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On mathematics with distinction, a learner-centered conceptualization of challenge and choice-based pedagogies[1]

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Abstract: The main argument of this article is that "challenging mathematics for all" can be more than just a nice slogan, on condition that all students are empowered to make informed choices of: a challenge to be dealt with, a way of dealing with the challenge, a mode of interaction, an extent of collaboration, and an agent to learn from. Pedagogies supporting such choices are called choice-based pedagogies. The article begins from a theoretical discussion of the relationships between the notions 'mathematics with distinction,' 'giftedness', 'challenge' and 'choice'. As a result, the learner-centered conceptualization of mathematical challenge is proposed. Then two examples of choice-based pedagogies enabling students with different background and abilities to be engaged in mathematics with distinction are presented. Implications are drawn.

Keywords: challenging mathematics; equity; learner centered mathematics; choice based pedagogies

Introduction

Our reasoning is stipulated by words, and sometimes a new word or collocation becomes an impetus for a new line of reasoning. This is what happened to me when Ioannis Papadopoulos (personal communication, May 21 2014) introduced me to notion, which in translation from Greek sounds as *mathematics with distinction*. Ioannis explained me that the word *distinction* in Greek has a positively-colored aspect and a negatively-colored aspect. The positive aspect is related to achievements that deserve recognition and

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[1] This article is based on a plenary address given at the 6th *Conference of the Greek Association for Research in Mathematics Education* (Koichu, 2015a). The conference took place in Thessaloniki in December 4-6 2015. The topic of the conference was "Mathematics with distinction and without discrimination."

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acknowledgment (as in *getting a top result in a mathematics Olympiad is a distinction*), and a negative aspect is related to the lack of equity (as in *being gifted is a distinction as many think that only the gifted can get top scores in mathematics*).

Equipped by this explanation and, inevitably, by my beliefs and personal history in mathematics education research and practice, I inquired whether there is room for another collocation, *mathematics with distinction for all.* I inquired whether *mathematics with distinction for all* is an oxymoron or a notion that can characterize some emerging mathematics education practices. This article stems from this inquiry. It presents theoretical argument and examples from research I am involved in. The (hoped for) contribution of this article is in introducing a reconceptualization of *mathematics with distinction* and *challenge* notions, and in presenting a particular family of pedagogies, *choice-based pedagogies*, as a tool for progressing towards a situation in which *mathematics with distinction for all* would neither be an oxymoron nor just a nice slogan.

Three reconceptualizations

Reconceptualization of *mathematics with distinction*

A positive aspect of *mathematics with distinction* notion alludes to recognizable achievements in studying or doing mathematics. Clearly, each achievement has the achiever, and recognition must be attributed to him or her by somebody, for instance, a board of authorities or people of importance to the achiever. Consequently, *mathematics with distinction* can be regarded as a *labeling phenomenon*: it is what people recognize and label as such. In addition, *mathematics with distinction* is what Sternberg and Zhang (1995) would name an *implicit theory*: a non-formal intellectual construction that comprises people's intuitions about a phenomenon of importance. Sternberg and Zhang

(1995) argue that “implicit, not explicit theories have the most influence on actual life and practices.” (p. 89).

Since each mathematical achievement has the achiever, *mathematics with distinction* notion can be considered in relation to *mathematical giftedness* notion. Sternberg and Zhang (1995) argue that *giftedness* is a labeling phenomenon and offer the *implicit theory of giftedness*. This theory substantiates the following statement. People tend to label a person *gifted* if and only if he or she: is excellent at something, possesses a high level of some uncommon trait, is (at least potentially) productive and can demonstrate the trait through superior performance that must be in an area valued by society and within the culture of that society.

The absence of a broadly accepted explicit definition of giftedness (cf. Davis & Rimm, 2004, for a review of definitions of giftedness, and Leikin, Berman & Koichu, 2009, for a collection of approaches to defining mathematical giftedness) provides a fruitful ground for sustaining implicit theories. For example, in a recent review of research on mathematically gifted students, Leder (2012) indicated the overwhelming diversity of explicit definitions of mathematical giftedness and decided upon the following reviewing strategy: “[not to attempt] to provide a unique definition of mathematically gifted students or its pseudonyms, [but to accept] the diversity of definitions used in different publications…at face value” (Leder, 2012, p. 389)” In a response to the Leder review, I observed that it included publications that, generally speaking, were concerned with supporting or characterizing those students who seemed to be insufficiently challenged by regular mathematical curricula in their countries (Koichu, 2012).

I now use this observation in order to re-conceptualize *mathematical giftedness* and *mathematics with distinction* notions as implicit theories. *Mathematical giftedness* can be regarded to be a label that people use in order to acknowledge and recognize a person's ability to be productively challenged by more advanced mathematics than it is needed in order to challenge other individuals that belong to the age cohort or community of that person. Accordingly, *mathematics with distinction* can be seen as that "more advanced mathematics." Note that this reconceptualization is compatible with Sternberg and Zhang's (1995) theory. Note also that the reconceptualization is detached from "objectively" measured mathematical achievements and attached to subjective experiences of individuals engaged in studying or doing mathematics. In addition, note that the above reconceptualization includes an assumption that challenging mathematics for some can be unchallenging (e.g., too easy or too difficult) for others.

Next, I transform the above assumption into a question: Is it possible to challenge mathematically gifted students of a particular age cohort by tasks or activities that would also be challenging and feasible for the rest of the students from that cohort? In other words, is it possible to challenge all students by the same mathematical tasks and expect that some students would take them further than others?

For me, a positive answer to the above question would mean that "challenging mathematics for all" notion is neither an oxymoron nor just a slogan. Moreover, the hope that the answer to the above question can be positive is supported by various examples of activities that are known as being potentially challenging for all (e.g., Holton et al., 2009; Leikin, 2014). The word "potentially" is a troublesome, however. In the next section, the discussion of *challenging mathematics for all* is continued in light of recent theorizing on *challenge* and *studenting*.

Reconceptualization of *challenge* and *challenging mathematics for all*

The 16th ICMI Study Volume "Mathematical challenge in and beyond the classroom" offers the following definition:

> For the purpose of the Study, we will regard challenge as a question posed deliberately to entice its recipients to attempt its resolution while at the same time stretching their understanding and knowledge of some topic. Whether the question *is* a challenge depends on the background of the recipient; what may be a genuine puzzle for one person may be a mundane exercise or a matter of recall for another with more experience (Barbeau, 2009, p. 5).

This definition regards mathematical challenge as a question designed by its proposers to *entice* its recipients to act in accordance with the proposers' epistemological expectations, that is, expectations about the recipient solution moves and knowledge-seeking actions. The definition also implies that for any question, the challenge proposers' epistemological expectations can be fulfilled or not depending on the recipient's background. It is, however, silent about the recipient's intentions, and in particular, about his or her intention to accept or not the challenge. It is also silent about a competition aspect of challenge, an aspect that makes challenges potentially praiseworthy but also potentially dangerous for their recipients' self-esteem. This aspect is, however, put forward in a definition provided by the Oxford English dictionary. That definition states:

"*Challenge* [is] a call to someone to participate in a competitive situation or fight to decide who is superior in terms of ability or strength."[3]

The above comments set a stage for the following query: Why, and under which circumstances, do individuals accept challenges? A related query in the context of mathematics education is: Why and under which circumstances are our students inclined to accept or not the requests to invest intellectual effort in doing mathematical tasks that we, their teachers, attempt to challenge them by? The professional literature provides us with quite general suggestions regarding this query. For instance, Barbeau (2009) formulates (with the reference to Danesi, 2002) what he calls "optimistic message" (p. 5). Namely, Barbeau suggests that we can expect a latent willingness for people to accept challenges when a suitable stimulus is provided. Further, Harel (2008) postulates (with reference to Aristotle), that all humans "possess the capacity to develop a desire to be puzzled and to learn to carry out mental acts to solve the puzzles they create" (p. 894).

A salient feature of these views is that they emphasize human wishes and desires. This emphasis implies that humans, all humans, are in position to choose when and which challenges to accept and when and which – to reject or circumvent. In other words, an individual's general inclination to accept challenges does not mean that he or she is ready to accept every challenge offered; on the contrary, Harel (2008) takes a deliberately recipient-centered position when alluding to challenges that people create for themselves. Consequently, students are always in position to choose whether or not to accept what teachers or textbooks mean to be a challenge for them. This is true even when the norm "students must do what they are told" is implanted and the teacher feels "in control."

[3] See http://www.oxforddictionaries.com/definition/english/challenge

Moreover, this is true for all categories of students. In support of this point, consider results of three empirical studies conducted with mathematically disadvantaged, regular and gifted students.

First example: Koichu and Orey (2010) conducted an inquiry into computation strategies of mathematically disadvantaged high school students. The students were given a test consisting of a series of arithmetic tasks. We expected that the study participants would not remember standard computation procedures and invent their own methods and shortcuts. Most of the students indeed did not remember the standard procedures, but only some of them acted in accordance with our epistemological expectations. The in-depth interviews following the test revealed that some students chose not to accept the challenge, but circumvent it, for instance, by using a calculator, which was not allowed, or by answering only those questions, which were easy for them. The study resulted in realization that the students were continuously engaged in a multi-step decision-making process driven by a self-imposed question: “Is it praiseworthy to attempt solving the given task?” Only some of the student behaviors corresponded to the expected epistemological behaviors. The other behaviors were indicative of the students’ choices made in accord with what Goldin et al. (2011) called *stay-out-of-trouble* affective structure.

Second example: Liljedahl and Allan (2013a, 2013b) explored *studenting behaviors*[4] of normative high school students who were offered different mathematical problems to solve. They found that the majority of the students exhibited behaviors subverting the

[4] Fenstermacher (cited in Liljedahl & Allan, 2013a) conceptualizes *studenting* as student behaviors in learning situations, including what students do in order to ‘psych out’ teachers, figure out how to get certain grades, ‘beat the system’, deal with boredom so that it is not obvious to teachers etc.

intentions of the teachers to challenge them and engage in activities aimed at enhancing understanding or problem-solving skills. The identified studenting behaviors included *stalling*, *faking* and *mimicking*; the latter was the most frequently observed behavior. By *mimicking* Liljedahl and Allan meant avoiding problem-solving activity by mechanically following a previously presented solution pattern. According to the students who exhibited mimicking, this was what the teachers wanted them to do. In sum, the teachers in Liljedahl and Allan (2013a, 2013b) studies provided students with various opportunities to be challenged, but the majority of the students chose to reject the challenge.

Third example: Koichu and Berman (2005) documented cases when exceptionally gifted students preferred to circumvent challenges by solving the given geometry problems by brute force of algebra. It was evident that the knowledgeable students were not proud of their actions. This behaviour was explained as an instantiation of an epistemological version of the principle of intellectual parsimony. The principle states:

> When achieving a goal, for instance, when solving a problem, one intends not to make more intellectual effort than the minimum needed. In other words, one makes more effort only when forced to do so by the evidence that the problem cannot be solved with less effort (Koichu, 2008, p. 274).

The above argument and examples are provided in order to substantiate the following point. On one hand, it is reasonable to assume that humans in general and mathematics students in particular have latent willingness to be challenged by intellectual matters. On the other hand, what is regarded as a challenge by its proposer (e.g., a teacher or a textbook) is frequently not regarded as such by its recipients (e.g., students).

Consequently, conceptualizations of challenging mathematics solely in terms of epistemological expectations of its proposers for its recipients[5] can be regarded as necessary, but insufficient ones for educational needs. A learner-centered conceptualization of *challenging mathematics* notion is required. A suggestion for such a conceptualization is presented in the next paragraph.

Challenging mathematics for a learner is mathematics that he or she chooses to deal with in a way that requires putting intellectual effort in understanding it and in resolving related questions, for which the solution method cannot be readily recalled or reiterated. Sometimes the learner choice of a mathematical challenge is made under the influence of his or her teachers, and sometimes it is made under other influences, such as the influence of fellow learners, a book or a real-life situation. The learner choice to be challenged or not is stipulated by psychological and social factors, including considerations of profit-investment nature and of intellectual parsimony.

An optimistic message is that challenging mathematics, as re-conceptualized above, *can* exist "for all". However, it may remain latent. One's chances to discover challenging mathematics increase when appropriate pedagogies are used. An important educational mission is to construct such pedagogies. One of them, a *choice-based pedagogy*, is presented in the next section.

Reconceptualization of *choice* and an introduction of *choice-based pedagogies*

As argued, one's engagement in challenging mathematics is essentially a matter of his or her choice. An immediate corollary of this stance is that pedagogies supporting student

[5] For instance, Barbeau (2009) characterizes "a good challenge" (p. 5) as a challenge that often involves explanation, multiple approaches, conjecturing, evaluation of solutions for effectiveness and elegance, and construction and evaluation of examples.

engagement in challenging mathematics must include opportunities for each student to choose suitable challenges. The roots of this idea can be traced to seminal work of Dewey (1938/1963), who reasoned that students must be involved in choosing what they learn.

A choice of a challenge is, however, only one type of choices that people make when doing mathematics. Choices of additional types are made when a chosen challenge is pursued. For instance, when a student becomes truly engaged in a challenging for him or her problem, he or she makes choices regarding the solution moves or, more generally speaking, regarding the use of mathematical knowledge and strategies. This type of choices has been in the focus of research on problem solving for several decades (e.g., Mamona-Downs & Downs, 2005; Schoenfeld, 1992). As a result, conditions and teaching practices for facilitating problem solving in a classroom have been identified in various contexts (e.g., Schoenfeld, 1985; Koichu, Berman & Moore, 2007; Leikin, 2014; Liljedahl, in press).

This section is devoted to pedagogies supporting additional types of student choices as well, Choice-Based Pedagogies[6] (hereafter, CBPs). CBPs are flourishing in business schools and art education (e.g., CBAE, 2008, Douglas & Jaquith, 2009), but are not common in mathematics education. A central premise of the proponents of the CBPs in the field of art education is: “The student is an artist…In an authentic choice-based environment, students have control over subject matter, materials, and approach” (CBAE, 2008, p. 6). A classroom functions as a studio with different activity centers working in parallel, and students make “real choices” about in which activity to take part and how (Douglas & Jaquith, 2009).

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[6] Note the difference between Choice-Based Pedagogy and Pedagogy of Choice notions. The latter notion is usually used as a name of pedagogies, in which *teachers’* choices are considered (Cummins, 2009).

A possible transposition of this approach to the context of mathematics education is as follows. The student is a problem-solver, who is involved in choosing a challenge and is in position to choose who, when and how to interact with when pursuing it. Choosing and handling the challenges occurs in a studio-like learning environment, in which different activities and discourses take place, sometimes in parallel. In particular, each student is in position to choose the most appropriate to him or her:

- ✓ extent of collaboration, from being actively involved in exploratory discourse with peers of his or her choice to being an independent solver;
- ✓ mode of interactions, that is, whether to talk, listen or be temporary disengaged from the collective discourse, as well as whether to be a proposer of a problem-solving idea, a responder to the ideas by the others or a silent observer;
- ✓ agent to learn from, that is, the student can decide whose and which ideas are worthwhile his or her attention, if at all.

Examples of learning situations and environments having such characteristics are presented in the next section.

Theory in action: Two examples

Example 1: An online discussion forum as a case of CBP[7]

A 10th grade of 17 regular (that is, not identified as gifted) students and their teacher took part in an study, in which Olympiad-style geometry problems were given for solution during 5-7 days each, in an environment combining classroom work and work from home. The work from home was supported by an online discussion forum at Google+. Realization that this learning environment is a CBP came to me during the analysis of the

[7] This section consists of a modified section from Koichu (2015b).

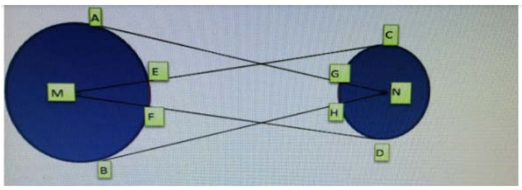
data, which were collected from the following sources: content of the forums, the teacher diary, interviews with selected students, and reflective questionnaires that have been filled in by all the students after working on each problem.

The students indicated in the reflective questionnaire that they had worked collaboratively for about 40% of time that had been devoted to solving each of the given problems. (On average, the students worked on a typical problem of the project for about 3 hours distributed during 1-3 days). As a rule, the students chose to appear at the forum when they were stuck and sought for new ideas or for the feedback on their incomplete ideas. Some students additionally sought for chat with their classmates.

The exposition below focuses on one student, Marsha[8], who solved one of the problems in a particularly original way. It is of note that, according to the teacher, Marsha has neither been an active student in a classroom nor a successful student in terms of mathematics tests. The problem (Sharygin & Gordin, 2001, No. 3463, with the reference to Figure 1a), was as follows:

> Two extrinsic circles are given. From the center of each circle two tangent segments to another circle are constructed. Prove that the obtained chords (GH and EF – see the drawing) are equal.

[8] All the student names are pseudonyms.



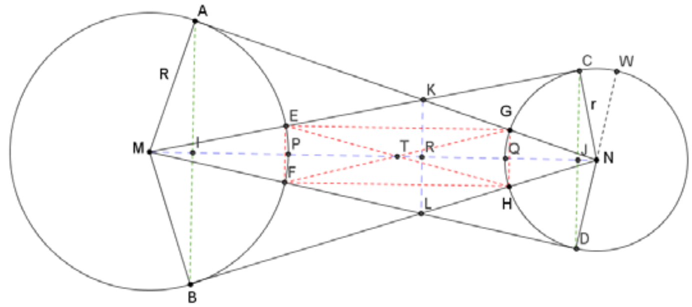


Figure 1a: The given drawing Figure 1b: Summary of auxiliary constructions

Figure 1: Drawings for the two-circle problem

The anticipated challenge was that the problem could hardly be solved by including the chords in some pair of congruent triangles or in a parallelogram. The intended solution was based on consideration of two pairs of similar triangles, $\Delta MEP \sim \Delta MCN$ and $\Delta NCQ \sim \Delta NAM$ (see Figure 1b). $EP = \frac{r \cdot R}{MN}$ from the first similarity, and $GQ = \frac{r \cdot R}{MN}$ from the second similarity, which concludes the proof. Two solutions based on this idea were posted by two students by the end of the forum, two days after Marsha's solution.

The key idea of Marsha's solution was that $KL \parallel EF \parallel GH$. This was particularly difficult for her to prove. Her proof of this fact consisted of 24 "claim - justification" rows. She then used this fact in order to prove similarity of two pairs of triangles, $\Delta KNL \sim \Delta GNH$ and $\Delta MKL \sim \Delta MEF$. She concluded the proof by consideration of proportions stemming from these similarities, in conjunction to a proportion stemming from a "bridging" pair of similar triangles, $\Delta MAK \sim \Delta NCK$.

The actual problem-solving process was not straightforward at all. In brief, at the beginning Marsha focused her attention on how to use MN and the radiuses of the circles. She considered these objects for some time with one of the classmates, Sarah. Then data show when Marsha and Sarah's solution pathways departed and why. Marsha then attempted to prove that $EGHF$ was a rectangle. This attempt was in line with the reasoning of another participant, Mary. After the exchange of ideas and a period of being silent, Marsha announced at the forum that she completely changed the direction and that she was “several moments from the proof”. However, her initial explanation of the proof had a logical flaw. Marsha succeeded to produce a mathematically valid proof after polishing her reasoning in reply to clarification questions by two other participants of the forum.

Marsha was very active at the forum, but indicated in her questionnaire that she worked alone for about three out of five hours that she devoted to solving the problem. It was also observed that Marsha had been highly selective when choosing with whom and how to communicate.

One point of the example is that the described learning environment, in which Marsha accepted the challenge and succeeded, was indeed a choice-based environment for her: she was in position to choose and change when needed an extent of collaboration, a mode of interaction, and an agent to learn from. Another point is that Marsha’s performance in the described situation can be qualified as dealing with mathematics with distinction, in the meaning specified above. To recall, Marsha and her classmates were not gifted students by any “formal” criterion.

Example 2: Student mathematics research projects as a case of CBP[9]

The case of interest occurred in the framework of a project entitled "Open-ended mathematical problems." The project is conducted in one of the Israeli schools. The goal of the project is to create opportunities for the students to develop algebraic reasoning

[9] This section focuses on an aspect of a Ph.D. study of Alik Palatnik (Palatnik, 2016). I compare in the section two cases which were separately presented in two publications (Palatnik & Koichu, 2014; 2015).

and exploration skills through long-term solving challenging problems in the context of numerical sequences[10].

At the beginning of a yearly cycle of the project, a 9th grade class is exposed to about 10 challenging problems in an introductory lecture. The students choose a particular problem to pursue and work on it in teams of two or three. The students work on a chosen problem practically daily during the leisure hours at home and during their enrichment classes. Weekly 20-minute meetings of each team with the teacher take place during the enrichment classes. When the initial problem is solved, the students pose and explore related questions. By the end of the project the teams present their research at the workshop at the Technion. One of the project's problems is presented on Figure 2.

Every straight cut divides pizza into two separate pieces. What is the largest number of pieces that can be obtained by n straight cuts? A. Solve for n = 1, 2, 3, 4, 5, 6. B. Find a recursive formula for the case of n. C. Find an explicit formula. D. Find and investigate other interesting sequences.

Figure 2: Pizza Problem

Five student teams chose the Pizza Problem since 2012. The exposition below focuses on two teams: a team of Ron (briefly, TR) and a team of Eli (briefly, TE). These teams are chosen to be presented here as particularly informative about the differences between the project learning trajectories and outcomes. The beginnings however were similar: both teams approached the Pizza Problem empirically, by drawing tens of circles representing

[10] It is of note that 9th graders in Israel, as a rule, do not possess any systematic knowledge on sequences; this topic is taught in 10th grade.

a pizza, cutting them by straight lines and counting the pieces. Then the team trajectories departed.

The TR's progress was associated with gradual developing a set of abstract representations for the problem (from the drawings to number strings, to number-matchings and to two-column tables, see Figure 3), invention of useful notation, and discovery of the patterns in the tables.

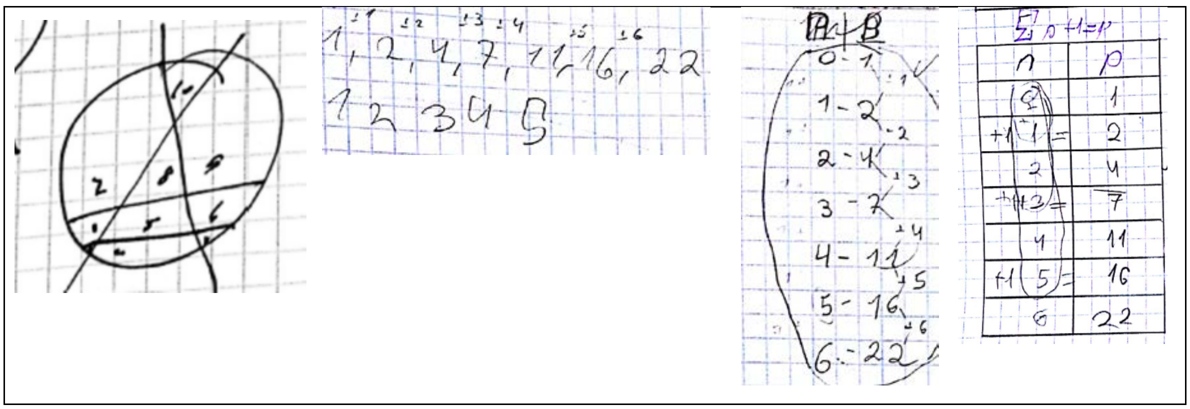


Figure 3: Selected drafts by TR in the order of their appearance

TR succeeded to produce a recursive formula for the sequence ($P_n = P_{n-1} + n$) during the first week, but discovering an explicit formula appeared for them to be a true challenge. After two weeks of unsuccessful attempts, Ron eventually handled the challenge in quite a serendipitous manner. He managed to find a regularity connecting number 5 in the left column of the table with corresponding number 16 in the right column. (The discovered regularity was $(5 \div 2) \times 6 + 1 = 16$, see Palatnik & Koichu, 2015, for details of Ron's discovery). After additional attempts, Ron translated the regularity into an explicit formula ($P_n = \frac{n}{2}(n+1) + 1$), and thus accomplished Item C of the Pizza Problem. TR realized that the formula was found "by chance" and was eager to find a safer way of finding such formulas as well as to learn how such formulas can be proved. These two self-imposed questions became a focus of the second part of the TR project. Eventually, TR produced a "proof" of the explicit formula by algebraically connecting it with two additional formulas that they found for so-called "open" and "closed" pieces of the pizza (e.g., pieces No. 2 and 10 on Figure 3 were considered "open," and pieces No. 5 and 8 – "closed"). This "proof" was the culmination of the project for TR.

In contrast, TE relatively easy (though not quickly), produced both the recursive and explicit formulas for the sequence. Specifically, TE discovered the regularity $P_n = (1 + 2 + \cdots + n) + 1$ (as TR also did) and then used their familiarity with the formula for computing a sum of the first n integers (TR was not familiar with this formula). As a result, obtaining the explicit formula for the sequence ($P_n = \frac{n}{2}(n+1) + 1$) was not perceived by TE as an achievement to be proud of. TE's main exploration took another direction. It should be noted at this point that two month before the beginning of the project Eli wrote an essay about *triangular numbers*[11]. In the essay, Eli noted that $T_n = 1 + 2 + 4 + \cdots + n = \frac{n}{2}(n+1)$ and wrote: "A factorial $[n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n]$ is very similar to a triangular number. Only there is multiplication instead of addition." The involvement of the sum $1 + 2 + 3 + \cdots + n$ in a sequence for the Pizza Problem and in a sequence of triangular numbers caught Eli's attention. As a result, he proudly introduced at the final workshop a new mathematical sign, $n\underset{+}{!}$, which he called "factorial of addition" (note "+" in $n\underset{+}{!}$ vs. "$\cdot$" in $n!$). Justification of the usefulness of this sign for "future calculators" became the culmination of the project for TE.

[11] See http://mathworld.wolfram.com/TriangularNumber.html

The first point of the presented example is that the described learning environment appeared to be very rich with the opportunities for the students to choose. The second point is that the presented task became a source of different challenges for the less knowledgeable students (TR) as well as for the more knowledgeable ones (TE).

Summary and implications

Theoretical argument presented in the first part of the paper consists of the following steps. (i) Reconceptualization of *mathematical giftedness* as a label that people use in order to acknowledge one's ability to be productively *challenged* by more advanced mathematics than it is needed in order to challenge his or her peers. *Mathematics with distinction* is equated with that "more advanced mathematics." (ii) *Mathematical challenge* is re-conceptualized from a notion emphasizing epistemological expectations of the challenge proposers to a notion that puts forward the student choice to invest intellectual effort in dealing with what he or she sees as a challenge. It is argued that such mathematical challenges can exist *for all.* (iii) The *choice* notion is extended to embrace not only a choice of a challenge, but also additional choices, including a choice of a social mode of dealing with the challenge and of an agent to learn from. Characteristics of *choice-based pedagogies* (i.e., pedagogies supporting diverse student choices) are introduced.

The second part of the paper presents two examples of situations, in which middle-school students of different abilities were challenged by the same tasks in different ways. The first example overviews a situation, in which the students have been empowered to choose: (1) to attempt solving the given problem or respond to the solution produced by somebody; (2) to work independently or with the peers; (3) who communicate with and when; (4) which ideas to discuss and which to use. In the second example, the students could choose: (5) a problem to solve; (6) a way of dealing with the problem; (7) a direction for a follow-up exploration; (9) the most important for them and consequently worthwhile sharing results of their research. It is up to the reader to decide whether the student mathematics results in the described situations can qualify as *mathematics with distinction*. In my personal view (and consistently with the presented theoretical argument), all the presented products of student explorations are distinct.

Let me conclude by saying that re-conceptualizations and examples presented in this paper in order to make a point that there is room (and even the need) for the *mathematics with distinction for all* notion, may have implications for research and practice. For instance, let us imagine a practicum for pre-service or in-service mathematics teachers, which is entitled exactly as this paper is entitled. The participants of the practicum systematically learn to analyze their teaching in terms of challenges and choices that they provide their students with, and reflect on the student achievements by reasoning to which extent they are *distinct*. The participants also learn to compile their students' individual profiles, in which the cases of experiencing *mathematics with distinction* are valued. The participants fully realize that *mathematics with distinction* for each student is different even when the same tasks are used. Furthermore, the analysis of teaching through the lenses of challenges and choices becomes for the participants a guideline for task design and for adapting appropriate technological means. It may be a bit naïve, but I believe that the participants of such a practicum and their students would jointly learn meaningful mathematics and enjoy it.

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