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Marek Svetlik

Marija Radojicic

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Justifying Euler's formula through motion in a plane

Marek Svetlik
Faculty of Mathematics, University of Belgrade, Serbia

Marija Radojičić
Faculty of Technical Sciences, University of Novi Sad, Serbia

Slaviša Radović
Faculty of Mathematics, University of Belgrade, Serbia

Ksenija Simić-Muller
Mathematics Department, Pacific Lutheran University, USA

ABSTRACT: In this paper we consider motion of an object in a plane to provide a mechanical interpretation of Euler's formula. We believe that this approach contributes to a deeper and more intuitive understanding of Euler's formula and can especially be useful for students learning it and encountering complex numbers for the first time. Euler's formula can be introduced in many ways and using various approaches, but not all of these approaches help develop an intuition for where it comes from and why it works. We believe that the emphasis on physical interpretations and connections to motion can contribute to a more natural introduction and easier understanding of this fascinating formula. In this paper, we describe and consider motion in a plane, using it to give a detailed explanation of Euler's formula. Also, our paper points to the need for an integrated approach to teaching mathematics and physics. In our opinion the interpretation of mathematical results based on the physical phenomena and processes has an important methodological and motivational role in the process of learning. We believe students will be more successful in using this and other formulas of mathematics if they understand them first.
Keywords: Euler's formula, motion, complex numbers

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1 Introduction

Some mathematicians insist that Euler's identity, $e^{i\pi} + 1 = 0$, is one of the most beautiful formulae ever derived, as it connects five of the most important mathematical constants, 0, 1, π, e, and i, in one short expression. Montano [Mon] emphasizes the aesthetic role of Euler's identity, and Feynman [Fey] refers to it as "the most remarkable formula of mathematics" and "our jewel". Also, in 1988, Mathematical Intelligencer's poll elected Euler's identity as the most beautiful theorem [Wel]. Furthermore, Zeki et al [ZRB], examining the relationship between the human sense of beauty and mathematics content, found that the human brain reacts in the same way when listening to pleasant music or viewing a masterpiece as when admiring Euler's identity.

However, beauty is not everything, and this beautiful identity is also an important topic in the study of mathematics. Because of its broad applications, Euler's formula (see formula (1.1) below), of which Euler's identity is the direct consequence, is often an integral part of school curricula, yet explaining and understanding it remains a challenge for both teachers and students. This is corroborated by the statement of the famous Harvard professor Benjamin Pierce: "Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means. But we have proved it, and therefore we know it must be the truth" [Nah].

Euler's identity can be derived from Euler's formula

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y), \quad z = x + iy \in \mathbb{C}. \tag{1.1}$$

Euler obtained formula (1.1) using Taylor expansions. Euler did not give a geometric interpretation of the formula; in fact, a geometric interpretation was first given fifty years later by Brenner and Lutzen [BL], who considered complex numbers as points in the complex plane. However, neither of these approaches answers the question of what the formula essentially means.

In this paper we consider motion of an object in a plane to provide a mechanical interpretation of Euler's formula, inspired by the book *Visual Complex Analysis* by Tristan Needham [Nee]. We believe that this approach contributes to a deeper and more intuitive understanding of Euler's formula and can especially be useful for students learning it and encountering complex numbers for the first time.

We assume only the very basic knowledge of complex numbers in this paper. In particular, the reader should not assume any knowledge of properties of complex-valued functions. Instead, our explanation of Euler's formula will be based on properties complex-valued functions should have, if they are to be analogous to real-valued functions.

The formula (1.1) is somewhat miraculous, as it is not at all obvious at first sight that a connection exists between complex exponential functions and real trigonometric functions. On the other hand, it is natural (by analogy with real exponential functions) for complex exponential functions to be differentiable, and that for all $z = x + iy \in \mathbb{C}$, the following should hold:

- $e^z \in \mathbb{C}$;
- $e^{x+iy} = e^x e^{iy}$;
- $(e^{iy})' = ie^{iy}$.

In this paper, we start from 1), 2) and 3) to derive formula (1.1). Based on 1) and 2), to show Euler's formula holds, it will be enough to show that for all $t \in \mathbb{R}$

$$e^{it} = \cos t + i \sin t. \tag{1.2}$$

We provide an explanation of the formula (1.2) using motion of an object in two dimensions (in a plane). Roughly speaking, we will interpret the variable t as time, the function $f : \mathbb{R} \to \mathbb{C}$ defined by $f(t) = e^{it}$ will be the position function of the object, and f' will be its velocity function. Based on 3), $f'(t) = if(t)$ holds for all $t \in \mathbb{R}$ and the crucial point in our justification will be this: For all $t \in \mathbb{R}$ the vectors $f'(t)$ and $f(t)$ are mutually orthogonal and $|f'(t)| = |f(t)|$.

In order to give a mathematical interpretation of motion in two dimensions, we adapt the approach from [MS], which gives a mathematical interpretation of motion in one dimension.

2 Motion in a plane

Motion of an object is its displacement in relation to another, referent object. Motion is completely determined if we know the time interval during which the motion happens and the position of the object relative to the referent object at every moment during that time interval.

Let us consider motion in a plane (motion in two dimensions), where the plane is interpreted as the set of complex numbers, $\mathbb{C}$. We take the point corresponding to the complex number 0 to be the referent object, and assume that the motion begins at $t = 0$ and ends at $t = T$, i.e. that the motion is occurring during the time interval $[0, T]$. The position of the moving object M is defined by the position function (position vector) $f : [0, T] \to \mathbb{C}$. More precisely, the position of object M at time t is the point corresponding to the complex number $f(t)$ (Figure 1). Further, we can assume that the function f is continuously differentiable on $[0, T]$.

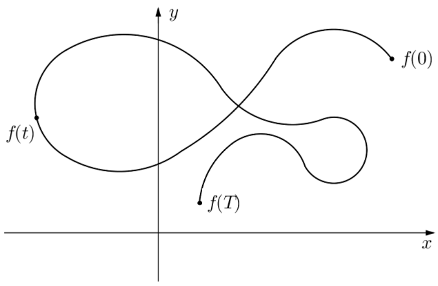

Figure 1: Motion of an object on a plane

In order to determine the path of the object between times 0 and T, $t \in [0, T]$, let $0 = t_0 < t_1 < ... < t_i < ... < t_n = t$ be an arbitrary partition of the interval $[0, t]$. Denote that partition by Π. If the length of the interval $[t_i, t_{i+1}]$ is sufficiently small, we can assume that the path of the object during that time interval is close to a straight line. Hence, the path length during the time interval $[t_i, t_{i+1}]$ approximately equals $|f(t_{i+1}) - f(t_i)|$. The path length from the moment 0 until the moment t approximately equals

$$\sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|. \tag{2.1}$$

In order to determine the path length from moment 0 to moment t, transform (2.1) as follows[1]:

$$\begin{aligned}
& \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)| \\
= & \sum_{i=0}^{n-1} \frac{\sqrt{(\Re f(t_{i+1}) - \Re f(t_i))^2 + (\Im f(t_{i+1}) - \Im f(t_i))^2}}{(t_{i+1} - t_i)} (t_{i+1} - t_i) \\
= & \sum_{i=0}^{n-1} \sqrt{\frac{(\Re f(t_{i+1}) - \Re f(t_i))^2 + (\Im f(t_{i+1}) - \Im f(t_i))^2}{(t_{i+1} - t_i)^2}} (t_{i+1} - t_i) \\
= & \sum_{i=0}^{n-1} \sqrt{\frac{((\Re f(t_{i+1}) - \Re f(t_i))^2}{(t_{i+1} - t_i)^2} + \frac{(\Im f(t_{i+1}) - \Im f(t_i))^2}{(t_{i+1} - t_i)^2}} (t_{i+1} - t_i).
\end{aligned} \tag{2.2}$$

The Lagrange Mean Value Theorem implies that there exist $\xi_i, \zeta_i \in (t_i, t_{i+1})$ so that (2.2) is equal to

$$\sum_{i=0}^{n-1} \sqrt{(\Re f'(\xi_i))^2 + (\Im f'(\zeta_i))^2} (t_{i+1} - t_i).$$

[1] Here $\Re z$ and $\Im z$ denote, respectively the real and imaginary part of a complex number z.

If the parameter of partition Π tends to 0, then the previous sum tends to

$$s(t) = \int_0^t |f'(\tau)| d\tau,$$

(see [Rud] for details). The function s is called a path function.

In addition to the position and path of the object, other important quantities associated with motion are velocity and speed.

The average velocity of our object during time interval $[t, t + \Delta t]$ will be defined as

$$u_{avg}(t, t + \Delta t) = \frac{f(t + \Delta t) - f(t)}{\Delta t},$$

and the instantaneous velocity (Figure 2) at time t will be defined as

$$u(t) = \lim_{\Delta t \to 0} u_{avg}(t, t + \Delta t) = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = f'(t).$$

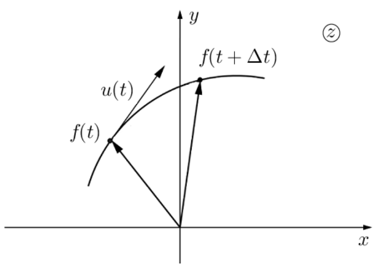

Figure 2: Current velocity

Speed is observed as a scalar quantity, as the quotient of the path length and the elapsed time. More precisely, we define average speed during the time interval $[t, t + \Delta t]$, in the following way

$$v_{avg}(t, t + \Delta t) = \frac{s(t + \Delta t) - s(t)}{\Delta t},$$

and the instantaneous speed at time t as

$$v(t) = \lim_{\Delta t \to 0} v_{avg}(t, t + \Delta t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t).$$

It is not difficult to show that instantaneous velocity and instantaneous speed are related as follows:

$$v(t) = |u(t)|. \tag{2.3}$$

Note that the path length, speed and velocity of an object are completely determined by its position function.

To help the reader understand the different functions associated with motion in a plane, we provide the following example.

Example: Let $r \in (0, 3]$. Consider the motion of an object M whose position function is $f_r : [0, 2\pi] \to \mathbb{C}$, where $f_r(t) = (r^2 \cos 2t + 2r \cos t) + i(r^2 \sin 2t - 2r \sin t)$ (Figure 3). The path function $s_r : [0, 2\pi] \to \mathbb{R}$ is determined by

$$s_r(t) = \int_0^t \sqrt{(-2r^2 \sin 2\tau - 2r \sin \tau)^2 + (2r^2 \cos 2\tau - 2r \cos \tau)^2} d\tau$$

i.e.

$$s_r(t) = 2r \int_0^t \sqrt{r^2 - 2r\cos 3\tau + 1}\, d\tau.$$

The instantaneous velocity $u_r : [0, 2\pi] \to \mathbb{C}$ is determined by

$$u_r(t) = (-2r^2 \sin 2t - 2r \sin t) + i(2r^2 \cos 2\tau - 2r \cos \tau),$$

and the instantaneous speed $v_r : [0, 2\pi] \to \mathbb{R}$ is determined by

$$v_r(t) = 2r\sqrt{r^2 - 2r\cos 3\tau + 1}.$$

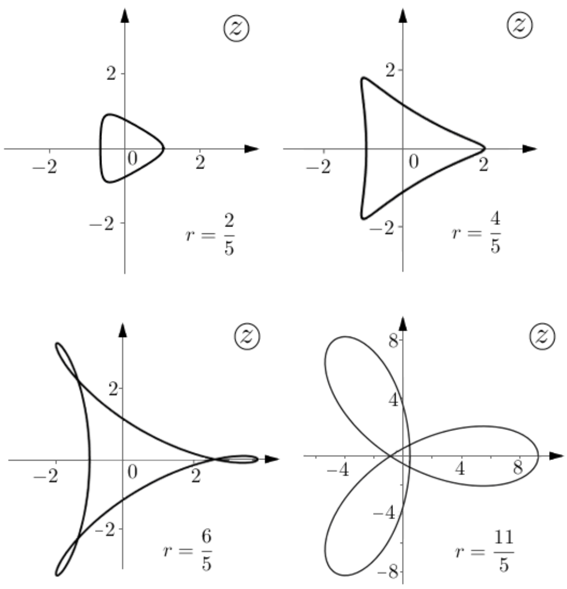

Figure 3: Motion of object M for various values of parameter r.

3 Euler's formula

We are now able to investigate the values of e^{it}, where t is an arbitrary real number. Notice that $t \mapsto e^{it}$ is a function from $\mathbb{R}$ to $\mathbb{C}$: we will denote this function by f. We do not know, as yet, what values the function f takes, or what properties it has. In fact, determining the values of f will be the goal of this section. On the other hand, since i is a constant, if we formally differentiate function f, we obtain $f'(t) = ie^{it}$ i.e. $f'(t) = if(t)$. Hence it is natural to assume that f is continuously differentiable and that $f'(t) = if(t)$ holds for all t.

Notice that

$$f(0) = e^{i\cdot 0} = e^0 = 1. \tag{3.1}$$

Bearing in mind that the multiplication of a complex number with imaginary unit i is a rotation by $\frac{\pi}{2}$ about the origin, we conclude that for all $t \in \mathbb{R}$,

$$\angle(f(t), f'(t)) = \frac{\pi}{2}. \tag{3.2}$$

Notice also that

$$|f'(t)| = |if(t)| = |i||f(t)| = |f(t)|. \tag{3.3}$$

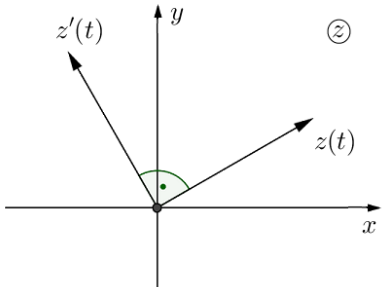

Figure 4: Angle between $z(t)$ and $z'(t)$

We can observe the restriction of the function f on the interval $[0, 2\pi]$ as a position function of an object M. Denoting $z = f|_{[0,2\pi]}$, $x = \Re z$ and $y = \Im z$ we obtain the following parameters of object M's motion.

The path function is $s(t) = \int_0^t |ie^{i\tau}|d\tau = \int_0^t |i||e^{i\tau}|d\tau = \int_0^t |e^{i\tau}|d\tau$. The instantaneous velocity is $u(t) = ie^{it}$, and instantaneous speed is $v(t) = |ie^{it}| = |i||e^{it}| = |e^{it}|$. Based on (3.2) we conclude that at each moment, the angle between the position of object M and its instantaneous velocity is $\frac{\pi}{2}$ (Figure 4). Also, based on (3.3) we conclude that the absolute value of the position of object M is equal to its instantaneous speed.

Hence, we deduce that the dot product of position function $(x(t), y(t))$ and instantaneous velocity $(x'(t), y'(t))$ is 0 i.e.

$$\langle (x(t), y(t)), (x'(t), y'(t)) \rangle = 0 \text{ for all } t \in [0, 2\pi]$$

i.e.

$$x(t)x'(t) + y(t)y'(t) = 0 \text{ for all } t \in [0, 2\pi]. \tag{3.4}$$

By integrating equality (3.4) we obtain that for all $t \in [0, 2\pi]$

$$\int x(t)x'(t)dt + \int y(t)y'(t)dt = C,$$

i.e.

$$\frac{(x(t))^2}{2} + \frac{(y(t))^2}{2} = C,$$

where C is a real constant. Since, $z(0) = 1$ (formula (3.1)), we obtain $x(0) = 1$ and $y(0) = 0$. Hence we obtain $C = \frac{1}{2}$. Therefore, for all $t \in [0, 2\pi]$ it is true that

$$(x(t))^2 + (y(t))^2 = 1,$$

i.e.

$$|z(t)| = |e^{it}| = 1. \tag{3.5}$$

From equality (3.5) we obtain that for all $t \in [0, 2\pi]$ point $z(t)$ belongs to the unit circle. By interpreting the function z as a position function of object M, we deduce that the object M moves along the unit circle. We also conclude that the path function is $s(t) = \int_0^t |e^{i\tau}|d\tau = \int_0^t 1d\tau = t$, and the instantaneous speed is $v(t) = 1$.

We can therefore conclude that there is a function $\varphi : [0, 2\pi] \to \mathbb{R}$, such that

$$x(t) = \cos\varphi(t)$$

and

$$y(t) = \sin\varphi(t)$$

i.e.

$$z(t) = \cos\varphi(t) + i\sin\varphi(t), \tag{3.6}$$

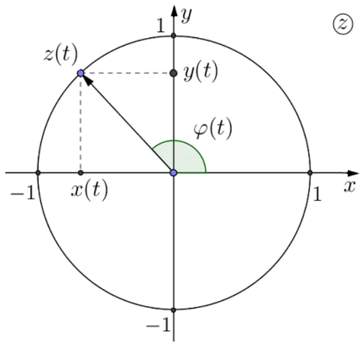

Figure 5: $z(t) = \cos\varphi(t) + i\sin\varphi(t)$

for all $t \in [0, 2\pi]$ (Figure 5).

On the other hand, since

$$z(t) = e^{it},$$

it follows

$$e^{it} = \cos\varphi(t) + i\sin\varphi(t). \tag{3.7}$$

Let us determine the function φ. Assume that the function φ is continuously differentiable. Using $z'(t) = iz(t)$, we get

$$(-\sin\varphi(t))\varphi'(t) + i(\cos\varphi(t))\varphi'(t) = i(\cos\varphi(t) + i\sin\varphi(t)),$$

i.e.

$$(-\sin\varphi(t))\varphi'(t) = -\sin\varphi(t) \quad \text{and} \quad (\cos\varphi(t))\varphi'(t) = \cos\varphi(t), \tag{3.8}$$

for all $t \in [0, 2\pi]$.

Bearing in mind that for all $t \in [0, 2\pi]$, at most one of $\cos\varphi(t)$ and $\sin\varphi(t)$ is 0, using equalities (3.8) we get

$$\varphi'(t) = 1. \tag{3.9}$$

By integrating (3.9) we get

$$\varphi(t) = t + C,$$

where C is a real constant. Since $z(0) = 1$ (formula (3.1)), by using equality (3.7) we get that $\cos\varphi(0) = 1$ and $\sin\varphi(0) = 0$. Hence we obtain $\varphi(0) = 2k\pi$ and conclude that $C = 2k\pi$, for some $k \in \mathbb{Z}$. It follows that

$$\cos\varphi(t) = \cos(t + 2k\pi) = \cos t \quad \text{and} \quad \sin\varphi(t) = \sin(t + 2k\pi) = \sin t.$$

Finally, based on (3.7) we get that for all $t \in [0, 2\pi]$

$$e^{it} = \cos t + i\sin t,$$

i.e.

$$z(t) = \cos t + i\sin t.$$

Specially, when $t = \pi$, we have

$$e^{i\pi} = -1,$$

i.e.

$$e^{i\pi} + 1 = 0.$$

4 Conclusion and Remark

Euler's formula can be introduced in many ways and using various approaches, but not all of these approaches help develop an intuition for where it comes from and why it works. We believe that the emphasis on physical interpretations and connections to motion can contribute to a more natural introduction and easier understanding of this fascinating formula.

In this paper, we describe and consider motion in a plane, using it to give a detailed explanation of Euler's formula. Our explanations and proofs are geometrically and mechanically motivated, which allows to us to see and intuit Euler's formula instead of just taking the word of other mathematicians that it is true.

Our explanation relies on the function $f : [0, 2\pi] \to \mathbb{C}$ defined as $f(t) = e^{it}$, which we interpret as the position function of an object M moving in a plane during the time interval $[0, 2\pi]$. This motion has the following properties:

a) at time 0, the object M and the referent object share the same position;

b) the position vector and instantaneous velocity of object M are orthogonal to each other at each moment;

c) the norm of the position vector and the instantaneous speed of object M are equal at each moment.

Our paper points to the need for an integrated approach to teaching mathematics and physics. In our opinion the interpretation of mathematical results based on the physical phenomena and processes has an important methodological and motivational role in the process of learning. We believe students will be more successful in using this and other formulas of mathematics if they understand them first.

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Marek Svetlik, Faculty of Mathematics, University of Belgrade, Studentski trg 16, 11000 Belgrade, Serbia

Email address: `svetlik@matf.bg.ac.rs`

Marija Radojičić, Faculty of Technical Sciences, University of Novi Sad, Trg Dositeja Obradovia 6, 21000 Novi Sad, Serbia

Email address: `marija.radojicic@uns.ac.rs`

Slaviša Radović, Faculty of Mathematics, University of Belgrade, Studentski trg 16, 11000 Belgrade, Serbia
Email address: radovic.slavisa@gmail.com
Ksenija Simić-Muller, Mathematics Department, Pacific Lutheran University, 1010 122nd St S., Tacoma, WA 98418, USA
Email address: simicmka@plu.edu

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