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Forms of Knowledge of Advanced Mathematics for Teaching

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Abstract**:** In this paper, we explore in more detail *why* knowing advanced mathematics might be beneficial for teachers, specifically in relation to their classroom practice. Rather than by listing courses or specific advanced topics, as though those were the agents of change, we do so by considering advanced mathematical content for teachers in terms of more general *forms of knowledge*. In particular, we identify five forms of knowledge of advanced mathematics for teaching: peripheral, evolutionary, axiomatic, logical, and inferential. These categories were derived from analysis of an extensive mapping process linking K-12 content to relevant advanced mathematics. We connect these forms of knowledge to particular practices that teachers engage in so as to clarify the perceived relations to classroom practices. We view such work as important to and productive for teacher educators, particularly in conceptualizing and structuring mathematics courses for teachers so that content that truly informs the work of K-12 teaching can be highlighted, and in a manner that facilitates teachers' formation of those connections.

*Keywords***:** Advanced mathematics; Content knowledge for teaching; Teacher preparation; Forms of knowledge

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1. Introduction

Teachers, particularly at the middle and secondary level, are frequently required to take advanced mathematics courses (e.g., real analysis, abstract algebra, etc.) as part of their teacher preparation program. This is based on a general sense of the importance of strong content knowledge for teachers. However, many teachers at some point question the relevance of these advanced courses for their future teaching careers (e.g., Zazkis & Leikin, 2010). Indeed, as Monk (1994) noted, the quantity of advanced mathematics preparation does not guarantee teaching quality; that is, one's own mathematical understanding does not necessarily translate into an ability to enhance the understanding of others. With the recent focus on practice-based approaches to conceptualizing teacher's mathematical knowledge (e.g., Ball, Thames, & Phelps, 2008; Petrou & Goulding, 2011) – which has led to documenting specific ways teachers use their knowledge of mathematics in their professional work – much more needs to be done to identify, define, and document how *advanced* mathematics may inform the actual work of teaching. Although this problem has been observed for a long time – Felix Klein (1932) made comments about this in the early 20th century – little progress has been made in this regard.

In contrast to defining specific content areas in advanced mathematics or undergraduate course recommendations (e.g., CBMS 2001; 2012), this paper focuses instead on more general *forms of knowledge* of advanced mathematics that may be productive for the teaching of school mathematics. Following a grounded theory approach (e.g., Strauss & Corbin, 1990), we developed a framework of five forms of knowledge of advanced mathematics for teaching that, while not necessarily an exhaustive or exclusive list, moves the discussion of mathematical knowledge for teaching – particularly of advanced mathematics – beyond just listing *what* content teachers need to know, and toward a more general conception of *how* knowing advanced mathematics could positively interact with the work of teaching. We begin by reviewing related literature and situating this work within it.

2. Literature

2.1. Teachers' Knowledge of advanced Mathematics

Recent efforts to conceptualize the mathematical knowledge required for teaching have incorporated practice-based approaches to teacher knowledge – that is, the content knowledge teachers need should be relevant for the practices and work of teaching. In particular, researchers have used this perspective to conceptualize different "domains" of knowledge, at both the elementary and secondary levels (e.g., Ball, Thames, & Phelps, 2008; McCrory, et al., 2012; Heid, Wilson, & Blume, 2015). Notably, many of these – and others' (e.g., Zazkis & Leikin, 2010) – allude to the importance of knowing advanced mathematics. For example, the Mathematical Knowledge for Teaching (MKT) framework (Ball, Thames, & Phelps, 2008) included the domain Horizon Content Knowledge (HCK), which is related the conversation about advanced mathematics (e.g., Wasserman & Stockton, 2013; Jakobsen, Thames, & Ribeiro, 2013; Zazkis & Mamolo, 2011); and in translating to secondary mathematics teaching, McCrory, et al. (2012) included knowledge of advanced mathematics as a domain of their Knowledge of Algebra for Teaching (KAT) framework. Heid, Wilson, and Blume (2015) described Mathematical Understandings for Secondary Teaching (MUST), which, although not connected explicitly to advanced courses, have connections to larger mathematical practices in the discipline which are often honed in courses such as abstract algebra, real analysis, or an introduction to proofs course. Thus, many regard advanced mathematics as important for teaching.

However, finding explicit connections to practice has been more difficult. This fact is perhaps captured best by the "provisional" nature of the HCK domain – it was left less-developed and less-defined than other parts of the MKT framework. Some have questioned whether HCK is in fact a subdomain of knowledge itself (Fernández & Figueras, 2014), while others use the terminology "knowledge of the mathematical horizon" (KMH) instead of HCK – some differentiating between the two, others not (Guberman & Gorev, 2015; Zazkis & Mamolo, 2011). The discourse and provisional nature of this domain are indicative of the difficulty in meaningfully connecting this knowledge to teaching practice – indeed, very little has been done to explicitly connect such advanced mathematical knowledge and the tasks and work of teaching. Although courses in advanced mathematics (e.g., abstract algebra, number theory) cover ideas that can be related to the *content* of school mathematics (e.g., McCrory, et al., 2012; CBMS, 2012), such connections do not necessarily suggest an explicit need for advanced mathematics in relation to the tasks of *teaching* that content. These conversations about advanced mathematics echo Klein's (1932) observation of a "double discontinuity" for teachers in their education. The first discontinuity was that the study of university mathematics did not develop from or suggest the school mathematics that students (i.e., future teachers) knew. The second discontinuity was a disconnect for these future teachers in returning back to school mathematics, where the university mathematics appeared unrelated to the tasks of teaching. The first is a comment about the *teaching of advanced mathematics*; the second is a statement about the *(advanced) mathematical preparation of teachers*. While perhaps both discontinuities still exist, it is this second discontinuity that is of particular interest in this paper – in exploring how knowing advanced mathematics might influence the teaching of school mathematics.

We make explicit two additional comments before moving on. In the context of teachers, what counts as advanced mathematics might differ depending on the specific content and/or grade level at which a teacher teaches. Like Wasserman (2016), we situate advanced mathematics as *outside* the local (epsilon) neighborhood (i.e., nonlocal) of the content a teacher teaches. By a local (epsilon) neighborhood, we mean those concepts and ideas that are proximally close to the content a teacher teaches – where "close" includes both the degree to which mathematical ideas are closely connected but also temporally close in relation to when mathematical ideas are typically developed. For the purposes of this study, however, we made an arbitrary cut as to what counts as "advanced" mathematics, more in line with (although slightly different from) Zazkis & Leikin's (2010) conceptualization of AMK and McCrory et al.'s (2012) subdomain, knowledge of advanced mathematics.

In particular, for the purposes of this work we defined advanced mathematics as that content that is beyond standard K-12 mathematics content, as represented by the Common Core State Standards in Mathematics (CCSS-M, 2010)[2], which also positions this as knowledge outside the typical scope of what a school mathematics teacher would likely teach. Second, when we consider ways that advanced mathematics might influence teachers' practice – i.e., consistent with a practice-based approach to conceptualizing teachers' knowledge of advanced mathematics – we are *not* including the teaching of say, abstract algebra, to secondary students. We view the work of school mathematics teachers as teaching school mathematics – not advanced mathematics. Thus, our perspective about relating to and being relevant for teaching is aligned with what Wasserman (2016) termed a transformational approach: that is, that knowledge of advanced mathematics can transform teachers' own "understanding about and perception of the

[2] Within the CCSS-M, there are some "optional" standards, designated with a (+); because that content is not *required* in K-12, we assume it may not be covered in many school settings and so consider content that arises in the (+) standards as advanced mathematical content according to our starting definition.

content they teach in ways that influence their teaching" (pp. 30). Based on this understanding, we now look more closely at how we approached connecting advanced mathematics to teachers' practices.

2.2. Teachers' Moves in Relation to Practice

In order to assess the relevance of studying particular advanced mathematical topics, it was important to identify concrete ways teachers could leverage that knowledge during the course of their teaching. As part of McCrory et al.'s (2012) Knowledge of Algebra for Teaching framework, they described three activities that teachers often undertake that make use of their content knowledge in teaching: *trimming* (removing complexity while maintaining mathematical integrity), *decompressing* (unpacking a topic's mathematical complexity in ways that make it comprehensible), and *bridging* (making connections across topics, assignments, representations, and domains). More recently, Wasserman (2015) expanded on and clarified two of these notions by considering the neighborhood (local or nonlocal – in relation to the content being taught) of the mathematical complexities involved. In particular, based on teachers either highlighting or hiding local or nonlocal complexities, Wasserman laid out four teacher moves: *unpacking* complexity, *foreshadowing* complexity, *abridging* complexity, and *concealing* complexity. Unpacking (similar to McCrory et al.'s (2012) decompressing, and Ball and Bass's (2000) unpacking) is a response that intentionally highlights and describes some local complexity within a topic – a choice to make explicit to students some of the inherent complexities within a topic being taught (e.g., making clear our base-ten numeral system for expressing numbers). Foreshadowing is a similar response but to some nonlocal complexity – a choice to add complexity to a current idea in order to prepare students for a future mathematical transition.

Abridging (similar to McCrory et al.'s (2012) trimming) has the opposite response to a nonlocal complexity – the shortening and condensing of an idea that intentionally hides the complexity, retaining the essence of the concept without going into any unnecessary details and complexities. Concealing is a similar response to abridging, but in the context of some local complexity – a temporary hiding of some local complexity for the sake of clarity and emphasis (e.g., initially not using irrational numbers that make it difficult see a conceptual connection to units and area). Along with bridging, these four ideas describe some of the practical actions that teachers make that are informed by their content knowledge. We use these as one way, though not the only, of connecting advanced mathematics to the work of teaching.

Yet in contrast to describing how teachers use and apply their knowledge – like bridging, unpacking, foreshadowing, abridging, and concealing – in this paper we explore the *forms* that knowledge may take so that a teacher can accomplish those various teaching acts.

2.3. Forms of Knowledge

As a mathematics education community, scholars have at different points distinguished between various forms of knowledge, including conceptual versus procedural knowledge (e.g., National Research Council, 2001), intuitive versus analytical knowledge (e.g., Fischbein, 1999), knowing that, knowing how, knowing why and knowing to (e.g., Mason & Spence, 1999), etc. The overarching claim in these various works is that the forms represent distinct facets of knowledge – that is, they describe some meaningful knowledge structure. In many cases, it is that the forms exist independently of one another (although they may develop simultaneously) – these forms could be represented by disjoint sets. So one can have procedural knowledge when it comes to computing area but simultaneously no conceptual knowledge about what is being computed – or vice versa. At other times there may be cognitive structures inherent within the different forms – a progression of sorts describing how knowledge builds, where the forms could be represented by nested sets. So one can "know that" without "knowing why" but "knowing why" necessitates "knowing that."

Similar to previous efforts, this paper distinguishes different forms of knowledge. However, the knowledge being described is specific in nature. These are forms of knowledge of advanced mathematics for teaching. In other words, the different forms describe different ways that one could know ideas about advanced mathematics. These are different in nature from the descriptions and characterizations of advanced mathematical thinking (e.g., Tall, 1991) – they do not describe a general process for thinking about mathematics in an advanced way, but rather portray different ways that one could know the advanced mathematics. That is, for any idea in advanced mathematics, there are many different ways that it can be known to and cognitively structured within an individual. This paper focuses specifically on describing those forms of knowledge of advanced mathematics that are then also relevant to the practices of teaching of K-12 mathematics. By relevant, we do not just mean that the advanced mathematical ideas are connected to the *content* of school mathematics, but rather that these forms of knowledge of advanced mathematics are in some way productive for the *teaching* of school mathematics content. Frequently, we use examples of specific considerations and moves (e.g., unpacking) in teaching to connect the discussion of these forms of knowledge to practice – but our key categorizations are not of how knowledge gets used, but rather the forms in which one needs to know that knowledge in order to be able to use it productively in teaching.

3. Methodology

In this paper, we address the following research question: what forms should teachers' knowledge of advanced mathematics take that are useful for and relevant to the teaching of school mathematics? The process we used to answer this question had two distinct stages – elaborated on below.

3.1. Advanced Mathematics from a School Perspective

The first stage of our process had to do with systematically specifying aspects of advanced mathematics that might be useful for the teaching of school mathematics. In order to do so, we used the content of school mathematics as a starting point from which we could look ahead to advanced mathematics, to capture specific aspects of such knowledge that would be relevant for the work of teaching. This approach contrasts with others, including Klein's (1932) "elementary mathematics from an advanced perspective" and the more recent *Mathematical education of teachers (I/II)* reports (CBMS, 2001; 2012), which have arisen mostly from the perspective of mathematicians reflecting *back* on the school mathematics arena. Given the relatively broad adoption in the United States and their benchmark to other international mathematics standards, the CCSS-M (2010) were used as the source of mathematical content and standards for analysis, roughly representative of the types of school mathematics content teachers need to be prepared to teach. We state this explicitly because the phrasing of the standards themselves may have been influential on identifying connections to teaching; however, the general mathematical ideas drawn from this analysis are not unique to the CCSS-M – indeed, the resulting mathematical ideas are commonplace in international mathematics education.

Two researchers worked together to analyze all of the standards – treating them, in totality, as representative of school (K-12) mathematics. As we read each one, we considered the question: “Teaching of this CCSS-M standard may be informed by teachers’ knowledge of which (if any) advanced mathematical content?” Each author answered this question individually and then worked collaboratively to develop consistent descriptions of the hypothesized-as-relevant advanced content knowledge for each standard. For example, from the specific standard 8.EE.5: *Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways. For example, compare a distance-time graph to a distance-time equation to determine which of two moving objects has greater speed* (CCSS-M, 2010), we agreed that teachers would likely benefit from knowing the advanced topic of average and instantaneous rates of change in relation to derivatives (from Calculus). We proceeded in a collaborative coding process (e.g., Harry, Sturges, & Klingner, 2005) following the constant comparative method of a grounded theory approach (e.g. Strauss & Corbin, 1990). Our overarching categories included mathematical ideas from areas such as Set Theory, Geometry and Measurement, Algebraic Structures, Mathematical Foundations, Number Theory, Analysis of Number Systems, Calculus of Functions, Vectors and Matrices, and Probability and Statistics. Some findings from this stage of the analysis can be found in Wasserman (2016).

This process was naturally informed by the two researchers’ own mathematical and educational backgrounds; it would be hard to identify connections to advanced mathematical content we ourselves were unacquainted with, for example. Our mathematical experiences at the undergraduate and graduate levels, however, more than cover the typical range of advanced mathematical content school mathematics teachers would likely be required to study. The two researchers both have Ph.D.s in Mathematics Education, and both have the equivalent coursework of a Master's degree in Mathematics. One researcher works in an Education Department (following 6 years of secondary teaching), teaching pre- and in-service teachers in graduate level mathematics content courses for teachers, and having previously taught undergraduate mathematics courses. The other researcher works in a Mathematics Department, teaching mathematics courses across the entire undergraduate curriculum, including mathematics content courses for pre-service teachers and upper level courses for mathematics majors. Between our educational backgrounds and professional experiences, including 16 years of combined teaching experience, we felt well-positioned to identify connections between school mathematics and advanced mathematical content presented at the undergraduate and early graduate levels.

3.2. Identifying Forms of Knowledge

Through this process of specifying facets of advanced mathematics that might be useful for teachers' work in teaching these standards, we began to identify specific kinds of connections that kept recurring. The coded set of standards linked to the advanced mathematical content therefore served as our corpus of data for the next stage of analysis, presented in this paper. We analyzed the linkage of the school mathematics content with the advanced mathematics content for specific patterns – *why* was it that the advanced content had been identified as useful for teaching the school mathematics content described in the standards? In answering this question – looking across all of the connections we had made in the first stage – we began to identify consistent forms of knowledge of advanced mathematics that, through our iterative coding process conversations and discussions, we found important for teaching. Via a grounded theory approach (e.g., Strauss & Corbin, 1990), we sought to characterize and categorize the distinct forms of knowledge of advanced mathematics for teaching that emerged from this process. We reviewed and revised the advanced content connected to each standard, generating examples to justify ways that the advanced mathematics would be helpful or even necessary for the work of teaching. We iteratively refined our coding and categories, until we developed an emerging framework that categorized the connections between advanced mathematical content and the work of K-12 teaching into five forms of knowledge of advanced mathematics for teaching.

4. Results: Forms of Knowledge of Advanced Mathematics for Teaching

The forms of knowledge described below attempt to capture more general reasons *why* certain advanced content would be relevant for the teaching of certain K-12 content. By "knowledge of advanced mathematics *for teaching*", we refer to a teacher's knowledge and ways of thinking, which can then be translated into appropriate teaching actions for conveying the related K-12 mathematics content to students. As a reminder, we are not suggesting that the teacher should be introducing his/her K-12 students to the advanced mathematical material directly, but rather that such knowledge might inform their practices for teaching school mathematics. In the sections that follow, we describe each of these five forms of knowledge of advanced mathematics for teaching – *peripheral* knowledge, *evolutionary* knowledge, *axiomatic* knowledge, *logical* knowledge, and *inferential* knowledge – and attempt to illuminate each with a selection of examples representing a cross-section of content strands and grade levels.

4.1. How simple things become complex later on (Peripheral Knowledge)

Several mathematical ideas are quite straightforward at an elementary level but gain more nuanced complexity as one progresses to advanced mathematical instantiations of the topic. Teachers regularly abridge content in their explanations to help crystalize the essence of an idea. To help avoid false oversimplifications and misstatements in this process it would be beneficial for teachers to have some awareness of the advanced versions of the topic and how that eventual complexity is related to the simplified cases under consideration. For example, common misstatements from teachers (or students) may include "multiplication makes things larger", "you can't subtract a larger number from a smaller number", "anything to the zero power is one" or "the even numbers make up half of the whole numbers so there are half as many of them."

Statements that may be true within the limited scope being considered at that grade level often fail to hold true under all conditions. For example, exponents are often first introduced as "repeated multiplication". This conceptual approach holds for whole number exponents, but breaks down when dealing with rational, radical, complex, or integer exponents later on. Therefore, when introducing exponents teachers would benefit from an understanding of the complications their students will face in later grades (including ways that radicals, limits, Euler's formula, and DeMoivre's theorem, for example, inform further understanding of exponents). This is not to say that describing exponents as repeated multiplication is inappropriate, but rather that understanding the limitations of this description due to future complexities has potential benefits for instruction, reducing over- and inappropriate- reliance on this relatively simplistic and limited description.

Another example of how middle- and secondary-school material grows complex in later mathematics relates to functions. From the first formal introduction of functions through the analysis of functions in high school, certain assumptions are made about the functions' characteristics, domain, range, etc., that apply at the time but may fail to hold in future function analysis in advanced contexts. Many of the introduced analytic 'techniques' for functions only really apply to rectangular graphing scenarios – the vertical line test, for example, fails to hold the same meaning in a polar-coordinate graph or graphing on the complex plane. Therefore, it is important for teachers to understand not just the trick or shortcut of a technique but also to understand, and be able to convey to students, the real mathematical meaning and reasoning behind such a 'test'. Without recognition for how the concept of function gets more complex and abstract, teachers would likely be unable to convey the essence of such 'tests' to their students, relying, inappropriately, on tricks that hold in one context but fail to do so in another.

Specifically, a teacher's peripheral knowledge could be used to better foreshadow future developments, setting students up for success later on and avoiding inappropriately abridging and oversimplifying concepts in ways that will hinder students' ability to expand their conception of a topic or operation to new scenarios. Thus a teachers' in-depth understanding of how future developments "complicate the picture" can support a more-nuanced treatment of the initial subject, reflecting one key form of knowing advanced mathematics for teaching.

4.2. How mathematical ideas evolve(d) (Evolutionary Knowledge)

This category highlights mathematics as an ongoing process, driven by the desire to answer open questions and resolve unsettled issues. When teachers understand the process by which a particular mathematical idea developed over time, or how an elementary idea is drawn to completion at advanced levels of mathematics, they are better prepared to set students along a fruitful mathematical path that paves the way for both the development of mathematical questions of their own and the resolution of those questions later on.

For example, consider the concept of slope as rate of change that is introduced in algebra classes. A series of relevant mathematical questions following on from this idea might include "what if the rate isn't constant over time?" or "what if the graph is curved instead of straight – what is meant by rate of change then?" Indeed, these and other similar questions could be considered a driving force behind the development of Calculus, leading to ideas such as tangent lines, limits, infinitesimals, instantaneous rate of change, and derivative. Knowing the overarching mathematical direction that particular early ideas point toward could therefore help teachers set students up for further study, planting the seeds of those "next" questions in an idea's evolution.

One example from the CCSS-M standards that calls for this form of knowing is the development of rational and irrational numbers as extension of number sets beginning with whole numbers. When introducing each "new" number set to students, teachers could motivate the extension into new mathematical territory as informed by the advanced mathematical concept of group closure: i.e. how closure under subtraction leads to extension to integers; then closure under division leads to extension to rational numbers; then further extension to irrationals via exponents and historically motivating questions such as doubling the cube. Each new issue or problem type (e.g. wanting to find inverse elements under addition) calls for a new number concept (e.g. negative integers) and the successive series of questions draws the elementary idea of number from strictly whole numbers to the entire real number line.

The development of ideas through mathematical questioning could also inform the teaching of some probability and statistics material, such as how we measure error in terms of the distance between points of data and a line of best fit. For example, why have statisticians and mathematicians *decided* to use the vertical distance from a point to the line of best fit, as opposed to another common measure in geometry, that of the perpendicular distance from the point to the line? When teachers understand the evolution of mathematical and statistical ideas as a process, they can better introduce students to the premise that, mathematically, we often have a *choice* in these situations. Students can then be encouraged to explore the ramifications of various choices in order to better understand and appreciate the accepted mathematical norms and definitions.

One particularly prominent subcategory of this form of knowledge is understanding *how mathematical ideas developed historically* – which also echoes Heid, Wilson, and Blume's (2015) inclusion of historical and cultural knowledge in their framework. For teachers, knowing how mathematical concepts have evolved historically can be particularly useful for gaining student interest (e.g., story-telling), developing concepts by following their historical trajectory, and situating mathematics in relevant social, political, and cultural contexts. For example, when high school students study Euclidean geometry – in particular, the parallel postulate – it could be useful for their teachers to have a deeper understanding of the "unsettled" nature of Euclid's Fifth Postulate for centuries, and how attempts to resolve that question over time led to the development of various non-Euclidean geometries such as spherical and hyperbolic. While a teacher may not introduce their students to those alternate geometries, their own understanding of the attempts to prove or refute the parallel postulate could inform how they introduce certain properties in Euclidean geometry that are based on the necessity of assuming the parallel postulate as an axiom (such as the interior sum of angles in a triangle) (e.g., Wasserman & Stockton, 2013), compared to those that do not require this assumption.

Indeed, knowing mathematics in this way provides motivation for the more general practice of mathematical questioning, providing students with some insight into what it means to 'do' mathematics, both in terms of identifying the arc of questioning that has led to further and further refinement of a single idea over time and in terms of situating mathematical exploration as a human endeavor. In other words, in such classrooms, students might be more inclined to ask "what-if-not" (e.g., Brown & Walter, 2004) kinds of questions, and be charged with exploring the possible ramifications. Knowing mathematics in this way also opens the door to important notions in mathematical modeling (as emphasis in the CCSS-M (2010) standards) – of making choices and assumptions from a given situation, and then examining the results of the model based on such choices.

4.3. How mathematical systems are rooted in specific axiomatic foundations (Axiomatic Knowledge)

A necessary precursor to teachers' development of students' understanding of mathematical ideas is the ability to see the development of an idea from its base principles (axioms). Here we use the phrase 'axiomatic foundations' in the more modern sense of the word 'axiom', which is to say a building block or starting point of reasoning, rather than the classical sense (indicating a self-evident statement for which no proof is necessary). Students themselves are regularly asked to consider how a mathematical idea can be broken down into basic component pieces, or how definitions and core ideas serve as a starting point from which to build additional mathematical results. This process occurs quite visibly in the study of geometric systems, for example, when the ideas of perpendicular or parallel lines are developed from the building block notions of point, line, plane, distance, etc. Since teachers are required to guide students in the development of these richer mathematical ideas, they should therefore have the knowledge themselves of the process via which more complex definitions and concepts are constructed from those foundational building blocks.

As another example, consider the process of determining fraction products as rectangular areas calculable by tiling the rectangle with appropriately-sized unit squares – albeit ones split into rectangular pieces. Leading students through this process requires that teachers deeply understand that measurements require appropriate units and countable additivity, specifically thinking about determining length, area, volume, angle, etc. by adding and subtracting non-overlapping component portions. For teachers, the understanding of these concepts and processes could occur at an advanced level in the context of measurement theory or the introductory topological concept of metric spaces.

The relevance of this form of knowing is not limited to geometry and measurement. Throughout the majority of fields within advanced mathematics – such as real analysis, abstract algebra, etc. – teachers would be exposed to the mathematical practice of building from foundational axiomatic structures. Algebraic structures such as groups, rings, and fields form the underlying foundation of much of school mathematics content. For example, even in the simplest of equations, such as $x + b = \mathrm{c}$ and $ax = c$, algebraic structure is present. In these equations, the four group axioms underlay the solving process: it is necessary to assume that addition and multiplication on the reals are associative, have inverse elements and an identity element, and are closed (see Wasserman, 2014). For a teacher to point students toward important structures that underlie all algebraic reasoning and processes, the teachers should understand the axiomatic foundations that give rise to these similarities, across different objects and different operations. The teacher's formation of these connections will be particularly valuable when students encounter future objects which incorporate similar structures, such as functions and matrices or polynomials as compared to the ring of integers. Within the CCSS-M standards themselves, in fact, parallel structures are suggested: "HSA-APR.A.1. *Understand that polynomials form a* *system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials."* (CCSS-M, 2010) For teachers, then, more complete understanding of the axiomatic foundations that give rise to these similarities, across different objects and different operations, becomes important.

Fundamentally, we argue that familiarity with axiomatic systems would help teachers understand something of the foundation of mathematics. In particular, this form of knowledge is useful for providing an overarching framework within which teachers present mathematics; it informs teachers of particularly important ideas and concepts in mathematics, giving them eyes to "see" consistent structures in a topic (e.g., the countable additivity axiom for measurement, the collective importance of arithmetic properties – group axioms – for solving equations, the parallel structures between the ring of integers and of polynomials, the rigorous development of the real numbers and real-numbered functions in analysis, or the probabilistic underpinnings throughout statistics) in order to unpack them with students.

4.4. How mathematical reasoning employs logical structures and valid rules of inference (Logical Knowledge)

The standard for mathematical practice (CCSS-MP3): *construct viable arguments and critique the reasoning of others* highlights the importance of this form of knowledge for teachers of mathematics. Given how regularly students are asked to explain *why* an algorithm works or *how* they solved a problem, it is clear that teachers need to have a deep understanding of different processes for mathematical proof and the ability both to generate their own logically-sound explanations and to interpret and respond to arguments provided by students. All of these tasks are supported by a teacher's knowledge of valid logical rules of deductive inference and ability to apply logical structures to mathematical scenarios (such as use of precise mathematical language and definitions, for example). As early as elementary school, teachers have an opportunity to lay foundations for strong mathematical reasoning, such as pushing students to justify patterns and properties mathematically, based on definitions or conceptual models rather than just a few concrete cases. These foundations relate directly to a student's ability to *generalize* later on – to move from concrete examples to more abstract representations or from selected cases to more comprehensive mathematical explanations.

One important facet of assessing student work is recognizing when a student's reasoning is valid even though it doesn't take the same form of explanation the teacher had generated or expected. This form of knowing therefore highlights the ability for teachers to apply their knowledge of logical structures (e.g. equivalent arguments, contrapositive, counterexample, or negation of a false statement) and rules of deductive inference to identify alternative reasoning and argumentation schemes. For example, when teaching the Pythagorean Theorem and its converse, a teacher should be deeply familiar with multiple proofs of the Pythagorean Theorem, but also needs to recognize that the proofs of the theorem and its converse are substantially different. More generally, the teachers' understanding of these proofs would be enhanced by their understanding that in many circumstances, just because an "if… then…" statement is true does not mean its converse is true. Therefore, we posit that teachers should know the meaning and use of logical operators and concepts (e.g. and, or, not, if, then, contrapositive, converse, inverse, logical equivalence, existence, uniqueness, if and only if, structure of mathematical definitions, etc.).

An important foundation for logical reasoning in mathematics is the use of precise mathematical language and the ability to work with mathematical definitions. Indeed, defining is a mathematical activity – one that students frequently struggle with (e.g., Zandieh & Rasmussen, 2010). For students, these mathematical skills are called upon as early as elementary school, when they are asked to name and categorize various shapes according to specific attributes. It would therefore be helpful for teachers to understand the nuances and precision of various mathematical definitions and how such definitions form a basis for further logical reasoning about shapes and their properties. In particular, to know that definition statements have an "if and only if" structure, but that the same object can have multiple definitions, where definitions only determine the starting point of reasoning. For example, while the more common definition of a parallelogram is two pairs of opposite sides that are parallel, another appropriate definition would be a quadrilateral in which the diagonals bisect each other. Both statements are bi-implications.

Even in statistical settings, logical ideas are important considerations. One example is a common confusion about conditional probabilities (i.e., confusion of the inverse – Falk, 1986), informed by Bayesian ideas. Given a statistic that, say, 75% of smokers will get lung cancer, the common incorrect extrapolation that the chance of someone who has lung cancer having been a smoker is similarly high stems from a misunderstanding of the logical structure of implications. That is, if you are a smoker, then you have a high chance of getting lung cancer, but it is not necessarily the case that if you have lung cancer, then you have a high chance of having been a smoker. It is precisely the logical structure of an implication (if…then) that is not a bi-implication (if and only if) that informs this distinction.

The fundamental concepts underlying logic and mathematical proof serve as yet another disciplinary cornerstone that crosses content areas and relates directly to ways of doing, knowing, and communicating mathematics. They continue to inform teachers' conceptions of what mathematics is as a discipline and what the practice of doing mathematics entails, especially in ways that inform how teachers need to respond to students as they develop their own sense of mathematical reasoning and justification.

4.5. How statistical inference differs from other forms of mathematics reasoning (Inferential Knowledge)

As a field of study, mathematics has historically inquired into and made conclusions about *certainties*. The Pythagorean theorem, for example, gives a certain and specific answer to the length of a right triangle's hypotenuse (in Euclidean geometry); the law of cosines similarly extends this conclusion about side lengths to more general triangles. In fact, rigorous deductive proof provides absolute certainty about these conclusions. So, for much of history, when dealing with *uncertainties*, and *uncertain events*, mathematicians did not feel that there were sufficient mathematical tools to explain events of chance and variability. This changed with the rigorous development of probabilistic, and later statistical, ideas. Although mathematics teachers are frequently charged with teaching probability and statistics, statistics educators argue that the two fields are fundamentally different (e.g., see Franklin et al.'s (2005), *Guidelines for Assessment and Instruction in Statistics Education (GAISE) K-12 Report*). One particular difference in statistics has to do with the non-deterministic nature of conclusions drawn from data, due to the omnipresence of variability. Now, that is not to say that you cannot make *certain* conclusions about *uncertain* events – you can. But this only reiterates our point: given the need to disentangle the conclusions and interpretations you can and cannot make about results in probability and statistics, we found it especially important for teachers to attend to how statistical inference differs from other forms of mathematical reasoning.

For example, particular attention should be paid when considering lines of best fit. Although in mathematics, there is a deterministic answer for the missing value for a function, in statistics, lines of best fit frequently provide a non-deterministic estimate for missing values. This is contrasted by the fact that, regardless of whether a set of data appears linearly related, or whether there appear to be outliers, there *is* a deterministic answer for the least squares regression line (a mathematical question) – even if that line may not be considered well-representative of the given set of data (a statistical question). For teachers, being able to differentiate between these various mathematical and statistical aspects is important – though, as Casey & Wasserman (2015) report, teachers frequently struggle to do so.

Similar issues arise when discussing the Central Limit Theorem and confidence intervals, such as when making inferences about a population from sample statistics. Indeed, the Central Limit Theorem provides a deterministic result that we can claim with certainty: regardless of the original distribution, the distribution of *n*-sized sample means approaches a normal distribution (for n large enough) of $N\left(\mu, \frac{\sigma^2}{n}\right)$. Confidence intervals leverage this fact to infer the true mean of a population, based on desired levels of probabilistic certainty. Yet we cannot know, with absolute certainty, that the true population mean in fact lies in this confidence interval – by definition, it would lie outside the interval some of the time. In addition, many of the deterministic results that are important in statistics, such as the "68-95-99.7 rule" for standard deviations, rely on particular assumptions (often regarding the normality of data). While many phenomena can be modeled by normal distributions, teachers should be particularly aware of these assumptions in order to recognize and point out limitations appropriately with students. Awareness of useful distributions that model different situations, such as Poisson or Bernoulli distributions, may also provide an important perspective on inferences.

The inherent intermingling between both statistical and mathematical reasoning requires teachers to be particularly attentive to how statistical ideas are presented and treated. Probability and statistics are largely about understanding and interpreting events of chance, whereas mathematical reasoning aims to conclude about events of certainty. The duality present with regard to interpreting probabilistic and statistical ideas and inferences represent an important form of knowledge of advanced mathematics for teaching.

5. Discussion

Because we chose to focus this paper's analysis solely on identified advanced mathematical content useful for teaching, our forms of knowledge do not necessarily incorporate the many other types of content a teacher should be know. Many of these other types of knowledge (e.g. Specialized Content Knowledge (Ball et al., 2008)) naturally form a cornerstone of teacher preparation programs, and seem to have already been more clearly defined. We opted to retain our focus on the *how* or *form* of knowing for this paper, rather than (again) producing a list of *what* exact content teachers should know, since these forms of knowledge can inform a wide variety of teaching actions more generally. Several of the forms of knowledge focus on understanding mathematical processes and modes of reasoning, highlighting some critical aspects of mathematics as a discipline rather than specifying exactly *which* advanced topics a teacher might need to know. It may well be the case that some of the forms of knowledge we identified for advanced mathematical content therefore also apply to non-advanced material. So we do not claim the forms are exclusive, but hope that they do inform the work of teacher educators responsible for engaging pre- and in-service teachers with advanced mathematical content.

Additionally, these forms of knowledge could contribute to a more robust understanding of Ball et al.'s (2008) horizon content knowledge as an aspect of mathematical knowledge for teaching. In other words, not just *any* understanding of advanced mathematics is useful for teaching – rather, by specifying these five forms of knowledge, we get at some of the kinds of understandings of mathematics that make up a teacher's mathematical horizon and are correspondingly particularly useful for teaching. Simon (2006) described a related notion of *key developmental understandings* (KDU): characterized by conceptual advances – changes in an ability to "think about and/or perceive particular mathematical relationships" (p. 362). We argue that the five forms of knowledge detailed here may serve to foster KDUs for teachers, but in a specific sense: knowledge of advanced mathematics serving as a KDU for school mathematics in ways that inform instruction.

5.1 Envisioned Teaching Actions

Part of the process of identifying these forms of knowledge of advanced mathematics for teaching consisted of envisioning teaching practices that would rely on a teacher's advanced content knowledge at some point in the process of planning, teaching, or responding to student thinking on a particular topic. For example, one action that knowing "how simple things become complicated later on" (i.e., Peripheral Knowledge) would especially prepare teachers to undertake is *foreshadowing* or *abridging* (Wasserman, 2015). This form of knowledge should help the teacher provide clear descriptions and generalizations of content that are appropriate for current student learning but that also leave room for and/or point toward correct conceptual development in the future. When teachers focus on "how mathematical ideas evolve" (i.e., Evolutionary Knowledge) and are drawn to completion, or how different concepts draw on similar axiomatic foundations (i.e, Axiomatic Knowledge), they are helping students construct *bridges* (McCrory, et al., 2012) across related content areas in mathematics. Another teaching action entails *unpacking* (Ball & Bass, 2000; Wasserman, 2015) complexity for students – a task informed by a teacher's deep understanding of both the foundational underpinnings of the topic (i.e, Axiomatic Knowledge) and logical reasoning processes (i.e., Logical Knowledge) that was followed to reach mathematically valid conclusions from those starting axioms. The five forms of knowledge also suggest potential implications for teachers' mathematical exposition (e.g., Peripheral Knowledge and avoiding mis-statements), as well as on classroom discourse (e.g., Logical and/or Inferential Knowledge and hearing students' justifications).

5.2 Implications for teacher educators

Previous efforts to define mathematical content requirements for teacher education seem to have focused more on the course level – i.e., teachers should take Abstract Algebra, or Statistics, or a course in Geometry. There has also been a growing trend within some of the recommendations, such as the MET II report (CBMS, 2012), toward recommending courses such as "Mathematics for Elementary School Teachers" or "Secondary Mathematics from an Advanced Perspective". Defining courses and possible content to cover is helpful, yet how, precisely, should teacher educators go about presenting this content to future teachers? In other words, how is it that teachers should understand the content from an "abstract algebra" course or a "secondary mathematics from an advanced perspective" course – so that the knowledge they gain is relevant to their future professional practice? This is where we see the forms of knowledge discussed in this paper as potentially productive in teacher education. Beyond any content area recommendations, these forms of knowledge of advanced mathematics might inform the structure and approach of teaching any mathematics content course for teachers.

As an example, teacher educators might examine a particular standard or topic in K-12 mathematics, and then ask pre-service teachers: "How does this idea become complex later?"; "How did this or does this mathematical idea evolve?"; "What axiomatic foundations undergird this idea?"; "What logical or inferential structures are present in this topic?"; or "How is this representative of statistical or mathematical inference?" These questions are at the heart of having pre-service teachers develop their knowledge of advanced mathematics in ways that can become productive for teaching. Doing so helps situate the study of advanced mathematics in relation not only to school mathematics content, but also its teaching. For example, when looking at exponents, the question "how does this idea become complex later on?" points to some very interesting mathematics – what does 5^{π} mean? Or how about $(5+2i)^{2/3}$ or $e^{i\pi}$? These questions bring out some important conceptions of real numbers, such as limits of sequences of rational numbers. Some additional layers of complexity involve the extension to complex numbers, such as DeMoivre's theorem and polar representations, and also include Euler's famous historical equation, $e^{i\pi}+1=0$. As teachers begin to understand advanced mathematics in this light – as related to how a simple ideas becomes complex later on – there are specific advantages in relation to their actual teaching. As mentioned previously, this form of knowledge might be useful for helping teachers *abridge* mathematical ideas in ways that are both cognitively appropriate for students and also mathematically correct.

Taking this same content area, exponents, and looking at its axiomatic foundations also helps clarify that fulfilling exponent laws (e.g., $a^b a^c = a^{b+c}$) captures how mathematicians decided on the value for 3^{-1}, etc. Although the notation of exponents started as a way to represent repeated multiplication, trying to understand 3^{-1} as "3 multiplied by itself -1 times" is nonsensical – it was fulfillment of certain properties of exponents that ultimately led to our current definitions and conceptions of exponent values. And as teachers begin to understand the axiomatic foundations on which ideas are built, there are natural connections to teaching, such as helping appropriate proper emphasis on things that are conceptually important, either within their own mathematical explanations or as evident with tasks they design for students. In addition, such knowledge can also convey a sense of how mathematics has and does evolve. And there are plenty of other examples that we might discuss. However, more generally, approaching the teaching of advanced mathematics content to teachers using these questions – related to the five forms of knowledge – as a lens could help make the links between K-12 and the advanced mathematics more clear, and in specific ways that are potentially useful for teaching.

Teacher educators have the two-pronged task of making advanced mathematics content relatable for their students (i.e., pre-service teachers) and assisting the development of the skills necessary for effective teaching. One way to make the advanced material feel accessible and gain teacher buy-in for the need to study such content is highlighting its relevance both the to K-12 content itself, and also – perhaps more importantly – to the teaching of said content. An emphasis on the forms of knowledge of advanced mathematics that could prove fruitful in the course of the students' day-to-day work of teaching would help teacher educators address both of their primary goals.

5.3 Limitations

All identifications of advanced content connections necessarily stem from our own depth of knowledge and experience with advanced mathematical content (as well as with K-12 standards). Our results are therefore necessarily dependent on our own mathematical and teaching experiences, but we hope that by extending the discussion on forms of knowledge of advanced mathematics for K-12 teaching to the mathematics education community we may generate an expanded set of examples as others bring their own sets of experiences to bear on the question at hand.

6. Conclusion

Following an extensive mapping process linking the CCSS-M standards to relevant advanced mathematical content, we subsequently characterized several forms of knowledge of advanced mathematics for teaching:

1. Peripheral Knowledge: Understanding how simple things become complex later on
2. Evolutionary Knowledge: Understanding how mathematical ideas evolve
3. Axiomatic Knowledge: Understanding how mathematical systems are rooted in specific axiomatic foundations
4. Logical Knowledge: Understanding how mathematical reasoning employs logical structures and valid rules of inference
5. Inferential Knowledge: Understanding how statistical inference differs from other forms of mathematics reasoning

Through this process, we aimed to justify more fully *why* teachers should know advanced mathematics, by framing an approach to advanced mathematical content for teachers in terms of more general *forms of knowledge* rather than focusing solely on a laundry list of courses or topic sequences. We connect these forms of knowledge to particular practices that teachers engage in, such as abridging, concealing, foreshadowing, bridging, and unpacking. Additionally, we posit that these forms contribute to a more robust understanding of our conception of teachers' horizon content knowledge, as they describe particularly productive ways of developing key developmental understandings for teachers. Analyzing advanced mathematical content with this lens may help teacher educators conceptualize and structure mathematics courses for teachers highlighting content that truly informs the work of K-12 teaching and in a manner that facilitates teachers' formation of those connections.

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