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Coherence and enrichment across the middle and secondary levels: Four mathematically authentic learning experiences

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Abstract. This article discusses four mathematically rich settings with origins in the elementary, middle, and secondary school curricula. Depending on the questions asked and the connections made within each setting, the problem spaces allow the instructor to import tools leading to sophisticated extensions appropriate for college-level study. These topics include the Heaviside function, randomness, symmetry, modular arithmetic, the generalized Pythagorean Theorem, and the theory of groups. Given the potentially extensive ground covered by these settings, they serve to reward those students who are inherently curious while highlighting the coherence in the curriculum as one progresses through the grades. The mathematical experiences invite disciplinary and interdisciplinary connections and encourage discourse and productive struggle.

Keywords: *Mathematics Instruction, Mathematics Curriculum, College Mathematics, Secondary School Mathematics*

INTRODUCTION

The Oliver Heaviside function. Fermat's Last Theorem. Modular arithmetic. Dihedral groups. These areas are commonly studied in college, perhaps in an undergraduate course in engineering or a graduate-level seminar in mathematics or computer science. Is there a place for these advanced topics in the high school curriculum? Can teachers of 9-12 connect content to these areas? Most importantly, can students benefit by engaging in problem settings where these ideas are introduced? In this article, we argue that the answer to each of these questions is yes.

Teachers of mathematics undoubtedly see important connections in their subject area but these connections may not be evident to students. By blurring the boundary between the high

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school and college curriculum, we offer a number of benefits to the high school experience. First, exploring challenging content can whet students' appetites to the investigative nature of mathematics. Even if these investigations prove difficult, students confront the limits of their mathematical knowledge and are encouraged to learn more; this is a valuable lesson. Second, there are beneficial ties to areas such as engineering, physics, biology, or art. In very simple cases, innocent questions can carve a pathway to research-oriented mathematics. Third, students may come to realize that pure mathematics can be interesting and engaging—lest we say this is often shocking to students! Finally, these experiences may provide some much needed enrichment to those who need it. Interesting problems that use novel tools can leave a lasting impression on students.

This article is principally about introducing young students to mathematical ideas and practices *not* typically explored at the high school level. We discuss four problems with appropriate origins in high school settings. The problems then flourish, extend, and enlighten by using sophisticated tools typically reserved for college mathematics. In **The Parking Garage**, we introduce a modeling problem suitable for a standard algebra course (Grade 8-10) but we use the Heaviside function as the centerpiece. Students learn about piecewise and discontinuous functions—important objectives in the high school curriculum—but via a tool typically used by engineers. In **Clock Arithmetic**, students operate in the familiar modulo 12 as this builds upon previously learned elapsed time problems that surface in CCSSM in Grade 4 (CCSSI, 2010). Then through student-created informal definitions and generalizations, discussions in modular arithmetic lead to non-trivial extensions in number theory and randomness. In **Pythagoras meets Fermat and Newton**, students begin with a ubiquitous mathematical theorem (Grade 8) and engage in the interplay of informal reasoning (conjecturing, experimenting) and formal reasoning (generalizing, extrapolating). This experience mirrors that of professional mathematicians and finishes with the ultimate prize: a generalized mathematical proof. In **Star Polygons**, we present the definition of a star polygon, a construction that naturally lends itself to line and rotational symmetry explorations, recurrent topics in the K-12 curriculum (Grades 4, 9-10). However, further investigation uncovers rich connections to group theory, usually reserved for undergraduate mathematics students but accessible to high school students given this context.

In each of these cases, we import tools and practices more prevalent to college mathematics into the high school experience. Doing so supports mathematics as a unified body of knowledge with clear connections from grade to grade (CCSSI, 2010). As NCTM (2000) reminds us, "Mathematics is an integrated field of study….The notion that mathematical ideas are connected should permeate the school mathematics experience at all levels" (p. 64).

THE PROBLEMS

The Parking Garage. One of the most important concepts in all of mathematics is that of *function*. The unequivocal mastery of function is the goal—constructing functions from data, evaluating functions, using functions as rules, graphing functions, translating functions, and studying their various properties (Oehrtman, Carlson, & Thompson, 2008). One such context that opens the door to each of these competencies is what we call **The Parking Garage Problem**. Simply ask students to find data on a parking garage of their choice. Airport parking garages are a good place to look. Such data are plentiful on the web and are almost always provided in table form. For example, parking rates at one of the many area garages surrounding Chicago O'Hare (current as of July 2015) are provided in the table shown in Figure 1:

HOURLY PARKING	MONDAY - FRIDAY
First 10 minutes	$0.00
1 hour or less	$2.00
2 hours or less	$4.00
3 hours or less	$4.00
4 hours or less	$10.00
5 hours or less	$30.00
6 hours or less	$36.00
7 hours or less	$42.00
8 hours or less	$49.00
9 hours or less	$55.00
Over 9 - 24 hours	$60.00

Figure 1. Parking Rates at Chicago O'Hare airport (CDA, 2015)

Using time as input and parking fee as output, nontrivial ideas such as *piecewise*, *discontinuous*, and *step-function* resonate in perfect harmony here. Beginning around grade eight (CCSSI, 2010), many of the following competencies can be introduced, reinforced, and supported:

(1) Functions as organizational tools so data are clearly represented,
(2) Students connecting the vagaries of mathematical symbols ($<$ and $\leq$), verbal expression (e.g., less than), and graphical conventions ($\bullet$ and $\circ$)
(3) The vertical line test and the violation of one-to-oneness.

The above competencies—even if abstract in the pages of a mathematics book—have concrete meaning in the context of a parked car. Once this foundation has been set, this is the ideal place to introduce a function not typically seen in high school curricular documents—the Heaviside step function. The Heaviside function $H(x)$ is named after the engineer/physicist Oliver Heaviside and is defined as

$$H(x)=\begin{cases}0, & x<0\\ 1, & x>0\end{cases}$$

This function has value zero for a negative argument and value 1 for a positive argument. Its value at $x=0$ can be defined based on context. Teachers can share with students the use of $H(x)$ in the modeling of physical phenomena; common settings in engineering and physics include the edge of a material or an on/off switch in signal processing. Students learn that piecewise functions, in effect, are switches. In the above situation, $H(x)=0$ may indicate the OFF position while $H(x)=1$ may indicate the ON position. As an introduction, the teacher could first ask students to explore something simple, such as $H(x-2)$. After some time, students should see this is identical to $H(x)$ with the switch located at $x=2$. Where the discussion turns interesting is in constructing the graph for $H(x)+H(x-2)$. This is specifically where the ON/OFF mechanism proves useful for those new to Heaviside functions. Since the switches are located at $x=0$ and $x=2$, constructing a number line model (Figure 2) is helpful:

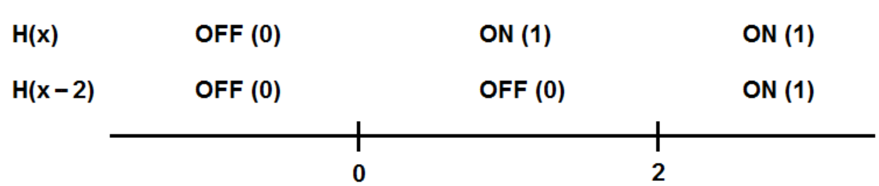


Figure 2. Heaviside with two switches

A discussion may highlight how both $H(x)$ and $H(x-2)$ are OFF when $x<0$, $H(x)$ switches ON and $H(x-2)$ remains OFF when $0<x<2$, and both are ON when $x>2$. The graph of $y=H(x)+H(x-2)$ is seen in Figure 3.

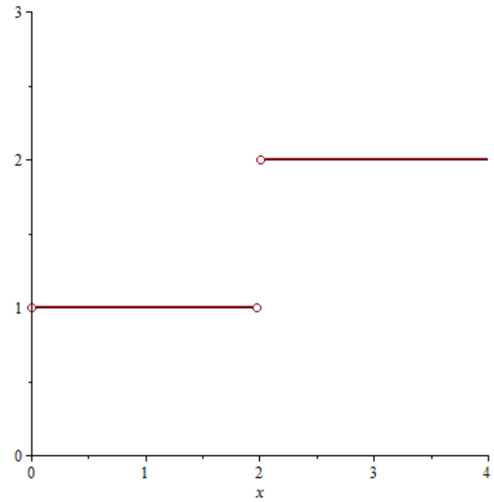


Figure 3. Sketch of $H(x)+H(x-2)$

The above analysis presents an opportunity to discuss two important features of the Heaviside function. First, the function has “memory” in that once the function is ON, it remains ON. This is why, for $x>2$, $H(x)+H(x-2)=1+1=2$, where the individual 1’s reflect the ON position for each function. Second, we may take full advantage of the sum operation by applying coefficient “weights” to the individual functions, thus increasing graphing flexibility. A challenging activity is to ask students to construct an algebraic function that yields the parking garage graph for the first four hours or so. After some thinking and perhaps a few adjustments, we come to the formula $y=2H\left(x-\frac{1}{6}\right)+2H(x-1)+6H(x-3)$. This graph is shown in Figure 4.

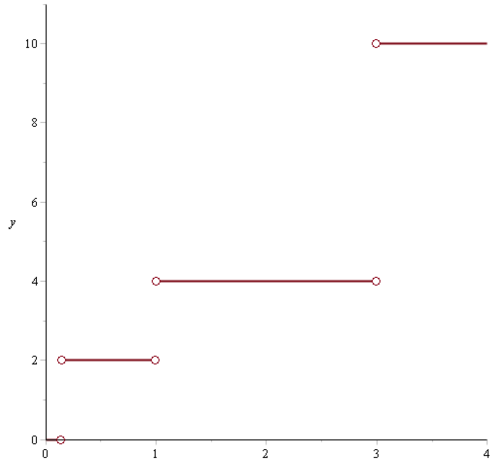


Figure 4. The graph of $y = 2H\left(x - \frac{1}{6}\right) + 2H\left(x - 1\right) + 6H\left(x - 3\right)$

Of course, we may now appropriately define this function at the specific hours (e.g., when $x = 2$ hours) and make further adjustments as needed. Most important, the mathematical seeds for future experiences have been sowed:

(a) Students get meaningful practice operating on functions, including translations and sums. Instead of simply graphing variations of a parent function, the translations and weights have specific meaning in the context of time and parking fee.
(b) The number line analysis in Figure 2 foreshadows models used in calculus for examining monotonicity and concavity. Moreover, both *memory* and *accumulation* prove to be important ideas in the study of calculus.
(c) Heaviside presents a simpler, more compact representation of our model (e.g., compare this with a piecewise function having eleven pieces). Finding simpler but equivalent representations is a major pillar of mathematical enculturation.

In sum, introducing the Heaviside function to students not only allows reinforcement of previously learned material but promises ties to future mathematical topics in a way that is challenging, accessible, and interesting.

Clock Arithmetic. Number theory topics are weaved throughout the K-12 curriculum and provide a foundation for multiplicative thinking in higher level mathematics. For instance, a student's ability to decompose and compose numbers through factorization, develop divisibility rules, and identify prime numbers in grade four (CCSSI, 2010) are directly connected to more abstract applications in high school algebra and their related proofs in undergraduate mathematics. The study of clock arithmetic offers a unique opportunity for high school students' engagement while eliciting many connections to number theory and randomness.

Clock arithmetic, a special case of modular arithmetic, can be defined as the arithmetic of congruences. "In modular arithmetic, numbers 'wrap around' upon reaching a given fixed quantity, which is known as the *modulus* (which would be 12 in the case of hours on a clock)" (Modular Arithmetic, 2016). For example, if the current time is 10 am, and 3 hours elapse, the time is not 10 + 3 = 13 o'clock; instead, the time becomes 1 o'clock. This can be represented mathematically as $10 \oplus 3 = 1 \pmod{12}$.

Elementary students are first exposed to telling time on an analog clock in first grade. In subsequent grade levels, students continue to build fluency in foundational time-telling skills, and in fourth grade, students solve elapsed time problems (CCSSI, 2010). Thus, clock arithmetic provides a familiar yet rich context for a problem situation. We have found that by introducing the conventional notation and vocabulary, students can create their own informal definitions, generalize, and extend to other modulo. We list some possible tasks and discussion questions here:

> <u>Task 1</u>. Use clock arithmetic to add, subtract, and multiply in mod 12 using natural numbers. Examples include $10 \oplus 3 = 1 \pmod{12}$, $10 \oplus 4 = 2 \pmod{12}$, $9 \oplus 3 = 0 \pmod{12}$, $2 \ominus 3 = 11 \pmod{12}$, $5 \ominus 7 = 10 \pmod{12}$, $4 \otimes 6 = 24$ or $0 \pmod{12}$, $5 \otimes 5 = 25$ or $1 \pmod{12}$.
>
> <u>Task 2</u>. Introduce equivalence notation $\equiv$ and use a clock to visualize, for example, how 1:00 + 12 hours elapsed is equivalent to 1:00. Discuss congruences, such as $1 \equiv 13 \pmod{12}$, $0 \equiv 12 \pmod{12}$, and $4 \equiv 2 \pmod{12}$.

Task 3. Help students develop an informal definition for equivalence. For instance, $25 \equiv 1$ (mod 12) since $25 - 24 = 1$, and this is true because "you can add or subtract any multiple of 12 and you're back where you started" (in the words of a typical middle school student).

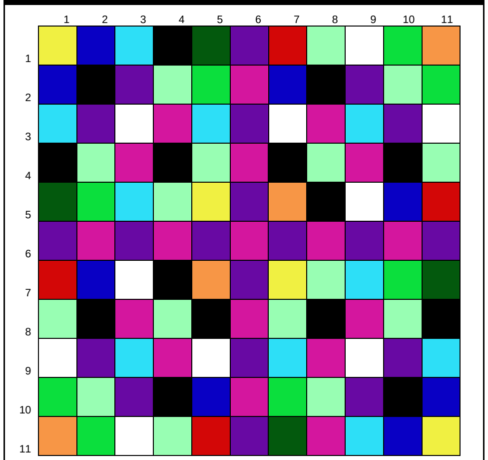
When students engage in these equivalence tasks and discuss why certain numbers are equivalent to each other, such as $1 \equiv 13$ (mod 12) or $25 \equiv 1$ (mod 12), students are actually working with the formal number theoretic definition of *divides*: If $m \mid a - b$ (which means $a - b = mc$ for some $c \in \mathbb{Z}$), then $a \equiv b$ (mod m). In fact, when students realize that the sum of a number and any multiple of 12 are equivalent, the students are saying that in mod m, a and $a + mc$ are indistinguishable (Niven, Zuckerman, & Montgomery, 1991).

Furthermore, if $x \equiv y \pmod{m}$, y is called the *residue* of mod m. Thus, the *complete residue system* mod 12 of the number 1 is the infinite set $\{13, 25, 37, \ldots\}$ because all of these numbers are congruent to 1 in mod 12 (Niven, Zuckerman, & Montgomery, 1991). Although the instructor may not define these notions formally in middle school or high school, students are certainly able to understand and discuss these concepts.

Once students gain comfort and fluency in modulo 12, we ask the students to change their "clocks" from a 12-hour clock to clocks using mods other than 12. Students can then practice modular addition, subtraction, and multiplication in mods such as 5 or 7. The website http://www.shodor.org/interactivate/activities/ClockArithmetic/ assists students in visualizing modular arithmetic in the form of a clock. This site has an interactive clock that can be set between modulo 2 and 26.

A surprising interdisciplinary connection can be made between mathematics and art. To create mathematical artwork based on modular arithmetic, have students complete tables of results for addition and multiplication in mod 5 and 12 (and other modulo if the students want a challenge). We suggest omitting the zero row and column for aesthetic reasons on the multiplication charts. Color code the charts so that each value represents a color (then omit the numbers) to create designs. Figure 5 is an example of a middle school student's mod 12 multiplication chart.

Fig. 5 Mod 12 multiplication chart.



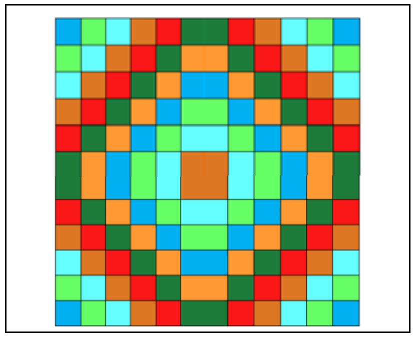
A modular addition table will have clear reflection symmetry along the diagonal. But a modular multiplication table such as the one in Fig. 5 does not seem to have a clear pattern. But if students examine a sufficiently large table, symmetry becomes discernible. See Fig. 6 for an example, or see the applet on http://britton.disted.camosun.bc.ca/modart/jbmodart.htm. Interestingly, the quest to establish connections between the computational and the visual bears a striking similarity to modern investigations of deterministic models with chaos and complexity—bringing to mind *A New Kind of Science* (Wolfram, 2002). Any classroom experience that mirrors the nature of science or mathematics is valuable for students. Students should be allowed to experience the uncertainty and ambiguity in mathematical situations; sometimes we discern patterns easily and sometimes we do not, but most often *we do not know* without further study.

Fig. 6 Patterns in a mod 24 multiplication chart, before color coding.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
24	0	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
23	0	23	21	19	17	15	13	11	9	7	5	3	1	24	22	20	18	16	14	12	10	8	6	4	2	0
22	0	22	19	16	13	10	7	4	1	23	20	17	14	11	8	5	2	24	21	18	15	12	9	6	3	0
21	0	21	17	13	9	5	1	22	18	14	10	6	2	23	19	15	11	7	3	24	20	16	12	8	4	0
20	0	20	15	10	5	0	20	15	10	5	0	20	15	10	5	0	20	15	10	5	0	20	15	10	5	0
19	0	19	13	7	1	20	14	8	2	21	15	9	3	22	16	10	4	23	17	11	5	24	18	12	6	0
18	0	18	11	4	22	15	8	1	19	12	5	23	16	9	2	20	13	6	24	17	10	3	21	14	7	0
17	0	17	9	1	18	10	2	19	11	3	20	12	4	21	13	5	22	14	6	23	15	7	24	16	8	0
16	0	16	7	23	14	5	21	12	3	19	10	1	17	8	24	15	6	22	13	4	20	11	2	18	9	0
15	0	15	5	20	10	0	15	5	20	10	0	15	5	20	10	0	15	5	20	10	0	15	5	20	10	0
14	0	14	3	17	6	20	9	23	12	1	15	4	18	7	21	10	24	13	2	16	5	19	8	22	11	0
13	0	13	1	14	2	15	3	16	4	17	5	18	6	19	7	20	8	21	9	22	10	23	11	24	12	0
12	0	12	24	11	23	10	22	9	21	8	20	7	19	6	18	5	17	4	16	3	15	2	14	1	13	0
11	0	11	22	8	19	5	16	2	13	24	10	21	7	18	4	15	1	12	23	9	20	6	17	3	14	0
10	0	10	20	5	15	0	10	20	5	15	0	10	20	5	15	0	10	20	5	15	0	10	20	5	15	0
9	0	9	18	2	11	20	4	13	22	6	15	24	8	17	1	10	19	3	12	21	5	14	23	7	16	0
8	0	8	16	24	7	15	23	6	14	22	5	13	21	4	12	20	3	11	19	2	10	18	1	9	17	0
7	0	7	14	21	3	10	17	24	6	13	20	2	9	16	23	5	12	19	1	8	15	22	4	11	18	0
6	0	6	12	18	24	5	11	17	23	4	10	16	22	3	9	15	21	2	8	14	20	1	7	13	19	0
5	0	5	10	15	20	0	5	10	15	20	0	5	10	15	20	0	5	10	15	20	0	5	10	15	20	0
4	0	4	8	12	16	20	24	3	7	11	15	19	23	2	6	10	14	18	22	1	5	9	13	17	21	0
3	0	3	6	9	12	15	18	21	24	2	5	8	11	14	17	20	23	1	4	7	10	13	16	19	22	0
2	0	2	4	6	8	10	12	14	16	18	20	22	24	1	3	5	7	9	11	13	15	17	19	21	23	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**0**

Creating artwork from modular arithmetic also affords the students opportunities to explore or revisit transformations. For instance, although a single table may stand on its own as artwork (Fig. 5), students may choose to use transformations to create a 4-table modular quilt. One way is to reflect a color-coded modular table over the *x*- and *y*-axes (Fig. 7). Another idea is to start with one table, and rotate it 90° clockwise, then again, and again to produce a different version of a 4-table modular quilt. Transformations such as these mirror (pun intended!) the types of activities in high school geometry when translating, reflecting, and rotating figures on the coordinate plane.

Fig. 7 Mod 7 addition chart, reflected over the x- and y-axes.



In summary, we see that clock arithmetic has surprising connections to number theory, art, and randomness. These connections begin as early as first grade and follow the student all the way through college.

Pythagorean meets Fermat and Newton. Students learn many important theorems in their schooling but perhaps none more memorable than the Pythagorean Theorem. Given any right triangle with sides of length *a*, *b*, and *c*—where *c* represents the length of the hypotenuse—the areas of the squares constructed on each side of the triangle are related by the equation $a^2 + b^2 = c^2$. See the figure for the case when *a*, *b*, and *c* are 3, 4, and 5, respectively.

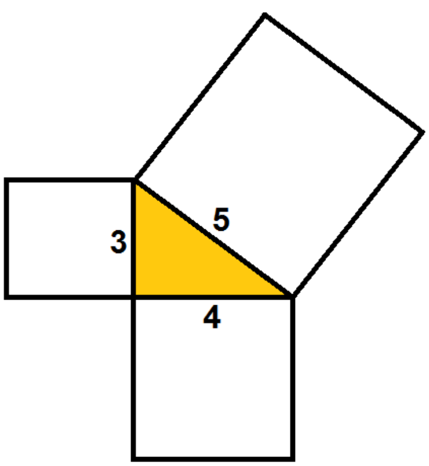


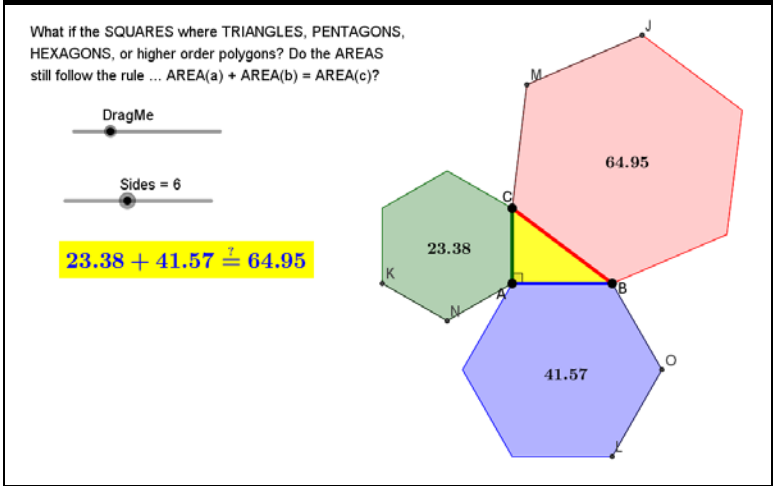
Figure 8. The Pythagorean Theorem

In time, it is no secret that students may eventually discard the geometric significance of the theorem in lieu of the algebraic statement. If we attempt to generalize in either direction, we get two very different conjectures.

(1) (Algebraically) Since the theorem is valid for squares, might it work for cubes? How about powers of 4? Or 5?
(2) (Geometrically) Since the theorem is valid for squares, might it work for other shapes such as triangles or pentagons? How about semicircles?

Of course, should we consider option (1) with a, b, and c as positive integers, we get a taste of mathematical history and a *very* challenging problem indeed (Fermat's Last Theorem). This path may be unreasonable mathematically but it is interesting nonetheless. On the other hand, option (2) presents students with a worthwhile investigation. As different geometric shapes are used to test this conjecture, more sophisticated mathematics may need to be incorporated. Dynamic software is an excellent way to test the reasonableness of conjectures (see Figure 9 for regular pentagons) and an eventual proof of the conjecture would fuse the algebraic, relational, and geometric qualities of the theorem. Bending this theory until it breaks is a valuable experience to share with our students.

Fig. 9 GeoGebra exploration of regular polygons (Edgar, 2016).



At some point, the following question is inevitable. Can we adjoin three similar figures to sides *a*, *b*, and *c* (call the figures *A*, *B*, and *C*, respectively) and conclude $\text{Area}(A) + \text{Area}(B) = \text{Area}(C)$? See Figure 10.

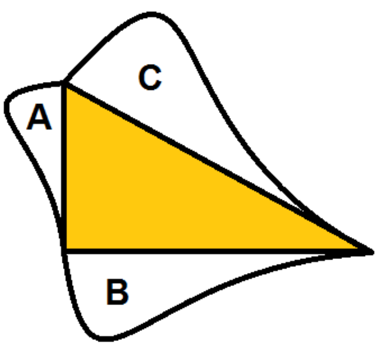


Figure 10. Does $\text{Area}(A) + \text{Area}(B) = \text{Area}(C)$?

This question lives in the realm of calculus yet much of the accompanying mathematics is instructive. To begin, students should harness the power of functions in their symbolic form. That is, if we designate *C*'s outer function as $y = f(x)$, a discussion of how to construct the other functions can begin. This is a question of transformational algebra. Then in order to explore the areas themselves, we may use integral calculus. We discuss this presently.

To start, we can think of the function $y = f(x)$ as defined on the interval $[0, c]$. In a similar way, we can define functions *g* and *h* to trace the outer boundary of figures *A* and *B*, respectively. Then we have the conventions as seen in the figure.

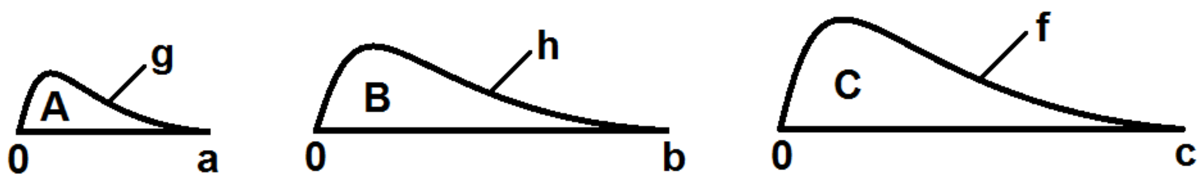


Figure 11. Functions f, g, and h and their associated intervals

In order for the figures to be similar, students must utilize some basic principles from algebra—functional transformations. Since functions g and h are variations of the parent function $y = f(x)$, it can be shown that $g(x) = \frac{a}{c} f\left(\frac{c}{a}x\right)$ and $h(x) = \frac{b}{c} f\left(\frac{c}{b}x\right)$. In brief, while we modify the input of f to resonate with the smaller intervals of g and h (using $\frac{c}{a} > 1$ and $\frac{c}{b} > 1$ as horizontal shrinking parameters), we must modify (reciprocally) the outputs of f to adhere to the analogously smaller outputs of g and h. Then we may ask, does

$\text{Area}(A) + \text{Area}(B) = \text{Area}(C)$ or, does $\int_0^a g(x)\,dx + \int_0^b h(x)\,dx = \int_0^c f(x)\,dx$? Beginning with the left-hand side, we get

$$\int_0^a g(x)\,dx+\int_0^b h(x)\,dx=\int_0^a \frac{a}{c} f\left(\frac{c}{a}x\right)dx+\int_0^b \frac{b}{c} f\left(\frac{c}{b}x\right)dx$$
$$=\frac{a}{c}\int_0^a f\left(\frac{c}{a}x\right)dx+\frac{b}{c}\int_0^b f\left(\frac{c}{b}x\right)dx$$
$$=\frac{a^2}{c^2}\int_0^a f\left(\frac{c}{a}x\right)\frac{c}{a}dx+\frac{b^2}{c^2}\int_0^b f\left(\frac{c}{b}x\right)\frac{c}{b}dx$$

Continuing, we let $u=\frac{c}{a}x$ and $v=\frac{c}{b}x$ so $du=\frac{c}{a}dx$ and $dv=\frac{c}{b}dx$. Finally,

$$\frac{a^2}{c^2}\int_0^a f\left(\frac{c}{a}x\right)\frac{c}{a}dx+\frac{b^2}{c^2}\int_0^b f\left(\frac{c}{b}x\right)\frac{c}{b}dx=\frac{a^2}{c^2}\int_0^c f(u)\,du+\frac{b^2}{c^2}\int_0^c f(v)\,dv$$
$$=\left(\frac{a^2}{c^2}+\frac{b^2}{c^2}\right)\int_0^c f(r)\,dr,$$

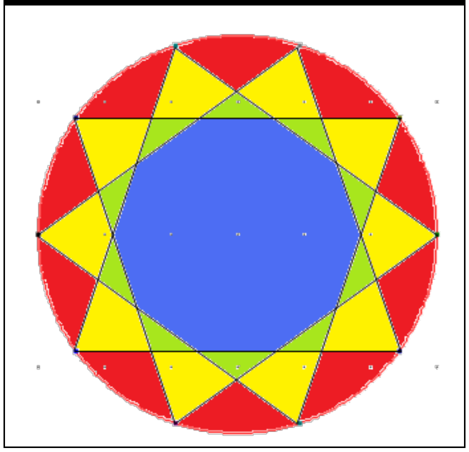
using r as a general variable of integration. At this point, the "ordinary" Pythagorean Theorem tells us $a^2+b^2=c^2$ so the result follows. That is, we have the remarkable statement $\int_0^a g(x)\,dx+\int_0^b h(x)\,dx=\int_0^c f(x)\,dx$ for similar curves f, g, and h adjoined to the sides of any right triangle.

What is the lesson here? Students are introduced to the standard Pythagorean relationship early in their schooling so the problem context requires little background knowledge to start. It then connects effortlessly to geometric similarity, makes full use of algebraic representation through functional similarity, and harnesses the power of technology to conjecture and explore. While generalizing this well-known theorem may not be common in high school, it is a reasonable challenge with several advantages. First, conjecturing with the use of software goes hand-in-hand with mathematical proof. A noteworthy reminder is the proof of the Four Color Map Theorem due to Kenneth Appel and Wolfgang Haken (as well as subsequent computer-assisted proofs). Second, horizontal mathematizing (conjecturing, seeking structure) and vertical mathematizing (formalizing, proving) are common dualities in mathematical activity (Rasmussen, Zandieh, King, & Teppo, 2005). For students, this makes explicit the link between uncertainty (*Is this true?*) and certainty (*I have a completed proof!*). This is, generally speaking, how the mathematical agenda is pushed forward. Finally, given a seemingly basic mathematical idea (the Pythagorean Theorem), research mathematics may be lurking on the other side of the door (Fermat's Last Theorem). All of these realizations are accessible to high school students and paint an authentic picture of mathematical work.

Star Polygons. The study of star polygons, though not routinely a part of the middle level or high school curriculum, naturally lends itself to higher level mathematical discussions. The rudiments of group theory are hiding in many levels of mathematics and star polygons represent one way to elicit group theory concepts. (Connections to number theory are also relevant.) Overall, the exploration of star polygons provides students the opportunity to create several mathematical generalizations from their constructions, and even use the mathematics as a basis to create original artwork.

First, a star polygon is defined as a non-convex polygon which looks like a star. A star polygon $\{p, q\}$ with positive integers p and q, is a figure formed by connecting with straight lines every qth point out of p regularly spaced points lying on a circumference (Star Polygon, 2016). Figure 12 is an example of a computer-generated {10, 3} star polygon created by a student using equally spaced points on a circle and the Paint software.

Fig. 12 {10, 3} Star polygon.



After students have constructed and accurately named a few of their own star polygons, we begin asking them to consider specific cases to elicit some general conclusions. We list some possible discussion questions and results here:

> Investigation 1: Consider the case of {5, 1}, {7, 1} and {8, 1}.
> What do all of these have in common? Can we generalize?
> Answer: They are all regular polygons with p sides.
>
> Investigation 2: Consider the case of {5, 2} and {5, 3}, or {8, 5} and {8, 3}.
> What is happening here? Can we generalize?
> Answer: Star polygons in the form $\{p, q\}$ and $\{p, p - q\}$ produce congruent star polygons.
>
> Investigation 3: Consider the case of {10, 5} and {8, 4}.
> What do these have in common? Can we generalize?
> Answer: Star polygons in the form $\{p, \frac{p}{2}\}$ result in a single segment.

More in-depth questions can also be posed which invoke topics from number theory. For example, when constructing the star polygon $\{p, q\}$, how do you know when you will end up at the same point at which you started *and* you've used up all the points? To help students organize their thinking, we have found it helpful to provide a table with three columns—star polygon name, all points used (yes/no), all points not used (yes/no)—to record their observations for this experiment. By doing so, students can more readily make a conjecture based on their data. (It turns out that only when *p* and *q* are relatively prime does this occur! In fact, these types of star polygons are called *regular star polygons*.)

Furthermore, star polygons provide an appealing context to investigate symmetry, and this leads to elementary group theory. Students can readily describe both the reflection (line) symmetry and rotational symmetry of their star polygons. For example, consider the star polygon example {10, 3} (Figure 12). This star polygon has 10 lines of symmetry. The rotational symmetry can be described as either $\frac{1}{10}$ and all multiples of $\frac{1}{10}$, or 36°n where $\{n \in \mathbb{Z} | \ 1 \leq n < 10\}$. The set of rotational symmetries of any regular star polygon forms a *finite cyclic group*. Consider {10, 3}, which forms the cyclic group C_{10}. This means there are 10 ways to map the star polygon to itself by a rotation about its center. This group is isomorphic to $\mathbb{Z}_{10}$. (Herstein, 1990).

A more accurate way to describe this set of symmetries of regular star polygons is that the set forms a *finite dihedral group*, which means that the figure has both reflection *and* rotational symmetry. For example, {10, 3} forms the dihedral group D_{10}. This means there are 10 reflection symmetries and 10 rotational symmetries. At this point, students can be asked to investigate the possible relationships between figures that can be categorized as having both symmetry type C_N and D_N. For instance, are all figures that are elements of the cyclic group C_N also elements of the dihedral group D_N? Is the converse true? In general, if a figure has rotational symmetry, does it also have reflection symmetry? Under what conditions would this be true? (Murawska, Wilders, & Van Oyen, 2013). Similar questions can also be found in Farmer's (1996) *Groups and Symmetry: A Guide to Discovering Mathematics.*

To sum up, what begins as an engaging geometric construction eventually provides a context in which students can conjecture, test their conjectures, and create mathematically valid conclusions. Even though the students are not formally proving these conclusions, these types of experiences foreshadow the thinking required for formal proofs they will encounter later in their educational journey.

CONCLUSION

The four problems discussed here share two prominent features. First, they reach beyond what is commonly taught in middle and high school classrooms. The problem contexts will be familiar to students even if the questions asked and the subsequent investigations are new. Thus, the problems serve to enrich and enlighten. With this enrichment comes an authentic experience based on the curiosities of the students with appropriate scaffolding by the teacher. Second, the tasks highlight the subject matter coherence in K-16 mathematics by offering complementary topics across many grade levels. It is important to reinforce previous learning while promoting new learning, and these problem settings make this coherence explicit. We elaborate on each of these features, enrichment and enlightenment, presently.

Enrichment. Educators typically discuss remediation over enrichment (for good reasons, we might add). In a curriculum short on time and space, rarely are students given the opportunity to generalize or extrapolate on the content they are learning. The four problems discussed here embrace layer upon layer of situational complexity while still addressing the core idea(s) students should learn. The problems offer opportunities to abstract, generalize, or dabble in topics not typically addressed in the high school curriculum. To add, the problems allow the learner to move in many different directions (often with careful teacher guidance). Tasks such as these are exactly the kinds of activities where students could easily stumble upon mathematics not yet known to the community.

Coherence in K-16 mathematics. Some topics and themes are omnipresent in the study of K-16 mathematics. These include pattern recognition, function, conjecturing, symmetry, elegance, counting, proving, and generalizing. These are skills needed from our earliest encounters in mathematics to our most advanced of encounters. Some problems connect ideas within mathematics while some even connect different branches of mathematics. This has multiple benefits, including helping to illuminate the web of connections in the subject matter, serving as referents for ideas that can be later revisited, and adding an element of surprise, wonder, or curiosity to the subject. These are the healthy byproducts of mathematical engagement in rich problem contexts and we find these dispositions to be just as valuable as the mathematics learned.

In summary, some tasks in mathematics have the potential to connect ideas, serve as platforms to introduce new ideas, and reward students for their curiosities. A classroom environment that combines rich tasks and good questioning allows student curiosity to run wild, all while supporting the logical consistency and regularity seen in K-16 mathematics. The problems we pose and the questions we ask set no limits for our expectations for engagement and success. In the end, we can only hope our students will thank us for this.

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