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ABSTRACT: We construct nine lines in the plane over the integers having the property that for any three general points in the plane over any field, at last one of the constructed lines avoids the three points. We then show that nine lines is the fewest number possible having this property.

Keywords: Lines in the plane, finite fields, collection of three points

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Introduction

Elementary problems, that is problems that can be stated, understood, and solved by only knowing elementary mathematics, are sometimes appealing and interesting because of their simplicity. An example of such a geometric problem is the Sylvester-Gallai Theorem that states that given any n points in the plane, not all on a line, there exist at least one line passing through exactly two of the points [AZ][Chapter 10].

We will in this note discuss a similar problem concerning configuration of lines in the plane defined over different fields. Let $\mathbb{P}^2$ denote the projective plane over the integers. A line in the plane is given as the zero locus of a non-zero linear form L with integer coefficients. For a given field F we specialize the coefficients of the linear form L and get, assuming the specialization is non-zero, a line in the projective plane $\mathbb{P}^2(F)$ defined over the field F.

Recall also that three points are said to be general if they do not lie on a line, that is the three points are not aligned. The problem we are about to discuss in this note is the following: Does there exist a collection of lines in $\mathbb{P}^2$ such that for any field F, any three general points in the plane $\mathbb{P}^2(F)$, at least one of the specialized lines avoids the three points?

It is a bit surprising that the stated configuration problem, unarguably elementary, has, apparently, not been asked earlier. One possible reason for why the problem has been overlooked is that without the assumption on the points being general, the posed problem does not have a solution.

The problem arose concretely when we were trying to give effective and explicit examples of certain moduli problems (see [GS]). Strictly speaking we were searching for a collection of hyperplanes with the property that at least one of them avoiding a given $n+1$ general points in projective n-space, over arbitrary fields. But, at that time even the case with $n+1=3$ points in the plane appeared to be out of reach.

By this short introduction I hope to have gotten the reader interested, and such a reader I would then encourage not to read any further and instead spend some time working out a solution to the above mentioned problem. After all, the main purpose with this note was introduce the reader with an elementary, but interesting problem.

1 Avoding three points

1.1 Why not seven lines?

A collection $\mathcal{L}$ of lines in the plane are said to be general if the intersection of any three lines in the collection is empty. Clearly, if one has seven general lines in the projective plane over a field, then at least one of the lines will avoid any given three points defined over that field. However, over finite fields there does not always exist seven general lines.

For instance, consider the projective plane over the field F_2 of two elements, pictured below

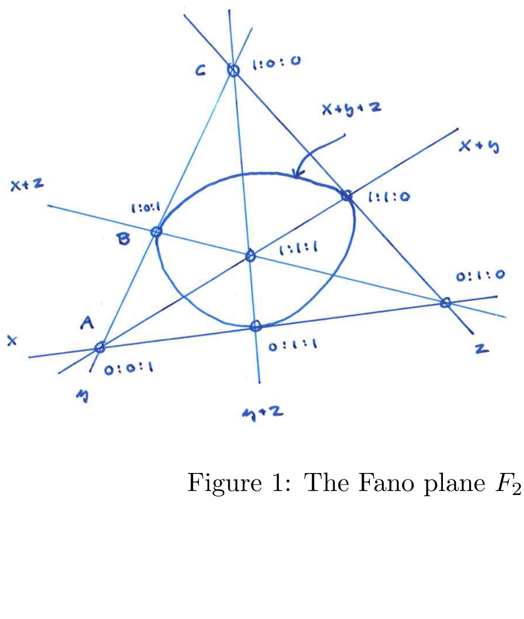

Figure 1: The Fano plane $F_2[x, y, z]$.

This particular plane is called the Fano plane. There are only a total of seven lines in the Fano plane, and these lines are not general. In fact, any three of these seven lines have a non-empty intersection (see Figure (1)).

Moreover, one can find three points in the Fano plane, as for instance the points A, B and C in Figure (1), that together occupy all seven lines. In particular there exist no collection of lines $\mathcal{L}$ in the projective plane $\mathbb{P}^2$ such that at least one line avoids any given three points defined over some field. Because, at most the collection $\mathcal{L}$ would specialize to the seven lines in the Fano plane, and none of these lines avoids the three points $\{A, B, C\}$.

However, by further inspection one realizes that the any three points that occupy all seven lines in the Fano plane, are always *aligned*. Thus, for any three general points in the Fano plane at least one of the seven lines avoids the three points.

Lemma 1.1. *Let $\mathcal{L}$ be a collection of lines in the projective plane $\mathbb{P}^2(F_3)$ over the field of three elements. Assume that for any three general points at least one line in $\mathcal{L}$ avoids these three points. Then the cardinality of $|\mathcal{L}| \geq 9$.*

Proof. We will show that if $|\mathcal{L}| = 8$ then we can find three general points occupying all the lines in $\mathcal{L}$. Let ℓ be one line in $\mathcal{L}$. As the base field is F_3 the remaining seven lines will intersect ℓ in the four points of ℓ. If all seven lines intersect ℓ in the same point A we are done as the point A alone lies on all lines in $\mathcal{L}$. We will consider the most critical situation where the multiplicities of the intersections is the lowest possible. That means that three pairs of lines intersect three different points of ℓ and that the seventh line intersects the fourth point. Moreover, this will be situation for any of the eight lines in $\mathcal{L}$. That means that we can rearrange the lines in four pairs $\{\ell_i, \ell_{i+1}\}$ with $i = 1, 3, 5, 7$ such that $\ell_i \cap \ell_{i+1} \cap \ell = \emptyset$ for any $i = 1, 3, 5, 7$, and any $\ell \in \mathcal{L} \setminus \{\ell_i, \ell_{i+1}\}$. Moreover, any other triple intersection is non-empty.

There are $56 = \binom{8}{3}$ ways to form a triple intersection of 8 lines. With our assumptions it follows that $\frac{8 \cdot 6 \cdot 4}{3!} = 32$ of these triple intersections are non-empty. Thus, with our assumptions there should be 32 different triple intersections, which is a number that exceeds the 13 points available in the projective plane over F_3.

Because of that there is at least one line ℓ in $\mathcal{L}$ such that three other lines in $\mathcal{L}$ intersect ℓ in one point A. The remaining four lines of $\mathcal{L}$ intersect ℓ in at most three other points. If the remaining four lines intersect ℓ in on point B, then the two points $\{A, B\}$ alone occupy all eight lines of $\mathcal{L}$. If the four lines do not intersect ℓ in one point, then one can chose arrange the four lines $\ell_1, \ldots, \ell_4$ such that $B = \ell_1 \cap \ell_2$ and $C = \ell_3 \cap \ell_4$ are not aligned with the point A. Then the three points $\{A, B, C\}$ occupy all eight lines. The remaining cases with even more lines intersecting ℓ at one point are left for the reader.

$\square$

1.2 A collection of sufficiently general lines

The projective plane over the integers is given by the graded polynomial ring $\mathbb{Z}[x, y, z]$. The nine linear forms

$$x, \quad y, \quad z, \quad x+y, \quad y+z, \quad x+z, \quad x+y+z, \quad y-z, \quad x-z$$

gives us nine lines $\mathcal{L}$ in $\mathbb{P}^2$. We group the intersection points after their multiplicities, that is how many of the lines in $\mathcal{L}$ that pass through the points. The intersections of the nine lines $\mathcal{L}$ are a total of 14 points. The two points

$$P_1 = [0:1:0] \quad \text{and} \quad P_2 = [1:0:0]$$

are both given as the intersection of four of the lines in the collection $\mathcal{L}$. The point P_1 is the intersection of the lines $\{x, x-z, x+z, z\}$ and P_2 is the intersection of the lines $\{y, y+z, y-z, z\}$. The intersection points

$$\begin{array}{ccc} Q_1 = [0:0:1], & Q_2 = [0:1:-1], & Q_3 = [1:-1:1] \\ Q_4 = [-1:1:1], & Q_5 = [1:0:-1], & Q_6 = [1:-1:0] \end{array}$$

are each contained in three of the lines in the collection $\mathcal{L}$. The remaining six intersection points

$$\begin{array}{ccc} R_1 = [0:1:1], & R_2 = [1:0:1], & R_3 = [1:1:-1] \\ R_4 = [1:1:1], & R_5 = [1:-2:1], & R_6 = [-2:1:1] \end{array}$$

are each contained in only two of the lines in $\mathcal{L}$. In Figure (2) we have pictured these nine lines, together with their intersection points.

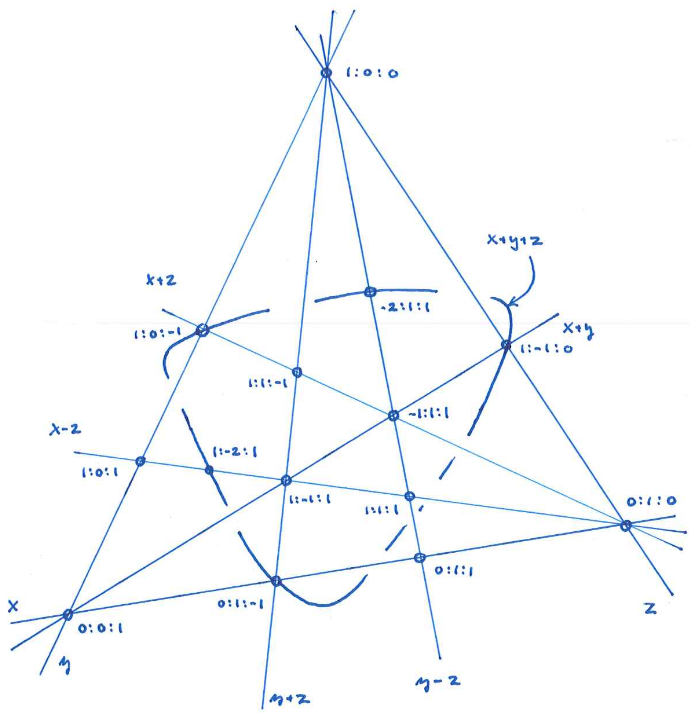

Figure 2: Our 9 lines and their 14 intersections

Proposition 1.2. *Let $\mathcal{L}$ be the collection* (1.2) *of nine lines in the projective plane $\mathbb{P}^2$. For any three general points in $\mathbb{P}(F)$, with F a field, at least one of the lines in $\mathcal{L}(F)$ avoids the three points. In particular we have that collection $\mathcal{L}$ is a solution to our avoidance problem, and we have that the smallest possible cardinality of such a collection is nine.*

Proof. Given three general points in $\mathbb{P}^2(F)$, with F a field. Arguing as the devils advocate, the worst case scenario is when our three points are placed on as possibly many lines in $\mathcal{L}(F)$ as possible. The intersection points of the lines $\mathcal{L}(F)$ are specializations from $\mathbb{P}^2$ over the integers, and it follows that we may assume that $F = \mathbb{Z}/(p)$ where p is a prime number, or $F = \mathbb{Q}$.

In characteristic different from 2 we have that the collection $\mathcal{L}(F)$ consists of nine lines. Furthermore, when the characteristic is neither 2 nor 3, then the lines $\mathcal{L}(F)$ intersect in the 14 points given as specialization of the 14 points in the projective plane $\mathbb{P}^2$ over the integers. That means that if we show the claims in the proposition over F_3 then we also have proven the claims for all characteristics different from 2. And, if $F = F_2$ then the nine lines of $\mathcal{L}$ specializes to the seven lines of the Fano plane, where we already have noted that at least one line avoids a given three general points.

Thus, we need only to verify the claim over F_3. We use the same notation as above for the intersection points, but note that $R_2 = R_4 = R_6$ and this point we label as Q_7. The point Q_7 is a point contained in three of the lines $\mathcal{L}(F_3)$, the other intersection points have the same multiplicity as over the integers.

A picture of lines and their intersection points are given below.

If two of our three points are $A = P_1$ and $B = P_2$, then these two points occupy seven of the nine lines in $\mathcal{L}(F_3)$. The remaining two lines are given by $x + y$ and $x + y + z$ whose intersection is the point Q_6. However, the point Q_6 lies on the line $z = 0$ that already contains P_1 and P_2, and can therefore not be the third point C in our collection. Therefore at least one of the lines $x + y = 0$ or $x + y + z = 0$ will not contain C, which then also avoids all three points $\{A, B, C\}$.

The remaining possibilities are that we choose three points from the collections $Q = \{Q_1, \ldots, Q_7\}$ and $\{P_1, P_2\}$, but not both P_1 and P_2. Note that of the three lines through a given point in Q, one line passes through P_1 and another through P_2. Thus, if we choose two points $\{A, B\}$ from Q, then these

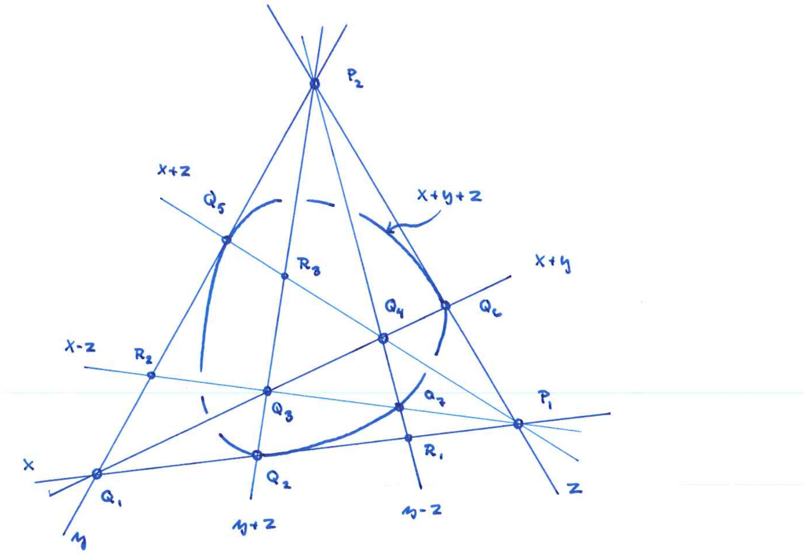

Figure 3: The $F_3[x, y, z]$-situation

two points occupy at most 6 lines, and the third choice with $C = P_1$ or $C = P_2$ would then only occupy another 2 lines, leaving us with one ninth line avoiding all three points $\{A, B, C\}$.

Thus the only possibility is that all three points are chosen from $Q = \{Q_1, \ldots, Q_7\}$. Now we note that the three lines through a given point in Q also go through all but, at most, one of the remaining points of Q. The point Q_6 is special here as the three lines through it also will go through the remaining points of Q. Anyhow, that means that the lines going through three points in Q is at most eight in number. Again giving us a ninth line that avoids all three points. □

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