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A COMMENTARY ON FREUDENTHAL'S DIDACTIC PHENOMENOLOGY OF THE MATHEMATICAL STRUCTURES ASSOCIATED WITH THE NOTION OF MEASUREMENT

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Abstract. This paper discusses Freudenthal's didactical phenomenology for the mathematical structures related to measurement. Freudenthal starts with the set G of all objects having the attribute of weight, an attribute that is to be measured. He proposes two operations, the first one among the measurements themselves, called "addition". Addition is interpreted in terms of the mental actions associated with measurement and, in the case of weights, it consists on the process of placing weights on one of the two dishes of a balance in order to balance them with a predetermined gauge, its fractions or multiples thereof, which are placed on the other. Their total weight is obtained by comparison with a predetermined gauge. The second operation, between weights and positive integers, allow weights to be amplified (by adding as many times as desired their weight measure). A related operation to amplification is that of contraction (by dividing the weight into as many equal parts as desired.) Freudenthal then extends his definitions in order to assign a meaning to the multiplication of a measure by, first, any positive rational number, and, second, by completeness, to any real number. Freudenthal's didactical phenomenology is based on the historical phenomenology of measurement, fractions, decimals and percentages. It is a central tenet of Realistic Mathematics Education (RME) that didactical phenomenologies of mathematical structures constitute proposals from which to make cognitively advantageous teaching sequences for the relevant mathematical structures. We discuss how Freudenthal's phenomenology of the mathematical structures for measurement gets translated into the mathematical models associated with the rational numbers. Finally, we make reference to a paradigmatic teaching unit which incorporates coherently Freudenthal's didactical phenomenology.

Keywords: Realistic Mathematics Education, fractions, percentages, decimals, models, didactical phenomenology.

1. Introduction

[1] In [1, Freudenthal (1973)] a phenomenology for the mathematical structures relevant to measurement is presented. Freudenthal's exposition is paradigmatic of the process of "modelization" at two levels. First, he starts with the description of the mental actions relevant to the phenomena, in this case, measurement of weights, (horizontal mathematization) to then begin a process of a more abstract formalization which subsumes his original description of the "phenomena" (vertical mathematization). Hence, Freudenthal's original model in terms of the mental actions germane to measurement constitutes a *descriptive* model of his didactical phenomenology, and he transforms it into a *prospective* model for the mathematical structures of measurement. We would like to discuss this point since it illustrates that the vertical mathematization implicit in evolution of descriptive models into prospective models, both in general and in the study of measurement in particular, not only introduces new formal elements into the descriptive model to transform it into the prospective one, but it also, incorporates the corresponding mental proceses and associated logical interconnections. This can be regarded as part of the verticalization process and, in our view, it is, in part, an ingredient of what is generally called "number sense." Implicit in this verticalization there are elements of automatization of structural model features (formal features) as well as mental processes. Such automatizaton empowers students to add to their knowledge efficiently as they are able to muster the resources automatized and use them as basic or given elements in the subsequent use of models to *describe* new

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mathematical situations. We will give examples of this using the Realistic Mathematics Education (RME) approach to fractions, percentages and decimals, as presented, for instance, in [3, Keijzer et all (2006)]. Since in RME, didactical sequences are to be based on an initial proposal suggested by a didactical phenomenology of the appropriate mathematics, we begin by discussing first Freudenthal's didactic phenomenology for measurement, as discussed in [1, Freudenthal (1973), pp. 193-212].

2. Mathematical formulation of magnitudes according to Freudenthal

Definition 2.1. A set of magnitudes is a set G with and operation $+ : G \times G \to G$ (called "addition") and an order relation $< \subseteq G \times G$ such

i. $<$ **satisfies trichotomy:** for all $a, b \in G$ one and only one of the following conditions hold: $a < b$ or $a = b$ or $b < a$.
ii. $<$ **satisfies transitivity:** for all $a, b, c \in G$, if $a < b$ and $b < c$ then $a < c$.
iii. $+$ **is associative:** for all $a, b, c \in G$, $(a + b) + c = a + (b + c)$.
iv. $+$ **is commutative:** for all $a, b \in G$, $a + b = b + a$.
v. $+$ **satisfies the cancellation law:** for all $a, b, c \in G$, if $a + c = b + c$, then $a = b$.
Also,
vi. For all $a, b \in G$, $a < b$ if and only if for some $c \in G$ (necessarily unique) $a + c = b$.
vii. Measurements can be amplified by whatever integral factors desired:
The symbol "$n \cdot a$", for n a positive integer and $a \in G$, is defined by induction as $1 \cdot a = a$ and for each positive integer n, $(n + 1) \cdot a = n \cdot a + a$. We call $n \cdot a$ the amplification of a by the factor n.
viii. Measurements can be divided into as many equal parts as desired:
For all $a \in G$ and each integer $n \geq 1$ there is $b \in G$ (uniquely determined) so that $n \cdot b = a$. We write

$$b = \frac{1}{n} \cdot a.$$

A straightforward argument using the definition proves that $\frac{1}{n} \cdot (\frac{1}{m} \cdot a) = \frac{1}{nm} \cdot a$, for all n, m positive integers and all $a \in G$.

Freudenthal (ibidem) describes an example of a set G consisisting of all objects having the attribute of weight. We can define two relations as follows: two objects a and b are the equivalent if they both can be used to level the balance; we write $a \approx b$ in such a case; the weight a is smaller than the weight b if a is the lighter of the two; we write $a < b$. Clearly $\approx$ and $<$ are relations on G. The relation $\approx$ is an equivalence relation and $<$ satisfies the conditions of Definition 2.1.

An important observation is the following:

Proposition 2.2. If n and m are positive integers, then

$$\frac{1}{m}(n \cdot a) = n\left(\frac{1}{m} \cdot a\right),$$

for all $a \in G$.

Proof. First of all, it is true and straightforward to check that $p \cdot (q \cdot a) = (p \cdot q) \cdot a$ for all integers p, q with $p \geq 1$, $q \geq 1$ and for all $a \in G$ (note that the various occurrences of the symbol " $\cdot$ " can have different meanings in this relation. So suppose that $a \in G$ and that n, m are integers with $n \geq$ and $m \geq 1$. Let $b = \frac{1}{m} \cdot a$ be the unique element of G satisfying $a = mb$. It is also clear that for all integers $p \geq 1$ and $a \in G$,

$a = \frac{1}{p} \cdot (p \cdot a)$. Then

$$\begin{aligned}
\frac{1}{m} \cdot (n \cdot a) &= \frac{1}{m} \cdot (n \cdot (m \cdot b)) \\
&= \frac{1}{m} \cdot ((n \cdot m) \cdot b) \\
&= \left(\frac{1}{m} \cdot (m \cdot n)\right) \cdot b \\
&= \left(\left(\frac{1}{m} \cdot m\right) \cdot n\right) \cdot b) \\
&= n \cdot b \\
&= n \cdot \left(\frac{1}{m} \cdot (m \cdot b)\right) \\
&= n \cdot \left(\frac{1}{m} \cdot a\right).
\end{aligned}$$

□

All this goes to show that for $a \in G$ and integers n and such that $n \geq 1$ and $m \geq 1$, the expression $\frac{n}{m} \cdot a$ is well defined, either as $n \cdot (\frac{1}{m} \cdot a)$ or $\frac{1}{m} \cdot (n \cdot a)$

Remarks 2.3.

a. Using the axioms and mathematical induction, it is easy to ascertain that the following properties hold for positive integers n, m and $a, b \in G$:
   i. $(n + m) \cdot a = n \cdot a + m \cdot a$;
   ii. $n \cdot (a + b) = n \cdot a + n \cdot b$;
   iii. $n \cdot (m \cdot a) = (nm) \cdot a$

b. It can also be shown that for each positive integer m,

$$\frac{1}{m} \cdot (a + b) = \frac{1}{m} \cdot a + \frac{1}{m} \cdot b.$$

   Using this fact it is easily verified that ai-aiii also hold for when n and m stand for positive rational numbers.

c. A consequence of the above considerations is the relation for the addition of fractions. Of course, "addition" does not refer to some previously defined algebraic rule, but rather to the addition of weights in G as postulated by Freudenthal. Students have a hard time understanding why not define addition of fractions by

$$\begin{aligned}
&\frac{n}{m} + \frac{n'}{m'} = \frac{n + n'}{m + m'} \text{ rather than the, perhaps, unexpected} \\
&\frac{n}{m} + \frac{n'}{m'} = \frac{nm' + n'm}{mm'}.
\end{aligned}$$

   It is straightforward to prove from the above properties that $n \cdot a = \frac{1}{m}(n \cdot m) \cdot a$ for every pair of positive integers n, m and every $a \in G$. But of course we can take $G = \mathbb{Q}^+$ and then, if $s \in G$ we have, for positive integers n, n', m, m',

$$\begin{aligned}\left(\frac{n}{m}+\frac{n'}{m'}\right)\cdot s &= \frac{n}{m}\cdot s+\frac{n'}{m'}\cdot s\\ &= \frac{1}{m}(n\cdot s)+\frac{1}{m'}(n'\cdot s)\\ &= \frac{1}{m}\left(\frac{1}{m'}\cdot(n\cdot m')\cdot s\right)+\frac{1}{m'}\left(\frac{1}{m}\cdot(n'\cdot m)\cdot s\right)\\ &= \frac{1}{m\cdot m'}\cdot\Big(n\cdot m'+n'\cdot m\Big)\cdot s\\ &= \frac{n\cdot m'+n'\cdot m}{m\cdot m'}\cdot s.\end{aligned}$$

Taking $s = 1$ the formula follows.

The next point that Freudenthal discusses is the introduction of a gauge, or a unit of measurement $\mathbf{e} \in G$. We know $\mathbb{Q}^+ \cdot \mathbf{e} \subseteq G$ but perhaps equality does not hold. Since a system of measurement, as we know it, admits arbitrary additions of the unit (that is, arbitrary multiplication of the unit by positive integers) we expect that such additions will eventually exceed any given measurement. In other words, no matter what weight we place on a plate of the balance, if we keep placing units on the other plate, the balance will tilt and the plate containing the units will be lowered. Similarly we expect to have infinite divisibility of the weights. We can divide them in halves, thirds, fourths, etc, without having to stop. These are universal features of measuring systems, be it length, areas on the plane, volumes in space etc. Freudenthal incorporates these properties into his phenomenology of the mathematics of measurement by enunciating the archimedean property of the systems of weights. Before stating the property we make a straightforward observation and that is that for every $c \in G$, $\mathbb{Q}^+ \cdot c$ cannot have a smallest or a largest element. In fact if $r \in \mathbb{Q}^+$ and $s \in \mathbb{Q}^+$ satisfy $s < r$, then as we shall see $s \cdot c < r \cdot c < (r+1) \cdot c$. The property then, as stated by Freudenthal is:

Principle 2.4 (**Archimedean Principle (AP)**)**.** For a given $c \in G$, G has no elements smaller than all the elements of $\mathbb{Q}^+ \cdot c$ nor larger than all the elements of $\mathbb{Q}^+ \cdot c$. In other words, G contains neither infinitesimal weights nor infinite weights[2].

In order to state the archimedean principle in Freudenthal's setting we need tho following discussion (the easy proofs are left to the interested reader):

Proposition 2.5. Suppose $r, r' \in \mathbb{Q}^+$ and $a, a' \in G$. The following statements are valid:

i. If $r > r'$, then $r \cdot a > r' \cdot a$.
ii. If $r > r'$ and $a > a'$ then $r \cdot a > r' \cdot a'$.

Remark 2.6. Now we can rephrase the AP as: For every $a, b \in G$ there are $r, s \in \mathbb{Q}^+$ such that

$$\begin{aligned}&rc < a \text{ and}\\ &sc > a.\end{aligned}$$

This is clear in view of previous remarks, since the first inequality says that a cannot be smaller than all the elements of $\mathbb{Q}^+ \cdot c$ and the second inequality says that a cannot be larger than all the elements of $\mathbb{Q}^+ \cdot c$.

Freudenthal then proves:

Theorem 2.7. *G satisfies the AP if and only if for all $a, b \in G$ there is $n \in \mathbb{Q}^+$ such that $na > b$.*

[2]This point is, in our view, more controversial, since there are historical phenomenologies of non standard mathematical structures useful for analysis and for a great diversity of mathematics, including measurement. For instance, a non standard didactical phenomenology of the mathematical structures of the XVII century would yield, in our view, improvements in the teaching of the calculus. But the historical phenomenology for this attempt is at odds with the history of mathematics in which non-standard analysis comes to being in the twentieth century, after the work of Abraham Robinson. The historical phenomenology, however, is not to be ignored. The calculus can be presented by means of non standard analysis in very much the same way its originators thought it out, but the presentation of the non standard theoretical foundations of Calculus took more than 200 years after the invention of the calculus to consolidate, so there might very well be a cognitive advantage to study the calculus itself with infinitesimals, but the study of its theoretical foundations is another matter altogether. But we leave the discussion of this interesting point for a future time.

This is, of course, the usual statement of the AP.

Proof. The proof depends on a property of $\mathbb{Q}^+$ that we now state: for every $r \in \mathbb{Q}^+$ there are positive integers n, m such that $\frac{1}{n} < r < m$ (in fact, we can choose a positive integer k such that $\frac{1}{k} < r < k$). By the well ordering of the positive integers $k = \min\{p \,|\, p$ is a positive integer and $r < p\}$ exists. A similar argument using $\frac{1}{r}$ instead of r yields a positive integer p such that $\frac{1}{r} < p$ and p is the smallest such. In particular, we have,

$$\frac{1}{p} < r < k.$$

Taking $q = \max\{p, k\}$ we have

$$\frac{1}{q} < r < q.$$

So suppose G satisfies AP, and let $a, b \in G$. Since b is not larger than all of the elements of $\mathbb{Q}^+ \cdot a$, there is $r \in \mathbb{Q}^+ \cdot b$ such that $r \cdot a > b$. By our remark, there is a positive integer n such that $n \cdot a > b$.
Conversely, if the conclusion of the theorem holds, then it is clear that there are $r, s \in \mathbb{Q}^+$ such that $r \cdot a > b$ (taking $r = n$) and $s \cdot b > a$ (taking $s = \frac{1}{n}$). $\square$

Remarks 2.8.

i. We now ask about the meaning of $x \cdot a$ when $x \in \mathbb{R}$ and $a \in G$. It should be remarked that, most likely, the set of weights consists only of rational numbers. In practice you can only made so many fractional subdivisions of the unit so, since weighting is a comparison with the unit and there are only so many fractional subdivisions of it, we expect to have only rational number as weights. The same is true about other measurements; in fact all of what is being said here is that given a measuring instrument based on a given gauge, you can measure accurately until the last subdivision mark of your scale and then estimate the next one. For instance, in measuring length we have some calibrated stick and the calibration is in multiples of a certain unit and fractions of the same unit. So when measuring a given length we count all of the marks and estimate the next marking not on the stick to get, again, a rational number for the length. But rational numbers are not enough for a detailed analysis of measurement. Measurement in the case of lengths leads to the matric number line model, and to have the full power of the analysis resources, we must have real numbers and not just rational numbers. For instance, calculus and analysis are not possible on the rational number line, since most of the results that constitute the pillars of the calculus are not valid on this line. In fact, the main theorems are false (like the Mean Value Theorem, the Fundamental Theorems of the Calculus or the Intermediate Value Theorem). So it is important to be able to define the operation implicit in the symbol $r \cdot a$ for $r \in \mathbb{R}$ and $a \in G$. We indicate a way to do this, a little different than the one used by Freudenthal, but equivalent to it.

ii. Suppose now that $x \in \mathbb{R}^+$ and $a \in G$. By the completeness of the real numbers we know that $x = \sup\{r \,|\, r < x\}$ so it would be "natural" to define

$$x \cdot a = \sup\{r \cdot a \,|\, r \in \mathbb{Q}^+ \text{ and } r < x\}. \tag{2.1}$$

The set on the right is a non empty and bounded set of elements of G and must have a least upper bound; we leave the easy details of this fact to the interested reader. However, the least upper bound in (2.1) need not, necessarily, belong to G. In other words, there is the possibility that for some $a \in G$ and $x \in \mathbb{R}$ the real number defined as $x \cdot a$ does not belong to G. Freudenthal call for one additional axiom:

iii. For all $x \in \mathbb{R}$ and all $a \in G$, $\sup\{r \cdot a \,|\, r \in \mathbb{Q}^+ \text{ and } r < x\} \in G$. With this axiom the least upper bound that we write as $x \cdot a$ is an element of G as expected. In other words, we suppose that $\mathbb{R}^+ \cdot a \subseteq G$ for all $a \in G$.

We can use (2.1) to show that the properties listed in the Remarks 2.3 hold when n and m are real numbers. We leave the details to the reader.

Finally, Freudenthal proposes the following definition.

Definition 2.9. Suppose $e \in G$ is a gauge. The "measure function with respect to the gauge e" is defined as the function $\nu : G \to \mathbb{R}^+$, such that for every $a \in G$,

$$\nu_e(a) = \sup\{r \in \mathbb{Q}^+ \,|\, r \cdot e < a\}.$$

We have the following straightforward result:

Theorem 2.10. *With the notation of Definition 2.9 we have for all $a, b \in G$ and all $t \in \mathbb{Q}^+$*

i. *If $a < b$, then $\nu_e(a) < \nu_e(b)$;*
ii. $\nu_e(a+b) = \nu_e(a) + \nu_e(b)$;
iii. $\nu_e(e) = 1$;
iv. $\nu_e(t \cdot a) = t\nu_e(a)$.

In this way Freudenthal makes the set of measures into a collection of real numbers.

3. Essential elements for a teaching sequence based on Freudenthal's didactic phenomenology of the mathematical structures related to measurement and rational numbers

The reader will, surely, have no difficulty in rephrasing in his own words the contents of each one of the above statements. From Freudenthal's Didactic phenomenology some important features that a teaching sequence for fractions percentages and decimals must have. He list them here:

i. Descriptive models should be proposed to illustrate the addition operation of magnitudes. For instance, in the case of length, addition presupposes de placing of a length juxtaposed to another one, without overlap, in order to determine the total length resulting (by comparison to the gauge). In the case of weights, addition is the placing of two weights on the plate of a balance to determine a second weight, again, by comparison to the weight gauge.
ii. Since $G = \mathbb{Q}^+$ is an example, in fact a canonical one, we must not lose sight of the fact that Freudenthal's didactical phenomenology of measurement applies to the teaching of fractions themselves. So all of the other comments about possible descriptive/ prospective models for measurement are applicable to fractions. This is the most ignored guideline in fraction curricular design[3].
iii. Descriptive models should be proposed to illustrate the multiplication of the unit (measure of the gauge) as well as its infinite subdivisions of the gauge, first by fractions of positive integers.
iv. The didactical phenomenology of measurement has built into it the operation of measures by rational numbers. Prospective models should be appropriate for the representation of fractions themselves and not only fractional parts of other measurements. It is also clear from Freudenthal's phenomenology for measure, that prospective models should also be appropriate for placing the fractions themselves on the line and the representation of fractional whole-part relationships.
v. Other miscellaneous models serve to integrate and consolidate the notions developed through the previous models.

We will know illustrate the descriptive and prospective models used by RME to create teaching sequences for fractions, decimals and percentages. We will draw from some of the Mathematics in Context units, particularly from [3, Fraction Times (2006)]. We organice the discussion according to the previous list.
Notable models in the implementation of the didactical phenomenology of measurement.

i. **Paper strip folding models** (for measuring using a given gauge for length.)
Typically, in using these models, students are asked to prepare paper strips of arbitrary length to measure lengths of objects in the classroom. In measuring lengths larger than a unit students juxtapose without overlapping the strip and get some integer multiple of the length of the strip. Then, to get the unaccounted part of the total length, if any, the student folds the strip in sub-unit lengths to get measurements of 1/2, 1/4, 1/8, etc. Since the folding in half process leads naturally to the diadic or base two representation decimals. Since our representation of numbers uses ten or decimal notation, the process of folding by halving, which is a practical descriptive model, gets transformed into "mental" models of folding ten fold (which is not so easy to do manually). When the model is used in situations where, say, liquids in tanks are measured, then the model obtained is a double line, calibrated in capacity units on one end (like gallons or liters) and in fractions on the other. Such double lines can be varied

[3] According [4, Wu (2014), p. 4] "For grades 5 and up, there is no choice: there has to be a definition of a fraction. By and large, school mathematics (if textbooks are any indication, regardless of whether they are traditional or reform) does not provide such a definition, so that teachers and students are left groping in the dark about what a fraction is." It is Wu's view that fractions should be defined not as possible results of fair division processes, but rather as numbers on the line. This is a valuable suggestion that follows directly from Freudenthal's didactical phenomenology for measurement

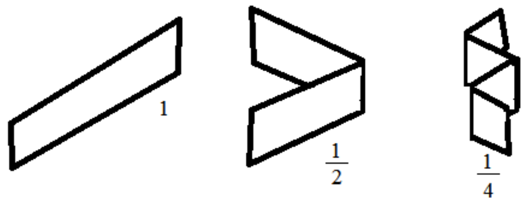

FIGURE 1. Folding paper strips for measuring

to include markings that do not correspond to a folding mark in order to help students estimate the multiples and the sub-parts of the unit (see Figures 2, and 3:

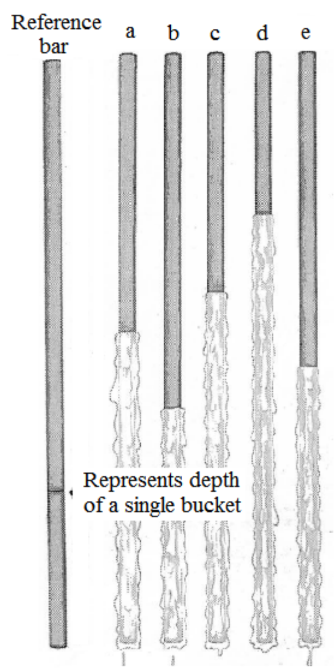

FIGURE 2. Bars for measuring liquids

In this process the student are given markings on a strip that corresponds to none of the folds of the strip and asked to give the length. For instance,

The above descriptive models help students learn the mental actions involved in the estimations of the size of fractional parts not represented by markings. This is a very important skill in actual measurement and in understanding the decimal representation of numbers. Also, students are able to appreciate the whole-part interpretation of fractions, when a double line is calibrated in fractions on one edge and in fractions on the other.

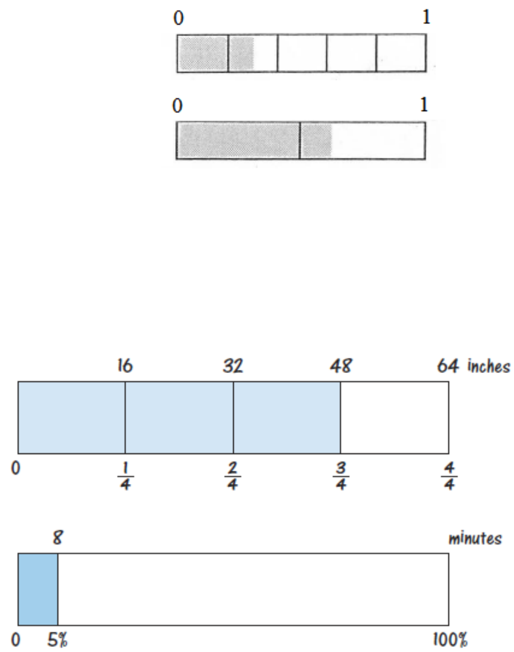

Figure 3. Estimating fractional divisions

ii. **Double line models** To introduce these, students are given contexts where they can read scales in two different ways, as in Figure 4.

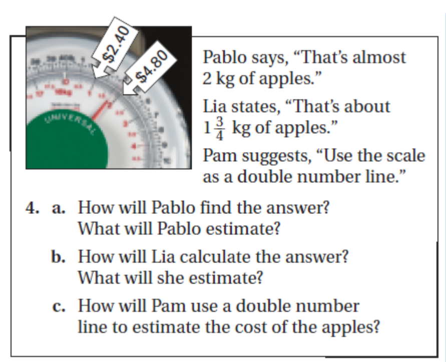

Figure 4. Double scales context

Also, pie charts initially used to represent fractional data and solve problems of fair divisions, progressively turn into pie charts with scales and double lines; see Figure 5. Note that in this Figure the pie chart is "unfolded" double line in which the sections show the whole-part relations for the corresponding context.

Calibrated double line models are introduced based on the paper strip models as in Figure 7. Here gauges mesuring volumes of liquids in tanks, look suspiciously similar to folded strips.

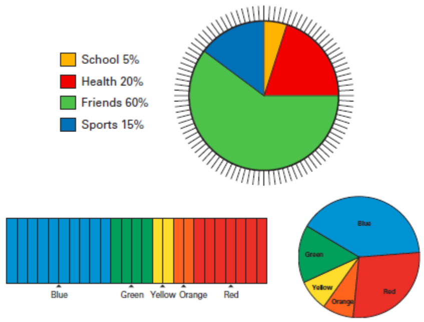

FIGURE 5. Double scales context

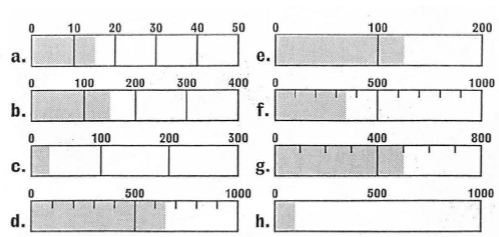

FIGURE 6. Gauges to measure liquids in tanks

Similarly, the context of maps provide interesting double lines as in Figure 7.

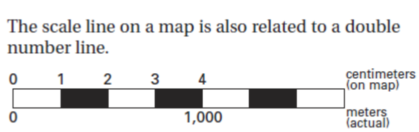

FIGURE 7. Scale on a map

Division	Decimal	Fraction
$1 \div 2$		
	0.25	
		$\frac{1}{10}$
		$\frac{1}{20}$
$3 \div 4$		
	0.15	
		$\frac{2}{5}$

FIGURE 8. Money tables

iii. **Money models** Money is a very useful context to get the basic equivalences between fractions decimals and percentages for the fractional parts of the coins in which the dollar brakes. Also, for such fractional parts, one gets also the equivalent formalisms of fractions, decimals and percentages. For instance, the tabulation of Figure 8 is required of students to link long division with decimals and fractions. For instance, the formalisms of percentage and decimals for the fractions $\{\frac{1}{20}, \frac{1}{10}, \frac{1}{4}, \frac{1}{2}\}$ become clear in the context of the values of coins (in cents) and the fractional parts represented by such coins. Interesting exceptions are fractions like $\frac{1}{3}$. Long division duly interpreted yield after a few of the steps are carried out the following (exact) expressions:

$$\frac{1}{3} = 0.3 + \frac{1}{10};$$
$$\frac{1}{3} = 0.33 + \frac{1}{100};$$
$$\frac{1}{3} = 0.333 + \frac{1}{1000}.$$

Hence, to write $\frac{1}{3} = 0.333\cdots$ takes a limiting process which in effect expresses the facts that the finite decimals $0.3, 0.33, 0.333$, etc will get arbitrarily close to the actual value of fraction $\frac{1}{3}$. Hence "long division" will be the basis for establishing that every fraction is actually a decimal, finite or infinite, but if it is infinite, it must be a repeating decimal. There is, to be sure, something to be said about the "meaning" of an infinite decimal. Infinite non repeating decimals are not rational numbers. in fact, the non completeness of the rational numbers can be stated by saying that there are digits (integers from 0 to 9), say, $d_1, d_2, d_3, \cdots$ so that the corresponding decimal is not the decimal expansion of any rational number. Also the finite decimals are precisely the decimals that admit two expressions as decimals, a finite one and another which is infinite (and periodic). At first, these discussions about infinite decimals are informal in nature, but the "explanation" is basically the same, in other words a paradigmatic situation of decimal expressions. Using the infinite divisibility of the unit of the real line successively divided in ten parts, $0.999\cdots$ is is clear that $1 = 0.999\cdots$. In fact 0.9 lies in the last interval when the unit is divided in ten equal parts, 0.99 lies in the last interval when the unit divided in 100 equal subintervals, and so on. Hence, for example, $0.44 = 0.43999\cdots$, since $0.00999\cdots = \frac{0.999\cdots}{10^2} = \frac{1}{10^2} = 0.01$, so that $0.44 = 0.43 + 0.01 = 0.43 + 0.00999 = 0.43999\cdots$. Similarly $0.25 = 0.24999\cdots$, etc. All this initially must be discussed informally until the notion of limit is discovered. Furthermore, the long division process can never yield the infinite decimal expression of a finite decimal.

iv. **Automatization of arithmetic competencies related to the notational formalisms of rational numbers** There are strong relations between the various formalisms for rational numbers and there are exercises to automatize the arithmetic operations and the interrelations of the various formalisms. Consider the following one:
Complete the following table only using the fact that

$$36 \times 75 = 2\,700.$$

Then,

$18 \times 75 =$	
$72 \times 75 =$	
$9 \times 150 =$	
$36 \times 25 =$	
$36 \times 150 =$	
$3.6 \times 1.50 =$	
$7.2 \times 7.5 =$	
$0.9 \times 1.5 =$	
$0.36 \times 1.25 =$	

v. **Jumps of ten line model** Since the decimal notational system capitalize on translations on the line of 1, 10, 100 etc. (supra unit) or 1, $\frac{1}{10}$, $\frac{1}{100}$ etc. (sub unit) descriptive models can be supplied to reflect these strutural features of numbers; see Figure 9.

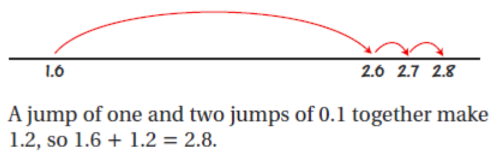

FIGURE 9. Jumps of ten line model

vi. **Infinite divisibility line models** Our decimal number system is based on the iterated amplification or contraction by factors of ten. In fact, a decimal like 0.96 represents the number x that lies in the tenth interval when $[0, 1]$ is divided in ten equal intervals. It also lies in the seventh interval when $[0.9, 1]$ is divided in ten equal subintervals. Similarly for infinite decimals, the only exception being finite decimals. When the decimal ending in an infinite sequence of nines is chosen for a decimal fraction, then the number lies in the tenth subintervals from a certain point on of the subdivisions. In any case, the infinite divisibility models are useful in understanding the density on the real line of the decimal fractions. An example appears in Figure 10

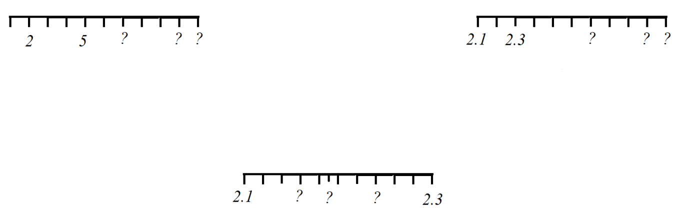

FIGURE 10. Infinite divisibility segments

vii. The present didactical sequence for rational numbers (fractions, percentages and decimals) leads, as it should to the study of algebra where a systemic view "number" is needed. The number referred to is, of course, real numbers, a set that consists of the collection of all decimals. Rational numbers are periodic decimals, infinite of otherwise. Irrational numbers are infinite non periodic decimals. In this setting, the completeness axiom states that any sequence of digits (integers from 0 to 9) consists of the digits of a decimal expansion of a real number. But all of these latter considerations are nowadays matter of study at the university level.

4. CONCLUSION

In a recent special issue of The Mathematics Enthusiast [2, (2014), pp.197-200], a knowledge base related to seminal areas of study in elementary school mathematics (such as fractions, decimals, algebra, geometry and measurement) is discussed. According to the exposition, the example of measurement is central in this knowledge base, specially as it is addressed in some depth by the Common Core State Standards. Measurement, fractions, decimals and percentages are part of the mathematically intertwined list of mathematical topics of study for primary school, which are in urgent need of coherent and integrated presentations linking the multifaceted mathematical structures that give coherence to the topics. The present work presents a plan for curriculum development in these areas which does precisely this, and illustrates it by means of exemplary didactical units.

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