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Towards an alternative approach to modelling in school mathematics

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ABSTRACT: This article analyses the philosophical underpinnings of mathematical modelling with the focus on the modelling cycles in mainstream mathematics education. Due to its background of epistemological representation, the main focus is set on Kantian philosophy. I argue that representational thinking leads to a problematic view on mathematics and its relation to reality. Finally, I outline an alternative approach to mathematical modelling according to Deleuze's differential ontology.

Keywords: Deleuze, Kant, Modelling, Mathematics Education, Representation

Introduction

Let's start with a story. In the 1930s, two cartographers, Otto G. Lindberg and Ernest Alpers, marked a spot on a map called Agloe published by Esso (now Exxon). However, Agloe was not a real place such as a street, a town, a lake, or a river. It was an anagram of the initials of the two cartographers that should function as a copyright trap. This was and is a very common way to ensure that no other company can easily copy a map they have not fabricated themselves. This works as follows: Imagine all the work necessary to design a new map. So, in case that cartographers see their fake place on a different map it is evidence that such map must be a pirate copy. Even today, cartographers like Google or Apple use the same kind of trap. For example, on Google maps, there used to be the non-existing place Argleton which was removed after it had been discovered.

So what happened to Agloe? It appeared on a map of Esso's business rival Rand McNally. Consequently, Esso threatened to sue them. But interestingly, a store named

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Agloe General Store had been built on the spot where Agloe was marked on Esso's map. This means that Agloe had become a real place and therefore no copyright infringement could be established. The store was closed down in the 1990s and Agloe practically seemed to disappear as easily as it had been established. Nevertheless, the story was not over yet: Agloe had been assigned a key role in the novel *Paper Towns* written by John Green. A 'paper town' is an alternate name for those copyright traps, i.e. towns that only exsist on paper. One of the main characters of the book, a girl called Margo, runs away from home and hides herself in Agloe. The title of the book, *Paper Towns,* alludes to several so-called paper towns that Margo discovers while running away. Nowadays, it happens that fans of the book or of the later film adoption go on a pilgrimage to Agloe. The place has become, so to say, a cultural site. Just have a look at Google Maps and you will see all the recommendations by fans of the book.

Modelling cycles in mainstream mathematics education

Even though "there does not exist a homogeneous understanding of modelling and its epistemological backgrounds within the international discussion on modelling" (Kaiser & Sriraman 2006, p. 302), several epistemological assumptions about modelling are widely shared and accepted among researchers focusing on modelling in mathematics education. In alliance to these shared assumptions, this article focuses on what can be called the mathematical modelling in mainstream mathematics education. This understanding of mathematical modelling in mainstream mathematics education will now be outlined.

Against the background of the international debate on modelling in school mathematics, it is remarkable that "authors and researchers, as an aid to understand student behaviours, often represent the modelling process as a cyclic activity" (Haines 2009, p. 146). For example, PISA[2] (OECD 2009) includes the following conception in its mathematical framework (see figure 1). There, a cyclic structure is assumed; likewise reality and mathematics are represented as two separated 'worlds'. Following this point of view, mathematical modelling has to be considered as a process which organises real world problems according to (pre-established) mathematical concepts.

[2] Programme for International Student Assessment

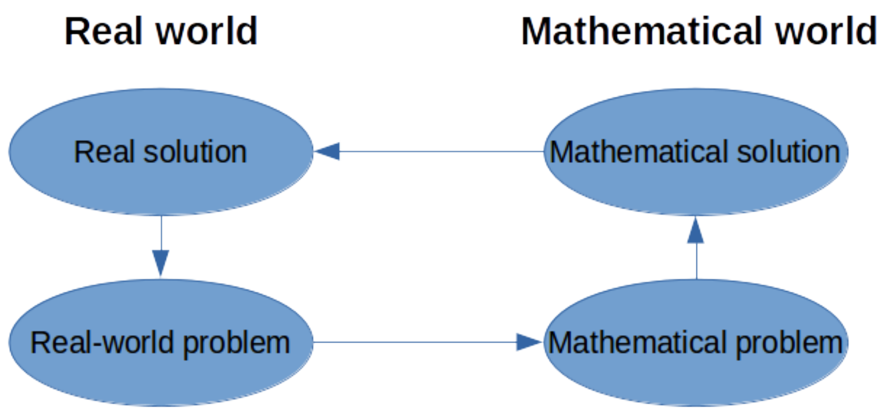

Fig. 1 Modelling cycle used in PISA's theoretical framework (OECD 2009, p. 105)

In the introduction of the 14th ICMI[3] study on modelling and applications (Blum, Galbraith, Henn, & Niss 2007), a basic modelling cycle is shown. It also distinguishes between "mathematics" and an "extra-mathematical world". Additionally, many more modelling cycles can be found in the ICTMA[4] proceedings. Figure 2 shows the modelling cycle designed by Blum and Leiß (2006). It was presented at the ICTMA 12 Conference as well as at the CERME[5]4. This cycle is probably the most cited one in German mathematics education research and is well known in the international debate as well. It is, for example, used in several contributions to the ICTMA proceedings (e.g. Biccard & Wessels 2011, Zöttl, Ufer & Reiss 2011, Vos 2013).

[3] International Commission on Mathematical Instruction
[4] International Community of Teachers of Mathematical Modelling and Applications
[5] Conference of the European Society for Research in Mathematics Education

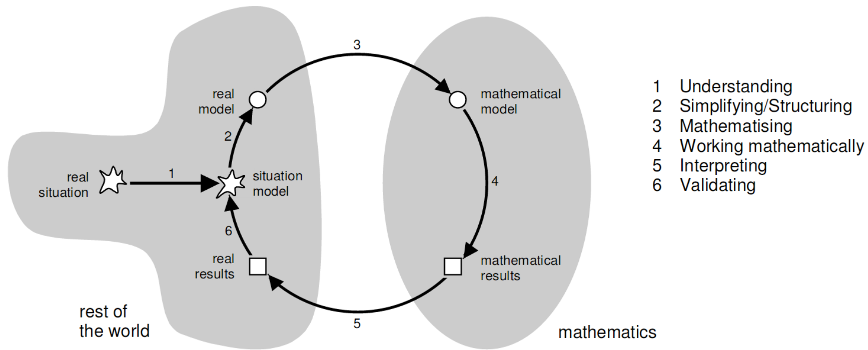

Fig. 2 Modelling cycle by Blum and Leiß (2006)

Figure 2 shows a modification of Blum's modelling cycle which was already published in 1988 (see figure 3). The current modelling cycle of Blum and Leiß has been modified by several authors. For example, Greefrath (2011) modified Blum's and Leiß's modelling cycle by adding a distinct phase for using technology, Saeki and Matsuzaki (2013) modified the cycle by inventing a dual modelling cyclic framework, and Ludwig and Reit (2013) modified the cycle by adding particular modelling competencies to any single step. Beside these modifications, other authors use quite a similar cyclic structure in their descriptions of modelling processes (e.g. Kaiser 1996, Kaiser & Stender 2013, Henn 2011, Girnat & Eichler 2011).

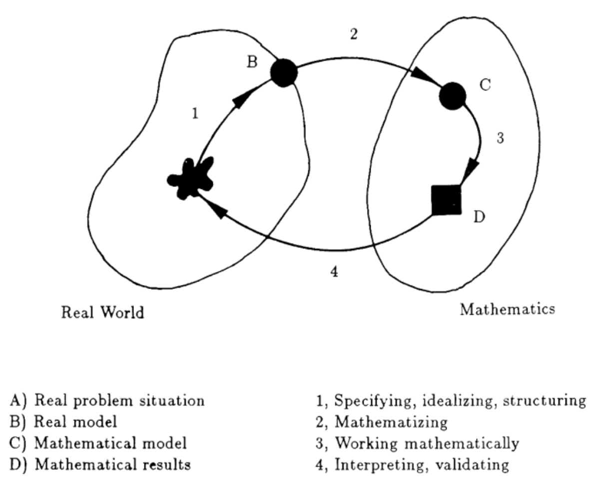

Fig. 3 Modelling cycle by Blum (1988, p. 278)

Quite often, these so-called modelling cycles strengthen the belief that mathematics is separated from reality or at least the rest of the world. This kind of separation can be found in nearly every modelling cycle. Indicated by arrows, modelling is described as a process of translation or mapping between distinct epistemic levels, i.e. modelling in mainstream mathematics education is embedded in a tradition of representation within history of philosophy. Moreover, objects in reality and concepts in mathematics both are described as distinguishable identities. Additionally, an identitarian subject is assumed within the modelling cycle which is the author of any translation between two epistemic levels. Later on, I argue how this kind of identitarian thinking is an outcome of philosophy's tradition of representational thinking.

Critique on mathematical modelling in mainstream mathematics education

In the past, on an empirical level, some of the assumptions related to modelling cycles in mainstream mathematics education have already been challenged. For example, by analysing the work of students engaged in solving so called realistic Fermi problems, Ärlebäck (2009) questions the cyclic structure of the modelling process assumed by Blum. Frejd and Bergsten (2016) interviewed scientists who are professional modellers. In the cause of the analysis of their interviews, they identified a gap between modelling in schools and the outside world. This finding led them to claim that descriptions of the modelling process in mathematics education are inadequate compared to the work done by expert modellers in real life situations. Biehler, Kortemeyer, and Schaper (2015) analysed students' processes of solving problems in engineering courses. According to their study, the separation between mathematics and the "rest of the world" as well as the division into two separate phases (setting up the model, mathematical solution), as the modelling cycle suggests it, are inadequate.

From a more theoretical perspective, Jablonka (2007) argues that it is hard to identify what is called 'modelling competencies' in case that the diversity of contexts is taken seriously. More likely, a variety of practices which do not have much in common is meant by the construct 'mathematical modelling'.

Aim of this article

This article challenges the modelling cycle, including its underlying assumptions, from a philosophical perspective. With the Agloe story in mind, I outline the discussion on modelling in mainstream mathematics education driven by epistemological assumptions well known from philosophy, especially from Immanuel Kant's (1724–1804) epistemology. I further argue that these underpinnings are problematic to some extent and hence advocate for an alternative approach to mathematical modelling based on Gille Deleuze's (1925–1995) works.

Below, modelling cycles are taken as a paradigmatic example to illustrate that modelling is embedded in the tradition of scientific representation as well as, to some extent, driven by Kant's epistemological philosophy. Nevertheless, it has to be mentioned that not all descriptions of modelling processes can be classified as Kantian. Azarello, Pezzi and Robutti (2007, p. 130), for example, emphasize the social construction of knowledge and the semiotic mediation provided by cultural artefacts; they describe their general framework as Vygotskian. Jablonka (1996) discusses approaches to mathematical modelling as either structural, or functional; that is, as pertaining to a philosophy of structure or to a philosophy of process. While the first can be considered as embedded in a Kantian tradition, the latter rather stresses a non-Kantian point of view.

Furthermore, it has to be mentioned that the long tradition of representational thinking within western philosophy, addressed and analyzed by this article, cannot be limited to Kant's philosophy only (e.g. identitarian thinking, a consequence of the tradition of representation, can be found from Plato to Descartes and Kant).

Modelling and the limits of representational thinking

Scientific modelling in general and modelling in mainstream mathematics education belong to the long tradition within philosophy where (scientific) representation is seen as a relationship between scientific domains (e.g. theories) and their targets (e.g. objects in real-world systems or theoretical objects). The representational thought is located at the basis of scientific thinking in general. "Science provides us with representations of atoms elementary particles, polymers, populations, genetic trees, economies, rational decisions, aeroplanes, earthquakes, forest fires, irrigation systems, and the world's climate. It's through these representations that we learn about the world" (Frigg & Nguyen 2016). Even if representational thinking is not limited to modelling only, models are pivotal to what is at stake in the debate on scientific representation. "As philosophers of science are increasingly acknowledging the importance, if not the primacy, of scientific models as representational units of science, it's important to stress that how they represent plays a fundamental role in how we are to answer other questions in the philosophy of science" (ibid.). In agreement to this argument, mainstream mathematics education emphasizes the representational character of models. For example, Niss (2015, p. 67) describes the classical purpose of (descriptive) mathematical modelling as "to capture, represent, understand, or analyse existing extra-mathematical phenomena, situations or domains, usually as a means of answering practical, intellectual or scientific questions–and solving related problems–pertaining to the domain under consideration."

Representational thinking and the modelling cycle

Clearly, all modelling cycles belong to the representational way of thinking. This way of thinking is related to additional assumptions which are also inherent to modelling cycles:

1. Epistemology vs. Ontology: Models are seen as pure epistemic vehicles which rather deny ontological impacts of mathematical modelling.
2. Separation: Within the representation of a target system's objects with the help of concepts of a scientific domain, a separation between both of them is established. Therefore, mathematics and reality have to be considered as being separated to some extent. Furthermore, a translating ego, the 'I think', is needed because it is the subject of the representational process of modelling and is likewise separated from the model's domains 'reality' and 'mathematics'.
3. Identitarian thinking: Representation engenders identitarian thinking. Objects in reality and mathematical concepts have to be thought as identitarian entities such as the subject itself.

The first aspect of modelling in school mathematics mentioned above can be reformulated in terms of Kant's philosophy (Kant 1783, 1787). Kant turned general metaphysics, i.e. ontology, to epistemology. Kant's so-called Copernican revolution places the rational being in the centre of attention. Quite similar, the modelling cycle describes modelling in mathematics education as a purely epistemic relation between a situation or a problem in reality and the modeler, i.e. the subject of activity.

This turn in metaphysics is in alignment with the philosopher's distinction between phenomena and *noumena* (Kant 1787, B294–B315); the second assumption of representational thinking located in the discourse on modelling. While *noumena* are objects that exist independently from human perception, phenomena are objects of human perception and thinking. According to Kant, in principle, the domain of *noumena* is not accessible for human perception and thinking. At first sight, similar assumptions can be found in the discourse on modelling in school mathematics. For instance, Pollak (2011) uses the term "translate" which, in this case, indicates a gap between real world entities (*noumena*) and objects of individual perception (phenomena); for example, "you have an idealized version of the real-world situation which you translate into mathematical terms" (ibid.). By this means, another aspect of Kant's epistemology is inherent to modelling cycles, i.e. to see mathematics not only as independent from reality but as a condition of experience. Kant defines mathematical propositions as derived from categories of thinking and pure forms of intuition, i.e. space and time. First of all, here, it seems impossible to assume a connection between mathematics and experience; that is because mathematical propositions are true without a reference to any kind of experience. In fact, because they are prior to a human's experience, they entail the possibility to organize experience in order to generate empirical knowledge. Additionally, the separation between *noumena* and phenomena is also present in the modelling cycle in case that any situation model is a mental representation of a given situation. Consequently, reality or the things themselves are not assessable by cognition or perception. Furthermore, the modelling cycle gives an impression of how mathematics is perceived in the discourse on modelling. The modelling cycle interconnects two domains: mathematics and reality. There is a clear separation between mathematics and the „real world" or, at least, between the mathematical world and the "extra-mathematical world". Therefore, mathematics does not seem to be a part of reality but rather functions as an ocular for those who interpret objects in reality. Also, mathematics seems to be an epistemic vehicle rather than an object of ontology. To emphasize this thought, I would like to consider Kant's view on mathematics. Generalized, Kant sees propositions of mathematics as synthetic judgements a priori (Kant 1787, B14–B16). This means that they are prior to our empiric experiences and therefore constitute them. Thus, the implied separation between mathematics and reality is a narrative used to describe any kind of mathematical modelling activity. Here, it will be neither explained how a (mathematical) judgement can be synthetic and a priori at the same time, nor do I outline the vehement critique on this view on mathematics which was established with non-Euclidean geometry. Notably, Kant and mainstream mathematics education treat mathematics as a purely epistemic vehicle. Both proclaim that mathematics is separated from reality to some extent.

In addition to mathematics and reality as two distinct domains of the modelling cycle, another separation occurs in the epistemological setting of mathematical modelling. The process of mathematical modelling requires a subject that translates between the two worlds and that links objects in reality to concepts in mathematics. Here, we should focus on another aspect of Kant's philosophy. Kant's epistemology tries to avoid pure idealistic concepts such as Plato's world of ideas (Plato 2000, 514a–517a). Instead, he aims to formulate an epistemology which is maximally immanent to the experience of rational beings and, at the same time, free from transcendent ideas as they can be found in the idealistic tradition. At this point, I want to recall that the terms *transcendent* and *transcendental* not only sound slightly different but have an opposing meaning. While a transcendental idea or a concept signifies a condition of experience, a transcendent idea or concept is something outside of the domain of any possible experience. In order to describe the possibility of empirical experience, Kant postulates transcendental ideas. They demarcate, so to say, the border of the domain of all possible experience. Here, special attention should be paid to the 'I think' (Kant 1787, B131–B136). To Kant, empirical experience is only possible in case that the 'I think' can be added to all of a person's judgements. The 'I think' ensures the identity of a perception and itself is assumed to be a transcendental identity. That is why it is the very first condition of every kind of experience to Kant. This assumed status of the individual can be found in the modelling cycle as well. There, the knowing and acting individual is a precondition of any transition from one step to another. Within the process, the individual's decisions are crucial, and the 'I think' is involved in every single step of the modelling cycle. Conclusively, representational thinking has led to at least three separated epistemological levels in the modelling process; i.e. the acting subject, the objects in reality, and mathematics.

Finally, one of the consequences of representational thoughts is identitarian thinking. The concept of representation is based on the assumption that there are identical concepts which resemble identical objects in reality. From this point of view, concepts are static objects which are isolated to some extent, can be opposed to each other, and stay the same over a certain period of time. Similar assertions can be made in regard to the subject, i.e. Kant's 'I think'. It is presumed that it also does not differ over time.

Representational thinking and the Agloe story

At first glance, a map seems to be an ideal example of representational thinking. A city map, for example, represents the lanes, streets, and roads in and around a city by a set of lines of varying widths, lengths, and colours. Even though the real properties might be different, the structure of the city is represented by the map. Additionally, one favour of a geographic map is that it represents the properties of a landscape. Nevertheless, the short episode of Esso's map showing Agloe challenged the representational value of maps as models already. From a Kantian perspective, models are purely epistemic tools. They depend on concepts in a related theory, e.g. mathematics, on the one side and on any given situation in reality on the other side. However, several authors highlighted the independence of models due to the fact that models cannot easily be deducted from a superior theory in an easy and uncomplicated way. Sometimes, models include assumptions which are even incompatible with the related theory (Bartels 2005, p. 83–107). Moreover, the debate on scientific models challenges the view on models as pure representations of reality. To Bailer-Jones (2002, p. 3), for example, models tend to be vague, sometimes inconsistent and can only focus on a few aspects of reality. So, to some extent, models seem to be independent from circumstances of reality.

Nevertheless, in case that models are independent from superior theories as well as from reality, one would have to admit that the independence of the Esso map is in some respect different. Of course, to mark a place on a map which does not exist is incompatible to any superior geographic theory and loosens the model's dependence from circumstances in reality. Yet, this kind of independence is an epistemological one. Furthermore, the Agloe example illustrates how a model can be independent from an ontological perspective. Everything that happened in relation to the map had an ontological impact. The map became part of reality from the moment it had been published. The model/map changed the landscape, raised legal questions, and inspired literature. So, a map is not a tracing.

In addition, the weakness of identitarian thinking becomes apparent in this story. Even though the image of representation treats all places on the map as identitarian concepts, Agloe itself cannot be considered as such. Agloe has no traceable and defined territory; it is out of scale. Moreover, Agloe's entire identity remained undefined. At first, Agloe was a paper town which then became real; or put differently, Agloe had a quite fuzzy ontological status. It existed and did not exist at the same time. While the two cartographers would have denied Agloe's existence due to the fact that they invented the paper town, all the customers of the Agloe General Store would assert the opposite. In the end, it became a desirable destination to plenty of readers of a book. The example of Agloe illustrates how a model can change its nature multiple times by being transferred from one hand to another. More precisely, the example of Agloe shows that models do not inherit a single nature, essence, or substance. In objection to that thought one could argue that all this happened accidentally and without purpose. However, that is exactly the point: it happened even though it was not expected.

Towards an alternative approach to modelling in school mathematics

For Deleuze, "the aim of philosophy is … to find the singular conditions under which something new is produced" (Smith & Protevi 2015). By aspiring this, he replaces the epistemological question on what ensures representation by the ontological question of becoming. In the following, it is argued that this principal turn in thinking about the aim of philosophy might help to reconsider mathematical modelling in school and in educational research. As it has been emphasized already, representational thinking goes hand in hand with identitarian thinking. Deleuze challenges this way of thinking by investigating what he calls 'difference in itself', i.e. an inner principle of reality, prior to identities. Together with the turn from epistemology to ontology, his project can be described as 'ontology of difference'.

It is important to note that Deleuze applies mathematics, current mathematical problems of his time and history of mathematics, to develop his approach. He neither writes a philosophy of mathematics, nor does he do mathematics itself (for surveys on the use of mathematics in Deleuze thinking, see Duffy 2006). Deleuze argues that mathematical investigations challenge our view on reality. He uses them to illustrate that reality can be seen from different perspectives compared to the mainstream or, as Deleuze calls it, major tradition of western philosophy. Thus, the hope of mathematics education research is that Deleuze's philosophy can also be used to give new insights on the learning of mathematics, especially on modelling processes (for an overview of the use of Deleuze's philosophy in mathematics education research, see de Freitas & Walshaw 2016, pp. 93–120). Now, I outline some of Deleuze's insights about mathematics which might be productive in order to reformulate modelling in an ontological, non-representational manner.

Mathematical concepts as multiplicities

First of all, I want to mention Deleuze's distinction between problematic and axiomatic formalization in mathematics (Deleuze 2004, Deleuze & Guattari 2004). Problematic formalization is a minor strand in the history of mathematics or a "nomad" way of thinking. The axiomatic (or arithmetized) way of formalization is the major strand in the history of mathematics.

> Nomadic mathematics, according to Deleuze and Guattari, disrupted the regime of axiomatic through its emphasis on the event-nature of mathematics. In particular, nomadic mathematics attended to the accidents that condition the mathematical event or encounter, while the axiomatic attended to the deduction of properties from an essence of fundamental origin (de Freitas 2013, p. 583).

The tensions between axiomatic and problematic approaches surfaced at several points in the history of mathematics. In a nutshell, the major trend in mathematics history can be characterized as a shift from solving concrete problems to deducing mathematics from axioms (e.g. a shift from Archimedean to Euclidean geometry); so what happened was a substitution of illustrative geometry under the conditions of arithmetic and algebra (e.g. Desargues' projective geometry was replaced by Fermat's and Descartes' analytic geometry). A further change was the displacement of infinite and dynamic mathematical events by a finite and static image of mathematics (e.g., Leibnitz and Newton interpreted the calculus dynamically, but due to Weierstraß' static and finite epsilon-delta criterion this was not needed anymore). To Deleuze, the tension between both sides illustrates a general conflict line that can be found in both, western philosophy and society. "Deleuze's approach is always socio-political, always showing how political and the mathematical are entwined" (de Freitas & Walshaw 2016, p. 95).

To solely focus on the axiomatic side tends to be problematic in relation to modelling in school mathematics. Assuming, in alignment to Kant, that axiomatic mathematics is a condition of experience, modelling tends to be a process where real objects and mathematical concepts are treated as identities. Consequently, a model would be nothing new but another application of mathematics.

To develop a differential ontology, Deleuze tries to grasp reality as near to the real objects as possible. Therefore, his aim is to develop a philosophy of pure immanence. He doubts that Kant's transcendental approach is successful by indicating that abstract concepts like the 'I think' are still transcendent concepts. In opposition to Kant, he outlines conditions of the genesis of real experience (not necessarily possible experience) and the becoming of the new. His purpose is to avoid the identitarian thinking of western philosophy. Furthermore, he emphasizes that difference is prior to identity. Within major tradition of philosophy, that relationship was defined the other way around. Difference was thought as the difference between already divided and distinct concepts (A is not B). Again, Deleuze refers to mathematics, especially to the history of the calculus, to address this questionable definition and to show how difference can be a genetic principle prior to identity. In order to do so, he uses Leibniz's geometric interpretation of the calculus (Leibniz 1701). There, it is shown that the differential relation dy/dx continues to exist and has an expressible finite quantity, even in case that its terms have vanished.

> The relation dy/dx is not like a fraction which is established between particular quanta in intuition, but neither is it a general relation between variable algebraic magnitudes or

> quantities. Each term exists absolutely only in its relation to the other: it is no longer necessary, or even possible, to indicate an independent variable (Deleuze 2004, p. 219).

It is a pure relation which is prior to the lengths dy and dx. According to Deleuze, this serves as an example of what he calls the concept of 'difference in itself'. The differential relation is prior to the identities dy and dx and establishes them. From Leibniz's point of view, the lengths dy and dx are pre-established by the differential relation in any given point of a curve-line.

From this very first differential principle, the 'difference in itself', Deleuze derives several other related concepts to exemplify how the new comes into the world and how real experience can be generated. At first, he gives an alternative interpretation on what is called a singularity. Commonly, a singularity is defined in relation to a universal. Comparable to set theory, a concrete object is subordinated to a more general concept. So, the particular contrasts the universal. However, bearing the calculus in mind, it is now possible to rethink a singularity as an element just beside regular ones. An extremum of a curve-line where the differential relation dy/dx tends to be zero might serve as an example of a singularity beside regular points. Singularities can be interpreted as events (i.e. something is 'happening' at these points). This view on singularities as events in a continuum of surrounding points is very different from the 'traditional' hierarchical view on singularities as reification of a universal concept.

Deleuze replaces universal concepts by what he calls a multiplicity. A multiplicity is constituted by folding or twisting particular elements like the curve-line which is generated by differential relations prior to its singular points. The multiplicity is an assemblage of regular and singular elements, but contrary to a universal concept, it is open in principle and therefore fluent. To be open in principle means that every multiplicity is interwoven with different ontological levels, e.g. politics, music, economics, sciences, ecology, etc. (no order is assumed here). Thus, any mathematical concept can be seen as a multiplicity and therefore related to various fields of experience. To be precise, the relations are part of the concept itself (Deleuze 2004, p. 203).

With the concept of multiplicity in mind, it is possible to rethink some of the limits of representational thinking in regard to modelling processes. First of all, due to its relations to several fields and its openness in principle, a multiplicity is not just a concept in an epistemological sense or a tool to organize our perception. It rather exists on an empirical level of pure immanence and tends to have a life on its own rather than being an identical universal concept. This is precisely what happened to the Esso map in the introductory story. The map was a part of the real world, so its very own character cannot be described adequately as a pure epistemic tool. Furthermore, to limit a problem to a singular field of knowledge, like static mathematics, undermines the multiplicity structure of the problem. Concepts that do not only apply to mathematical problems and propositions should not be treated in a Kantian way of thinking; i.e. as pre-established tools to organize our perception. In fact, the problem-oriented way of approaching mathematics exemplifies the possibility to generate concepts from problems in a way that problems lead to concepts. More precisely, concepts indicate a problem. In consequence, reality and mathematics should no longer be perceived as separated entities. From Deleuze's perspective, mathematics is a part of reality just like anything else. About the application of mathematics he says that "there is no reason to question the application of mathematics to physics: physics is already mathematical, since the closed environments or chosen factors also constitute systems of geometrical co-ordinates" (Deleuze 2004, p. 3; for a historical and philosophical analysis of the separation of mathematics and reality against the background of Kant's philosophy, see Schürmann 2016).

Conditions of real experience

The questions on the conditions of real experience and the conditions of the new have to be explicated in closer detail. In Kantian thinking, the distinction between possible and real experience is omnipresent. Above, the philosophical question of how real experience is possible and how to think the becoming of the new as an open question within the space of possible experience is sketched. Therefore, Deleuze replaced the dichotomy of the possible and the real with the terms of the virtual and the actual. The virtual should not be confused with any kind of virtual reality and it is clearly distinguishable from the concept of the possible. "The virtuality of the Idea has nothing to do with possibility" (Deleuze 2004, p. 240). The only categories that limit Kant's definition of the possible are thinking and pure forms of intuition. This makes a Minotaur, a bull-headed man, part of possible experience because it is thinkable. However, the space of the possible is much larger than the real; therefore, the transition from the possible to the real cannot be described in Kantian terms. Also, the transcendental/transcendent character of the possible is highly problematic. The possible seems to be pre-existent to the real. “With the concept of possibility, in short, everything is already given; everything has already been conceived” (Smith 2007, p. 16). In comparison, the virtual and the actual both describe the real, hence, they are both moments of real experience. “The virtual is opposed not to the real but to the actual. The virtual is fully real in so far as it is virtual” (Deleuze 2004, p. 260). The two concepts can be described by the related terms ‘problems’ and ‘solutions’. A problem is situated within the real and an actualization is just one of the solutions to this problem. Always, any given actualization is surrounded by a halo of virtualities. In other words, the virtual signifies possible actions to a particular problematic situation. Real experience is described by Deleuze as a zigzag trajectory between virtualities and actualization.

> The virtual is the condition for real experience, but it has no identity; identities of the subject and the object are products of processes that resolve, integrate, or actualize (the three terms are synonymous for Deleuze) a differential field (Smith & Protevi 2015).

Again, the Agloe story can exemplify this. The very first objective of cartographing is to successfully gain orientation in a new and undiscovered terrain. This problem immediately sets up a virtual space. A map can be the adequate solution for navigate oneself in a new terrain. Therefore, every map is a particular actualization of this distinct virtuality. Once actualized, every map is already surrounded by its virtual space and can be used, for example, to navigate, to decorate a living-room, or to wrap a sandwich. Of course, it can be used in a novel to describe the journey of a character. All of these examples are actualizations which come along with the halo of virtualities. This is why the process of actualization can be described as a process of a never ending becoming.

Due to the fact that very different ways of actualization are thinkable, the real can be very different at any point. Deleuze and Guattari (2004) describe how this has led or is still leading to various formations within the real world. Again, it is mathematics that serves to describe different kinds of actualisation. Deleuze and Guattari, for example, describe two different kinds of space, the smooth and the striated, as parts of the real.

These terms refer to Riemann's concept of space, i.e. the (smooth) Riemannian manifold (Deleuze & Guattari 2004, pp. 532–534) and space as a set of points or elements which is comparable to Cantor's set theory. While in the latter case the measurement of magnitudes, especially distances, is only possible from a global perspective, i.e. by a metric of the hole set, the metric for measuring a distance between two points can differ depending on where the points are located in the Riemannian manifold. Consequently, that implies that in "striated space, lines or trajectories tend to be subordinated to points: one goes from one point to another. In the smooth, it is the opposite: the points are subordinated to the trajectory" (Deleuze & Guattari 2004, p. 528). For Deleuze and Guattari, these different ways of mathematical thinking give an idea about what is happening in the socio-political space. While a striated space is, for example, used by the state to count, sort, and measure in order to control society by means of normalization and processes of in- and exclusion, the smooth space gives more opportunities of creativity and unpredictable events. However, according to Deleuze and Guattari, these two spaces do not oppose each other. In fact, even if they are not of the same nature, their relations to each other are more complex than being the opposite of each other:

> No sooner do we note a simple opposition between the two kinds of space than we must indicate a much more complex difference by virtue of which the successive terms of the oppositions fail to coincide entirely. And no sooner have we done that than we must remind ourselves that the two spaces in fact exist only in mixture: smooth space is constantly being translated, transversed into a striated space; striated space is constantly being reversed, returned to a smooth space (Deleuze & Guattari 2004, p. 524).

To some extent, mapping a terrain can be seen as a way of striation. Deleuze and Guattari explicate that by referring to the mapping of the sea: The sea once was "a smooth space par excellence", but later on "the striation of the sea was a result of navigation on the open water" (Deleuze & Guattari 2004, p. 529). Additionally, they describe a contrary way of mapping a terrain; the nomad way of navigation to exemplify the possibility to map or even establish a smooth space: "The variability, the polyvocality of directions, in an essential feature of smooth spaces … and it alters their cartography" (Deleuze & Guattari 2004, pp. 421–422). The nomad life always is an intermezzo, once reached a spot; the nomad is immediately going to leave it. That means that the nomad map is spread with individual spots, but these points are subordinated to routes of travelling. There are no paths through given points, but the points appear and are established only by the trajectory itself. The roads that arise from the travelling of nomads are not sedimentary roads. They are fluent because they are the outcome of continuous moving. This gives an impression of how the socio-political space can be a smooth one.

At least three different types of striation need to be considered when it comes to modelling in the mathematics classroom. First, the teacher observes and measures the individual, e.g. through tests and marks. Additionally, educational research measures the individual. Secondly, striation takes place by connecting certain teaching methods with mathematical modelling tasks. The openness of modelling tasks fits very well with certain open methods of teaching. Some of these open methods promote the idea that individuals can be observed and evaluated at different times and in different activities (for an example of how a classroom situation can be smooth or striated in the same lesson, see de Freitas & Walshaw 2016, pp. 107–118).

While these two ways of striation mentioned so far come from the outside of the modelling process, the third way of striation is located on the inside of the modelling process itself. This way of striation is related to the epistemological assumptions underlying the modelling process. As discussed above, to limit a problem to a singular field of knowledge, like static mathematics, already undermines the multiplicity structure of a problem. Nowadays, there is a constant tendency to focus only on those solutions or actualizations of a problem which tend to be measurable. These can be subordinated to the conditions of pre-given mathematics concepts.

Conclusion and Perspectives

Deleuze's philosophical investigations facilitate us to rethink modelling processes in a non-representational manner. From this point of view, mathematical concepts, and concepts in general, can be grasped only in relation to problems. Furthermore, there is no need to separate mathematics from reality. Mathematical concepts are inherent to the same plane of immanence as problems. Deleuze's concept of 'difference in itself' elicit the possibility to think about concepts in a non-hierarchical way. Singularities and regular points establish a multiplicity that connects very different levels of experience while it is always open to new and unexpected connections.

Real experience can be grasped by the terms 'virtual' and 'actual'. While the virtual signifies possible actions to an actual situation, the actual is one of these actions, i.e. a part of an already actualized virtuality. Deleuze and Guattari point out that actualisation rather tends to striate a space than to smooth it in many occasions.

It has been argued that the modelling cycle is limited to representational thinking and thus promotes an inadequate image of both mathematics and reality. Therefore, perspectively, it should be replaced by another model of modelling. Deleuze and Guattari use the botanic rhizome as a model to describe a multiplicity; non-representational, and non-identitarian way of thinking. According to them, a rhizomatic structure is defined by several characteristics (Deleuze & Guattari 2004, pp. 7–28). Any point of a rhizome can be connected to another arbitrary point; the rhizome itself is a multiplicity; a rhizome may be broken, but could start up again; finally, a rhizome is map, not a tracing.

> What distinguishes the map from the tracing is that it is entirely oriented toward an experimentation in contact with the real. … The map is open and connectable in all of its dimensions, it is detachable, reversible, susceptible to constant modification. It can be torn, reversed, adapted to any kind of mounting, reworked by an individual, group or social formation. It can be drawn on a wall, conceived of as a work of art, constructed as a political action or as a meditation (Deleuze & Guattari 2004, pp. 13–14).

So possibly, the rhizome can serve as a model to grasp modelling processes in school.

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