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Teacher Development and Seventh Graders' Achievement on Representing and Solving Equations

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Abstract

We analyze the impact of a teacher development program based on a functions approach to algebra on 7th graders understanding of equations and examine how students' score gains during the academic year relate to their teachers' initial level of mathematical knowledge of algebra, functions, and graphs. Students from participating teachers' and their control peers completed a mathematics assessment at the start and at the end of the school year the teachers were taking the program. We determined teachers' initial levels of mathematics knowledge through a written assessment given at the start of the program. Although both groups of students improved from the start to the end of the school year, the students from participating teachers showed significantly greater improvement. Moreover, among control students', improvement in creating, solving, and interpreting equations was positively correlated with their teachers' initial levels of mathematical knowledge. Improvement among students of cohort teachers in the same

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items was high regardless of their teachers' initial performance in the assessment, with students of teachers in the low level group showing the highest gains.

Keywords: teacher development, seventh grade students, functions approach to algebra, word problems, mathematics teacher knowledge, algebra equations, student achievement

Introduction

Algebra, a central topic in the mathematics curriculum (Common Core State Standards Initiative, 2010; Moses & Cobb, 2001; National Council of Teachers of Mathematics, 2000; Sharpe & Schliemann, 2014), has been a gatekeeper for higher education and a roadblock preventing access to careers in the STEM (Science, Technology, Engineering, and Mathematics) fields (Kaput, 1998; Moses & Cobb, 2001; Pearson & Miller, 2012; Sharpe & Schliemann, 2014). Students need to take mathematics and science courses to graduate from high school, a goal that requires learning and understanding algebra from, at least, the middle school years. However, many students lose interest in mathematics in middle school, when expressions and equations are introduced for the first time (Common Core State Standards Initiative, 2010). Given the high rate of students who perform poorly in mathematics in middle or high school (NAEP, 2013), teachers need to find opportunities to improve their ways of teaching mathematics and, more specifically, algebra.

In our view, improving instruction on algebra requires that teachers understand algebra from a broader perspective that relates algebra to functions and their multiple representations. It also requires that teachers understand students' ways of reasoning and how to take these as a starting point in teaching and learning. As Ball, Thames, and Phelps (2008) state, teachers need to know if students' ideas are mathematically viable and should have flexibility around the content. This study examines the impact of a teacher development program centered on a functions approach to algebra and on teachers' understanding of students' reasoning on (a) 7th grade students' ability to understand and create equations to solve word problems, and (b) how students' score gains during the academic year relates (or does not relate) to their teachers' initial levels of knowledge about algebra, functions, and their multiple representations.

Background

Research on middle and high school students' learning of algebra has repeatedly documented difficulties that are often attributed to the inherent abstractness of algebra and to students' level of cognitive development (see Collis, 1975, Kuchemann, 1981, MacGregor, 2001). For example, students often view the equals sign as a unidirectional operator that produces an output on the right side from the input on the left (Booth, 1984; Kieran, 1981, 1985; Vergnaud, 1985, 1988), focus on computing specific answers (Booth, 1984), find difficult to use mathematical symbols to express relationships between quantities (Bednarz, 2001; Bednarz & Janvier, 1996; Vergnaud, 1985; Wagner, 1981), do not use of letters as generalized numbers or as variables (Booth, 1984; Kieran, 1984; Kuchemann, 1981; Vergnaud, 1985), and do not operate on unknowns (Bednarz, 2001; Bednarz & Janvier, 1996; Filloy & Rojano, 1989; Kieran, 1985, 1989; Steinberg, Sleeman, & Ktorza, 1991).

Generating equations from word problems and using those equations to solve a problem are difficult for algebra students in the middle grades and lower secondary grades (11-15 years) (Kieran, 2007). Moreover, 6th and 7th grade students who can generate an equation to represent a word problem often use methods other than the syntactic manipulation of symbols to solve the equation (Koedinger & Nathan, 2004; van Amerom, 2003). These other methods may be suitable for some problems and equations but, in more complex cases, the manipulation of symbols may be required. Here, research has often found that most students produce wrong answers, failing to apply equivalent operations to both sides of the equation (Bednarz, 2001; Bednarz & Janvier, 1996; Filloy & Rojano, 1989; Kieran, 1985, 1989; Steinberg, Sleeman, & Ktorza, 1991) or refusing to operate on unknowns (Sfard & Linchevsky, 1994).

However, recent approaches to algebra in high school (Chazan, 2000), as well as studies of algebra in the elementary grades (Cai & Knuth, 2011; Carraher & Schliemann, 2007, 2016; Kaput, Carraher, & Blanton, 2008; Schliemann, Carraher, & Brizuela, 2007, 2012) show that even elementary school children can understand basic algebraic principles and representations. Such findings strongly support Booth's (1988) proposal that students' difficulties with algebra in middle and high school, rather than due to developmental limitations, may result from the traditional computational approach to arithmetic and to a focus on algebra that prioritizes the manipulation of equations.

Mathematicians and mathematics education researchers (Kaput, 1998; Schoenfeld, 1995; Schwartz & Yerushalmy, 1992) have argued that a functional approach to algebra has the potential to better prepare students for a deep understanding of algebra (Sharpe & Schliemann, 2014). A functional approach to algebra has been advocated and explored by many (see Bethell, Chazan, Hodges, & Schnepp, 1995; Chazan, Yerushalmy, & Schwartz, 1993; Kieran, Boileau, & Garancon, 1996; Rubio, 1990; Schwartz & Yerushalmy, 1992). This approach intertwines functions and equations as the comparison of functions throughout the teaching of algebra and is exemplified by Schwartz and Yerushalmy's (1992) use of multiple software environments (Visualizing Algebra: The Function Analyzer, The Function Supposer: Explorations in Algebra environment, and The Function Comparator). Within these environments, students can explore equations of one or two variables through symbolic, graphical, and tabular representations of functions.

A number of teacher development programs showed positive effects from a broader view of algebra and a focus on functions and their representations on teachers' learning. For example, Lloyd and Wilson (1998) observed and interviewed an experienced high school teacher first implementation of an NCTM (1989, 2000) standards-based curriculum (the Core-Plus curriculum; see http://wmich.edu/cpmp/) during a six-week unit on functions. The teacher was able to combine features of his own conceptions of functions with the Core-Plus approach to functions, focusing on graphical representations and on co-variation, to support meaningful discussions with his students. Herbel-Eisenmann and Phillips (2005) asked teachers to examine and sort 16 problems (based on characteristics they noticed such as types of representation, context of the problem, linear versus non-linear), to solve the problems, and to examine students' work on the problems. Through discussion during each step of the activity, they found that teachers start "to see algebra as a study of relationship between variables and to understand the importance of multiple representations in the study of algebra" (p. 66). Hough, O'Rode, Terman, and Weissglass (2007) used concept maps to explore teachers' growth in understanding algebra. They report that use of concept maps with reflective writing and discussion contributed to teachers' more complex and connected knowledge of algebraic structures and increased subject matter knowledge in terms of breadth, depth and connectivity. Concerning student learning, Schoen, Cebulla, Finn, and Fi (2003), also using the Core-Plus curriculum, analyzed the relationship between teacher variables and student achievement with a sample of 40 teachers and 1466 students in 26 schools. Their results show that student achievement growth was positively related to teacher behaviors that were consistent with the standards' recommendation and that reflected high mathematical expectations.

From a pedagogical point of view, studies on teacher development in different areas of mathematics consider a teachers' understanding of student's ideas as a first step in designing learning opportunities for students (Carpenter et al, 1989; Jacobs et al, 2010). These approaches have their roots in a constructivist perspective inspired by Piaget's theory of cognitive development. Children create their own knowledge by acting and reflecting upon the results of their actions on objects. Therefore, we need to find "the most adequate methods for bridging the transition between (...) natural but nonreflective structures to conscious reflection upon such structures and to a theoretical formulation of them" (Piaget, 1970, p. 47).

The goal of our program was to help teachers deepen their own understanding of mathematics and their understanding of students' mathematical reasoning and, ultimately, to enhance their students' mathematical learning. It included providing teachers with opportunities to (a) develop a rich and flexible knowledge of the mathematical content they teach, (b) understand how student's ideas about mathematics develop and the connections between their ideas and core concepts in middle and high school mathematics, (c) listen to students talk about their reasoning and use students' responses to assess their understanding and foster discussions, and (d) gain a strong professional learning community by collaborating with their peers to examine and improve their practices. As such, the program matched Borko's (2004) description of high-quality teacher development initiatives.

Most previous research on the impact of teacher development programs has focused on changes on teachers' attitudes, learning, or ways of teaching. Fewer studies documented the impact of teacher development on their students' learning. Examples of these are Carpenter et al.

(1989), Carpenter et al. (1999), and Franke and Kazemi (2001), on students solving word problems, Saxe, Gearhart, and Nasir (2001), on students understanding of fractions, and Hill, Rowan, and Ball (2005), on teachers' mathematical knowledge for teaching as a predictor of first and third graders yearly gains in mathematics achievement. Our goal here is to further explore the impact of teacher development on their students' learning and understanding, taking into account the teachers' initial level of mathematics understanding.

The Teacher Development Program

Teachers in this study completed three semester-long online graduate level courses (see Teixidor-i-Bigas, Schliemann, & Carraher, 2013 for details). The courses covered algebra and functions, their multiple representations, modeling and applications, and findings from research on students' understanding of algebra and functions. Course 1 dealt with functions and relations and the representation of functions on the real line and on the plane. Course 2 focused on fractions and divisibility as they relate to functions, transformations of the line, transformations of the plane, and the use of transformations to analyze graphs of functions and for solving equations as the comparison of two function represented in the plane. Course 3 included the representation of verbal statements as equations, work on linear equations, quadratic and higher order, the relation between factoring and roots of equations, and slope and rate of change. Activities in Course 1 alternated weeks of study of mathematical content with weeks where teachers examined classroom lessons related to the topics. Courses 2 and 3 were each structured across four three-week units plus two weeks dedicated to a final project. The first two weeks of each unit were dedicated to mathematics content, modeling applications, and educational perspectives on student learning of specific content. In each of the two weeks in a unit, participating teachers had online access to written notes, videos, and software demonstrations about mathematics and mathematics education issues. Each week they were asked to solve five to six problems, posting drafts of their work online, discussing each other's work, and receiving feedback from instructions. In the third week of each unit the teachers worked in groups of three to five teachers to design a learning activity (course 2) or to interview individual students to explore their ways of representing and reasoning about mathematics topics (course 3). The final project for courses 2 and 3 consisted of the design, implementation, and analysis of a lesson by each teacher. The program didn't include explicit instructions on how to teach and did not provide lesson plans for teachers to implement. Instead, it aimed at developing teachers' deeper understanding of the mathematics they teach and of their students' ways of thinking and learning and at nurturing opportunities for jointly discussing, planning, implementing, and analyzing their own lessons and videotaped lessons produced by research studies (Sharpe & Schliemann, 2014).

The Functional Approach to Algebra

Within a functional approach to algebra, the main initial focus of the courses was not solely on learning the syntactic rules for solving equations, but rather on conceiving equations as comparisons between two functions. The two functions being compared must of course have the same domain and codomain and the solutions of the equation are the values in the domain that make the comparison true. Other values of the variable do not satisfy the equation, but are still of interest since they would lead to inequalities. An additional advantage of this presentation is that it can provide a unified approach to the study of inequalities along with the study of equations.

In the functional approach to algebra, students are introduced early on to variables as placeholders for sets of values and to the analysis of relations between sets of numbers, before they start considering equations. They are also introduced to multiple representations of functions: verbal statements, number lines, data tables, Cartesian graphs, and algebraic notation.

The goal is to help students move smoothly between multiple representations of functions and use these representations to further understanding mathematics content. Once students are comfortable with multiple representations of functions, they can consider equations as the comparison between them. This idea is best conveyed through the graphs of the functions in the Cartesian plane. If the graphs of two functions go through the same point on the Cartesian plane, that is, if they intersect, the solutions to the equation are the x-values of the points of intersection of the two graphs.

While the graphing of equations is often taught in middle and beginning high school, alongside (or sometimes before or after) solutions of equations, it is quite rare to invite students to solve the equation graphically. This leads to a number of misunderstandings. For example, students often think that the solutions of the equation obtained in the intermediate steps of the solution process bear no relation to the solution of the initial equation.

Figure 1 gives an example of how to present a word problem as a comparison of two functions. In this question, analyzed in this paper, two children Liam and Tobet participate in a fundraiser walking the same number of miles and raising the same amount of money, despite the fact that their sponsors pledged different lump sums and per mile amounts. The amounts of money raised can be represented as functions of the number of miles walked. This leads to an equation that compares the two functions. The number of miles that Liam and Tobet walked to raise the same amount of money can be determined by solving the equation or by examining the value of x that corresponds to the intersection of the two graphs.

Liam and Tobet are going to walk in a fund-raising event to raise money for their school.

- *Liam's mother promised to donate to the school $4 per mile that Liam walks, plus an additional $30.*
- *Tobet's father promised to donate to the school $6 per mile that Tobet walks, plus an additional $20.*

After the event, Liam and Tobet compared their results. Liam had walked the same number of miles as Tobet. Liam's mother had donated the same amount of money as Tobet's father.

Functions: **$y = 4x + 30$** and **$y = 6x + 20$**
(where y represents total amount of money donated and x represents miles walked)

Equation: **$4x + 30 = 6x + 20$**

Using Functional Notation: **$L(x) = 4x + 30$** and **$T(x) = 6x + 20$**
(where L is for Liam and T is for Tobet)

Domain for both functions would be the number of miles walked **$[0, \infty)$**
Co-domain for both functions would be the total amount of money donated **$[0, \infty)$**

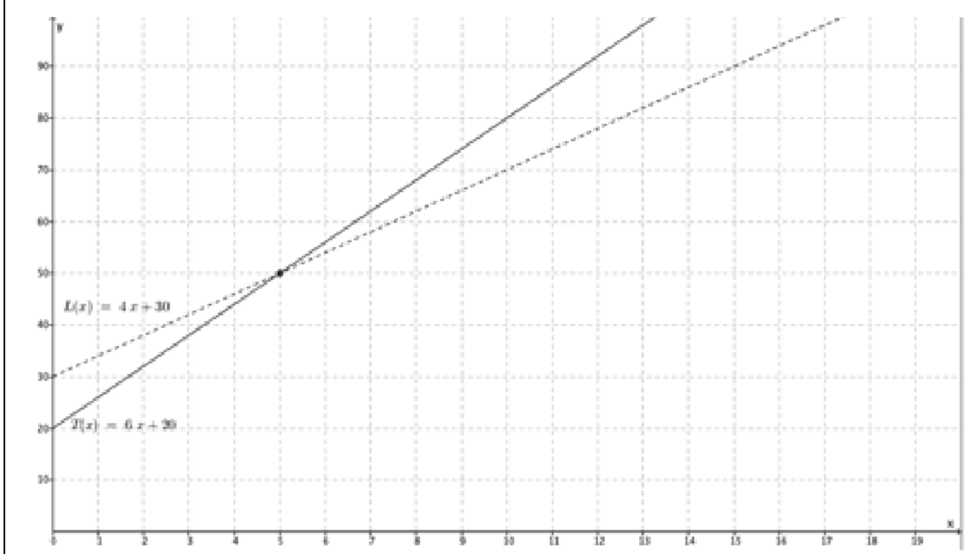


Figure 1 An example of a problem represented as two functions and as an equation

The implementation of a functional approach to teaching algebra represents a departure from the traditional path of teaching algebra solely by focusing on the manipulation of equations. As such, it requires the preparation of teachers for doing so and close evaluation of the impact of this preparation on students' learning (Sharpe & Schliemann, 2014). In the teacher development program, in addition to emphasizing equations as comparisons of functions and encouraging the use of multiple representations, we tried to make teachers aware of the fact that the algebraic manipulation of the equations can be matched to transformations of graphs in the plane (see Yerushalmy & Schwartz, 1993, for work with a computer software on this approach). In particular, we introduced them to transformations of the plane and invited them to look at the fact that only transformations that leave invariant the x-coordinate of the points of intersection for the two graphs being compared can be considered in the solution of equations. Such approach, aimed at deepening teachers' understanding of functions and equations, has the potential to better prepare teachers to help students represent and solve word problems and to better understand algebra in general.

The Three Semester-Long Online Graduate Level Courses

Teachers were first introduced to the idea that graphs of functions could be a way to solve a verbal problem by the end of Course 1, after extensive work on linear functions and their representation during the previous 13 weeks. This was achieved by asking them to examine and discuss a videotaped lesson and students' work from a grades 3 to 5 longitudinal classroom study on early algebra (see Carraher, Schliemann, and Schwartz, 2008, for details on the classroom intervention and lesson implementation). In the lesson, fifth grade students were first asked to show on paper what they knew about the following problem:

> Mike and Robin each have some amount of money.
>
> Mike has $8 in his hand and the rest of his money is in his wallet.
>
> Robin has, altogether, exactly three times as much money as Mike has in his wallet.

Note that no question was asked about Mike and Robin's amounts at this point. Since students were already familiar with use of variables to represent non-specified amounts in verbal problems, many of them represented Mike's amount as N + 8 and Robin's amount as 3N, 3 x N, or N + N + N. Should the instruction have aimed at algebra as learning how to solve equations, the next step would be to ask: "How much money is in the wallet if Mike and Robin have the same total amount of money?" In this case, students could then have proceeded to instruction on how to solve the equation N + 8 = 3N. Instead, in keeping with a functional approach to algebra, starting from fifth graders' representations of the problem, the classroom discussions shown in the video led students to discuss "Who would have more money?" In the discussion, they first considered possible values for the amount in the wallet and represented these in a table where each line showed, for each possible value in the wallet, the total amounts Mike and Robin each would have. Students were also familiar with how to plot functions in the Cartesian space and, discussing with the instructor, they came to produce, in their handouts and on the board, the graphs of the two functions. From inspection of the table and of the graph, they concluded that, if there were four dollars in the wallet, Mike and Robin would have equal amounts, namely, 12 dollars each.

In Course 2, after extensive work on transformations of the line and of the plane, teachers analyzed and discussed mathematics notes and applets on the correspondence between changes in the equation towards its solution and the translations and dilations of the graphs of the functions compared by the equation. They did so for equations comparing linear and non-linear functions.

In Course 3 teachers further analyzed how transformations of functions correspond to equations and to inequalities. The work on inequalities, in the third unit of the course, built upon the previous work on solving equations and was introduced by pointing out that:

> "We saw at the beginning of Week 1 of this course that we can view any inequality as a "function inequality" just as we do with equations. In this section, we will explore the algebra of solving both linear and absolute value inequalities.

> Instead of starting from scratch, let's use what we already know about solving equations to guide us. We know that, when we solve an equation f(x) = g(x) in one variable by seeing the solution set in the Cartesian plane, we actually see the solution set to two inequalities f(x) < g(x) and g(x) < f(x) as well."

From teachers' online and face-to-face comments we gathered that solving problems and equations by comparing functions in the Cartesian space and analyzing the correspondence between changes in the equation and changes in the graph were new ideas to most of them.

The purpose of the analysis presented here compares the results for students of teachers in the program to those of a control group and aimed at (a) examining the impact of the teacher development program on their students' assessment performance and (b) analyzing how students' score gains over one academic year in creating, solving, and interpreting equations relate to teachers' initial levels of knowledge about algebra, functions, and graphs.

Method

We analyzed changes in 7th grade students' answers to a written assessment questions involving creating, solving, and interpreting equations. Data come from a larger data collection from grades 5 through 9. We focus on 7th grade results due to the clearer connection between the content of the program and the content of the 7th grade curriculum.

The six 7th grade teachers in the teacher development program, like participating teachers in other grades, volunteered to enroll in the program for the 18 months of its duration. Their results and their students' results are compared to those of seven other 7th grade teachers (the non-cohort teachers) who didn't enroll in the program. The cohort and non-cohort teachers worked in the same schools, within the same school districts. Five hundred and eighty six 7th grade students (319 from cohort teachers' classrooms and 267 from non-cohort teachers) were given the written assessment at the start and at the end of a full school year during which the cohort teachers were taking the second and third of the three-course series. The non-cohort students received the traditional curriculum used in the schools. It’s important to note that we didn’t provide any prescribed lessons for the cohort teachers, but they may have incorporated some of the examples given in their course assignments into their classrooms lessons.

The teaching experience of non-cohort teachers ranged from three to 34 years, with a mean of 11.75 years and mode/median of 5 years teaching mathematics. The teaching experience for cohort teachers ranged from four to 11 years, with a mean of 6.17 years and mode/median of four years. Teachers’ backgrounds in both groups included Mathematics and Education, Science and Education, only Science, or only Education, with a higher proportion of education majors among the cohort teachers. Note that background data was available for only ten of the thirteen teachers.

The cohort and non-cohort teachers took an online mathematics assessment on algebra, functions, and graphs at the start of the teacher development program (January 2011). The teacher assessment consisted of 24 problems, some of them with multiple parts.

This paper focuses on the analysis of cohort and non-cohort students’ answers to the four problems in Figures 2 to 5. These problems were included in a 15-problem student written assessment given in September 2011 and repeated in June 2012. This design allowed us to control for the initial between-group differences and to control for lack of independence between the pretest and posttest. We selected these four problems, out of the 15-problems in the assessment, because they required creating, solving and interpreting equations. As such, they were closely related to the program’s functions approach to algebra.

The first problem, *the Liam and Tobet Problem,* described above and in Figure 2, is a relatively complex multi-step algebraic word problem previously included in the 2007 Massachusetts Comprehensive Assessment System (MCAS) test for 10th graders (problem #20). Part a of the Liam and Tobet problem asks for the total amount of money donated if Liam walks 15 miles at $4 per mile, with an additional amount of $30 donated; parts b and c ask to generate an equation for each of the two functions, where '*y*' represents the amount of money donated and '*x*' represents the number of miles walked by Liam and by Tobet, respectively; part d states that Liam and Tobet walked the same number of miles and donated the same amount of money, and asks the student to use the functions generated in parts b and c to determine how many miles they each walked; finally, part e asks to determine how much money was donated by Liam and Tobet's parents. In the analysis, each part of this problem constitutes an item to be scored.

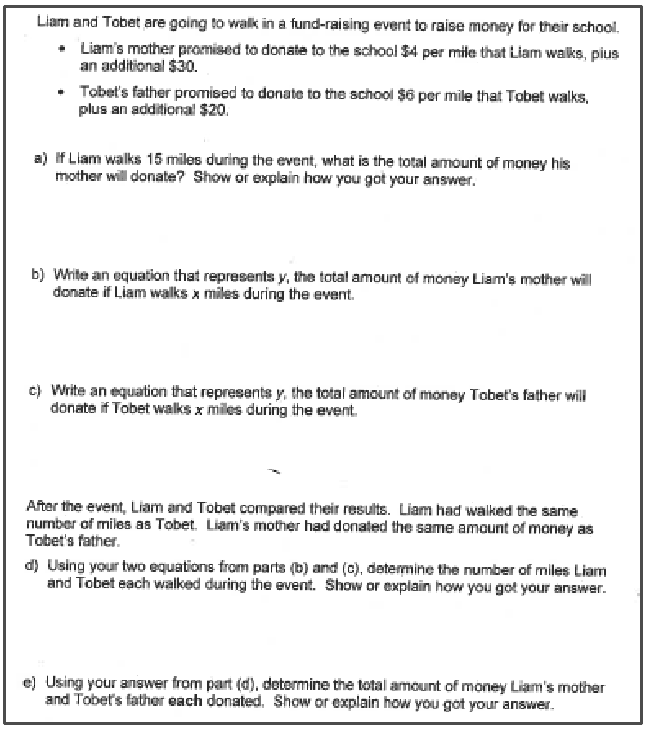
The second problem, the *Amusement Park Problem* (Figure 3), is a multiple-choice problem, previously included in the 2007 MCAS test for 8th graders (problem #32). It addresses the relationship between the dependent and independent variables in an equation that represents the total cost of a visit to a park, as a function of a flat entrance fee and the number of rides one pays.

The third problem, the *Cases Problem* (Figure 4), also a multiple-choice problem, was previously included in the 2003 National Assessment of Educational Progress (NAEP) test for 8th graders (problem #27). It requires determining which equation satisfies all of three sets of '*x*' and '*y*' listed values.

The fourth problem, the *Finding x Problem* (Figure 5), previously included in the 1999 Trends in International Mathematics and Science Study (TIMSS) test for 8th graders (problem

L17), asks students to find the value for '*x*' in an equation where the variable appears in both sides of the equality.

The Liam and Tobet word problem allows us to examine how students represent the problem as functions and as an equation, how they solve the equation, and how they interpret the equation solution. The other three problems allow examining the students' knowledge of equations in terms of (a) the relationship between the variables (the Amusement Park Problem), (b) finding an equation that satisfy three sets of *x* and *y* values (the Cases Problem), and (c) the manipulation of an equation to find the values that satisfies the equation (the Finding *x* Problem).



Liam and Tobet are going to walk in a fund-raising event to raise money for their school.

- Liam's mother promised to donate to the school $4 per mile that Liam walks, plus an additional $30.
- Tobet's father promised to donate to the school $6 per mile that Tobet walks, plus an additional $20.

a) If Liam walks 15 miles during the event, what is the total amount of money his mother will donate? Show or explain how you got your answer.

b) Write an equation that represents *y*, the total amount of money Liam's mother will donate if Liam walks *x* miles during the event.

c) Write an equation that represents *y*, the total amount of money Tobet's father will donate if Tobet walks *x* miles during the event.

After the event, Liam and Tobet compared their results. Liam had walked the same number of miles as Tobet. Liam's mother had donated the same amount of money as Tobet's father.

d) Using your two equations from parts (b) and (c), determine the number of miles Liam and Tobet each walked during the event. Show or explain how you got your answer.

e) Using your answer from part (d), determine the total amount of money Liam's mother and Tobet's father **each** donated. Show or explain how you got your answer.

Figure 2 Liam and Tobet Problem

Andrea went to an amusement park.

- The cost of admission was $5.
- The cost for each ride was $0.75.

The equation below shows *c*, Andrea's total cost to go to the amusement park and go on *r* rides.

$$c = 5 + 0.75r$$

Based on the equation, which of the following statements is true?

A) As the value of *r* increases, the value of *c* increases.
B) As the value of *r* decreases, the value of *c* stays the same.
C) As the value of *c* decreases, the value of *r* increases.
D) As the value of *c* increases, the value of *r* stays the same.

Figure 3 The Amusement Park Problem

	x	y
Case A	0	-3
Case B	1	-1
Case C	2	1

Which equation is true for the values of **x** and **y** shown in Cases A, B, & C above?

A) $3x + 2 = y$
B) $3x - 2 = y$
C) $2x + 3 = y$
D) $2x - 3 = y$
E) $x - 3 = y$

Figure 4 The Cases Problem

Find the value of *x* if $12x - 10 = 6x + 32$

Show your work.

Answer:____________________________

Figure 5 The Finding X Problem

Results

The five parts of the Liam and Tobet problem and each of the three other problems were scored as “0” when answers were missing or incorrect, and as “1” when it was correct; hence, the minimum score a student could receive for the four problems was 0 and the maximum was 8. The teachers’ scores in their 24 assessment problems could range from a minimum of 0 to a maximum of 47 points.

We analyzed the effect of the teacher development program on the 7th grade students’ performance on the four problems about creating, solving, and interpreting equations and examined how teachers’ initial mathematics knowledge, as revealed by their assessment results, relates to their students’ score gains on the problems (Sharpe & Schliemann, 2014). A second analysis, focused on each of the eight-assessment item, in the four problems, provides a deeper view of the students’ performance when creating, solving, and interpreting equations.

General Results

Table 1 provides a general view of the students’ results in the four problems and of the teachers’ results in the full assessment they took at the start of the program. The table shows that teachers’ assessment scores, collected at the start of the teacher development program, were higher for cohort teachers. It also shows that students of non-cohort teachers showed a slightly higher mean of correct answers at the start of the academic year, in comparison to those of students of cohort teachers. From the start to the end of the school year, the mean of correct answers for non-cohort students increased from 2.92 to 4.10 while those for the cohort students increased from 2.85 to 4.88.

Table 1

Descriptive Statistics

		N	Min	Max	Mean	SD
Teacher Score (January 2011)	Non-Cohort	6	20	43	33.51	9.194
	Cohort	7	32	44	37.76	4.230
	Total	13	20	44	35.82	7.256
September 2011 Student Score	Non-Cohort	267	0	8	2.92	2.247
	Cohort	319	0	8	2.85	2.319
	Total	586	0	8	2.88	2.285
June 2012 Student Score	Non-Cohort	267	0	8	4.10	2.485
	Cohort	319	0	8	4.88	2.536
	Total	586	0	8	4.53	2.541

Teachers' assessment scores were recoded (using the 33.3 and 66.6 percentiles) as low, medium or high, where a low corresponded to scores from 0 to 35, a medium to those from 36 to 40, and a high score to results from 41 to 47 points. Table 2 shows the teachers' scores distribution across the three levels.

Teachers' education background and teaching experience in mathematics were not related to their performance on the initial mathematics assessment about algebra, functions, and graphs. Namely, low performing teachers had education backgrounds in Mathematics and Education, Science, and Education with teaching experience ranging from 4 to 34 years. Medium performing teachers had education backgrounds also in Mathematics and Education, Science, and Education with teaching experience ranging from 4 to 10 years. And high performing teachers had educational background in Science, Science and Education, and Education with teaching experience ranging from 3 to 5 years.

Table 2

Teachers' Initial Scores Distribution across the Three Levels

Teachers' Initial Levels		N	Min	Max	Mean
Low	Non-Cohort	2	20	29	24.5
	Cohort	2	32	33	32.5
Medium	Non-Cohort	2	39	40	39.5
	Cohort	3	38	40	39.3
High	Non-Cohort	2	41	43	42.0
	Cohort	2	41	44	43.5

In the first analysis, we compared cohort and non-cohort students' scores at the end of the school year, to those at the start of the year. We also compared the cohort and non-cohort students' score gains, that is, the difference between the end and the start of the year scores, across the groups from classrooms of teachers with different initial levels of mathematics knowledge.

In the analysis of students' results by item, we computed the change in percent of correct answers for each item, from the start to the end of the school year, and compared the cohort and non-cohort changes across the groups from classrooms of teachers' with different initial levels of mathematics knowledge.

The next two sections focus, first, on comparisons between students' general average scores on the assessment and, then, on the percentage of students who answered each item.

Student Results by Cohort and Teachers' Initial Level

A Repeated Measures Mixed Design ANOVA revealed a significant interaction effect ($F(1, 583) = 13.63$, $p < .001$, $\eta^2_p = .023$) of time (September 2011 vs. June 2012) by cohort (Cohort vs. Non-Cohort teachers' students), on students' overall assessment scores (i.e., the 31 sub-items in the 15 assessment problems), after controlling for differences between the two groups at time 1 (September 2011). The increase in the students' math scores on all 31 items, from September 2011 to June 2012, was significantly higher for students of cohort teachers than for students of non-cohort teachers. These results support the conclusion that the teacher development program had a statistically significant impact on the 7th grade students' average mathematics scores for all items in the assessment (Sharpe & Schliemann, 2014).

Since the focus of this paper is on the four problems that more clearly relate to the main content of the teacher development courses (Liam and Tobet, Amusement Park, Cases, and Finding X problems), the analysis that follows will only deal with results for these problems, which required representing word problems as functions and as equations, solving equations, and interpreting equations' solutions to word problems.

Similar to the results for all assessment items, at time 2 (June 2012), the 7th grade students with cohort teachers performed better on the four selected problems than those of non-cohort teachers (see Figure 6). After controlling for differences between the two groups at time 1 (September 2011), the ANOVA showed a statistically significant interaction effect of time and cohort on students' scores ($F(1, 583) = 22.93$, $p < .001$, $\eta^2_p = .038$), thus supporting the conclusion that the teacher development program contributed to the improved performance of students on the specific abilities of creating, solving, and interpreting solutions to equations (Sharpe & Schliemann, 2014).

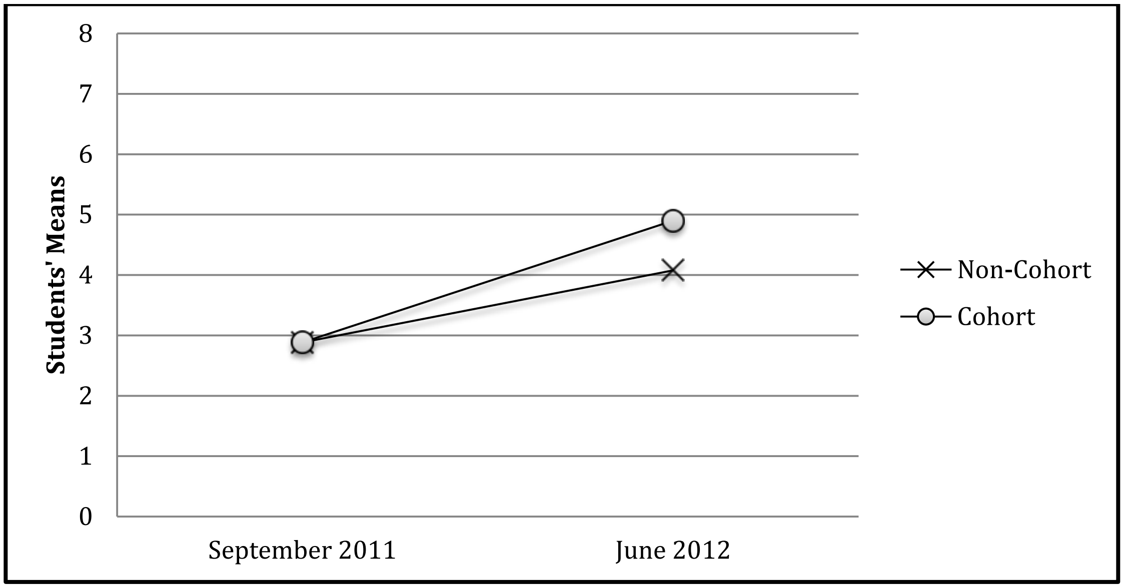


Figure 6 The interaction between Time and Cohort membership on students' scores for the four selected problems (reprinted from Sharpe & Schliemann, 2014).

Table 3 shows students' average scores in the eight selected items for cohort and non-cohort teachers, across levels of teacher knowledge of algebra, functions, and graphs at the start of the program.

Table 3

Students' Mean Score by Time, Cohort membership, and Teacher Initial Knowledge Level

Teacher Initial Level	Non-Cohort		Cohort	
	Sep-11	Jun-12	Sep-11	Jun-12
Low	2.86	3.45	2.78	4.97
Medium	2.43	3.79	2.66	4.45
High	3.31	5.15	3.14	5.25

Figure 7 displays students' average score gains per student in the eight selected items, by cohort and non-cohort teachers' initial level of knowledge of algebra, functions, and graphs. The gains at the end of the school year, that is, the difference between the average scores in September 2011 and June 2012, were higher for cohort students. Moreover, the average gains of cohort students do not vary much across teachers' initial knowledge levels, with students of teachers in the low level group showing the highest gains. In contrast, the score gains in the non-cohort group is related to teachers' initial levels, with students of non-cohort teachers with a low or medium initial level showing less gains than students of teachers with high initial level. A Univariate Analysis of Variance showed that the interaction effect of cohort membership by teacher initial levels on student score gains was statistically significant ($F(2, 585) = 6.53$, $p = .002$, $\eta^2_p = .022$). Students of cohort teachers showed somewhat similar score gains on the problems, regardless of their teachers' initial performance in the mathematics assessment on algebra, functions, and graphs, while students of non-cohort teachers' results were related to their teachers' initial level of mathematics knowledge. In this case, only students of teachers with high initial mathematical levels showed gains that approached those of students of cohort teachers.

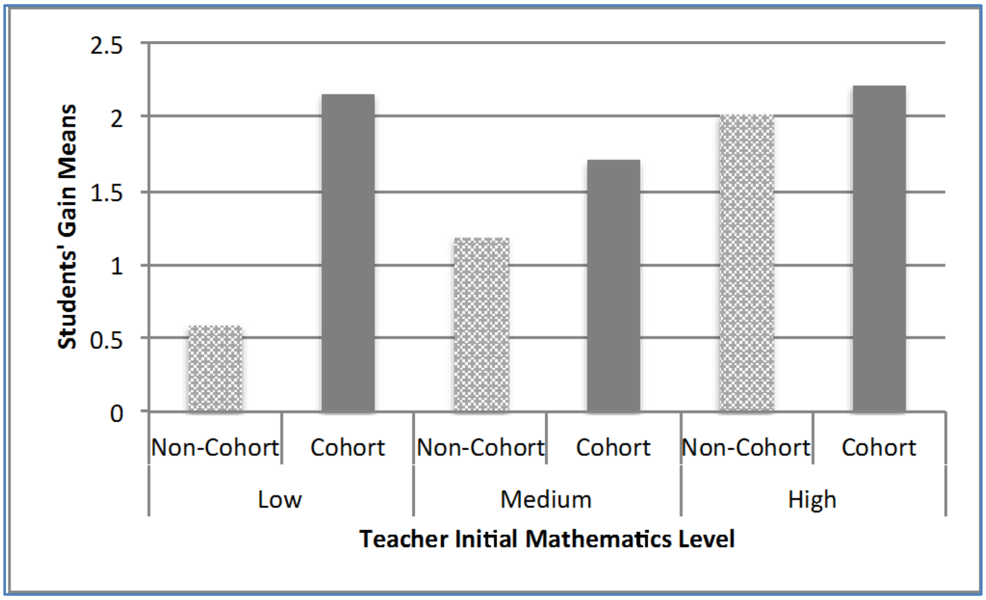


Figure 7 Student gains by Cohort membership and Teacher Initial Mathematics Levels (score gains range from -8 to 8) (reprinted from Sharpe & Schliemann, 2014).

Students' Results by Item

We take a deeper look at students' increased scores, from September 2011 to June 2012, by examining the percentage of students who correctly solved each of the eight items under analysis (five items in the first problem and one item for each of the other three problems).

Table 4 shows, for all students, the percentage of students who gave a correct answer to each item in September 2011. The items can be clustered in three groups.

Group 1, with more than 50% of students answering correctly, includes part a of the Liam and Tobet Problem (LTPa), the Amusement Park Problem (APP), and the Cases Problem (CP). These referred to, respectively, finding an arithmetic solution to the word problem, finding the relationship between the independent and dependent variables in an equation, and finding the equation that satisfy three sets of 'x' and 'y' values.

Group 2, with an average of 35% of students giving correct answers, includes parts c and part b of the Liam and Tobet Problem (LTPc and LTPb). These two items dealt with creating an equation from a word problem.

Group 3 includes parts d (LTPd) and e (LTPe) of the Liam and Tobet problem and the finding X Problem (FXP). The percent of students correctly answering these ranged from 11% to 16%, with the finding X problems being the most difficult of all.

Table 4

Percentage of students who correctly answered each item in September 2011

Items	Percent
Liam & Tobet Problem Part a	68%
Amusement Park Problem	56%
Cases Problem	51%
Liam & Tobet Problem Part c	36%
Liam & Tobet Problem Part b	34%
Liam & Tobet Problem Part d	16%
Liam & Tobet Problem Part e	16%
Finding X Problem	11%

Table 5 shows the percentage of cohort and non-cohort students who correctly answered each of the eight items in September 2011 and in June 2012 and, in the “Change” column, the increase in percentage points. This increase in performance was always higher for students of cohort teachers (ranging from 13 to 37 percentage points) than for those of non-cohort teachers (ranging from 3 to 27 points). A Repeated Measures Mixed Design Analysis of Variance showed that the interaction between time and cohort membership was statistically significant ($F(1, 14) = 6.61$, $p = .022$, $\eta^2_p = .321$).

Table 5

Percentage of Students who correctly solved each item in September 2011 and in June 2012

Items	Non-Cohort			Cohort			Difference
	Sep-11	Jun-12	Change	Sep-11	Jun-12	Change	In Change
LTP(a)	71	81	10	66	83	17	7
APP	55	58	3	56	69	13	10
CP	52	62	10	51	74	23	13
LTP(c)	36	63	27	35	68	33	6
LTP(b)	34	61	27	34	71	37	10
LTP(d)	16	33	17	16	44	28	11
LTP(e)	17	32	15	16	42	26	11
FXP	10	21	11	11	38	27	16

We also investigated the change in percentage of students in each cohort who correctly solved each of the eight items, across teachers' initial level of mathematical knowledge of algebra, functions, and graphs. An Univariate Analysis of Variance, where the unit of analysis was the eight items, revealed that the interaction between cohort and teacher initial levels on the percentage change approached statistical significant ($F(2, 42) = 3.19$, $p = .051$, $\eta^2_p = .132$). Figures 8 to 10 show, for each item and for students of cohort and of non-cohort teachers, the change in percentage of correct answers (from September 2011 to June of 2012).

For teachers with low initial level (see Figure 8), the percentage changes over all items were from 13% to 28% higher for students of cohort teachers, in comparison to those of non-cohort teachers. Students of non-cohort teachers showed modest improvements in seven of the items and a decline in the CP problem (picking an equation that satisfy three sets of x and y values). The two items with a larger difference in changes between cohort and non-cohort students were the CP item, on finding the equation that satisfy three sets of ‘x’ and ‘y’ values, and the FXP item, on computing the value for x in a given equation.

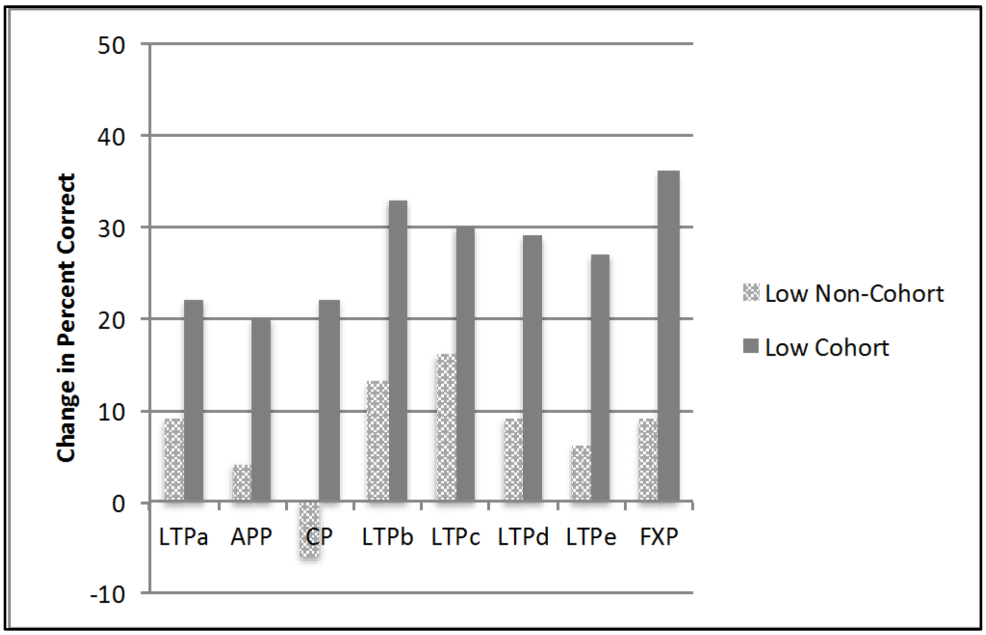


Figure 8 Change in Percent Correct by Item and Cohort membership for Students of Teachers with Low Initial Level.

For teachers with medium initial knowledge level (see Figure 9), the percentage change in five of the eight items (APP, LTPb, LTPc, LTPd, and LTPe) was from 5% to 20% higher for students of cohort teachers than for those from non-cohort teachers. The non-cohort students showed a decline in the APP problem (finding the relationship between the dependent and independent variables), equal increase in one problem (LTPa), and 2% to 6% higher improvement in the other two problems.

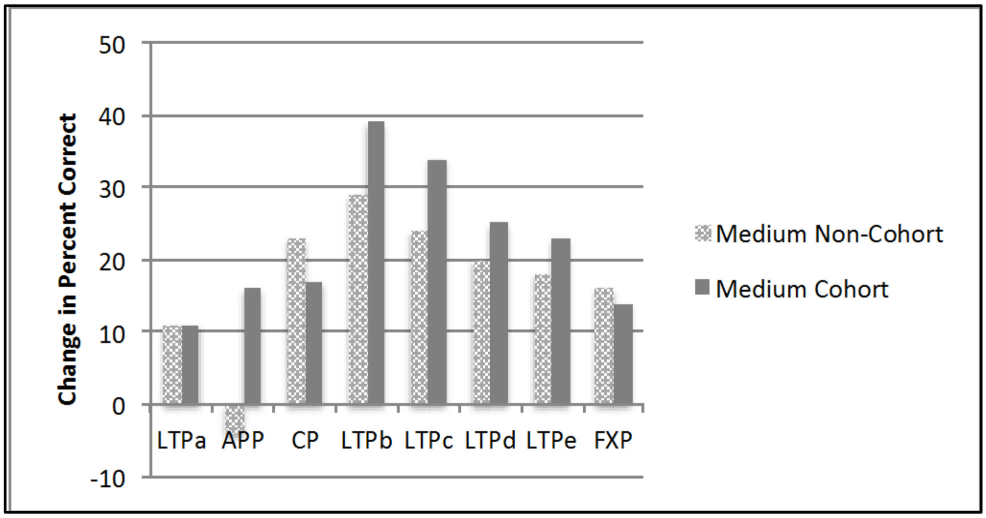


Figure 9 Change in Percent Correct by Item and Cohort membership for Students of Teachers with Medium Initial Level.

For teachers with high initial knowledge level (see Figure 10), the percentage change was 4% to 19% higher for students of cohort teachers on five of the eight items (CP, LTPa, LTPd, LTPe, and FXP) and from 3% to 9% higher for non-cohort students in the other problems.

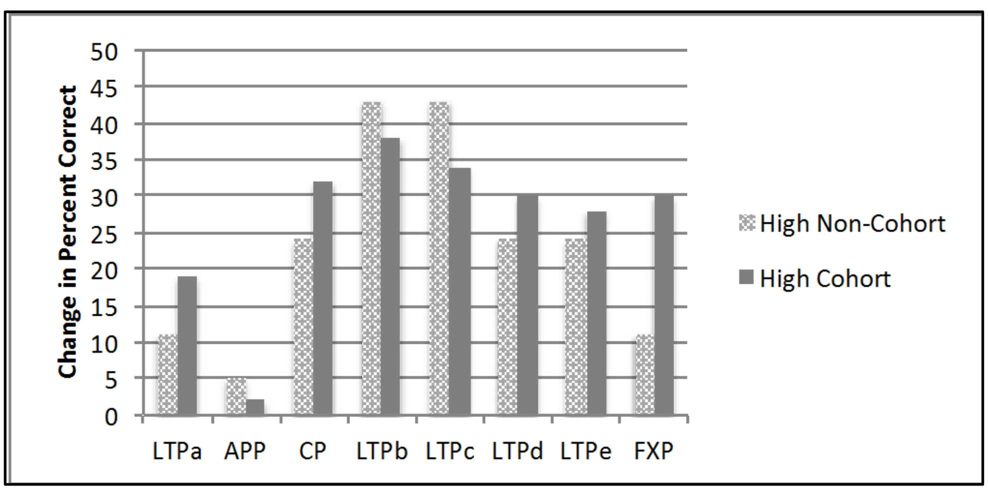


Figure 10 Change in Percent Correct by Item and Cohort membership for Students of Teachers with High Initial Level.

In summary, students of cohort teachers with low initial mathematics level showed higher improvement on the percentage of students giving correct answers for all eight items in comparison to their control counterparts. Moreover, regardless of the initial level of mathematics knowledge of their teachers, students of cohort teachers consistently showed a higher percentage of students giving correct answers, in comparison to non-cohort students, in items LTPd and LTPe, on using equations to solve the Liam and Tobet problem and on interpreting the solution to the problem.

Discussion

Our analysis strongly suggests that promoting teachers' understanding of a functional approach to algebra contributes to better teaching and learning of algebra among 7^{th} graders. Students of teachers who had participated in the teacher development program showed significantly higher scores in a written assessment in comparison to their control peers. More specifically, students of teachers in the program showed higher improvement in representing statements in a word problem as functions and as an equation that compares the two functions, of how to solve and interpret the solution to equations, and of how the elements in an equation relate to each other.

A most important feature of our results is that students of teachers with initial low levels of knowledge of algebra, functions, and graphs were those showing the highest improvement in the assessment scores. Such results are of utmost relevance, given the inequity of teachers' knowledge distribution across student populations. As data by Hill (2007) show, teachers with lower mathematics knowledge are usually assigned to lower performing students. Our results suggest that teacher development programs like the one described here may help teachers with a weak understanding of algebra, functions, and graphs better promote learning of algebra among low achieving students.

Regardless of the initial level of mathematical knowledge of their teachers, cohort teachers' students consistently outperformed their control peers in the two items students had initially showed the lowest percentage of correct answers. These were the items requiring writing, solving, and interpreting the equation corresponding to the Liam and Tobet problem. Changes in the percentage of correct answers by students of cohort teachers with initial low levels of knowledge were most notable in the Liam and Tobet items requiring algebraic representation, solution, and interpretation of the equation and in the FXP item on solving an equation where the variable appeared in both sides of the equal sign. Students of teachers at the medium and high initial levels of mathematical knowledge, in both cohort and non-cohort groups, showed most improvement on representing each function in the Liam and Tobet problem using algebra notation. Here, students of cohort medium level teachers improved more than those of their non-cohort controls, while an inverse trend appeared among students of high initial level teachers. It is interesting to note that students of low and high-level cohort teachers showed substantially higher improvement in solving an equation (FXP), in comparison to non-cohort students, while students of cohort medium-level teachers showed a modest decline to students of non-cohort medium-level teachers.

The achievements of 7th grade students taught by teachers who had participated in the teacher development program on algebra and functions are relevant, given the well-documented difficulties among middle and high school students with use of algebra notation to represent verbal problems, understanding of how the elements in an equation are inter-related, and solving and interpreting algebra equations. Our data contribute to further advance our understanding of the potential impact of teacher development programs on student learning and of how the contribution of such programs may interact with teachers' previous level of mathematical knowledge of algebra, functions, and graphs. However, we must keep in mind that the number of 7th grade cohort teachers in this study was low. As new cohorts of teachers join the program, further similar analyses will better evaluate the contribution of the program to student learning.

Besides the deeper understanding of mathematics afforded by a functions approach to algebra, other aspects of the teacher development program are likely to have contributed to the higher performance of cohort teachers' students. Among these we would like to highlight the programs' focus on: (a) use of mathematics to discuss everyday and science phenomena, (b) use of multiple representations, (c) teachers' analyses of students' thinking and learning, and (d) the online and face-to-face discussions of teachers among themselves and with mathematicians, scientists, and educators.

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