infinitesimals in teaching basic calculus to their 12th graders. After the class was over, the TA paid a special visit to the professor's office, so as to place a request that the following year, the course should be taught using the same approach. Furthermore, the TA volunteered to speak to the chairman personally, so as to impress upon him the advantages of non-standard calculus. The .999... issue was not emphasized in the class.[14]

Question 4.2. Non-standard calculus? Didn't Errett Bishop explain already that non-standard calculus constituted a debasement of meaning?

Answer. Bishop did refer to non-standard calculus as a *debasement of meaning* in his *Crisis* text [13] from '75. He clarified what it was exactly he had in mind when he used this expression, in his *Schizophrenia* text [15]. The latter text was distributed two years earlier, more precisely in '73, according to M. Rosenblatt [50, p. ix]. Bishop writes as follows [15, p. 1]:

> Brouwer's criticisms of **classical mathematics** [emphasis added–MK] were concerned with what I shall refer to as "the debasement of meaning".

In Bishop's own words, the *debasement of meaning* expression, employed in his *Crisis* text to refer to non-standard calculus, was initially launched as a criticism of *classical mathematics* as a whole. Thus his criticism of non-standard calculus was foundationally, not pedagogically, motivated.

In a way, Bishop is criticizing apples for not being oranges: the critic (Bishop) and the criticized (Robinson's non-standard analysis) do not share a common foundational framework. Bishop's preoccupation with the extirpation of the law of excluded middle (LEM)[15] led him to criticize classical mathematics as a whole in as vitriolic[16] a manner as his criticism of non-standard analysis.

[14] Hyperreal pedagogy is analyzed in the answer to Question 6.10.

[15] A defining feature of both intuitionism and Bishop's constructivism is a rejection of LEM; see footnote 35 for a discussion of Bishop's foundational posture within the spectrum of Intuitionistic sensibilities.

[16] Historian of mathematics J. Dauben noted the vitriolic nature of Bishop's remarks, see [22, p. 139]; M. Artigue [2] described them as *virulent*; D. Tall [61], as *extreme*.

Question 4.3. Something here does not add up. If Bishop was opposed to the rest of classical mathematics, as well, why did he reserve special vitriol for his book review of Keisler's textbook on non-standard calculus?

Answer. Non-standard analysis presents a formidable philosophical challenge to Bishopian constructivism, which may, in fact, have been anticipated by Bishop himself in his foundational speculations, as we explain below.

While Bishop's constructive mathematics (unlike Brouwer's intuitionism[17]) is uniquely concerned with finite operations on the integers, Bishop himself has speculated that "the primacy of the integers is not absolute" [12, p. 53]:

> It is an **empirical fact** [emphasis added–MK] that all [finitely performable abstract calculations] reduce to operations with the integers. There is no reason mathematics should not concern itself with finitely performable abstract operations of other kinds, in the event that such are ever discovered [...]

Bishop hereby acknowledges that the *primacy of the integers* is merely an *empirical fact*, i.e. an empirical observation, with the implication that the observation could be contradicted by novel mathematical developments. Non-standard analysis, and particularly non-standard calculus, may have been one such development.

Question 4.4. How is a theory of infinitesimals such a novel development?

Answer. Perhaps Bishop sensed that a rigorous theory of infinitesimals is both

- not reducible to finite calculations on the integers, and yet
- accomodates a finite performance of abstract operations,

thereby satisfying his requirements for coherent mathematics. Having made a foundational commitment to the *primacy of the integers* (a state of mind known as *integr-ity* in Bishopian constructivism; see [15, p. 4]) through his own work and that of his disciples starting in the late sixties, Bishop may have found it quite impossible, in the seventies, to acknowledge the existence of "finitely performable abstract operations of other kinds".

Birkhoff reports that Bishop's talk at the workshop was not well-received.[18] The list of people who challenged him (on a number of points) in the question-and-answer session that followed the talk, looks like the who-is-who of 20th century mathematics.

Question 4.5. Why didn't all those luminaries challenge Bishop's *debasement* of non-standard analysis?

Answer. The reason is a startling one: there was, in fact, nothing to challenge him on. Bishop did not say a word about non-standard analysis in his oral presentation, according to a workshop participant [40] who attended his talk.[19] Bishop appears to

[17]Bishop rejected both Kronecker's finitism and Brouwer's theory of the continuum.

[18]See Dauben [22, p. 133] in the name of Birkhoff [10, p. 505].

[19]The participant in question, historian of mathematics P. Manning, was expecting just this sort of critical comment about non-standard analysis from Bishop, but the comment never came. Manning wrote as follows on the subject of Bishop's statement on non-standard calculus published in the written

have added the *debasement* comment after the workshop, at the galley proof stage of publication. This helps explain the absence of any critical reaction to such *debasement* on the part of the audience in the discussion session, included at the end of the published version of Bishop's talk.

Question 4.6. On what grounds did Bishop criticize classical mathematics as deficient in numerical meaning?

Answer. The quest for greater numerical meaning is a compelling objective for many mathematicians. Thus, as an alternative to an indirect proof (relying on LEM) of the irrationality of $\sqrt{2}$, one may favor a direct proof of a concrete lower bound, such as $\frac{1}{3n^2}$ for the error $|\sqrt{2} - \frac{m}{n}|$ involved. Bishop discusses this example in [15, p. 18]. More generally, one can develop a methodology that seeks to enhance classical arguments by eliminating the reliance on LEM, with an attendant increase in numerical meaning. Such a methodology can be a useful *companion* to classical mathematics.

Question 4.7. Given such commendable goals, why haven't mainstream mathematicians adopted Bishop's constructivism?

Answer. The problem starts when LEM-extirpation is elevated to the status of the supreme good, regardless of whether it is to the benefit, or detriment, of numerical meaning. Such a radical, anti-LEM species of constructivism tends to be posited, not as a *companion*, but as an *alternative*, to classical mathematics. Philosopher of mathematics G. Hellman [28, p. 222] notes that "some of Bishop's remarks (1967) suggest that his position belongs in [the radical constructivist] category".

For instance, Bishop wrote [12, p. 54] that "[v]ery possibly classical mathematics will cease to exist as an independent discipline." He challenged his precursor Brouwer himself, by describing the latter's theory of the continuum as a "semimystical theory" [11, p. 10]. Bishop went as far as evoking the term "schizophrenia in contemporary mathematics", see [15].

Question 4.8. How can the elimination of the law of excluded middle be detrimental to numerical meaning?

Answer. In the context of a discussion of the differentiation procedures in Leibniz's infinitesimal calculus, D. Jesseph [30, Section 1] points out that

> [t]he algorithmic character of this procedure is especially important, for it makes the calculus applicable to a vast array of curves whose study had previously been undertaken in a piecemeal fashion, without an underlying unity of approach.

version [13] of his talk: "I do not remember that any such statement was made at the workshop and doubt seriously that it was in fact made. I would have pursued the issue vigorously, since I had a particular point of view about the introduction of non-standard analysis into calculus. I had been considering that question somewhat in my attempts to understand various standards of rigor in mathematics. The statement would have fired me up."

The algorithmic, computational, numerical meaning of such computations persists after infinitesimals are made rigorous in Robinson's approach, relying as it does on classical logic, incorporating the law of excluded middle (LEM).

To cite an additional example, note that Euclid himself has recently been found lacking, constructively speaking, by M. Beeson [7]. The latter rewrote as many of Euclid's geometric constructions as he could while avoiding "test-for-equality" constructions (which rely on LEM). What is the status of those results of Euclid that resisted Beeson's reconstructivisation? Are we prepared to reject Euclid's constructions as lacking in meaning, or are we, rather, to conclude that their meaning is of a post-LEM kind?

Question 4.9. Are there examples of post-LEM numerical meaning from contemporary research?

Answer. In contemporary proof theory, the technique of proof mining is due to Kohlenbach, see [35]. A logical analysis of classical proofs (i.e., proofs relying on classical logic) by means of a proof-theoretic technique known as proof mining, yields explicit numerical bounds for rates of convergence, see also Avigad [4].

Question 4.10. But hasn't Bishop shown that meaningful mathematics is mathematics done constructively?

Answer. If he did, it was by a sleight-of-hand of a successive reduction of the meaning of "meaning". First, "meaning" in a lofty epistemological sense is reduced to "numerical meaning". Then "numerical meaning" is further reduced to the avoidance of LEM.

Question 4.11. Haven't Brouwer and Bishop criticized formalism for stripping mathematics of any meaning?

Answer. Thinking of formalism in such terms is a common misconception. The fallacy was carefully analyzed by Avigad and Reck [5].

From the cognitive point of view, the gist of the matter was summarized in an accessible fashion by D. Tall [63, chapter 12]:

> The aim of a formal approach is not the stripping away of all human intuition to give absolute proof, but the careful organisation of formal techniques to support human creativity and build ever more powerful systems of mathematical thinking.

Hilbert sought to provide a finitistic foundation for mathematical activity, at the metamathematical level. He was prompted to seek such a foundation as an alternative to set theory, due to the famous paradoxes of set theory, with "the ghost of Kronecker" (see [5]) a constant concern. Hilbert's finitism was, in part, a way of answering Kronecker's concerns (which, with hindsight, can be described as intuitionistic/constructive).

Hilbert's program does not entail any denial of meaning at the mathematical level. A striking example mentioned by S. Novikov [44] is Hilbert's Lagrangian for general relativity, a deep and *meaningful* contribution to both mathematics and physics. Unfortunately, excessive rhetoric in the heat of debate against Brouwer had given rise to the famous quotes, which do not truly represent Hilbert's position, as argued in [5]. Hilbert's Lagrangian may in the end be Hilbert's most potent criticism of Brouwer, as variational principles in physics as yet have no intuitionistic framework, see [6, p. 22].

Question 4.12. Why would one want to complicate the students' lives by introducing infinitesimals? Aren't the real numbers complicated enough?

Answer. The traditional approach to calculus using Weierstrassian epsilontics (the epsilon-delta approach) is a formidable challenge to even the gifted students.[20] Infinitesimals provide a means of simplifying the technical aspect of calculus, so that more time can be devoted to conceptual issues.

5. Halmos on infinitesimal subtleties

Question 5.1. Aren't you exaggerating the difficulty of Weierstrassian epsilontics, as you call it? If it is so hard, why hasn't the mathematical community discovered this until now?

Answer. Your assumption is incorrect. Some of our best and brightest have not only acknowledged the difficulty of teaching Weierstrassian epsilontics, but have gone as far as admitting their own difficulty in learning it! For example, Paul Halmos recalls in his autobiography [27, p. 47]:

> ... I was a student, sometimes pretty good and sometimes less good. Symbols didn't bother me. I could juggle them quite well ...[but] I was stumped by the infinitesimal subtleties of epsilonic analysis. I could read analytic proofs, remember them if I made an effort, and reproduce them, sort of, but I didn't really know what was going on.

(quoted in A. Sfard [53, p. 44]). The eventual resolution of such pangs in Halmos' case is documented by Albers and Alexanderson [1, p. 123]:

> ... one afternoon something happened ... suddenly I understood epsilon. I understood what limits were ... All of that stuff that previously had not made any sense became obvious ...

Is Halmos' liberating experience shared by a majority of the students of Weierstrassian epsilontics?

Question 5.2. I don't know, but how can one possibly present a construction of the hyperreals to the students?

Answer. You are surely aware of the fact that the construction of the reals (Cauchy sequences or Dedekind cuts) is not presented in a typical standard calculus class.[21] Rather, the instructor relies on intuitive descriptions, judging correctly that there is no reason to get bogged down in technicalities. There is no more reason to present a construction

[20] including Paul Halmos; see [53], as well as the answer to Question 5.1 below.

[21] The issue of constructing number systems is discussed further in the answers to Questions 7.1 and 7.3.

of infinitesimals, either, so long as the students are given clear ideas as to how to perform arithmetic operations on infinitesimals, finite numbers, and infinite numbers. This replaces the rules for manipulating limits found in the standard approach.[22]

Question 5.3. Non-standard analysis? Didn't Halmos explain already that it is too special?

Answer. P. Halmos did describe non-standard analysis as *a special tool, too special* [27, p. 204]. In fact, his anxiousness to evaluate Robinson's theory may have involved a conflict of interests. In the early '60s, Bernstein and Robinson [9] developed a non-standard proof of an important case of the invariant subspace conjecture of Halmos', and sent him a preprint. In a race against time, Halmos produced a standard translation of the Bernstein-Robinson argument, in time for the translation to appear in the same issue of *Pacific Journal of Mathematics*, alongside the original. Halmos invested considerable emotional energy (and *sweat*, as he memorably puts it in his autobiography[23]) into his translation. Whether or not he was capable of subsequently maintaining enough of a detached distance in order to formulate an unbiased evaluation of non-standard analysis, his blunt unflattering comments appear to retroactively justify his translationist attempt to deflect the impact of one of the first spectacular applications of Robinson's theory.

Question 5.4. How would one express the number π in Lightstone's ".999...;...999" notation?

Answer. Certainly, as follows:

$$3.141\ldots;\ldots d_{H-1}d_H d_{H+1}\ldots$$

The digits of a standard real appearing after the semicolon are, to a considerable extent, determined by the digits before the semicolon. The following interesting fact might begin to clarify the situation. Let

$$d_{\min}$$

be the least digit occurring infinitely many times in the standard decimal expansion of π. Similarly, let

$$d^{\infty}_{\min}$$

be the least digit occurring in an infinite place of the extended decimal expansion of π. Then the following equality holds:

$$d_{\min} = d^{\infty}_{\min}.$$

This equality indicates that our scant knowledge of the infinite decimal places of π is not due entirely to the "non-constructive nature of the classical constructions using the axiom of choice", as has sometimes been claimed; but rather to our scant knowledge of the standard decimal expansion: no "naturally arising" irrationals are known to possess infinitely many occurrences of any specific digit.

[22]See the answer to Question 7.3 for more details on the ultrapower construction.

[23]Halmos wrote [27, p. 204]: "The Bernstein-Robinson proof [of the invariant subspace conjecture of Halmos'] uses non-standard models of higher order predicate languages, and when [Robinson] sent me his reprint I really had to sweat to pinpoint and translate its mathematical insight."

Question 5.5. What does the odd expression "H-infinitely many" mean exactly?

Answer. A typical application of an infinite hyperinteger H is the proof of the extreme value theorem.[24] Here one partitions the interval, say $[0, 1]$, into H-infinitely many equal subintervals (each subinterval is of course infinitesimally short). Then we find the maximum x_{i_0} among the $H+1$ partition points x_i by the transfer principle,[25] and point out that by continuity, the standard part of the hyperreal x_{i_0} gives a maximum of the real function.

Question 5.6. I am still bothered by changing the meaning of the notation .999... as it can be misleading. I recall I was taught that it is preferable to use the y' or y_x notation until one is familiar with derivatives, since dy/dx can be very misleading even though it can be extremely convenient. Shouldn't it be avoided?

Answer. There may be a reason for what you were taught, already noted by Bishop Berkeley[26] [8] nearly 300 years ago! Namely, standard analysis has no way of justifying these manipulations rigorously. The introduction of the notation dy/dx is postponed in the standard approach, until the students are already comfortable with derivatives, as the implied *ratio* is thought of as misleading.

Meanwhile, mathematician and leading mathematics educator D. Tall writes as follows [63, chapter 11]:

> What is far more appropriate for beginning students is an approach building from experience of dynamic embodiment and the familiar manipulation of symbols in which the idea of dy/dx as the ratio of the components of the tanget vector is fully meaningful.

Tall writes that, following the adoption of the limit concept by mathematicians as the basic one,

> [s]tudents were given emotionally charged instructions to avoid thinking of dy/dx as a ratio, because it was now seen as a limit, even though the formulae of the calculus operated as if the expression were a ratio, and the limit concept was intrinsically problematic.

With the introduction of infinitesimals such as Δx, one defines the derivative $f'(x)$ as

$$f'(x) = st(\Delta y/\Delta x),$$

where "st" is the standard part function.[27] Then one sets $dx = \Delta x$, and defines $dy = f'(x)dx$. Then $f'(x)$ is truly the ratio of two infinitesimals: $f'(x) = dy/dx$, as envisioned by the founders of the calculus[28] and justified by Robinson.

---

[24] See Appendix A, item A.9 for details.

[25] See Appendix A, item A.1.

[26] See footnote 8 and main text there.

[27] See Appendix A, item A.3 and item A.5.

[28] Schubring [52, p. 170, 173, 187] attributes the first *systematic* use of infinitesimals as a foundational concept, to Johann Bernoulli (rather than Newton or Leibniz).

6. A cardinal issue

Question 6.1. How does one relate hyperreal infinities to cardinality? It still isn't clear to me what "H-infinitely many 9s" means. Is it $\aleph_0$, $\aleph_1$, the continuum, or something else?

Answer. Since there exist *countable* Skolem non-standard models of arithmetic [54], the *short* answer to your question is "$\aleph_0$". Every non-standard natural number in such a model will of course have only countably many numbers smaller than itself, and therefore every extended decimal will have only countably many digits.

Question 6.2. How do you go from Skolem to *point, nine recurring*?

Answer. Skolem [54] already constructed non-standard models of arithmetic a quarter century before A. Robinson. Following the work of J. Avigad [3], it is possible to capture a significant fragment of non-standard calculus, in a very weak logical language; namely, in the language of primitive recursive arithmetic (PRA), in the context of the *fraction field* of Skolem's non-standard model. Avigad gives an explicit syntactic translation of the nonstandard theory to the standard theory. In the fraction field of Skolem's non-standard model, equality, thought of as a two-place relation, is *interpreted* as the relation $=_*$ of being infinitely close. A "real number" can be thought of as an equivalence class relative to such a relation, though the actual construction of the quotient space ("the continuum") transcends the PRA framework.

The integer part (i.e. floor) function $[x]$ is primitive recursive due to the existence of the Euclidean algorithm of long division. Thus, we have $[m/n] = 0$ if $m < n$, and similarly $[m + n/n] = [m/n] + 1$. Furthermore, the digits of a decimal expansion are easily expressed in terms of the integer part. Hence the digits are primitive recursive functions. Thus the PRA framework is sufficient for dealing with the issue of *point, nine recurring*. Such an approach provides a common extended decimal "kernel" for most theories containing infinitesimals, not only Robinson's theory.

Question 6.3. What's the *long* answer on cardinalities?

Answer. On a deeper level, one needs to get away from the naive cardinals of Cantor's theory,[29] and focus instead on the distinction between a language and a model. A language (more precisely, a theory in a language, such as first order logic) is a collection of propositions. One then interprets such propositions with respect to a particular model.

A key notion here is that of an internal set.[30] Each set S of reals has a natural extension S^* over $\mathbb{R}^*$, but also atomic elements of $\mathbb{R}^*$ are considered internal, so the collection of internal sets is somewhat larger than just the natural extensions of real sets.

A key observation is that, when the language is being applied to the non-standard extension, the propositions are being interpreted as applying only to internal sets, rather than to all sets.

[29] which cannot be used in any obvious way as individual numbers in an extended number system.

[30] See Appendix A, item A.2.

In more detail, there is a certain set-theoretic construction[31] of the hyperreal theory $\mathbb{R}^*$, but the language will be interpreted as applying only to internal sets and not all set-theoretic subsets of $\mathbb{R}^*$.

Such an interpretation is what makes it possible for the transfer principle to hold, when applied to a theory in first order language.

Question 6.4. I still have no idea what the extended decimal expansion is.

Answer. In Robinson's theory, the set of standard natural numbers $\mathbb{N}$ is imbedded inside the collection of hyperreal natural numbers,[32] denoted $\mathbb{N}^*$. The elements of the difference $\mathbb{N}^* \setminus \mathbb{N}$ are sometimes called (positive) infinite hyperintegers, or non-standard integers.

The standard decimal expansion is thought of as a string of digits labeled by $\mathbb{N}$. Similarly, Lightstone's extended expansion can be thought of as a string labeled by $\mathbb{N}^*$. Thus an extended decimal expansion for a hyperreal in the unit interval will appear as

$$a = .a_1 a_2 a_3 \ldots ; \ldots a_{H-2} a_{H-1} a_H \ldots$$

The digits before the semicolon are the "standard" ones (i.e. the digits of $\mathrm{st}(a)$, see Appendix A). Given an infinite hyperinteger H, the string containing H-infinitely many 9s will be represented by

$$.999\ldots;\ldots 999$$

where the last digit 9 appears in position H. It falls short of 1 by the infinitesimal amount $1/10^H$.

Question 6.5. What happens if one decreases $.999\ldots;\ldots 999$ further, by the same infinitesimal amount $1/10^H$?

Answer. One obtains the hyperreal number

$$.999\ldots;\ldots 998,$$

with digit "8" appearing at infinite rank H.

Question 6.6. You mention that the students have not been taught about $\mathbb{R}$ and lim before being introduced to non-terminating decimals. Perhaps the best solution is to delay the introduction of non-terminating decimals? What point is there in seeking the "right" approach, if in any case the students will not know what you are talking about?

Answer. How would you propose to implement such a scheme? More specifically, just how much are we to divulge to the students about the result of the long division[33] of 1 by 3?

[31] See discussion of the ultrapower construction in Section 7

[32] A non-standard model of arithmetic is in fact sufficient for our purposes; see the answer to Question 6.1

[33] This long division is analyzed in the answer to Question 8.1 below

Question 6.7. Just between the two of us, in the end, there is still no theoretical explanation for the strict inequality $.999\ldots < 1$, is there? You did not disprove the equality $.999\ldots = 1$. Are there any schoolchildren that could understand Lightstone's notation?

Answer. The point is not to teach Lightstone's notation to schoolchildren, but to broaden their horizons by mentioning the existence of arithmetic frameworks where their "hunch" that .999... falls short of 1, can be justified in a mathematically sound fashion, consistent with the idea of an "infinite string of 9s" they are already being told about. The underbrace notation

$$.\underbrace{999\ldots}_{H} = 1 - \frac{1}{10^H}$$

may be more self-explanatory than Lightstone's semicolon notation; to emphasize the infinite nature of the non-standard integer H, one could denote it by the traditional infinity symbol ∞, so as to obtain a strict inequality[34]

$$.\underbrace{999\ldots}_{\infty} < 1,$$

keeping in mind that the left-hand side is an infinite terminating extended decimal.

Question 6.8. The multitude of bad teachers will stumble and misrepresent whatever notation you come up with. For typesetting purposes, Lighthouse's notation is more suitable than your underbrace notation. Isn't an able mathematician committing a capital sin by promoting a pet viewpoint, as the cure-all solution to the problems of math education?

Answer. Your assessment is that the situation is bleak, and the teachers are weak. On the other hand, you seem to be making a hidden assumption that the status-quo cannot be changed in any way. Without curing all ills of mathematics education, one can ask what educators think of a specific proposal addressing a specific minor ill, namely student frustration with the problem of unital evaluation.

One solution would be to dodge the discussion of it altogether. In practice, this is not what is done, but rather the students are indeed presented with the claim of the evaluation of .999... to 1. This is done before they are taught $\mathbb{R}$ or lim. The facts on the ground are that such teaching is indeed going on, whether in 12^{th} grade (or even earlier, see [65]) or at the freshman level.

Question 6.9. Are hyperreals conceptually easier than the common reals? Will modern children interpret sensibly "infinity minus one," say?

Answer. David Tall, a towering mathematics education figure, has published the results of an "interview" with a pre-teen, who quite naturally developed a number system where 1, 2, 3 can be added to "infinity" to obtain other, larger, "infinities". This indicates that the idea is not as counterintuitive as it may seem to us, through the lens of our standard education.

[34] See answer to Question 8.3 for a more specific choice of H

Question 6.10. If the great Kronecker could not digest Cantor's infinities, how are modern children to interpret them?

Answer. No, schoolchildren should not be taught the arithmetic of the hyperreals, no more than Cantorian set theory. On the other hand, the study by K. Sullivan [57] in the Chicago area indicates that students following the non-standard calculus course were better able to interpret the sense of the mathematical formalism of calculus than a control group following a standard syllabus. Sullivan's conclusions were also noted by Artigue [2], Dauben [22], and Tall [58]. A more recent synthesis of teaching frameworks based on non-standard calculus was developed by Bernard Hodgson [29] in '92, and presented at the ICME-7 at Quebec.

Are these students greater than Kronecker? Certainly not. On the other hand, Kronecker's commitment to the ideology of finitism[35] was as powerful as most mathematicians' commitment to the standard reals is, today.

Mathematics education researcher J. Monaghan, based on field studies, has reached the following conclusion [43, p. 248]:

> [...] do infinite numbers of any form exist for young people without formal mathematical training in the properties of infinite numbers? The answer is a qualified 'yes'.

Question 6.11. Isn't the more sophisticated reader going to wonder why Lightstone stated in [38] that decimal representation is unique, while you are making a big fuss over the nonuniqueness of decimal notation and the strict inequality?

Answer. Lightstone was referring to the convention of replacing each terminating decimal, by a tail of 9s.

Beyond that, it is hard to get into Lightstone's head. Necessarily remaining in the domain of speculation, one could mention the following points. Mathematicians trained in standard decimal theory tend to react with bewilderment to any discussion of a strict inequality ".999 . . ." < 1. Now Lightstone was interested in publishing his popular article on infinitesimals, following his advisor's (Robinson's) approach. There is more than one person involved in publishing an article. Namely, an editor also has a say, and one of his priorities is defining the level of controversy acceptable in his periodical.

Question 6.12. Why didn't Lighstone write down the strict inequality?

Answer. Lightstone could have made the point that all but one extended expansions starting with $999\ldots$ give a hyperreal value strictly less than 1. Instead, he explicitly reproduces only the expansion equal to 1. In addition, he explicitly mentions an additional expansion–and explains why it does not exist! Perhaps he wanted to stay away from the

[35]As a lightning introduction to Intuitionism, we note that Kronecker rejected actual (completed) infinity, as did Brouwer, who also rejected the law of excluded middle (which would probably have been rejected by Kronecker, had it been crystallized as an explicit concept by logicians in Kronecker's time). Brouwer developed a theory of the continuum in terms of his "choice sequences". E. Bishop's Constructivism rejects both Kronecker's finitism (Bishop accepts the actual infinity of $\mathbb{N}$) and Brouwer's theory of the continuum, described as "semimystical" by Bishop [11, p. 10].

strict inequality, and concentrate instead on getting a minimal amount of material on non-standard analysis published in a mainstream popular periodical. All this is in the domain of speculation.

As far as the reasons for elaborating a strict non-standard inequality, they are more specific. First, the manner in which the issue is currently handled by education professionals, tends to engender student frustration. Furthermore, the standard treatment conceals the power of non-standard analysis in this particular issue.

7. Circular reasoning, ultrafilters, and Platonism

Question 7.1. Since the construction of the hyperreal numbers depends on that of the real numbers, wouldn't it be extremely easy for people to attack this idea as being circular reasoning?

Answer. Actually, your assumption is incorrect. Just as the reals can be obtained from the rationals as the set of equivalence classes of suitable sequences of rational numbers (namely, the Cauchy ones), so also a version of the hyperreals[36] can be obtained from the rationals as the set of equivalence classes of sequences of rational numbers, modulo a suitable equivalence relation. Such a construction is due to Luxemburg [39]. The construction is referred to as the ultrapower construction, see Goldblatt [25].

Question 7.2. Non-standard analysis? You mean ultrafilters and all that?

Answer. The good news is that you don't need ultrafilters to do non-standard analysis: the axiom of choice is enough.[37]

Question 7.3. Good, because, otherwise, aren't you sweeping a lot under the rug when you teach non-standard analysis to first year students?

Answer. One sweeps no more under the rug than the equivalence classes of Cauchy sequences, which are similarly *not* taught in first year calculus. After all, the hyperreals are just equivalence classes of more general sequences (this is known as the ultrapower construction). What one does *not* sweep under the rug in the hyperreal approach is the notion of infinitesimal which historically was present at the inception of the theory, whether by Archimedes or Leibniz-Newton. Infinitesimals were routinely used in teaching until as late as 1912, the year of the last edition of the textbook by L. Kiepert [34]. This issue was discussed in more detail by P. Roquette [49].

[36] One does not obtain all elements of $\mathbb{R}^*$ by starting with sequences of rational numbers, but the resulting non-Archimedean extension of $\mathbb{R}$ is sufficient for most purposes of the calculus, cf. Avigad [3].

[37] The comment is, of course, tongue in cheek, but many people seem not to have realized yet that the existence of a free (non-principal) ultrafilter is as much of a consequence of the axiom of choice, as the existence of a maximal ideal (a standard tool in algebra), or the Hahn-Banach theorem (a standard tool in functional analysis). This is as good a place as any to provide a brief unequal time to an opposing view [45]: *We are all Platonists, aren't we? In the trenches, I mean—when the chips are down. Yes, Virginia, there really are circles, triangles, numbers, continuous functions, and all the rest. Well, maybe not free ultrafilters. Is it important to believe in the existence of free ultrafilters? Surely that's not required of a Platonist. I can more easily imagine it as a test of sanity: 'He believes in free ultrafilters, but he seems harmless'.* Needless to say, the author of [45] is in favor of eliminating the axiom of choice–including the countable one.

Question 7.4. I have a serious problem with Lightstone's notation. I can see it working for a specific infinite integer H, and even for nearby infinite integers of the form $H+n$, where n is a finite integer, positive or negative. However, I do not see how it represents two different integers, for instance H and H^2 on the same picture. For in this case, H^2 is greater than $H+n$ for any finite n. Thus it does not lie in the same infinite collection of decimal places

$$;\ldots 1 \ldots$$

so that one needs even more than a potentially infinite collection of sequences of digits

$$;\ldots;$$

to cope with all hyperintegers.

Answer. Skolem [54] already constructed non-standard models for arithmetic, many years before Robinson.[38] Here you have a copy of the standard integers, and also many "galaxies". A galaxy, in the context of Robinson's hyperintegers,[39] is a collection of hyperintegers differing by a finite integer. At any rate, one does need infinitely many semicolons if one were to dot all the i's.

Lightstone is careful in his article to discuss this issue. Namely, what is going on to the right of his semicolon is not similar to the simple picture to the left. At any rate, the importance of his article is that he points out that there does exist the notion of an extended decimal representation, where the leftmost galaxy of digits of x are the usual finite digits of $\mathrm{st}(x)$.

8. How long a division?

Question 8.1. Long division of 1 by 3 gives $.333\ldots$ which is a very **obvious pattern**. Therefore multiplying back by 3 we get $.999\ldots = 1$. There is nothing else to discuss!

Answer. Let us be clear about one thing: long division of 1 by 3 does *not* produce the infinite decimal $.333\ldots$ contrary to popular belief. What it *does* produce is the sequence $\langle .3, .33, .333, \ldots \rangle$, where the dots indicate the *obvious pattern*.

Passing from a sequence to an infinite decimal is a major additional step. The existence of an infinite decimal expansion is a non-trivial matter that involves the construction of the real number system, and the notion of the limit.

Question 8.2. Doesn't the standard formula for converting every repeated decimal to a fraction show that $.333\ldots$ equals $\frac{1}{3}$ on the nose?

Answer. Converting decimals to fractions was indeed the approach of [65]. However, in a pre-$\mathbb{R}$ environment, one can argue that the formula only holds up to an infinitesimal error, and attempts to "prove" unital evaluation by an appeal to such a formula amount to replacing one article of faith, by another.

To elaborate, note that applying the iterative procedure of long division in the case of $\frac{1}{3}$, does not by itself produce any infinite decimal, no more than the iterative procedure

[38]See the answer to Question 6.2 for more details.

[39]See Goldblatt [25] for more details

of adding 1 to the outcome of the previous step, produces any infinite integer. Rather, the long division produces the sequence $\langle .3, .33, .333, \ldots \rangle$. Transforming the sequence into an infinite decimal has nothing to do with long division, and requires, rather, an application of the limit concept, in the context of a complete number system.

Note that, if we consider the sequence

$$\langle .9, .99, .999, \ldots \rangle,$$

but instead of taking the limit, take its equivalence class

$$[.9, .99, .999, \ldots]$$

in the ultrapower construction of the hyperreals (see [39, 25]), then we obtain a value equal to $1 - 1/10^{[\mathbb{N}]}$ where $\langle \mathbb{N} \rangle$ is the "natural string" sequence enumerating all the *natural* numbers, whereas $[\mathbb{N}]$ is its equivalence class in the hyperreals. Thus the unital evaluation has a viable competitor, namely, the "natural string" evaluation.

Question 8.3. Isn't it odd that you seem to get a *canonical* representative for ".999 . . ." which falls short of 1?

Answer. The hyperinteger defined by the equivalence class of the sequence $\langle \mathbb{N} \rangle = \langle 1, 2, 3, \ldots \rangle$ only makes sense in the context of the ultrapower construction, and depends on the choices made in the construction. The standard real decimal $(.999\ldots)_{\mathrm{Lim}}$ is defined as the *limit* of the sequence $(.9, .99, .999\ldots)$, and the hyperreal $(.999\ldots)_{\mathrm{Lux}}$ is defined as the class of the same sequence in the ultrapower construction. Then

$$(.999\ldots)_{\mathrm{Lim}} = 1 \tag{8.1}$$

is the unital evaluation, interpreting the symbol .999 . . . as a real number, while

$$.\underbrace{999\ldots}_{[\mathbb{N}]} = 1 - \frac{1}{10^{[\mathbb{N}]}}. \tag{8.2}$$

is the natural string evaluation.

In more detail, in the ultrapower construction of $\mathbb{R}^*$, the hyperreal $[.9, .99, .999, \ldots]$, represented by the sequence $\langle .9, .99, .999, \ldots \rangle$, is an infinite terminating string of 9s, with the last nonzero digit occurring at a suitable infinite hyperinteger rank. The latter is represented by the string listing all the natural numbers $\langle 1, 2, 3, \ldots \rangle$, which we abbreviate by the symbol $\langle \mathbb{N} \rangle$. Then the equivalence class $[\mathbb{N}]$ is the corresponding "natural string" hyperinteger. We therefore obtain a hyperreal equal to $1 - \frac{1}{10^{[\mathbb{N}]}}$.

The *unital evaluation* of the symbol .999 . . . has a viable competitor, namely the *natural string evaluation* of this symbol.

Question 8.4. What do you mean by $10^{\mathbb{N}}$? It looks to me like a typical sophomoric error.

Answer. The natural string evaluation yields a hyperreal with Lightstone [38] representation given by $.999\ldots;\ldots 9$, with the last digit occurring at non-standard rank $[\mathbb{N}]$. Note that it would be *incorrect* to write

$$(.999\ldots)_{\mathrm{Lux}} = 1 - \frac{1}{10^{\mathbb{N}}} \tag{8.3}$$

since the expression $10^{\mathbb{N}}$ is meaningless, $\mathbb{N}$ not being a number in *any* number system. Meanwhile, the sequence $\langle\mathbb{N}\rangle$ listing all the natural numbers in increasing order, represents an equivalence class $[\mathbb{N}]$ in the ultrapower construction of the hyperreals, so that $[\mathbb{N}]$ is indeed a quantity, more precisely a non-standard integer, or a hypernatural number [25].

Question 8.5. Do the subscripts in $(.999\ldots)_{\mathrm{Lim}}$ and $(.999\ldots)_{\mathrm{Lux}}$ stand for "limited" and "deluxe"?

Answer. No, the subscript "Lim" refers to the unital evaluation obtained by applying the *limit* to the sequence, whereas the subscript "Lux" refers to the natural string evaluation, in the context of Luxemburg's sequential construction of the hyperreals (the ultrapower construction).

Question 8.6. The absence of infinitesimals is certainly not some kind of a shortcoming of the real number system that one would need to apologize for. How can you imply otherwise?

Answer. The standard reals are at the foundation of the magnificent edifice of classical and modern analysis. Ever since their rigorous conception by Weierstrass, Dedekind, and Cantor, the standard reals have faithfully served the needs of generations of mathematicians of many different specialties. Yet the non-availability of infinitesimals has the following consequences:

(1) it distances mathematics from its applications in physics, enginering, and other fields (where nonrigorous infinitesimals are in routine use);
(2) it complicates the logical structure of calculus concepts (such as the limit) beyond the comprehension of a significant minority (if not a majority) of undergraduate students;
(3) it deprives us of a key tool in interpreting the work of such greats as Archimedes, Euler, and Cauchy.

In this sense, the absence of infinitesimals is a shortcoming of the standard number system.

Appendix A. A non-standard glossary

The present section can be retained or deleted at the discretion of the referee. In this section we present some illustrative terms and facts from non-standard calculus [33]. The relation of being infinitely close is denoted by the symbol $\approx$. Thus, $x \approx y$ if and only if $x - y$ is infinitesimal.

A.1. Natural hyperreal extension f^*.

The *extension principle* of non-standard calculus states that every real function f has a hyperreal extension, denoted f^* and called the natural extension of f. The *transfer principle* of non-standard calculus asserts that every real statement true for f, is true also for f^* (for statements involving any relations). For example, if $f(x) > 0$ for every real x in its domain I, then $f^*(x) > 0$ for every hyperreal x in its domain I^*. Note that if the interval I is unbounded, then I^*

necessarily contains infinite hyperreals. We will sometimes drop the star * so as not to overburden the notation.

A.2. **Internal set.** Internal set is the key tool in formulating the transfer principle, which concerns the logical relation between the properties of the real numbers $\mathbb{R}$, and the properties of a larger field denoted

$$\mathbb{R}^*$$

called the *hyperreal line*. The field $\mathbb{R}^*$ includes, in particular, infinitesimal ("infinitely small") numbers, providing a rigorous mathematical realisation of a project initiated by Leibniz. Roughly speaking, the idea is to express analysis over $\mathbb{R}$ in a suitable language of mathematical logic, and then point out that this language applies equally well to $\mathbb{R}^*$. This turns out to be possible because at the set-theoretic level, the propositions in such a language are interpreted to apply only to internal sets rather than to all sets. Note that the term "language" is used in a loose sense in the above. A more precise term is *theory in first-order logic*. Internal sets include natural extension of standard sets.

A.3. **Standard part function.** The standard part function "st" is the key ingredient in A. Robinson's resolution of the paradox of Leibniz's definition of the derivative as the ratio of two infinitesimals

$$\frac{dy}{dx}.$$

The standard part function associates to a finite hyperreal number x, the standard real x_0 infinitely close to it, so that we can write

$$\mathrm{st}(x) = x_0.$$

In other words, "st" strips away the infinitesimal part to produce the standard real in the cluster. The standard part function "st" is not defined by an internal set (see item A.2 above) in Robinson's theory.

A.4. **Cluster.** Each standard real is accompanied by a cluster of hyperreals infinitely close to it. The standard part function collapses the entire cluster back to the standard real contained in it. The cluster of the real number 0 consists precisely of all the infinitesimals. Every infinite hyperreal decomposes as a triple sum

$$H + r + \epsilon,$$

where H is a hyperinteger, r is a real number in $[0, 1)$, and ϵ is infinitesimal. Varying ϵ over all infinitesimals, one obtains the cluster of $H + r$.

A.5. **Derivative.** To define the real derivative of a real function f in this approach, one no longer needs an infinite limiting process as in standard calculus. Instead, one sets

$$f'(x) = \mathrm{st}\left(\frac{f(x+\epsilon) - f(x)}{\epsilon}\right), \tag{A.1}$$

where ϵ is infinitesimal, yielding the standard real number in the cluster of the hyperreal argument of "st" (the derivative exists if and only if the value (A.1) is independent of the choice of the infinitesimal). The addition of "st" to the formula resolves the centuries-old paradox famously criticized by George Berkeley[40] [8] (in terms of the *Ghosts of departed quantities*, cf. [55, Chapter 6]), and provides a rigorous basis for infinitesimal calculus as envisioned by Leibniz.

A.6. **Continuity.** A function f is continuous at x if the following condition is satisfied: $y \approx x$ implies $f(y) \approx f(x)$.

A.7. **Uniform continuity.** A function f is uniformly continuous on I if the following condition is satisfied:

- standard: for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in I$ and for all $y \in I$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.
- non-standard: for all $x \in I^*$, if $x \approx y$ then $f(x) \approx f(y)$.

A.8. **Hyperinteger.** A hyperreal number H equal to its own integer part

$$H = [H]$$

is called a hyperinteger (here the integer part function is the natural extension of the real one). The elements of the complement $\mathbb{Z}^* \setminus \mathbb{Z}$ are called infinite hyperintegers, or non-standard integers.

A.9. **Proof of extreme value theorem.** Let H be an infinite hyperinteger. The interval $[0, 1]$ has a natural hyperreal extension. Consider its partition into H subintervals of equal length $\frac{1}{H}$, with partition points $x_i = i/H$ as i runs from 0 to H. Note that in the standard setting, with n in place of H, a point with the maximal value of f can always be chosen among the $n + 1$ partition points x_i, by induction. Hence, by the transfer principle, there is a hyperinteger i_0 such that $0 \leq i_0 \leq H$ and

$$f(x_{i_0}) \geq f(x_i) \quad \forall i = 0, \ldots, H. \tag{A.2}$$

Consider the real point

$$c = \mathrm{st}(x_{i_0}).$$

An arbitrary real point x lies in a suitable sub-interval of the partition, namely $x \in [x_{i-1}, x_i]$, so that $\mathrm{st}(x_i) = x$. Applying "st" to the inequality (A.2), we obtain by continuity of f that $f(c) \geq f(x)$, for all real x, proving c to be a maximum of f (see [33, p. 164]).

A.10. **Limit.** We have $\lim_{x\to a} f(x) = L$ if and only if whenever the difference $x - a \neq 0$ is infinitesimal, the difference $f(x) - L$ is infinitesimal, as well, or in formulas: if $\mathrm{st}(x) = a$ then $\mathrm{st}(f(x)) = L$.

Given a sequence of real numbers $\{x_n | n \in \mathbb{N}\}$, if $L \in \mathbb{R}$ we say L is the limit of the sequence and write $L = \lim_{n\to\infty} x_n$ if the following condition is satisfied:

$$\mathrm{st}(x_H) = L \quad \text{for all infinite } H \tag{A.3}$$

[40] See footnote 8 for a historical clarification

(here the extension principle is used to define x_n for every infinite value of the index). This definition has no quantifier alternations. The standard (ϵ, δ)-definition of limit, on the other hand, does have quantifier alternations:

$$L = \lim_{n\to\infty} x_n \Longleftrightarrow \forall\epsilon > 0\,, \exists N \in \mathbb{N}\,, \forall n \in \mathbb{N} : n > N \implies d(x_n, L) < \epsilon. \tag{A.4}$$

A.11. **Non-terminating decimals.** Given a real decimal

$$u = .d_1 d_2 d_3 \ldots,$$

consider the sequence $u_1 = .d_1$, $u_2 = .d_1 d_2$, $u_3 = .d_1 d_2 d_3$, etc. Then by definition,

$$u = \lim_{n\to\infty} u_n.$$

Meanwhile, $\lim_{n\to\infty} u_n = \mathrm{st}(u_H)$ for every infinite H. Now if u is a non-terminating decimal, then one obtains a strict inequality $u_H < u$ by transfer from $u_n < u$. In particular,

$$.999\ldots;\ldots\hat{9} = .\underbrace{999\ldots}_{H} = 1 - \tfrac{1}{10^H} < 1, \tag{A.5}$$

where the hat ˆ indicates the H-th Lightstone decimal place. The standard interpretation of the symbol $.999\ldots$ as 1 is necessitated by notational uniformity: the symbol $.a_1a_2a_3\ldots$ in every case corresponds to the limit of the sequence of terminating decimals $.a_1\ldots a_n$. Alternatively, the ellipsis in $.999\ldots$ could be interpreted as alluding to an infinity of nonzero digits specified by a choice of an infinite hyperinteger $H \in \mathbb{N}^*\backslash\mathbb{N}$. The resulting H-infinite extended decimal string of 9s corresponds to an infinitesimally diminished hyperreal value (A.5). Such an interpretation is perhaps more in line with the naive initial intuition persistently reported by teachers.

A.12. **Integral.** The definite integral of f is the standard part of an infinite Riemann sum $\sum_{i=0}^{H} f(x)\Delta x$, the latter being defined by means of the transfer principle, once finite Riemann sums are in place, see [33] for details.

Acknowledgments

We are grateful to Alexander Bogomolny, Norman Megill, and Fred Richman for many probing questions that have made this text possible.

References

[1] Albers, D. and Alexanderson, G., eds. Mathematical people. Profiles and interviews. Birkhäuser Boston, Inc., Boston, MA, 1985.

[2] Artigue, Michèle: Analysis, in Advanced Mathematical Thinking (ed. David Tall), Springer-Verlag (1994), p. 172 ("The non-standard analysis revival and its weak impact on education").

[3] Avigad, J.: Weak theories of nonstandard arithmetic and analysis. Reverse mathematics 2001, 19–46, *Lect. Notes Log.*, **21**, Assoc. Symbol. Logic, La Jolla, CA, 2005.

[4] Avigad, J.: The metamathematics of ergodic theory. *Ann. Pure Appl. Logic* **157** (2009), no. 2-3, 64–76.

[5] Avigad, J.; Reck, E.: "Clarifying the nature of the infinite": the development of metamathematics and proof theory. December 11, 2001. Carnegie Mellon Technical Report CMU-PHIL-120.

[6] Beeson, M.: Foundations of constructive mathematics. Metamathematical studies. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 6. Springer-Verlag, Berlin, 1985.
[7] Beeson, M.: Constructive geometry, in Arai, T. (ed.) Proceedings of the Tenth Asian Logic Colloquium, Kobe, Japan, 2008, World Scientific, Singapore, 2009.
[8] Berkeley, George: The Analyst, a Discourse Addressed to an Infidel Mathematician (1734).
[9] Bernstein, Allen R.; Robinson, A.: Solution of an invariant subspace problem of K. T. Smith and P. R. Halmos, *Pacific Journal of Mathematics* **16** (1966), 421-431.
[10] Birkhoff, G., editor. Proceedings of the American Academy Workshop on the Evolution of Modern Mathematics. Held at the American Academy of Arts and Sciences, Boston, Mass., Aug. 7–9, 1974. Edited by Garrett Birkhoff, with the assistance of Sue Ann Garwood. With a foreword by John Voss. *Historia Math.* **2** (1975), no. 4. University of Toronto Press, Toronto, Ont., 1975. pp. i–iv and 425–624.
[11] Bishop, E.: Foundations of constructive analysis. McGraw-Hill Book Co., New York-Toronto, Ont.-London 1967.
[12] Bishop, E.: Mathematics as a numerical language. 1970 Intuitionism and Proof Theory (Proc. Conf., Buffalo, N.Y., 1968) pp. 53–71. North-Holland, Amsterdam
[13] Bishop, E.: The crisis in contemporary mathematics. Proceedings of the American Academy Workshop on the Evolution of Modern Mathematics (Boston, Mass., 1974). *Historia Math.* **2** ('75), no. 4, 507-517.
[14] Bishop, E.: Review: H. Jerome Keisler, Elementary calculus, *Bull. Amer. Math. Soc.* **83** ('77), 205-208.
[15] Bishop, E.: Schizophrenia in contemporary mathematics. Errett Bishop: reflections on him and his research (San Diego, Calif., 1983), 1–32, *Contemp. Math.* **39**, Amer. Math. Soc., Providence, RI, 1985.
[16] Borovik, A.: Mathematics under the Microscope: Notes on Cognitive Aspects of Mathematical Practice. Manchester. American Mathematical Society, 2010.
[17] Bos, H. J. M.: Differentials, higher-order differentials and the derivative in the Leibnizian calculus. *Arch. History Exact Sci.* **14** (1974), 1–90.
[18] Bos, H. J. M.: Philosophical challenges from history of mathematics. New trends in the history and philosophy of mathematics, 51–66, Univ. *South. Den. Stud. Philos.* **19**, Univ. Press of South. Den., Odense, 2004.
[19] Cornu, B.: Limits, pp. 153-166, in Advanced mathematical thinking. Edited by David Tall. Mathematics Education Library, 11. Kluwer Academic Publishers Group, Dordrecht, 1991.
[20] Dauben, J.: Appendix (1992): revolutions revisited. pp. 72–82 in Revolutions in mathematics. Edited and with an introduction by Donald Gillies. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1992.
[21] Dauben, J.: Abraham Robinson. The creation of nonstandard analysis. A personal and mathematical odyssey. With a foreword by Benoit B. Mandelbrot. Princeton University Press, Princeton, NJ, 1995.
[22] Dauben, J.: Arguments, logic and proof: mathematics, logic and the infinite. History of mathematics and education: ideas and experiences (Essen, 1992), 113–148, *Stud. Wiss. Soz. Bildungsgesch. Math.*, **11**, Vandenhoeck & Ruprecht, Göttingen, 1996.
[23] Ely, Robert: Student obstacles and historical obstacles to foundational concepts of calculus. PhD thesis, The University of Wisconsin - Madison, 2007.
[24] Ely, Robert: Nonstandard student conceptions about infinitesimals. *Journal for Research in Mathematics Education*, to appear.
[25] Goldblatt, R.: Lectures on the hyperreals. An introduction to nonstandard analysis. Graduate Texts in Mathematics, 188. Springer-Verlag, New York, 1998.

[26] Gray, E.; Tall, D.; Duality, Ambiguity, and Flexibility: A "Proceptual" View of Simple Arithmetic, Journal for Research in Mathematics Education 25(2) (1994), 116-140.

[27] Halmos, P.: I want to be a mathematician. An automathography. Springer-Verlag, New York, 1985.

[28] Hellman, G.: Constructive Mathematics and Quantum Mechanics: Unbounded Operators and the Spectral Theorem, *Journal of Philosophical Logic* **12** (1993), 221-248.

[29] Hodgson, B.R.: Le calcul infinitésimal. In D.F. Robitaille, D.H. Wheeler et C. Kieran, dir., Choix de conférence du 7e Congrès international sur l'enseignement des mathématiques (ICME-7). Presses de l'Université Laval, 1994, pp. 157-170.

[30] Jesseph, D.: Leibniz on the foundations of the calculus: the question of the reality of infinitesimal magnitudes. Leibniz and the sciences. *Perspect. Sci.* **6** (1998), no. 1-2, 6–40.

[31] Kanovei, V.; Shelah, S.: A definable nonstandard model of the reals. *J. Symbolic Logic* **69** (2004), no. 1, 159–164.

[32] Katz, Karin Usadi; Katz, M.: A strict non-standard inequality $.999\ldots < 1$. See arXiv:0811.0164

[33] Keisler, H. Jerome: Elementary Calculus: An Infinitesimal Approach. Second Edition. Prindle, Weber & Schimidt, Boston, '86.

[34] Kiepert, L.: Grundriss der Differential- und Integralrechnung. I. Teil: Differentialrechnung. Helwingsche Verlagsbuchhandlung. Hannover, 12th edition. 1912. XX, 863 S.

[35] Kohlenbach, U.; Oliva, P.: Proof mining: a systematic way of analyzing proofs in mathematics. *Tr. Mat. Inst. Steklova* **242** (2003), Mat. Logika i Algebra, 147–175; translation in *Proc. Steklov Inst. Math.* 2003, no. 3 (242), 136–164.

[36] Lakatos, Imre: Cauchy and the continuum: the significance of nonstandard analysis for the history and philosophy of mathematics. Math. Intelligencer 1 (1978), no. 3, 151–161 (originally published in '66).

[37] Lakatos, Imre: Proofs and refutations. The logic of mathematical discovery. Edited by John Worrall and Elie Zahar. Cambridge University Press, Cambridge-New York-Melbourne, 1976.

[38] Lightstone, A. Harold: Infinitesimals. *Amer. Math. Monthly* **79** (1972), 242–251.

[39] Luxemburg, W.: Nonstandard analysis. Lectures on Abraham Robinson's Theory of infinitesimals and infinitely large numbers. Pasadena: Mathematics Department, California Institute of Technology' second corrected ed., 1964.

[40] Manning, Kenneth: private communication, july '09.

[41] Medvedev, F. A. Nonstandard analysis and the history of classical analysis. Patterns in the development of modern mathematics (Russian), 75–84, "Nauka", Moscow, 1987.

[42] Medvedev, F. A.: Nonstandard analysis and the history of classical analysis. Translated by Abe Shenitzer. *Amer. Math. Monthly* **105** (1998), no. 7, 659–664.

[43] Monaghan, J.: Young peoples' ideas of infinity. *Educational Studies in Mathematics* **48**, (2001), no. 2-3, 239-257.

[44] Novikov, S. P.: The second half of the 20th century and its conclusion: crisis in the physics and mathematics community in Russia and in the West. Translated from Istor.-Mat. Issled. (2) No. 7(42) (2002), 326–356, 369; by A. Sossinsky. Amer. Math. Soc. Transl. Ser. 2, 212, Geometry, topology, and mathematical physics, 1–24, Amer. Math. Soc., Providence, RI, 2004.

[45] Richman, F.: Confessions of a formalist, Platonist intuitionist. April 9, 1994. See http://math.fau.edu/Richman/HTML/Confess.htm

[46] Richman, F.: Is $0.999\ldots = 1$? *Math. Mag.* **72** (1999), no. 5, 396–400.

[47] Robinson, A.: Non-standard analysis. North-Holland Publishing Co., Amsterdam 1966.

[48] Robinson, A.: Non-standard analysis. Reprint of the second (1974) edition. With a foreword by Wilhelmus A. J. Luxemburg. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1996.

[49] Roquette, Peter: Numbers and Models, standard and nonstandard ('08). "Algebra days", may '08, Antalya.

[50] Rosenblatt, M., Editor. Errett Bishop: reflections on him and his research. Proceedings of the memorial meeting for Errett Bishop held at the University of California, San Diego, Calif., September 24, 1983. Contemporary Mathematics, 39. American Mathematical Society, Providence, RI, 1985.

[51] Sad, L. A.; Teixeira, M. V.; Baldino, R. B.: Cauchy and the problem of point-wise convergence. *Arch. Internat. Hist. Sci.* **51** (2001), no. 147, 277–308.

[52] Schubring, G.: Conflicts between generalization, rigor, and intuition. Number concepts underlying the development of analysis in 17–19th Century France and Germany. Sources and Studies in the History of Mathematics and Physical Sciences. Springer-Verlag, New York, 2005.

[53] Sfard, A.: Reification as the birth of metaphor. *For the learning of Mathematics* **14** ('94), no. 1, 44-55.

[54] Skolem, Th.: Über die Nicht-charakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen. *Fundam. Math.* **23**, 150-161 (1934).

[55] Stewart, I.: From here to infinity. A retitled and revised edition of The problems of mathematics [Oxford Univ. Press, New York, 1992]. With a foreword by James Joseph Sylvester. The Clarendon Press, Oxford University Press, New York, 1996.

[56] Stewart, I.: Professor Stewart's Hoard of Mathematical Treasures. Profile Books, October 2009.

[57] Sullivan, K.: The Teaching of Elementary Calculus Using the Nonstandard Analysis Approach, *Amer. Math. Monthly* **83** (1976), 370-375.

[58] Tall, D.: Mathematical intuition, with special reference to limiting processes, Proceedings of the Fourth International Congress on Mathematical Education, Berkeley, 170-176 (1980).

[59] Tall, D.: The psychology of advanced mathematical thinking, in Advanced mathematical thinking. Edited by David Tall. *Mathematics Education Library*, **11**. Kluwer Academic Publishers Group, Dordrecht, 1991.

[60] Tall, D.: Cognitive Development In Advanced Mathematics Using Technology. *Mathematics Education Research Journal* **12** (3) (2000), 210-230.

[61] Tall, D.: Natural and Formal Infinities, *Educational Studies in Mathematics* **48** (2-3) (2001).

[62] Tall, D.: Dynamic mathematics and the blending of knowledge structures in the calculus, pp. 1-11 in Transforming Mathematics Education through the use of Dynamic Mathematics, *ZDM* (june '09).

[63] Tall, D.: How humans learn to think mathematically. Princeton University Press, 2010.

[64] Tall, D; Schwarzenberger, R.: Conflicts in the Learning of Real Numbers and Limits. Mathematics Teaching 82 (1978), 44-49.

[65] Weller, K.; Arnon, I.; Dubinsky, E.: Preservice Teachers' Understanding of the Relation Between a Fraction or Integer and Its Decimal Expansion. *Canadian Journal of Science, Mathematics and Technology Education*, 1942-4051, Volume 9 (2009), no. 1, 5–28.

Department of Mathematics, Bar Ilan University, Ramat Gan 52900 Israel
E-mail address: katzmik@macs.biu.ac.il