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ABSTRACT: The *mathematical continuum* has a number of formulations and technical definitions. Two of these reference the geometric line and the real number system. This conceptual coupling of line and number has been an enduring source for mathematical invention and paradox. The continuum captures the monstrous desire of mathematics, a desire to re-assemble the point with the line, the discrete with the continuous, the finite with the infinite. This paper explores how the continuum is a source of fundamental ambiguity fueling our desires and fears about mathematics.

Key words: continuum, paradox, number line, Real number system, desire

The haunting

In set theory, the *mathematical continuum* refers to the power set of the natural numbers, but on a more intuitive plane, the continuum melds the geometric line with the real number system. This conceptual coupling of line with number has been an enduring source for mathematical invention and paradox. If the density of the real numbers – that being the fact that you can always find another real number between any other two - is *not* adequate to ensure that the reals are continuous and without gaps, then there seems to be a haunting absence that destabilizes the continuum. Concerns that Euclid's axioms could not, in principle, construct the continuity of the number line lead to various attempts to do so in the nineteenth century. Dedekind (1831-1916), intent on banishing all geometric "intuition" from mathematics, used classes and "cuts" to compose the infinite granularity needed for the continuum. Each real number – each 'part' of the continuum – was to be uniquely identified with a cut (Dedekind 1901). When the cut

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designated a rational number, rather than irrational, the number was then assigned to one of the two sets on either side of the cut. But in the case of an irrational number, the number belonged to neither. In such irrational cases, the number was always on the outside of both sets, which made it strangely unreachable and yet adequately defined in absentia. Irrational numbers were thus produced through this method, and thus the continuum was adequately defined or constructed. And yet the irrationals were somehow excluded as well, being outside either set, which for some mathematicians cast some small measure of suspicion on Dedekind's method.

Cantor (1845-1918) would offer a similar 'compositional' approach, proposing necessary and sufficient conditions for continuity that relied on set theoretic constraints (Ferreirós 1993). As set theory came to dominate the field in the nineteenth century, the materiality of the number line was left behind, and the continuum became simply the power set of the natural numbers. But haunting this historical development is the unresolved *continuum hypothesis*. This famous hypothesis refers to the proposition that there is no set of numbers with cardinality between that of the natural numbers (1, 2, 3, …) and that of the real numbers. In other words, the hypothesis can be characterized by the claim that the next biggest set, that cannot be counted by the natural numbers, is the set of real numbers. The hypothesis predicts that there is no set of numbers with cardinality between these two sets, which would mean that the set of Reals captured everything between, and that they were continuous. If there is no other set, then there is nothing missing from the Reals – there are no gaps haunting them, so to speak.

Over the centuries, the continuum seems to vibrate with traumatic desire, a desire to be both discrete and continuous, counted and uncountable, separate but connected. The mathematical continuum seems to function like a desiring machine, spurring mathematical inventiveness and ever new axiomatic endeavors. This ongoing concern with the continuum reveals the affective-material dimensions of mathematics. The traumatic investment in cutting up the continuum shows how mathematics taps an animal desire, a desire to fold continuously with the world, but also to cut oneself off as discrete individual (de Freitas 2016b). The mathematical continuum encapsulates our collective dilemma - we are all connected, but we are all individuals. The rumbling continuum captures the monstrous desire of mathematics, a desire to re-assemble the discrete with the continuous, the finite with the infinite, the point with the line.

The diabolical

More generally, measurement harbors a profound anxiety about the ontological status of continua (Serres 2017). For instance, the early calculus relied on the use of infinitesimals - *infinitely small continua* - to accurately calculate various kinds of quantities in problems and applications. These were named indivisibles because they were considered a kind of fundamental element, without discrete parts, that could be infinitely distorted, inflated or stretched. They had no 'parts' in the conventional sense of separable parts, but they possessed a potential for differentiation, a sort of *difference in itself*. In other words, they seemed to be changelings that could be used as discrete entities with definitive outlines, and yet open to stretching and inflating as need be. Thus their very status seemed to bridge the continuous and the discrete, object and relation. In some sense, they were both and neither, and thus they perplexed those who argued for atomism and also those who argued against it. Aristotle argued against the existence of these infinitesimal 'indivisibles'; while, Archimedes deployed indivisibles in his computation of areas and volumes in the second century BC.

Archimedes' calculation techniques were taken up and further developed in the 1600s, during a period of intense mathematical invention in which infinitesimal calculation flourished. And yet the very idea of a smallest interval that could not be further dissected was indeed the source of many paradoxes. The infinitesimal of Bonaventura Torricelli (1598-1647) and Evangelista Cavallieri (1608-1647) was considered so radical that the Jesuits outlawed it in European education institutions.

In the 1630s, Jesuit fathers in Rome banned the doctrine of infinitesimals, in part because of these paradoxes, declaring the idea to be dangerous and subversive, and denouncing those who taught it. Deleuze and Guattari (1987) describe the infinitesimal as always "diabolical" because it undermines the atomism and fixity of individuals, and binds number and matter through infinite variation (p. 109).

Various definitions have emerged over the years – perhaps the simplest is that the infinitesimal is an infinitely small interval. Leibniz (1646-1716) used the term infinitesimal to designate the distance between two numbers that are infinitely close (Alexander 2014). This strange idea – that a continuous interval could be infinitely small – runs counter to our intuitions about intervals *as lengths* that can always be divided into yet smaller lengths. In what sense could an interval be infinitely small? Infinitesimals are like continua "viewed in the small" as though one could zoom in and find the ultimate miniscule straight lines that composed the macro surfaces that we typically encounter. Others have described the infinitesimal as a quantity less than any finite quantity, a quantity that operates beneath the finite world (de Freitas 2016a). Such quantities don't play by the usual rules, however, being so small that their squares and other powers can be neglected. Perhaps the infinitesimal is an *intensive* magnitude rather than an *extensive* magnitude, and as such partakes in the material world in quite different ways, weaving the mathematical continuum together.

The paradoxical

A closer look at the seminal work of Cavalieri and Torricelli in the seventeenth century sheds light on why there was so much concern. But it also shows how the mathematical continuum, as source for the paradoxical, was pivotal in the development of the calculus. In other words, the paradoxical was the driving force of invention. Torricelli, in particular, created highly accessible treatises and offered "short, direct and positive proofs" using infinitesimals (or indivisibles). Unlike Cavailieri, whose work was burdened by attempts to avoid paradoxes, Toricelli delved into the paradoxes and put them to work in a new kind of calculus. The mathematician and historian Amir Alexander (2014) claims that Torricelli "reveled in paradoxes" (p.111) and tapped the contradictions that emerged when one assumed the continuum was composed of indivisibles, using them as tools for investigation (p.111). "The paradoxes were, in a way, Torricelli's mathematical experiments … For Torricelli, paradoxes … pushed logic to the extreme, thereby revealing the true nature of the continuum, which cannot be accessed by normal mathematical means" (Alexander 2014, p.112).

As a simple example, consider the task of calculating the area of a parallelogram (Fig. 1). We divide the parallelogram into two equal triangles, and imagine the space of the two triangles composed of lines with infinitesimal width (here shown as dotted lines), in one triangle they are vertical and the other horizontal. Following the methods of the early calculus, these infinitesimal lines can be added up to determine the area of each triangle (much like we might integrate under a curve by adding up the differential rectangles).

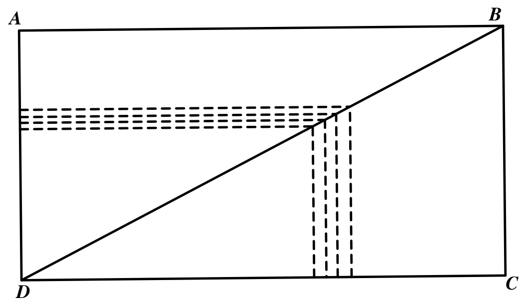

Fig. 1 Toricelli and paradox

But if we compare each infinitesimal line in one triangle with a corresponding line in the other, we see that the vertical line will always be shorter than the horizontal. It then follows, through pure logic, that since the vertical lines are always shorter than the horizontal to which they are compared, the result obtained after adding them will always be less. One triangle will have more area than the other! Contradiction! Cavalieri tried to avoid such paradoxes by not allowing indivisibles to be compared that were not parallel. But Torricelli would take up this simple paradox and delve into its potential for rethinking the mathematical continuum. Indeed, the reciprocity in this example, how we move back and forth between the vertical and the horizontal (lines of width *dx* and lines of width *dy*), will be used to huge advantage.

Torricelli introduced an entirely different way of thinking about composition and argued that there was a way that the longer lines could indeed add up to the shorter lines. The significance of this for my argument is that he was willing to break with the ruling doxa of the time, and did so by introducing a difference into a concept where there had been none before. He quite simply contradicted Euclid's definition of a line and claimed that *the short lines are wider than the long lines*. In other words, *lines are not all without width, nor are they all of equal width* (Alexander 2014). The idea that some lines were wider than others was a revolutionary idea, and broke with conventional definitions of the line. The same proposal was made for indivisible points that might inflate to varying sizes, and indivisible planes of varying thickness. It was as if Toricelli was carving out a new virtual dimension for these geometric objects, which suddenly allowed them to distort in convenient ways. Indivisibles were shown to possess a previously imperceptible dimension, which was allowed infinite variation in magnitude, and thus an infinitary calculus was born.

This controversial move allowed one to calculate various measures that had never before been attempted, extending mathematics reach and relevance, and re-assembling the relationship between mathematics and matter. If lines had infinitesimal width and planes had infinitesimal thickness, then geometry engaged with matter in new ways. These geometric concepts became physico-mathematical entities. Despite their awkward ontological status, people began to use infinitesimals in their calculations, calling them "linelets" and "timelets" and "evanescent quantities" and "inassignable quantities".

Concerns over their ontological status, however, would eventually lead to a theory of limits and an attempt to rid mathematics of *actually* infinite small quantities. Limit theory would eventually rule the day. But there have always been advocates for infinitesimals, and they continue to be of interest today. In the nineteenth century, for instance, the mathematician Paul du Bois-Reymond (1831-1889) argued on their behalf, stating, "The proposition that the number of points of division of the unit length is infinitely large produces with logical necessity the belief in the infinitely small." He advocated for a geometric number line composed of points *and* infinitesimal intervals. Charles Dodgson (1832-1898) and Charles Peirce (1839-1914) were also advocates for the infinitesimal. For Peirce, a continuous line contained no points, only continuous infinitesimal intervals. Wherever a point occurs, claimed Peirce, that point "interrupts the continuity" (CP 6.168). For Peirce, infinitesimals could be used for measurement *without* disrupting continuity. In other words, infinitesimals were measurements *intrinsic* to the continuous entity, and thus they avoided the perennial concern that measures of the continuous were always imposed from without (always the discrete fumbling to make sense of the robust continuous). Accordingly, infinitesimals were a "continuity-preserving method of measurement" (Buckley 2012, p. 149). The infinitesimal was finally given formal legitimacy (aside from its evident pragmatic value) in the 1960s when the mathematician Abraham Robinson produced a powerful and coherent foundation for the *hyperreal* numbers, which incorporated infinitesimals, transfinite numbers and the real numbers in one system (Bell 2013).

Abduction and indeterminacy

The marriage of the number line with the Real numbers brings hope and promise for an ultimate kind of synergy where measure and matter partake together of an onto-logical mixture. The continuum thus becomes a means of fusing connectives and quantifiers, geometry and arithmetic, the finite and the infinite. The paradoxes that haunt the continuum continue to be a driving force for new mathematical adventures today (Katz & Tall 2011, Katz & Poley 2017). The mathematician Fernando Zalamea (2012) demonstrates how contemporary developments in Category theory build on Peirce's work in the late nineteenth century on the mathematical continuum. Peirce merged modal logic with mathematics in novel ways, introducing the notion of *abduction* as a pivotal form of inference.[2] Peirce considered abduction "the process of forming explanatory hypotheses." and claimed that "It is the only logical operation which introduces any new idea" (CP 5.172). According to Peirce, abduction is a crucial form of reasoning in mathematics, and operates unlike induction and deduction, in that it involves *systematic guessing*. The guess becomes a key act of reason, a full-fledged form of operating mathematically.

Abduction is precisely how we hypothesize about an odd irregular event, and thus it plays a pivotal role in how we might respond to the weird number that refuses to obey our axioms, or the anxiety when faced with the apparent gaps in the continuum. We can't use deduction or induction in such instances, but must employ a different form of thinking. Abduction is at work when we devise means – as Dedekind did - of gluing the continuum back together wherever a cleavage or cut or gap is discerned. Speculating – or what Peirce would call guessing - is crucial in response to such tears in the fabric of life. In other words, a certain "explanatory continuum" is achieved through abduction, as it knits the torn threads of the mathematical continuum, darning the holes with speculation.

[2] For good introduction to Peirce on abduction, go to https://plato.stanford.edu/entries/abduction/peirce.html

An abduction is a process that one might say supplements induction and deduction in our efforts to 'explain' the continuum.

More specifically, abduction is characterized as the act of composing a hypothesis based on an observation of an irregularity that occurs in relation to expectations. This might be characterized as a "vague" deformation of syllogistic deduction that allows one to form a "retro-implicative inference" (Zalamea 2012, p.100). The table below, derived from that of Zalamea (2012), shows how abduction differs from deduction and induction, in terms of the kind of inference that is allowed:

Deduction	Induction	Abduction
All X is Y Some Z is X Some Z is Y	Some Z is X Some Z is Y All X is Y	All X is Y Some Z is Y Some Z is X

These examples help us discern the differences between different forms of reasoning, underscoring the ambition and weakness of each. In the case of abduction, the inference is deeply speculative, venturing to posit something that cannot be deduced or induced. Vagueness is precisely what makes abduction so powerful - guessing introduces possibilities of 'errors' but it also performs a kind of reasoning that cannot be achieved otherwise. In deduction, we know there are members of Z that are definitively part of Y, but in abduction we only know that members of Z *may* be part of a smaller set contained within the set to which they belong. Maybe so, maybe not. Note how terribly easy it is to go 'wrong' in abduction. The point here is that abduction plugs into our uncertainty – our not knowing - in a very particular and productive way. We use abduction regularly on a daily basis, whenever we form a hypothesis based on limited knowledge and generate a plausible explanation. Abduction tries to explain systematically a break or a breach in regularity or homogeneity or order, and is thus the very instrument needed when faced with the irruptions and gaps of the ever illusive continuum.

Thus the deep task of abduction may be seen as locally gluing breaks in the continuum, by means of an arsenal of methods which select effectively the "closer" explanatory hypotheses for a given break and which try to "erase" discontinuities from a new regularizing perspective:

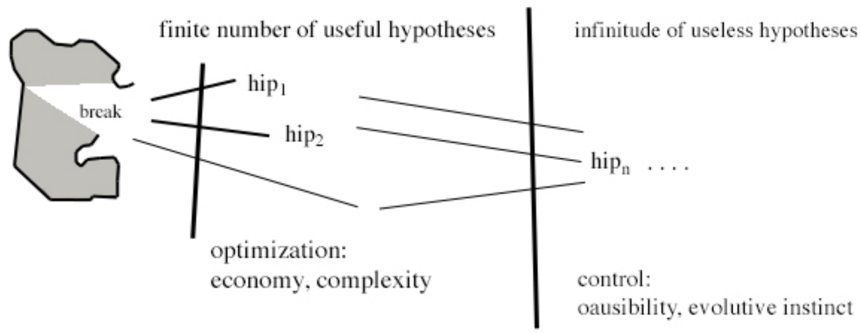

Fig. 2 Image from Zalamea, 2012, p. 102

Through the speculative act of abduction, we smudge the discontinuity and patch together both the mathematical continuum but also the cosmological continuum of life itself. Abduction, however, is not only a human faculty, but is an expression of a worldly synechism and tychism – the terms that Peirce used to describe his metaphysics of continuity and chance. For Peirce, continuity and chance are *the* two entangled metaphysical attributes of the world. Through abduction, *chance* is leveraged to mend the break in any continuum. Chance is a crucial term for Peirce, because it is through probability and modal logic that he will pursue abduction more systematically. There is a combinatorial logic in the act of guessing, a process of mapping the span of possibility, a habit of listing all possible outcomes, and then 'counting the chances' of one outcome occurring in the midst of that set of possibilities, more or less exactly. "What are the odds?" we ask, of finding an irrational number in this set, and we begin to weave hypotheses and stitch a covering of some kind, glimpsing and encountering provisionally the virtual realm of minuscule differentiations – this, and/or this, and/or this, and/or this, and/or this, and/or this, … in an iterative process that is at the heart of the concept of algorithm. Abduction engages with what we don't know, the essential not-knowing that spurs on a speculative hypothesis – a "plausible explanation" – selected from the differentiated sea of reason. This sea is so turbulent that we are made sea-sick, aghast at the vanishing indivisibles that subtend the threshold of imperceptibility. As Zalamea (2012) suggests, this realm is populated by an "infinitude of useless hypotheses" (p.102).

As the hallmark of our not-knowing, chance is the engine of a profound ontological indeterminism. Chance haunts the continuum, interrupting its smoothness, but also mending and stitching over the gaps. Peirce's proposal that abduction is the warp and weave of the mathematical continuum gives the continuum hypothesis all its cosmic and encyclopedic dimensions. Rather than concern ourselves with error and delusion, as the cogito would have us do, the mathematical continuum demands that we make the coupling of line and number a transcendental problem. *Chance and continuity together* becomes a generative problematics – still haunting, diabolical and paradoxical – but wisely at work for no purpose that we can fathom. As Deleuze (1990) suggests, the *interval* is more than a naïve measure, more than a hasty line drawn in the sand, but instead a gesture that affirms *all of chance*, a gesture that affirms indeterminacy as a plenitude rather than a lack. This profound indeterminism haunts the mathematical continuum like a hungry ghost.

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