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A Brief Look at the Evolution of Modeling Hyperbolic Space

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Abstract: Now considered one of the greatest discoveries of mathematical history, hyperbolic geometry was once laughed at and deemed insignificant. Credited most to Lobachevsky and Bolyai, they independently discovered the subject by proving the negation of Euclid's Fifth Postulate but neither lived long enough to see their work receive any validation. Several models have been constructed since its acceptance as a subject. I'm going to discuss three graphical models: the Klein Model, the Upper Half Plane Model and the Poincaré Disk Model. I will also discuss and construct three tape and paper models: the Annular Model, the Polyhedral Model and the Hyperbolic Soccer Ball Model. However, I will spend the most time discussing the construction and uses of crocheted models as well as some brief information on how these models are being used by Margaret and Christine Wertheim to promote global environmental effects on coral reef.

Key words: hyperbolic, model, crochet, Daina Taimina

Introduction

The original introduction of hyperbolic geometry, also known as non-Euclidean or Lobachevskian geometry, was not widely celebrated. At the time, discounting the work of Euclid was not only frowned upon, but for many years not accepted. Euclid published the *Elements* in 300 BC. Within this work was Euclid's Fifth Postulate, essentially stating that if two lines on the same plane are not parallel, they will eventually cross. A statement that sounds like a no brainer

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but in actuality, can't be proven. Mathematicians spent 2,000 years trying to prove Euclid's fifth postulate but no one was ever successful. In 1800 Lagrange was convinced that he had finally done the impossible and proven it to be true, but mid presentation he realized he had made a mistake (Bardi, 2009). Because the work on the fifth postulate was so diligent, it hadn't crossed many minds that the postulate may not be true. Euclid's *Elements* were accepted as part of reality so most mathematicians ridiculed and ignored the idea of "imaginary geometry" (Bardi, 2009). However, once this idea of a non-Euclidean space was accepted, mathematicians began searching for a complete hyperbolic plane to work with as well as trying to find a sufficient way to model this plane. This turned out to be a nearly insurmountable task. To understand why this was such an overwhelming task, we will explore what makes the hyperbolic plane so distinct from the Euclidean plane.

It should be noted that Nikolai Lobachevsky wasn't the only mathematician to work with hyperbolic space, although he is the better known of the mathematicians who did. Janos Bolyai published work on the subject independently of Lobachevsky. Upon his discovery of hyperbolic space, Bolyai's father submitted his work to Karl Friedrich Gauss who replied claiming to have already discovered the subject, just never publishing his work (Struik, 2008). This was a devastating response to receive, causing Bolyai to feel robbed of any priority status on the subject (Struik, 2008). As we've seen in mathematics before, independent discovery of the same subject often results in a feud and with three mathematicians working with this subject we'd expect to read about a dispute. But there wasn't one. Because the idea of hyperbolic geometry was so widely rejected, the discovery of the subject was less of an accomplishment and more of a burden. Since none of the three mathematicians could stir up any interest, no one needed to fight for the glory. There's been more dispute over who should be credited in the twentieth and twenty

first centuries than there ever was in the 1830's. Some scholars feel Gauss shouldn't be credited at all since he didn't publish anything, and some feel Bolyai and Lobachevsky should equally share the credit rather than Lobachevsky holding most of the acclaim (Bardi, 2009).

Hyperbolic geometry was developed as the result of these mathematicians' work with Euclid's Fifth Postulate. Many mathematicians before them had assumed the negation of the postulate in an effort to find a contradiction and in turn prove the postulate to be true, but all failed. Lobachevsky and Bolyai did more than assume; they proved the assumption that in the plane formed by a line and a point not on that line, it is possible to draw infinitely many lines through the point that are parallel to the original line, thus negating Euclid's Fifth Postulate (Bazhanov, 2014). This led to the Universal Hyperbolic Theorem which states, "In hyperbolic geometry, for every line l and every point P not on l there passes through P at least 2 distinct parallels. Moreover, there are infinitely many parallels to l through P " (Math Explorer Club [MEC], 2009). But are these parallels equidistant like parallels in Euclidean space? The answer is no. Parallel lines in the hyperbolic plane converge at one end and diverge at the other end (Castellanos & Darnell, 1994-2016). This, in turn, has repercussions on the idea of triangles within the hyperbolic plane. One of the first things we all learn in geometry is that the angles of a triangle always sum to 180 degrees. More technically speaking, the sum of the angles is equal to the sum of two right angles, as Euclid didn't use such a measure in his *Elements*. Do you think such a theorem exists in the hyperbolic plane? The answer again, is no. Two very important theorems exist concerning triangles in the hyperbolic plane. The first stating, "In hyperbolic geometry, all triangles have an angle sum less than 180 degrees," and the second stating, "In hyperbolic geometry, if two triangles are similar, they are congruent" (MEC, 2009). The differences between the hyperbolic plane and the Euclidean plane extend further than just these

three theorems so it's easy to understand why it was so difficult for mathematicians to accept hyperbolic geometry as a legitimate idea. Accepting all of these theorems meant accepting that there was another type of space, different than the space we live in. In his book, The Fifth Postulate, Bardi says, "To read Euclid was to know geometry, and to know geometry was to know reality." Mathematicians of the time weren't ready to question physical reality, nor were they ready to take the leap into the theoretical that Lobachevsky, Bolyai and Gauss had already taken. This discovery that should have brought all of them profound praise instead brought all of them nothing, a devastating event since a century later their work was regarded as one of the most drastic advances of mathematical history (Bardi, 2009).

These substantial differences exist between the geometries because the planes on which they exist are almost opposites. The Euclidean plane has no curvature. The hyperbolic plane has a constant negative curvature. This is easier to visualize if thought about in context to a sphere. A sphere has constant positive curvature so as to close in upon itself while hyperbolic space never behaves this way, always deflecting before it has the chance to close in on itself. This causes the space to extend infinitely (Wolf, 2011). In the words of Riemann in 1854, "hyperbolic geometry would be the intrinsic geometry of a surface with constant negative curvature that extended indefinitely in all directions" (Taimina, 2009).

Mathematicians searched long and hard for a physical representation of a hyperbolic plane but always failed because while we are surrounded by negative curvature in nature, none of these surfaces extend infinitely. Have you ever snorkeled in the ocean and seen the intricate coral reef? Maybe you looked at pictures of sea slugs in a high school biology class or you probably have lettuce in your kitchen or garden. All three of these objects have negative curvature. All three are finite, natural examples of a hyperbolic plane. It is important to realize that the

hyperbolic plane exists outside of the realm of mathematics and is an important space in our everyday lives.

Figure 1: Examples of Hyperbolic Space in Nature (Boggs, 2012) (Khoo, 2009) (Nickell, 2011)

Graphical Models of Hyperbolic Geometry

It was in 1866 when Eugenio Beltrami found the first mathematical model of the hyperbolic plane, finding it in the Euclidean three dimensional space (Margenstern, 2013). This held extensive significance. We know that the Euclidean plane is a consistent, contradiction free plane thus if the hyperbolic plane exists within Euclidean space, it is also a consistent, contradiction free plane. The finding of this plane helped solidify the idea of hyperbolic geometry as a whole. This discovery was followed by several more, including two models found by Henri Poincaré: the half-plane model and the disk model (Margenstern, 2013). These continued discoveries enhanced the significance of hyperbolic geometry, validating the subject more and more with every additional discovery.

There are several graphical models used to show hyperbolic geometry now and I am going to focus on three of them. The Klein Model, the Upper Half Plane Model, and the Poincaré Disk Model. All three of these models are in the two dimensional Euclidean space, although the disk model has a three dimensional application as well.

The Klein model of the hyperbolic plane defines the plane as the unit disk. Points in this model are defined as Euclidean points and lines are defined as portions of Euclidean lines that

intersect the disk. In order to differ this model from the Euclidean model, it is necessary to alternately define distance (Sanders, 1994). For the lines within this model to be infinite, distance is defined as follows: If (x, y) and (u, v) are the Euclidean coordinates of two points, then the hyperbolic distance between them is

$$\operatorname{arccosh} \left\{ \frac{1-xu-yv}{\sqrt{(1-x^2-y^2)(1-u^2-v^2)}} \right\}.$$

See Figure 2 for the Klein Model. Referencing the Universal Hyperbolic Theorem, I have labeled P with a red point and P and l with a blue cursive l , allowing you to see how the theorem holds within this model.

Figure 2: Demonstrating the Universal Hyperbolic Theorem with the Klein Model (Sanders, 1994)

To better understand this definition of distance, let's look at a few points. The first points we will define $(x, y) = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $(u, v) = \left(\frac{1}{4}, \frac{1}{3}\right)$. Calculating the distance in the Euclidean plane get us the following:

$$distance = \sqrt{\left(\frac{1}{4} - \frac{1}{2}\right)^2 + \left(\frac{1}{3} - \frac{1}{2}\right)^2}$$

$$distance = 0.3005$$

Calculating the distance in the Klein Model gets us the following:

$$distance = arccosh \left\{ \frac{1 - \frac{1}{2} * \frac{1}{4} - \frac{1}{2} * \frac{1}{3}}{\sqrt{\left(1 - \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right) \left(1 - \left(\frac{1}{4}\right)^2 - \left(\frac{1}{3}\right)^2\right)}} \right\}$$

$$distance = 0.4478$$

Now let's look at the points $(x, y) = \left(\frac{1}{2}, \frac{5}{8}\right)$ and $(u, v) = \left(\frac{1}{4}, \frac{11}{24}\right)$. We know on the Euclidean Plane that this is a parallel shift so the distance remains the same at 0.3005. But if you calculate this using the Klein model the distance is 0.5430. You can see from this that as the points get closer to the edge of the unit disk, the distance between points that are parallel gets greater.

The Upper Half Plane Model defines the plane to be the Euclidean upper half plane. In this model, lines are portions of circles whose centers lie on the boundaries of the plane and points are again Euclidean points (Sanders, 1994). Similar to the Klein Model, it is necessary to alter the definition of distance within this model to ensure that the distance from any point to the x-axis is infinite. In the Upper Half Plane Model, distance is defined as follows: The distance between two points with Euclidean coordinates (x, y) and (u, v) is

$$arccosh \left\{ 1 + \frac{(x-u)^2 + (y-v)^2}{2yv} \right\}.$$

See Figure 3 to see how the Universal Hyperbolic Theorem is shown within in this model. Figure 3 shows the same lines modeled in Figure 2. If we calculate the distance between our first two points in this model, the distance is as follows:

$$distance = arccosh \left\{ 1 + \frac{\left(\frac{1}{2} - \frac{1}{4}\right)^2 + \left(\frac{1}{2} - \frac{1}{3}\right)^2}{2 * \frac{1}{2} * \frac{1}{3}} \right\}$$

$$distance = 0.7203$$

This distance, like the distances found in the Klein model, is larger than the distance between the points in the Euclidean plane.

Figure 3: Demonstrating the Universal Hyperbolic Theorem with the Upper Half Plane Model
(Sanders, 1994)

To look at the Poincaré Disk Model we return to the unit disk. Again points are Euclidean points and here lines are portions of circles that intersect the boundary of the disk at a right angle. To find the distance between two coordinates z and w we also need z^* which is the complex conjugate of z (Sanders, 1994). In this model, the formula for distance is

$$2\operatorname{arccosh}\left\{\left|\frac{(z-w)}{(1-wz^*)}\right|\right\}.$$

Figure 4 uses the same lines as used in Figures 2 & 3, applying them to the Poincaré Disk Model.

Figure 4: Demonstrating the Universal Hyperbolic Theorem with the Poincaré Disk Model

All three of these models provide us with a clear demonstration of the Universal Hyperbolic Theorem however none of these models provide any ease in visualizing the true shape and behavior of the hyperbolic plane. For this reason, it was important to develop a physical, three dimensional model. Early developments are better categorized as attempts rather than successes as first models were clumsy and far from accurate.

Tape and Paper Models

Of all the attempted physical models, there are three tape and paper models of hyperbolic space that I found to be the most informative: the Annular Model, the Polyhedral Model and the Hyperbolic Soccer Ball. All three of these models are constructed as the section title suggests, with just tape and paper. I will state my opinion of these effectiveness of these models, and I encourage the reader to do the same. In determining the effectiveness of a model, the construction of the model should be taken into consideration. I have constructed each of these paper models which you can view in the following figures.

The first model I constructed was the Annular Model. This model is started by cutting out identical annular strips (the space between two concentric circles). The length of the strips does not matter but the inner and outer radius of all of the strips must be equal (Henderson).

Figure 5: Annular Strips of Varying Length

To construct the model, you just tape the inner circle of one strip to the outer circle of another strip. Sounds like a simple task, but I can tell you that it's actually quite frustrating. The finished product is only an approximation of the plane, however, the smaller the annular strips, the more accurate the model (Henderson). You can see my constructed model below.

Figure 6: Complete Tape and Paper Annular Model

Next I constructed the Polyhedral Model. This model is based around equilateral triangles. It was originally constructed by putting seven triangles together at every vertex but David Henderson altered this model by putting seven triangles at every other vertex and six triangles at the other vertices (Taimina, 2009). I chose to construct the Henderson alternative to this model. I started with strips of equilateral triangles as opposed to individual equilateral triangles.

Figure 7: Strips of Equilateral Triangles to begin construction of the Polyhedral Model

I suggest starting construction with your strips laid out as shown in Figure 7, this made construction fairly quick. This model is very pointy as the small paper triangles don't allow for much curve. Unlike the Annular Model, you cannot improve the accuracy of this model.

Decreasing the size of the triangles does not change the accuracy as no matter the size, the sum of angle degrees you are putting at each vertex is always the same. In comparison to the annular model, I found this model much less frustrating to construct but still unenjoyable. You can see my finished model below.

Figure 8: Complete Tape and Paper Polyhedral Model

The third, and final, paper model I constructed was the Hyperbolic Soccer Ball Model. This model is by far my favorite of the paper models as I found it the simplest to construct as well as the easiest for me to digest. This model is started by cutting out identical heptagons and identical hexagons.

Figure 9: Hexagons and Heptagons to begin construction of the Hyperbolic Soccer Ball Model

To construct this model, we take the tiling pattern of an ordinary soccer ball and replace the pentagons with heptagons. This replacement causes the surface to extend outward with negative curvature instead of closing in on itself and forming a sphere (Taimina, 2009). This model was

by far the easiest to construct. Keep in mind that due to the stiff paper you don't have a large amount of control in the directions your model goes. You can see in my Hyperbolic Soccer Ball Model, Figure 10, that one edge is hitting an opposite edge, an action that does not occur in actual hyperbolic space.

Figure 10: Complete Hyperbolic Soccer Ball Tape and Paper Model

All three of these models are stiff and fragile. While these models are all still acceptable, they are only approximations and, at least for me, are still lacking in providing a complete understanding of the hyperbolic plane. It is possible but difficult to describe coordinates and drawing parallel lines on them to demonstrate the Universal Hyperbolic Theorem is easier to try to visualize mentally than it is to physically do. However, these models do provide means for measuring distance. These models are interesting to look at but cannot be handled too much at risk of them starting to bend, fall apart or tear. These models are also quite difficult to transport as they can be easily crushed making them less appealing if you need to move them from office to classroom to lecture hall. If you are interested in constructing your own models, David W. Henderson provides templates on his Cornell University Website:

<http://www.math.cornell.edu/~dwh/books/eg00/supplements/models.html>

Crochet Models

It was 1997, David Henderson was lecturing on hyperbolic space and Daina Taimina was in the audience. Henderson was accompanying his lecture with a tape and paper model that Taimina could see was fragile and old and unable to be handled without extreme care (Henderson & Taimina, 2001). It was while sitting in that audience that Taimina began thinking of ways to construct a more durable model (Henderson & Taimina, 2001). Experienced in knitting and crochet, she determined that she could construct a model out of yarn. Her first attempt was a knitted model but she quickly realized that this would become too difficult (Henderson & Taimina, 2001). If you aren't familiar with knitting, all of your stitches are constantly contained on one of two needles. With the steady increase in the number of stitches as the model grows, it would eventually become impossible to fit all of the stitches on the needles. This is especially troublesome because a dropped stitch can cause the entire piece to eventually unravel. This caused her to switch to crochet. When crocheting, the number of stitches can grow as large as you would like as the crochet hook only loops one stitch at a time. In a theoretical math setting, crochet stitches can increase infinitely and without bounds while knitted stitches have a finite limit. As someone familiar with both crafts, I find much more freedom when crocheting.

Almost everyone has tape and paper at home but not many have a house always stocked with yarn and crochet hooks. What makes these models special enough to warrant learning a new skill? For starters, these models won't tear. You can fold them, squish them, and shove them in a bag without any repercussion. They are durable models that allow for manipulation and alteration with much more ease than the paper models. Remember my hyperbolic soccer ball that started crashing into itself? This isn't an issue with the crochet models as you can mold and

shape them however you would like with no risk of breaking the model. These models can be stitched directly onto, so viewing the distance between two points or the behavior of parallel lines is clear and easy (Wertheim, 2003). These models also allow you to approximately measure the radius.

To begin constructing a crocheted plane, you must first know two stitches: the chain stitch and the basic single crochet stitch (Taimina, 2009). After mastering these two stitches you can start your model. To determine the shape and curvature of your model you must decide on a ratio. As you crochet your model you must increase the number of stitches at a constant ratio of $N:N+1$. This ratio determines the curvature of your plane as $\frac{N}{N+1}$ determines the radius of your hyperbolic plane (Taimina, 2009). Different ratios will give you different models. You can experiment with whatever ratio you would like but it is important that each model stays with one, constant ratio. Changing the ratio in the middle of constructing a model will not result in an accurate model.

We know we need to decide on a ratio but how do we interpret this ratio in actually constructing the model? My first model crochet model has a ratio of 5:6, this means that I crocheted five stitches and increased on the sixth stitch. To increase while crocheting is to crochet a second stitch into the same loop as the previous stitch. If I start with 20 stitches, a 5:6 ratio would mean my next row has 23 stitches. Continuing with this ratio my next row would have 26 stitches, the next 30, then 35, 40, 46, 53, 61, 71 and so on. As you can see, the number of stitches increases exponentially, increasing faster and faster as each row goes on.

Figure 11: Complete Crochet Model with a Ratio of 5:6

For comparison, I would also like to briefly look at the rate of increase of a smaller and larger ratio to thoroughly demonstrate how much your selected ratio changes the shape of your plane. Let's start with a larger ratio of 15:16. If you begin with 20 chain stitches the number of stitches in each row will be as follows: 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 34, 36, 38, 40, 42, 44, 46, 48, 51, 54... This ratio requires 23 rows to reach 54 stitches while a 5:6 ratio reaches 53 stitches in just seven rows.

Figure 12: Complete Crochet Model with a Ratio of 15:16

Now let's look at one of our smallest possible ratios, 1:2. Starting with 20 stitches, the row counts would be: 20, 30, 45, 67, 100, 150, 225, 337... In eight rows, this ratio has produced a model with over six times as many stitches as the twenty third row of a 15:16 ratio.

Figure 13: Complete Crochet Model with a Ratio of 1:2

The only ratio smaller than 1:2 would be 0:1, which would mean increasing in every stitch thus doubling the number of stitches with every row. As I'm sure you've noticed, the closer the ratio is to zero, the faster the rate of expansion. The faster the increase in rows, the faster your model is going to start developing the ruffled edge of a hyperbolic plane and the smaller the radius of your plane. Alternatively, a larger ratio means a larger radius. As the radius of a hyperbolic plane approaches infinity, the hyperbolic plane approaches the Euclidean plane (Taimina, 2009).

Earlier in this section I mentioned that these models allow you to approximately measure the radius, but how? Your selected ratio helps determine your radius but to actually measure the radius you will need your model, a flat surface, an additional piece of yarn and a ruler. When you place your plane on a flat surface you will notice that arcs form. To measure the radius, place your piece of string around the formed arc and wrap it around until you complete a circle (Taimina, 2009). The radius of this circle is the radius of your hyperbolic plane.

Figure 14: Measuring the Radius of the 5:6 Ratio Crochet Model

In the first section of this paper, I stated two triangle theorems that exist on the hyperbolic plane. After spending my entire life thinking that all triangles have an angle sum of 180 degrees, I struggled to accept the first mentioned theorem of hyperbolic triangles to be true. It didn't seem possible to me that the angles of a triangle could sum to anything less than 180 degrees. That is until I had a crocheted model to work with. Stitching an ideal triangle onto one of my crocheted models instantly showed me why the angles don't sum to 180 degrees.

Figure 15: Ratio 5:6 Model with Stitched Ideal Triangle

As the vertices of the triangle get further and further away, the sum of the angles gets closer and closer to zero. When the vertices are at infinity, the sum of the angles is zero (Wertheim, 2005).

Further, the area of any ideal triangle will be r^2 where r is the radius, because of this all ideal triangles on a particular plane will be congruent (Wertheim, 2005).

While triangles are important, there is a topic arguably more important that we haven't discussed in terms of our crochet models yet: parallel lines. If these models can't clearly show the Universal Hyperbolic Theorem, then they are no better than the paper models. But rest assured, they can. It takes just four stitched lines to prove the negation of Euclid's Fifth Postulate (Taimina, 2009). This was nearly impossible with the paper models but it's somewhat trivial with crocheted models.

Figure 16: Ratio 15:16 Plane with Stitched Parallel Lines

At first glance, it is easy to think, "Well those lines aren't straight!" but they are. You can fold the model along any of the stitched lines and see that they are in fact straight lines.

Figure 17: Ratio 15:16 Plane Folded Along Various Parallel Lines

There is one other crochet model that I would like to briefly discuss, the symmetric hyperbolic plane. This model is not as simple as the other models as it requires calculations to ensure that the plane is indeed symmetric and has constant negative curvature. In her book Crocheting Adventures with Hyperbolic Planes, Daina Taimina provides a table in chapter 6 showing you how the varying calculations relate to increase ratios. Unlike our first models, to construct this model you must change your increase ratio several times. Additionally, this model begins with a loop of three stitches rather than a straight chain of 20 stitches. Taimina also provides you with the necessary equations to construct your own pattern, however I chose to follow the pattern she provided. This pattern started with three chain stitches in a loop. The first two rows have an increase ratio of 0:1, the third and fourth rows an increase ratio of 1:2, the fifth row 3:4, the sixth through eighth rows 4:5, the ninth through fourteenth 5:6 and the fifteenth on 6:7. The symmetric plane eventually stabilizes, increasing at the same rate until infinity which is crocheted as the 6:7 ratio crocheted from the fifteenth row on (Taimina, 2009). I crocheted a small symmetric plane, shown in Figure 18.

Figure 18: Crocheted Symmetric Hyperbolic Plane

As you can see, I used four different colors in my symmetric plane. This is an activity Taimina does in her above mentioned book to demonstrate area. I found it so fascinating that I had to do it myself. Looking at the model in Figure 18, which color do you think has the largest area? The answer is all of them. All four colors have the same area as all four sections started with exactly 30 feet of yarn. For a larger symmetric plane, just see the cover of Daina Taimina's crochet book. Her model is varying shades of purple, each made up of 100 meters of yarn.

The one downfall to these models is the time commitment. These models grow at such a high rate that a single row can take hours to complete. Taimina has spent up to eight months working intermittently on a single model (Taimina, 2009). Most mathematicians don't want to dedicate the necessary time themselves so they request models from Taimina herself (Wertheim, 2005). She has crafted models for mathematicians all over the world and uses them while teaching her classes at Cornell University (Wertheim, 2005). She stated in a 2005 interview with Margaret Wertheim that these models have really helped students "get a concrete sense of the properties of the hyperbolic plane" (Wertheim, 2005).

Further Application of Crochet Models

Sisters Margaret and Christine Wertheim have taken Taimina's models and created a nationwide project. They have taken these models and combined them with current marine

biology trends to create several massive museum displays of crocheted coral reef. These reef are prime examples of the hyperbolic plane in nature (Wertheim & Wertheim, 2003). These displays have been featured in the Andy Warhol Museum in Pittsburgh, the Hayward Gallery in London, the Science Gallery in Dublin, and the Smithsonian's Museum of Natural History in Washington D.C., among quite a few others. The Wertheim sisters were inspired by the work of Taimina, taking her techniques and further elaborating on them. The result has been a plethora of varying kelp, anemones, sea slugs and corals all depicting the constant negative curvature of the hyperbolic plane. Margaret and Christine Wertheim use varying algorithms and "iterative recipes" to develop each model (Wertheim & Wertheim, 2003).

The Wertheim sisters have often been surprised by the results of their algorithms, changing something small and getting a drastically different result. Because of this, they consider working with these models to be a form of experimental mathematics (Wertheim & Wertheim, 2003). Margaret and Christine Wertheim founded the Institute For Figuring which houses the Crochet Coral Reef Project. Taimina has given several lectures for the institute on her work with hyperbolic models (Wertheim & Wertheim, 2003). The Wertheim sisters are taking their project further and using their hyperbolic reefs to raise awareness about the toils currently faced in the world of marine biology due to global warming. They have constructed an entire Bleached Reef to promote to the world the effects environmental stress is having on the reef. They have also taken to crocheting plastic for an instillation they call the Toxic Reef.

Conclusion

When Daina Taimina was sitting in David Henderson's lecture in 1997, I bet she had no idea that the model she was constructing in her head would result in such an amazing achievement. Taimina has single handedly (and single hooked) figured out the only durable and

effective way to model hyperbolic space. Without her work we wouldn't have a tangible explanation of why Euclid's Fifth Postulate isn't always true. We wouldn't have a concrete way to view ideal triangles. I personally would still be trying to mentally picture how parallel lines behaved across my self-imploding hyperbolic soccer ball model. And beyond that, not only did Taimina create these astounding models, she inspired two women to build an empire of crocheted coral reef. There is nationwide awareness of environmental hazards and disappearing marine biology because two women took her hyperbolic models and have experimented with them with endless dedication. Over three million people have seen the Crochet Coral Reef, making it one of the largest participatory science and art endeavors in the world (Wertheim & Wertheim, 2003).

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