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Abstract: *After a short survey of the genesis of the ideas behind "functional reasoning" in mathematics education at the beginning of the 20th century, the development and implementation of these ideas in both parts of Germany and in Hungary in the second half of the 20th century are discussed and compared. The ideas of Felix Klein had a strong influence in both countries. However, from the 1950's to the 1970's (the period of "New Math") an attempt was made to teach functions and mappings in a more abstract sense, involving both elementary (real-valued) functions and geometric transformations. In the 1970's and 1980's there were lots of discussions about the notion and notation of functions. Finally, dynamic aspects of functions – one of the main ideas of the reforms at the beginning of the 20th century – came into focus again in the 1980's and remained essential to this day.*

Keywords: Function, Mapping, Functional reasoning, Set-theoretic pervasion, Germany, GDR, Hungary, New Mathematics, Felix Klein, Tamás Varga, Manó Beke

1. INTRODUCTION

Our main goal with this paper is to show common elements and significant differences in the history of the teaching of functions in Germany and Hungary, which goes back in both countries to the beginning of 20th century (see briefly in 2. Chapter). We focus on the period shortly after the Second World War (from the 50's to the 80's), which created a similar political situation in the GDR (German Democratic Republic – East Germany) and Hungary (it is surprising that there are yet foundational differences in the situation of mathematics education between the two socialist countries, e.g. in Hungary more room was given to the reforms and politics played smaller role than

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in the GDR). Therefore, the Hungarian reform resulted more significant changes both in curricula and in schools, including the introduction of functional thinking. While the development in West-Germany is well-known and documented (see e.g. Lenné 1969 and Besuden 2007), ideas and problems of mathematics education in East-Germany and Hungary are relatively unknown outside these countries (or the territory of the former GDR, respectively). Therefore, we will only briefly discuss the situation in the FRG and rather go into details about the development of mathematics teaching and especially the teaching of functions in Hungary and the GDR.

The central questions which we considered are the following:

- Which precedents from the late 19th/early 20th century are essential to understand the ideas of functional reasoning in mathematics teaching?
- What was the degree of generalization in the teaching of functions in mathematics education in Hungary, the GDR and the FRG in the 1960 -1990 period.
- Which concepts from the 1960-1990 period proved as successful, which approaches had to be rejected. In what ways were ideas of New Mathematics borrowed and implemented in the teaching of functions and with what results.

2. HISTORICAL BACKGROUND: INTRODUCTION OF FUNCTIONS INTO MATHEMATICS EDUCATION AT THE END OF THE 19TH AND THE BEGINNING OF THE 20TH CENTURY

Until the end of the 19th century in European schools functions were considered of lesser importance in math education. This changed fundamentally with the Meraner curriculum reforms at the beginning of the 20th century (see Schubring 2014, p.248f.). The leading representative of these reforms, Felix Klein, wrote in 1908:

> "We desire merely that the general notion of function, according to the one or the other of Euler's interpretations, should permeate as a ferment the entire mathematical instruction in the higher schools. It should not, of course, be introduced by means of abstract definitions, but should be transmitted to the student as a living possession, by means of elementary examples, such as one finds in large number in Euler." (Klein 1932, p.205)

Functions were implemented in the 1920's in German curricula for higher education (see e.g. Richert 1980) and the concept of function also gained important influence in other countries – supported by the fact that Felix Klein was the first chairman of the International Commission on Mathematical Instruction (ICMI), founded in 1908. His collaboration with Manó (Emanuel) Beke resulted in a special influence of Klein's ideas in Hungary.

Beke started his career as a high school teacher and he became a university professor in 1900. He was engaged in mathematics teaching in all possible levels, and also had an important work as a textbook writer. Because of his political activity, in 1922 Beke was deprived of his position at the university and his membership in the Hungarian Science Academy. Consequently, Beke began to work for a publisher and he could not take an active part in the teaching reform any more (Kántor 2014).

At the beginning of the 19th century, the Hungarian mathematics teaching reform was lead by Beke, and its' main objectives corresponded with the conceptions of Felix Klein's 1905 program. In 1906, the Hungarian Reform Commission was founded on the initiative of Beke and he became the

leader of this organization. Beke summarized the objectives of the mathematics teaching reform as follows:

> "Making arithmetic teaching more practical, development of the sense of economics, space-perception and stereoscopic understanding, graphic representation of relations between quantities, systematic development of the concept of functions, the introduction and utilization of the elements of differentiation and integration calculus." (Beke & Mikola 1909, p.200, translation G.A.)

The beginning of the mentioned systematic development of the concept of functions was suggested as early as in the elementary school (grades 5-8) – by the introduction and quantitative studying of the idea of dependence. In this subject, the use of real (and not formal) data, based on the different experience and measuring of pupils were emphasized (Goldzieher p.74, in: Beke 1909).

The notion of function appeared in the Hungarian high school textbooks in the middle of the 19th century, and the formation of the concept reflected Euler's interpretation. The main idea of this concept is that the function is given by an (algebraic) term. In the next almost hundred years, functions were given only by terms in the teaching materials for high schools, references to the concept of mapping were made only a few times (Katz 1989).

The enrichment of the knowledge of the concept and the use of functions can be followed in the new editions of the textbooks revised several times by Beke (analysis of empirical functions, solving problems using functions), and the number of tasks to solve is increasing too in the chapters on functions. The *connections between arithmetic/algebra and geometry* were emphasized (see Chapter "Relation of the algebra and geometry", Beke & Kőnig 1897, 1908), so as the *"dynamic" dependencies* between data in applications of mathematics, with the introduction of the coordinate system.

At the same time in Germany, the ideas of connections between arithmetic/algebra and geometry, "dynamic" dependencies between measures in geometric figures, applications of mathematics, especially in physics, and in general flexibility of thinking also gained importance (see Krüger 1999, pp.167-228). Students should obtain perception of functional dependencies from an early age on (Vollrath 1989, p.31). In high schools, *graphical representations* of dependencies were emphasized in Germany and Hungary as well, and functions also became important in connecting different fields of mathematics (arithmetic, algebra and geometry). As early as in 1908 Felix Klein underlined the intention of a "fusion of arithmetic and geometry" (Klein 1939, p.2), with functions (respectively mappings) playing a central role in the approach.

As the culmination of functional reasoning, *differential and integral calculus* was (at first only optionally) included into the curricula of German high schools at the grades 11 and 12. The introduction of differential and integral calculus into the teaching of mathematics in the high school was also an important demand of the Hungarian reform. Beke underlined that it was very important "just from the point of the development of (natural) scientific thinking". This topic was the subject of his keynote speech at the International Congress for Mathematics in Paris in 1914 which he gave at Felix Klein's request (Schubring 2014, p.191; Zuccheri & Zudini 2014, p.503).

Beke, and his collaborators took great pains to point out that the reform was much more than a matter of expanding the content of the mathematics curriculum. They emphasized that the teaching of infinitesimal calculus must be centered on the notion of functions. From grade 1 education

should endeavor to shape the pupils' way of reasoning and develop the ability to think in terms of functions. "The content of mathematics instruction at the secondary level should be regulated to include the most essential notions of contemporary natural science". …"We must transform the spirit of instruction, rather than simply tack differential and integral calculus on to the end of the curriculum" (Némethné 2006, pp.69-71) The differential and integral calculus became part of the official curriculum in 1924 (Némethné 2006, p.35).

In 1922, Engel wrote about the consequences of the focus on functional reasoning which was set by the Meraner reforms for *geometry education*:

> "The demand of education of functional reasoning brought movement into the traditional (static) rest of geometric education ... Lines, surfaces and bodies arise, grow and wane; they consist of movable elements. With the growing or waning size of one item grows or reduces the size of another one. The side changes the angle, the angle the side, the height changes the content, the radius the circle …"
>
> (Engel 1922, p.23, translation A. F.)

This statement describes clearly what was the purpose of functional reasoning in geometry. In fact, the intension of the Meraner Reform was to consider dependencies between geometric measures, but the examination of geometric transformations (which also can be considered as functions) was not in focus. However, especially in the second half of the 20th century, didactical discussions often referred to Felix Klein's "Erlangen program" (his inaugural lecture at the university of Erlangen in 1871), in which he stated:

> "As the generalization of geometry arises so the following comprehensive problem:
> Given a manifoldness and a group of transformations of the same; investigate the configurations belonging to the manifoldness with regard to such properties that are not altered by the transformations of the group." (Klein 1872, p.7)

Although this statement obtained the highest importance for the development of geometry as a scientific discipline, Klein considered it irrelevant for teaching geometry in schools (and at the time he developed the Erlangen program, he was not really concerned with mathematics education yet). Nevertheless, the idea of building the teaching of school geometry on the basis of mappings and the intension of a comprehensive understanding of functions and geometric mappings in school gained important influence on the development of mathematics education in the 1960's and 70's – often with reference to Klein's Erlangen program, see (Bender 1982, p.11).

3. FUNCTIONAL REASONING AND WORKING WITH FUNCTIONS IN GERMANY AND HUNGARY 1960–1990

3.1 International trends in mathematics education – the period of "new mathematics"

Beginning in the late 1950's, major reforms started to modernize mathematics education according to the development that happened in mathematics in the decades before. The ideas of the "Nicolas Bourbaki" group, based on a set-theoretic-axiomatic foundation of academic mathematics education, influenced mathematics instruction in schools. One of the leading characters of New Mathematics in schools was Jean Dieudonné, who gave an influential speech at a seminar on "new thinking in mathematical education", organized by OEEC (Organization for European Economic Co-operation, the predecessor of OECD), in 1959, (see Dieudonné 1961). In the following years,

the program of New Mathematics was introduced to mathematics education in the USA and several European countries, especially in France and Belgium and also in the Federal Republic of Germany.

The main characteristics of New Mathematics were the early introduction of set theory in elementary schools, the teaching of algebraic structures (especially group theory, beginning in high school) and a new approach to teach geometry on an vector-axiomatic and group-theoretic foundation (“Euclid must go!”, see Dieudonné 1961). The way of teaching functions was influenced by the set-theoretic foundation of math education (at all levels) which was a core concept of new mathematics. While in “traditional mathematics” (following the intentions of the Meraner reforms, see above) functions between measures and real-valued functions were studied respectively, the approach of New Mathematics was wider and involved functions in a generalized sense as mappings between sets. However, there are substantial differences between different countries concerning both the method and intensity of implementation of the new paradigm into the curricula and the teaching practice. The following sections will explore in more detail how this happened in both the German states and in Hungary.

3.2 Functions in mathematics education in the Federal Republic of Germany

While fundamental ideas of the Meraner reforms reappeared in the German school curricula after the Second World War (initially both in East and West Germany) and dominated until the early 1960’s. The 1960’s and the 1970’s were mainly characterized by the ideas of New Mathematics. An important document of implementing New Mathematics in West-Germany was the so called “Nürnberger Lehrpläne”, a kind of prototype curriculum published by the German Association for the Promotion of Mathematics and Science Education in 1965, see (Lenné 1969, p.313-325). This obtained considerable influence on mathematics education in the high schools of the Federal Republic of Germany for the following decade. According to this curriculum, mappings/functions were intended to be defined as unique assignments from arbitrary sets to others. Group theory was considered as a key concept of mathematics instruction as early as at lower secondary level and geometry should have been developed based on mappings. Groups of geometric transformations became an explicit part of the curriculum. It is written in the section “Contents of mathematics instruction” of the “Nürnberger Lehrpläne”:

> 0.2.4. “Mappings (Functions). They are, in the most general sense, unique assignments by which exactly one element of a set *B* (value range, image set) is assigned to each element of a set *A* (domain, original set); they are special relations from *A* to *B*. Mappings often relate to certain structures explained on *A* and *B*, e.g., the monotonic functions of real analysis with ordering structures, the continuous functions with the topological structure of the set of real numbers. Structure-maintaining mappings of algebraic structures are called homomorphisms … In geometry, the translations, reflections, rotations, and, in general, the congruence mappings are the characteristic structure-maintaining mappings (automorphisms) of the Euclidean plane; similarly, the similarity maps, affinities, projections, and others, represent the automorphisms of other geometries.” (Lenné 1969, p.317f., translation A. F.)

This statement shows clearly that, according to the intention of New Mathematics, students should obtain a much higher degree of abstraction concerning their understanding of functions, compared to the concepts of functional reasoning which characterized school mathematics since the Meraner reforms. However, the described intentions were visible in the definitions of mappings/functions and in the geometry curricula (both in the prototypical “Nürnberger Lehrpläne” and in the curricula of several federal states of West Germany which were completed in the following years after 1965),

but the way elementary (real-valued) functions were taught did not change essentially, (see Lenné 1969, p.260 and p.318–324).[1] The objectives pursued by the traditional teaching of functions were not abandoned in the period of New Mathematics", but new objectives, aspects and contents were added.[2] Unavoidably, a conflict arose between the intentions of the Meraner reforms and New Mathematics:

- Although in the "classical" approach of functional reasoning (which was essential for the Meraner reforms) the concept of functions was emphasized as a merging guiding principle for the arithmetic, algebraic, and analytical contents of mathematics education, in this concept, the general idea of a *dynamic dependence* between two (possibly measurable) variables was in the forefront.
- In the attempt to teach functions in a generalized sense as mappings between sets, which was an essential part of New Mathematics, functions/mappings necessarily have a *static character*.

To summarize, while on the one hand the structural approach of New Mathematics would (if successfully implemented) bring school mathematics closer to modern concepts of mathematics, on the other hand dynamic understanding of functions was impeded by this approach (as stated by Lenné already in 1969, p.259f.). It was not possible to resolve this conflict, and in the early 1970's the static character of functions was rolled back and thinking in dynamic dependencies gained importance again – in fact it never disappeared from the didactical discussion, not even at the time the ideas of New Mathematics" had great influence, (see e.g. Kirsch 1969).

It turned out in the 1970's that other main aspects of New Mathematics were also not successfully implemented and its approaches required an abstraction level that the majority of the students did not posess. The consequence of these problems was that abstract algebraic structures were removed from the curricula and the extent of set-theoretical considerations was also lowered. Nevertheless, in many schoolbooks functions were still *defined* in a general sense as mappings between sets until the late 1990's but these definitions were hardly *used* and therefore remained quite unimportant for students understanding of functions.

It turned out in the 1970's that the attempt to teach functions in a more general sense as mappings between sets wasn't successful (and after the decline of New Mathematics), the "classical" approach of functional reasoning – following the ideas of the Meraner reforms – became widely accepted again in the 1980's in the Federal Republic of Germany (see e.g. Kiesow & Spallek 1983 and Führer 1985). The approach of functions as dependencies between values also underlies the main idea of "functional relationship" in the current German educational standards. Three aspects of functional reasoning -identified by Vollrath (1989)- gained special influence, summarizing various publications and discussions from former decades. These aspects are:

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[1] Most changes in the curricula regarding to elementary functions were additions to the traditional contents of education and concerned algebraic-structural aspects of classes of functions, e.g. ring properties of polynomial functions and field properties of rational functions.
[2] This led to the problem that the scope of the subjects to be taught increased considerably, (see Lenné 1969, p.95–103).

- *Aspect of assignment*: a Function describes or creates assignments between ranges of values: every value of the range corresponds to one value of the domain – values of the range depend on corresponding values of the domain.
- The *aspect of variation* is concerned with the question of how the function value varies if the value of the domain is varied. This aspect includes properties like monotony and rate of change. It also characterizes the dynamic aspects of functions.
- *Functions as objects*: with a function, a given or created interrelation is regarded as an entity in its entirety.

Malle (2000) enhanced the second aspect in such a way that every change of x causes a specific change of $f(x)$ and vice versa (*aspect of covariation*). This aspect became of special interest in research to this day today. The investigation of forms of representation (pictures or verbal descriptions, graphs, tables, formulas) of functions and the changes between representations are both important tendencies in research and in practice of functional reasoning in schools. Models of functional reasoning – considering the aspects mentioned above as well as the transfers between forms of representation – were developed by different authors; see e.g. Höfer (2008). It has to be mentioned again that basic ideas of constitutional aspects of these models were already included in the ideas of Felix Klein and the Meraner reforms.

3.3 The guideline "mappings and functions" of mathematics education in the GDR

Before we examine the approaches to functional reasoning and working with functions in mathematics education in the GDR (German Democratic Republic – Eastern Germany) we need to highlight a significant difference between the educational systems of the GDR, West Germany and most West European countries:[3] All pupils in the GDR were educated according to the same curricula until grade 8 (until 1959) or until grade 10 (from 1959 on), while in West Germany three types of schools existed. While the previous section relates only to a minority of West German students (who attended "Gymnasium", the high schools with the most advanced level of education), the following considerations cover the education of all students in the GDR. From 1959 to 1990 all students in the GDR learned at polytechnic comprehensive schools which aimed to establish connections between scientific education and the acquirement of practical skills.

Mathematics curricula, which were introduced in the Soviet occupation zone of Germany (the later GDR) directly after the Second World War, were quite similar to those of the Weimar Republic (which existed until 1933) (see Neigenfind 19691970, part 1, p.649f.). Slightly modified schoolbooks were used from the 1920's (reprinted in 1946 by the GDR schoolbook publisher "Volk and Wissen") (see Neigenfind 1969–1970, part 2, p.731f.). But as early as in 1951 new curricula were introduced with the aim of the modernization of mathematics education (see Mader 1951). While there were no fundamental changes regarding the treatment of the real-valued functions, the geometry instruction was basically redesigned. Groups of geometric transformations were intended to form the core of geometry instruction from the grade 9 on, see Görke (1955, p.59f.) and Filler (2016, p.101ff.). It soon turned out that the curricula from 1951 couldn't be realized (see Görke 1955, p.61) and as early as in 1954 they were revised, a lot of contents were removed, especially

[3] For a more detailed description of the East German educational system see Birnbaum (2003).

everything about groups and geometric transformations, (see Neigenfind 1969–1970, part 4, p.15). Another revision was done in 1956 where additional content were removed (Neigenfind 1969–1970, part 4, p.16).

It has to be ascertained that the attempt of a fundamental modernization of math education (which had been undertaken with the curricula of 1951) ended with disappointment and was aborted after only three years. This could have been one of the reasons for the fact that the reforms which were undertaken in the 1960’s (see below) were carried out much more cautiously and mathematics education in the GDR was less affected by New Mathematics than in Western Germany and some other countries.

After the rapid changes of the 1950’s, it can’t be surprising that the situation of mathematics education was not satisfying at the beginning of the 1960’s. This was recognized by the political leaders, so in 1962 the Politbüro (executive committee) of the Socialist Unity Party of Germany (SED, the leading party in the GDR) and the government (Ministerrat) issued a directive (the so called “Mathematikbeschluss”, see SED-Politbüro 1962). In unusually clear words[4], the directive deplored deficits in almost all areas of mathematics education: lack of basic knowledge among students, unsatisfactory spatial perception, insufficient functional thinking, lack of qualified mathematics teachers, inadequate training of teachers and inadequate teaching materials.

> “The inadequate training of mathematical thinking is the main weakness in mathematics teaching. Moreover, the content of the mathematical knowledge and skills that is currently being taught to the pupils do not sufficiently reflect the development of mathematical science …”
>
> (SED-Politbüro 1962, p.142f., translation A. F.)

The “Mathematikbeschluss” included 14 courses of action to improve mathematics education, among them were improvements of pre- and in-service education of mathematics teachers and the establishment of an institute for school mathematics at the Humboldt-University in East-Berlin and a governmental commission for mathematics education[5], headed by the mathematician Klaus Härtig who became also one of the two chairmen of the institute for school mathematics. In these positions Härtig had great influence on the development of the plan for mathematics education which was worked out until the end of the 1960’s. This plan formed the most important foundation for mathematics education in the GDR until its end in 1990. Härtig himself worked particularly on concepts for functional reasoning and working with functions (detailed below).

Less than three years after the “Mathematikbeschluss” the governmental commission for mathematics education published a "Concept for mathematics teaching in the polytechnic comprehensive schools” (see Dietzel & Härtig 1965). This concept had great importance in the further development of mathematics education in the GDR because it formed the basis of new curricula which were introduced (stepwise) from 1967 to 1972 (see Weber 1968) and Weber 1970). Because the concept was worked out at the same time as the “Nürnberger Lehrpläne” for the

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[4] Usually very cautious formulations were used in political documents of the GDR to describe deficits; criticism had to be read “between the lines”.

[5] The Commission was dissolved in the early 1970’s, and its tasks were taken over by a research group on mathematics constituted in 1971 at the Academy of Pedagogical Sciences founded in 1970. (Akademie der Pädagogischen Wissenschaften – APW of the GDR, see Weber 1972).

Federal Republic of Germany (see above) these two documents make it easy to compare how New Mathematics influenced mathematics teaching in both countries. In this respect similarities and differences can be found:

- Set theory was considered to be of great importance in mathematics education both in the FRG and in the GDR. The "Concept for mathematics teaching in the polytechnic comprehensive schools" states:

  "Set theory and mathematical logic are used as principles to teach mathematics. Contentwise considerations ... allow a systematic development of the relation, representation, and function concepts from grade 1 on in geometry as well as in arithmetic and analysis."

  (Dietzel & Härtig 1965, p.442, translation A. F.)

- Algebraic structures (especially group theory) did not explicitly become a part of the curriculum in the GDR (in contrast to the FRG, see above), although the intention was present that "by collecting and comparing suitable examples, the understanding of structural aspects could be prepared" (Dietzel & Härtig 1965, p.442). Härtig described the reasons for the limitation on "classic" content areas of mathematics education (except set theory) as follows:

  "The object areas, which are studied in many areas of today's mathematics, go far beyond numbers and geometrical objects. The rising versatility and socially important applicability of mathematics is based on its high degree of abstraction and universality. … The fact that, on the other hand, the school curriculum is essentially limited to the "classical" subject areas is not caused only by tradition … The importance of the traditional subject areas have grown with the whole of mathematics. Solid knowledge in these fields remain a particularly efficient basis for a wide range of further special education … (APW 1975, p.41f., translation A. F.)"

In summary, it can be stated that the reform of mathematics education in the 1960's implemented some of the main ideas of New Mathematics, e.g. the set-theoretic foundation of the whole mathematics curriculum (including a set-theoretic concept of mappings and functions) but avoided other important parts of it – especially explicit teaching of algebraic structures. In this sense mathematics education in the GDR was less affected by New Mathematics than in Western Germany and some other countries. A reason for that could have been that the attempts of the1950's (see above) disastrously failed. Obviously the reforms of the 1960's were much more carefully prepared and considered. Another reason could have been – although, to the best of our knowledge, this has never officially been mentioned, – that the scientists and educators in charge for the development of the curricula had more realistic ambitions about the degree of abstraction that can be achieved under the conditions of a comprehensive school system.[6]

New mathematics curricula were introduced stepwise in the GDR based on the "Concept for mathematics teaching in the polytechnic comprehensive schools" between 1967 and 1972.[7] An

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[6] Obviously the achievement of a higher abstraction level (including understanding of basic parts of group theory) can rather be expected in high school education for a minority of students than in comprehensive schools attended by all students. Because of ideological reasons this was not officially expressed in the GDR – the superiority of the socialist educational system over the Western selective system was a political claim, stating that all students reach the highest level of education that was only reachable for a minority in the West.

[7] These curricula remained valid for 15 years and were not replaced until 1982/83 (for the grades 4 and 5) and 1985-1988 respectively (for the grades 6-10). The curricula which were introduced in the 1980's did not contain fundamental

important innovation was the introduction of main curricular concepts (“Lehrplanleitlinien” – “curricular guidelines”):[8]

> “As a very important change ... we consider the consistent structuring of the contents on the basis of specific guidelines. Paying attention on specific basic ideas in the selection, arrangement and presentation of the content allows a stronger concentration on the mathematically essential and a rationalization of the overall course.” (Weber 1970, p.10, translation A. F.)

The four content-related guidelines were:

- Set-theoretic pervasion
- Domains of numbers
- Mappings and functions
- Equations and inequalities (see APW 1975, p.58ff.).[9]

Both the treatment of elementary (real-valued) functions and the main part of the whole geometry instruction was subordinated under the guidelines “set-theoretic pervasion” and “mappings and functions”. A concept for the guideline “mappings and functions” was developed by Klaus Härtig and Dieter Ilse. They emphasized that (mainly) mappings between numbers and mappings between points have to be discussed. He highlighted the following aspects of the guideline (APW 1975, p.54f.):

- “Students familiarize themselves thoroughly with *special functions,* e.g. $f(a) = a + 1$ in the field of natural numbers at primary level (without calling it a function yet) or $y = \log x$ (explicitly) at secondary level.”
- “Some important *classes of functions* are examined, e.g. the classes of translations, rotations and reflections (combined: congruence mappings) … the class of the quadratic functions or the family of exponential functions $y = a^x$ ($a > 0$)…”
- “Functions can be derived from other functions. New functions can be defined by applying the arithmetical operators on functions or by the assigning inverse funtions to (biunique) functions or by the composition of functions (e.g. composition of congruence mappings, composition of three functions with the result $y = a \cdot \sin bx$). These cases should be examined.“
- “Among others, the *explicit set-theoretic definition* has to be acquired as a particularly important component of the concept of functions.”
- “Last but not least, the ability to see *assignments occurring in different contexts* is important,– e.g. the dependence of the results of a calculation or a geometric

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changes compared to those from 1967-1972 concerning mappings and functions, (see Ministerrat der DDR, Ministerium für Volksbildung 1982, 1987a, 1987b and Borneleit 2003, p.141).

[8] The intentions that were pursued with the introduction of main curricular concepts in the GDR can be compared to those behind the content and process standards in the USA (NCTM 2000) and the “guiding ideas” (“Leitideen”) in the German educational standards which were introduced in 2003.

[9] In addition to these guidelines two further guidelines were made, which would be described as process standards in today's terminology (see APW 1975, p.64ff.):

- Lines of linguistic-logical education (mathematical terminology and symbolism, defining, proving),
- Methods and tools of efficient working (algorithmic procedures, heuristic methods, work equipment).

> construction on the "input values". The continuous training of such "functional thinking" goes far beyond teaching mathematics."

It was intended to prepare functional reasoning from grade 1 on by using function tables in various ways, e.g. for summation, subtraction, multiplication and greatest common divisor. Direct and reciprocal proportionality were explored in grade 6, followed by the general (set-theoretic) definition of functions in grade 8 and the systematic study of linear, quadratic, power, exponential, logarithmic and trigonometric functions in grades 8-10 (APW 1975, p.59). As a special concern, Ilse and some other math educators in the GDR made significant efforts to integrate description of functions by functional equations (e.g. $f(x_1+x_2)=f(x_1)+f(x_2)$ for proportional functions and $f(x_1+x_2)=f(x_1)\cdot f(x_2)$ for exponential functions) into the curriculum (Ilse and Lehmann 1974). At high school level, students' knowledge of functions was deepened and extended by a comprehensive introduction into differential and integral calculus.[10]

Teaching of geometric transformations started in grade 4 to 6 in the GDR with translations, rotations, reflections and their composition, continuing with similarity mappings and projections, (see detailed in Filler 2016). As already mentioned, transformation groups were not taught explicitly but it was expected (or at least hoped) that students develop a basic understanding of algebraic structures and the idea of Felix Klein's "Erlangen program" (see above):

> "In geometry education students familiarizing themselves with examples of structures, and especially the treatment of movements and similarity transformations poses the question of invariants in respect of these transformation groups." (APW 1975, p.113, translation A. F.)

The described intentions of the guideline "mappings and functions" remained crucial for mathematics education in the GDR until its end in 1990, although some doubts arose in the 1980's especially about the usefulness of a mapping-based teaching of geometry, (see Filler 2016, p.124-129). Finally, it turned out that the idea that students would obtain integrated understanding of mappings on such an abstract level which was not successful at all, e.g. Stoye came to the conclusion (1990) after several studies that students do not integrate geometric transformations into their understanding of functions.

3.4 Functions in mathematics teaching in Hungary

In the first years after World War II the progressive ideas of fast democratization had an impact on education, too. The new high school textbook series – authored by the mathematicians and didacticians Rózsa Péter, Tibor Gallai and János Surányi brought in new contents and methods mostly in the spirit of the reform ideas of F. Klein. These progressive textbooks were way ahead of their time and the later didactics of mathematics research, but they did not become well-known in other countries, and their application were also difficult in Hungary due to the conservativism of the teachers (Deák 2002).

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[10] These high-schools („Erweiterte Oberschulen") were attended by only 10-15% of all students. While all students in the GDR were educated (from 1959 on) according to the same curricula until grade 10, there were special curricula for the grades 11 and 12 in these high schools.

Until 1965, functions were considered only given by terms in the Hungarian curriculum for high schools. The introduction of the new curricula started at that time, where a wider range of other functions appeared, as well as the buildup of geometry on the basis of transformations, and the central role of the function-concept in the teaching materials of mathematics. (Katz 1989, pp.26, 27)

Function oriented thinking appeared in some ways in other areas of school mathematics e.g. in geometry (transformations) in high school textbooks published from the 1950's (first in the textbooks of Rózsa Péter and Tibor Gallai). However, the function notion was used as "dependency of values from another values", similarly to Beke, even in these textbooks. There were "functions with one values" (to every value of the range corresponds one value of the domain) and "functions with several values" (i.e. $y = \pm\sqrt{x}$ considered as "function with several values").

The elementary school with eight schoolyear was introduced and became compulsory in 1945. A wider selection of the four-year high schools became available in the 1970's and they generally ended in final examination at the age of 18, with Mathematics being one of the compulsory exam subjects (along with Hungarian language and literature). The education system was centralized in Hungary until the 1990's. There was a unified curriculum and only one series of textbooks for each school. For the Hungarian educational system see also Szendrei (1996).

In elementary school classes, only arithmetic and measurement were taught (in fact, the name of the school subject was Arithmetic and Geometric measures) and teaching concentrated on basic knowledge until the end of the 1950's. With the elementary school curricula for 1958 and 1962, the teaching of algebra became more important, and the subject named "Mathematics" finally appeared, first in grade 8 with a wider content and with a textbook also named Mathematics. The graphical presentation of the direct and inverse proportionality appeared in the teaching materials in grade 7. "Function" was mentioned first in grade 8 as a relation between values, with the help of real world examples (e. g. the distance travelled by the train is "the function" of the journey time).

Functions as unique assignments between ranges of values was consistently defined in the mathematics curricula for high schools first in 1965, and it became possible to emphasize the central role of the functions in teaching. However, the realization of the curriculum's objectives was hindered, partly by deficiencies in students' knowledge of the elementary school's material, so usually they were only attained in advanced classes with extra mathematics lessons. (Katz 1989, p27)

Tamás Varga and the OPI[11] Project

Functionality became a central idea in Tamás Varga's mathematics concept. Tamás Varga was György Pólya's follower and had a significant working relationship with Zoltán Dienes. As a mathematics and physics teacher, he worked on the issues of changing, reforming mathematics education as early as in the 50's, and participated in the evaluation of textbooks as an expert of the Hungarian Ministry of Education.

Under the influence of the talks of the UNESCO Research Symposium on Mathematics Education (1962, Budapest), several small research projects started for the improvement of mathematics

[11] Országos Pedagógiai Intézet (OPI) [National Institute for Pedagogy]

education at elementary level in Hungary. Varga also intensified his work on the modernization of mathematics education. The trend represented by him and his colleagues was worked out and summarized in their Complex Mathematics Teaching Experiment - which was called as OPI Project in the international discussion (Szendrei 2007, Klein 1987). This project, set off in 1964, “basically set the same task – but with much more limited opportunity – as the Madison Project, namely ‘seeking the best experience with mathematics which can be provided for children at the pre-college level’ (Davis 1964)” (Klein 1987, p. 35).
“The research was well-known in the international community of mathematics education. The strongest channel for the relationship was the CIEAEM (Commission Internationale pour l’Étude et Amelioration de l’Enseignement des Mathematiques). The communication was oral, because it was not possible for Hungarian mathematics educators to publish their results internationally at that time.” – wrote Julianna Szendrei, one of the researchers in the research-group of Varga (Szendrei, 2007, p. 448). The teacher-researcher who implemented Varga’s concept in the classroom worked without textbooks, only with continuously developed worksheets (so called “Munkalapok”) which were acknowledged internationally e. g. by Freudenthal (1973 p. 244).

Varga was critical about the ideas of the New Math, he put students' active gaining of knowledge based on experiences and discovery before mathematical formalism and strict mathematical structures. In his ideas, the concept of function has a basic role throughout mathematics, a function (unique assignment) between numbers is only a special case of the more abstract mappings and we can find mappings not only in geometry, i. e. transformations, but in the whole school mathematics. For example, sorting results of combinatorics problems by parameters leads to function tables (Varga 1975, p.6). For notation, he preferred the form $x \longmapsto f(x)$ which stressed the dynamic dependence in opposite of the static form $y = f(x)$. He emphasized that not the *way* but the *fact* of the mappings is important and that students should recognize function type relationships; the intuitive concept of function should be a part of student’s way of thinking and this way it should become an operative concept outside mathematics, too (Varga 1983).

Varga considered the teaching of mathematics to be an “arc from grade 1 until the final exam in grade 12” similarly to Beke (Ambrus 2016) and his method was function centric. He considered the connections between functions and the areas of mathematics to be important in the education of mathematics. Summarizing, he wrote the following:

> “It is legitimate to view (consider) the curricula (1978), based on the complex education of mathematics as function oriented. In a sense this means the realization of the program from Felix Klein a century ago, but in a more consistent form where the present development of society, technology and mathematics is considered.” (Varga 1975, p.7, translation G. A.)

For Varga, the study of dynamic dependencies was very important since the grade 1, so as the slow preparation of the abstract definition of function (by mapping), which first occurred in the grade 8 in his method. He emphasized the importance of thinking in terms of connections, relationships and that examining the applicability of the mathematical models built using the discoveries of relationships is an essential part of the knowledge from early ages. A nice example of this can be found in one of his papers mentioning his granddaughter Eszter, who first finds a linear relationship (at age 5) between age and height of children. Later, when she tries to extend her “model” to all ages she finds the limits of this model (Varga 1983, p.74).

Based on the experience of the OPI Project, a new mathematics curriculum was introduced gradually from 1974 in grades 1-4 in elementary school, first for voluntary classes (C. Neményi 2002). The buildup of the function concept began already in grade 1 by simple mapping examples and rule-games. In the next years the pupils learned a lot about different sequences, functions and geometrical mappings and this way for 8 grade 8 they had a wide view about functions. By grade 8, where the function was first defined, pupils have already familiarized themselves with the elements of both function concepts: the set theory (arranged pairs) and the unique assignment based as well. They had some experience and knowledge about the elements of algebraic structures, transformations and compositions of functions (geometrical functions as well) and about some classes of functions; the teaching materials for functions were rich and manifold. (Katz 1989, pp. 28-29).

In the second part of the 1970’s the Project of Tamás Varga got more and more acknowleged in Hungary. “…the Ministry of Education ordered a study of the effectiveness of the OPI Project, and the results were impressively in favor of the experimental classes” (Szendrei, 2007, p. 448). In 1978 the new mathematics curriculum for primary schools was generally introduced in Hungary on the basis of Varga's experiment – despite the protest of the researcher, who stood up for a “gradual extension, not for the imposing of the reforms on teachers” (Szendrei 2007, Klein 1987). The curriculum included the following themes:

1. Sets and logics,
2. Arithmetic and algebra,
3. Relation, Functions, sequences
4. Geometry, measurement
5. Combinatorics, probability, statistics

and resulted in a complete change in mathematics teaching in grades 1-8.

Varga’s concept and the function centered teaching concept appeared only moderately in high school teaching (grades 9-12) in the late seventies and later on as well. There are documented projects as well to integrate his concept in the high school teaching materials (e. g. Mayerné, B. A. & Pálfalvi J. 1979, Katz 1989).
According to the Hungarian curricula for high schools in1979, the official definition of function was the following:

> Let A and B be two non-empty sets. If to each and every element of A we assign one or more elements of B, then we have mapped A into B. The assigment of elements in B to the elements of A may be conducted uniquely or non-uniquely. If to each element of A we assign a unique element in B, then the mapping is said to constitute a function. Set A is the domain for which the function is defined, while the values assigned to it are the range, which is not necessarily equivalent to set B. Set B is also known as the image set. (Mayerné - Bartal A. & Pálfalvi J. 1979 translation E.V.)

Although the new curricula and new textbooks in 1979 brought some positive changes, a lot of problems in teaching of functions remained unsolved in high schools (grades 9-12). The new materials were practically built on the previous teaching materials on functions in elementary schools (see Katz 1989).

The textbooks and worksheets for grade 1-8 for the 1978 curriculum were written by mathematics teachers (researcher-teachers) who took part in the OPI Experiment and had practical experience with this method, in harmony with Varga's concept. In spite of in-service teacher trainings and additional materials teaching according to Varga's method was too difficult for most of the teachers, and the 1978 curriculum had to be revised in the 1980's. The definition of functions as unique assignments appeared in the new curricula, together with the information that mappings between numbers are only a special case of more abstract mappings, also with stress on the dependence between data. Geometric transformations mappings were mentioned as well but in a more formal way than before.

The Complex Mathematics Teaching Experiment (OPI Project) led by Tamás Varga resulted in the the transformation of the former fragmented arithmetic and geometry teaching into an integrated mathematics teaching, completed with a number of other chapters of mathematics. Varga wanted to build on the student's individual thinking, and he expected the teacher to act as the student's colleague. Although his ideas were only partially implemented in Hungary, his influence was seen throughout Hungarian mathematics teaching and had a very important impact on teaching of functions at any level in Hungary in the next decades. After the last major political changes (1989) new reforms started in this field too, but these are beyond the scope of this paper.

4. CONCLUSION

Functional reasoning and working with functions in Hungary and in both German states had the same basic ideas which have been laid down at the beginning of the 20th century. From 1950 to 1990 attempts were made to modernize mathematics education in all three countries, especially in the subject of functions/mappings. A higher level of abstraction and generalization was targeted, according to the ideas of New Mathematics. The changes that were made differed considerably in Hungary, West- and East-Germany and were not primarily determined by the political systems (which were not identical but similar in Hungary and in the GDR).

Finally, it turned out in all three countries in the 1980's that the "classical" approach of functional reasoning, with the general idea of functions as dynamic dependencies, was the most successful and therefore it survived the changes made in the period of New Mathematics. The generalizations regarding the concept of functions in school mathematics (set-theoretic foundation, functions as mappings between arbitrary sets) turned out to work on an abstraction level that was too high for the vast majority of students. Considering the definitions of functions, the function-centered way of teaching mathematics and the teaching materials on the field of functions there were many characteristic traits in the three countries which are due to the mathematics educational traditions. Traditional teaching in the high school in Hungary hindered even the researches of new methods for higher grades not only in the 1950's but also in the time of the successful implementation of Varga's method in elementary schools. In Germany the three aspects of functional reasoning (Vollrath 1989) attained the utmost importance for the further development of working with functions in mathematics education. Basic approaches to these aspects were already included in the ideas of Felix Klein and the Meraner reforms.

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Appendix 1: Examples for the guidelines "mappings and functions" and "set-theoretic pervasion" in East German mathematics schoolbooks

Mappings explicitly appear in East German schoolbooks beginning at grade 6. The first examples (like the one below) refer to proportional assignments.

Proportional sequences of measures

An automatic cycle line processes a certain type of workpiece. In each case after one hour … 30 workpieces are finished. The following table shows how many workpieces are finished in this continuous operation after 1, 2, 3, … hours.

Arbeitszeit t in h	1	2	3	4	5	6	7	8
Anzahl n der bearbeiteten Werkstücke	30	60	90	120	150	180	210	240

Fig. 1: Proportional sequences of measures in a textbook for 6th grade (Bittner & Tietz 1969, p.70)

In this table are two sequences of measures. Each measure of one size is assigned the corresponding measure of the other size. …

The Cartesian coordinate system has been introduced in the same context to illustrate proportionality.

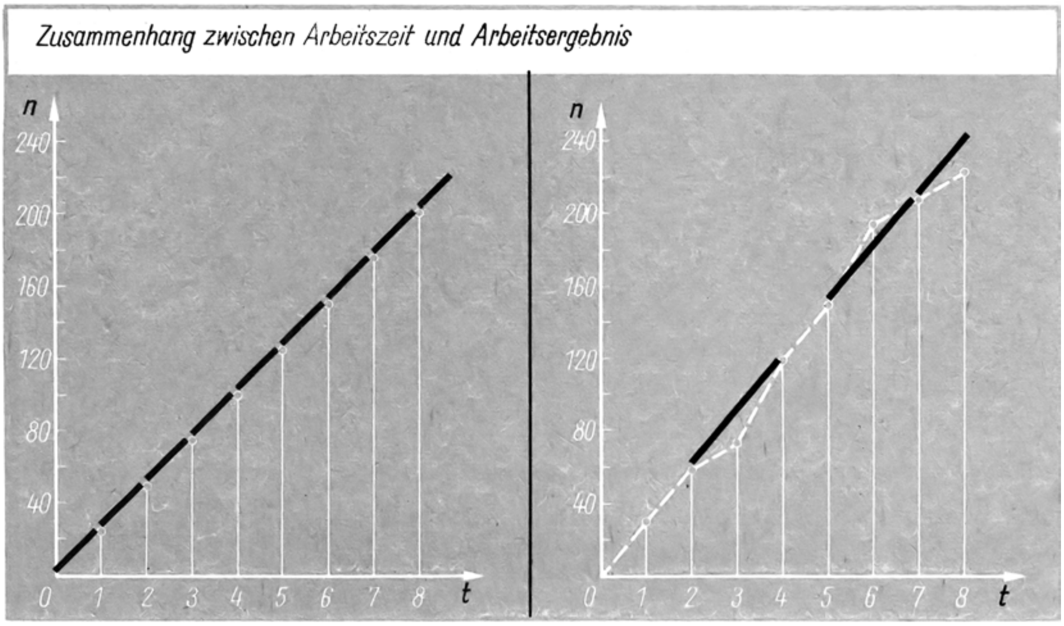


Fig. 2: Relationship between working hours and work results (Bittner & Tietz 1969, p.73)

The word "mapping" (Abbildung) is used for the first time in connection with geometric mappings. The following example shows the introduction of the term "mappings" in the same textbook (for 6th grade) from which the examples above were taken. In this introductory example a distinction is already made between unique (eindeutige) and biunique mappings (eineindeutige Abbildungen).

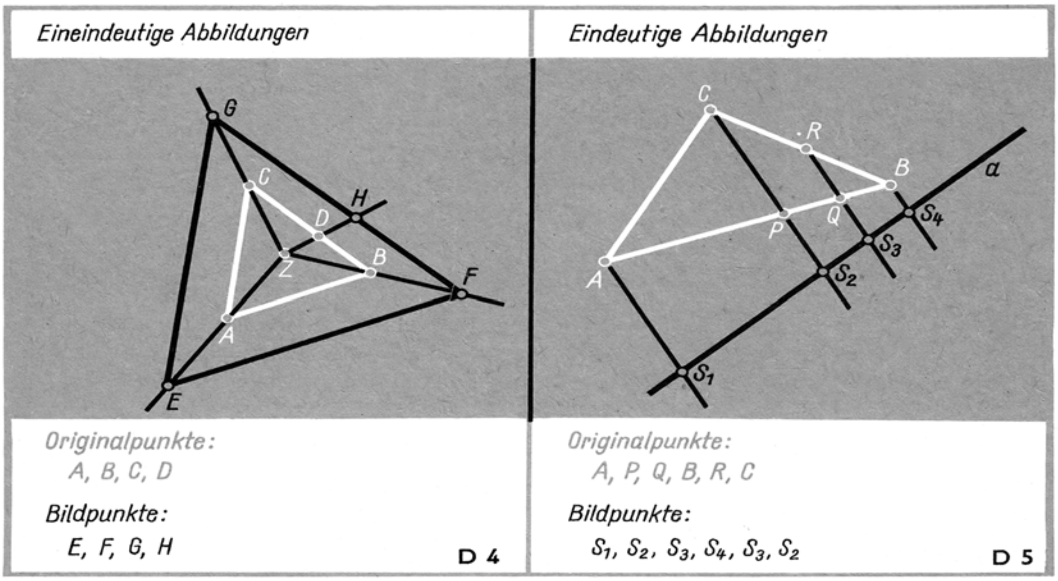


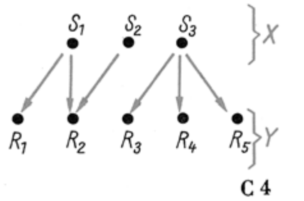
Fig. 3: Biunique and unique mappings (Bittner & Tietz 1969, p.73)

The following explanation is given to this example:

> In our example (D 4), each point of a geometric figure is assigned a point of another geometric figure. We are talking about a **mapping**.
>
> In mappings we distinguish between **originals** and **images**. In our example, each original point *P* is assigned **exactly one** point *P*' as an image. Conversely, every image point *P*' has exactly one original point *P*. Such a mapping is **reversibly unique** or **one-to-one**.
>
> In the example (D 5), each original point is again assigned exactly one point as an image. Conversely, however, there are image points that belong to **more than one** original point. Such a mapping is called **unique**. (Bittner & Tietz 1969, p.73)

Although sets and elements are not explicitly mentioned here (and instead points are referred to), the intention of the guideline "set-theoretic pervasion" becomes visible. Explicitly it forms the basis of the introduction of the term "function" in grade 8, where one of the introductory examples is the following.

Im Umkleidehäuschen eines Sportplatzes befinden sich 5 Räume, zu denen 3 Schlüssel gehören. Aus dem Bild C 4 geht hervor, wie die Schlüssel zu den Räumen passen. Wir sagen:
Den Schlüsseln S_1, S_2, S_3 sind die Räume R_1, R_2, R_3, R_4, R_5 zugeordnet, oder die Menge der Schlüssel $X = \{S_1, S_2, S_3\}$ wurde auf die Menge der Räume $Y = \{R_1, R_2, R_3, R_4, R_5\}$ abgebildet.



C 4

Fig. 4: Introductory example for functions (Bittner et al. 1971, p.60)

> In the changing room of a sports field there are 5 rooms, to which 3 keys belong. Picture C 4 shows how the keys fit to the rooms. We say:
>
> The keys S_1, S_2, S_3 are **assigned** the spaces R_1, R_2, R_3, R_4, R_5, or
>
> the **set** of keys $X = \{S_1, S_2, S_3\}$ has been **mapped** to the set of rooms $Y = \{R_1, R_2, R_3, R_4, R_5\}$.

As a generalization of this and other examples it is stated:

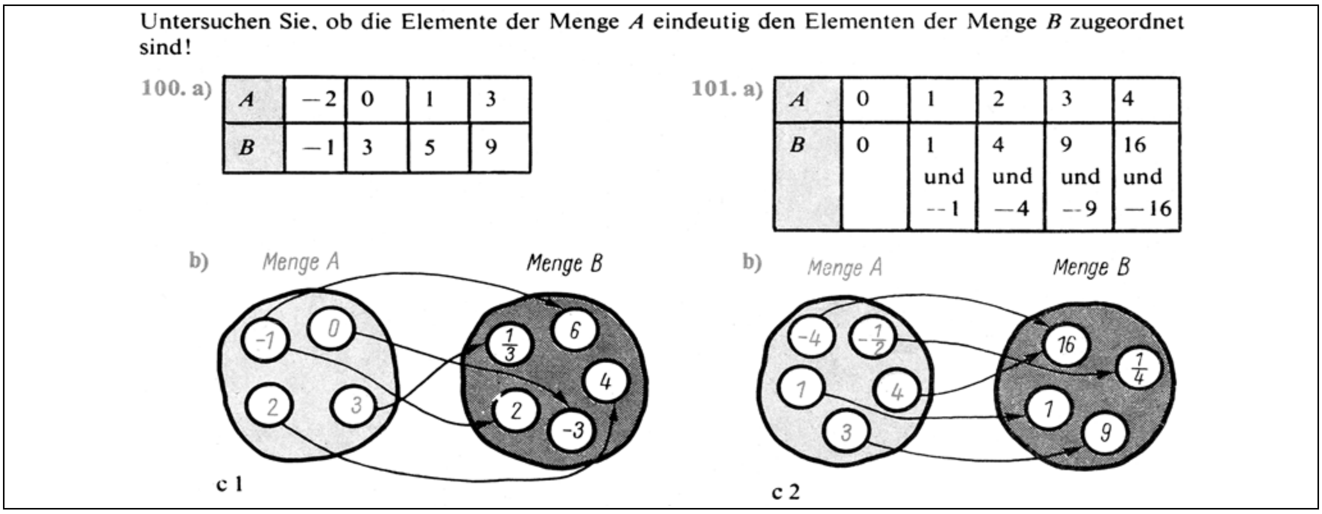
> In Examples C 4 and C 6, the mapping from *X* to *Y* is called unique. The condition for uniqueness is that each element of *X* is assigned **one and only one** (say **exactly one**) element of *Y*. (Bittner et al. 1971, p.60)

After these preparations the following definition of the term function is given (emphases as in the original):

> **DEFINITION: A set of ordered pairs [*x*;*y*] with $x \in X$ and $y \in Y$ which is a unique mapping of the set *X* to the set *Y*, is called *function*.**
>
> It is also said of these ordered pairs [*x*;*y*] that each $x \in X$ is **assigned** exactly one $y \in Y$.
>
> Such sets *X* as in Example C 4 […] are referred to as the **domain of definition** of the function concerned. In the following, we will always use the domain of rational numbers[12] as the domain of definition unless stated otherwise. The quantities *Y* of the assigned elements are referred to as the respective **value range**. The number *y* from the value range, which is assigned to the respective function of a number *x* from the domain of definition, is called the **function value** belonging to *x*. (Bittner et al. 1971, p.61)

After this definition and the explanations are given, linear functions are treated in detail (in a way that is common also today), with little reference to the set-theoretic terms. The set-theoretic foundation of the term "function" is repeated in the grades 9 and 10, when other classes of functions (quadratic, power, exponential, logarithmic and trigonometric functions) are discussed. The following example is taken from a schoolbook for grade 9:

> Examine whether the elements of set *A* are uniquely assigned to the elements of set *B*!



Untersuchen Sie, ob die Elemente der Menge *A* eindeutig den Elementen der Menge *B* zugeordnet sind!

100. a)

A	−2	0	1	3
B	−1	3	5	9

101. a)

A	0	1	2	3	4
B	0	1 und −1	4 und −4	9 und −9	16 und −16

Fig. 5: Exercise in schoolbook for grade 9 (Lemke et al. 1970, p.198)

[12] Real numbers are unknown to the students at grade 8; they are introduced at grade 9. Therefore the range of definition is restricted to the domain of rational numbers.

Appendix 2: Examples for "mappings and functions" in Hungarian mathematics schoolbooks

First we give a historical example with the help of an excerpt from the 1908 edition's chapter of the algebra textbook (Beke; Kőnig, 1908, Fig. 1) on linear functions, where the functions' practical application is emphasized.

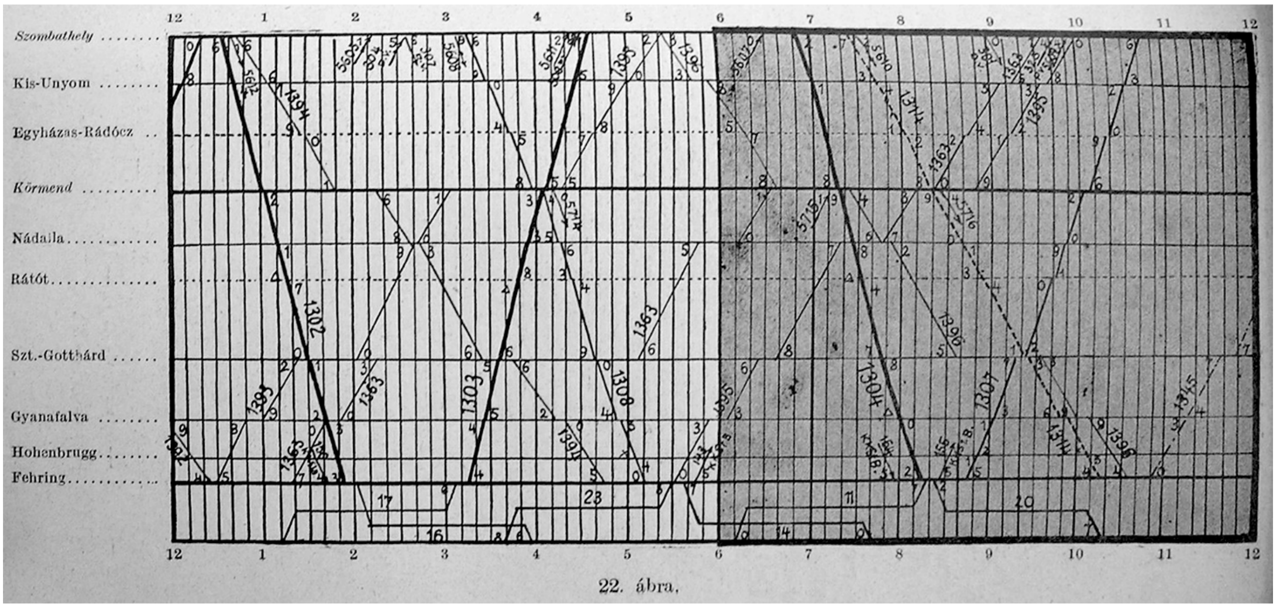


Fig 1: Timetable of Trains (Beke and Kőnig 1908, p.85)

Introducing the task - with the help of examples- Beke explains how to find one's way on a map, and then sets the following task:

Figure 1 shows the graphical time table of the trains from Szombathely (a Hungarian town in West) to Fehring (today in Austria, at that time belonged to Hungary). The thick line means express, the thinner passenger train and the very thin means goods train, finally the pointed line mixed train (passengers and goods). The time axis presents times from midday 12 o'clock to midnight 12 o'clock, all subintervals are 10 minutes long. Try to read from the figure when the afternoon express from Szombathely departures. When does the express from Fehring arrive in Szombathely? When does the evening express arrive in Fehring? Where does express 1303 meet the passenger train 1308? How long does goods train 1394 stay in Körmend (a Hungarian town in West)?

In the textbooks and worksheets according to the Method of Tamás Varga there are plenty of examples and exercises in connection with "mapping" from the 1st grade. The following examples are from the textbooks grade 5.-8.

The first example (Fig. 2) is from Textbook for 5th grade students[13] Chapter "Movement-graphs". There is an unconventional graphical form presenting the time-distance relation on the picture. The text says that a car starts from the 0 km point at 12 o'clock and arrives in a town, which is 400 km

away, at 8 pm. On his way the car was quicker sometimes, sometimes slower and made some stops as well. The picture shows where the car is at different points of time.

There are some questions in connection with the graph:

When was the car motionless? How many km did the car pass between 1 pm and 2 pm? In which period did the car cover more kms: between 5 pm and 6 pm or between 6 pm and 7 pm? In which period did the car cover more kms: between 3 pm and 4 pm or between 4 pm and 5 pm? The task is completed by two other graphical presentations and by an imagined discussion between two children about the graphs.

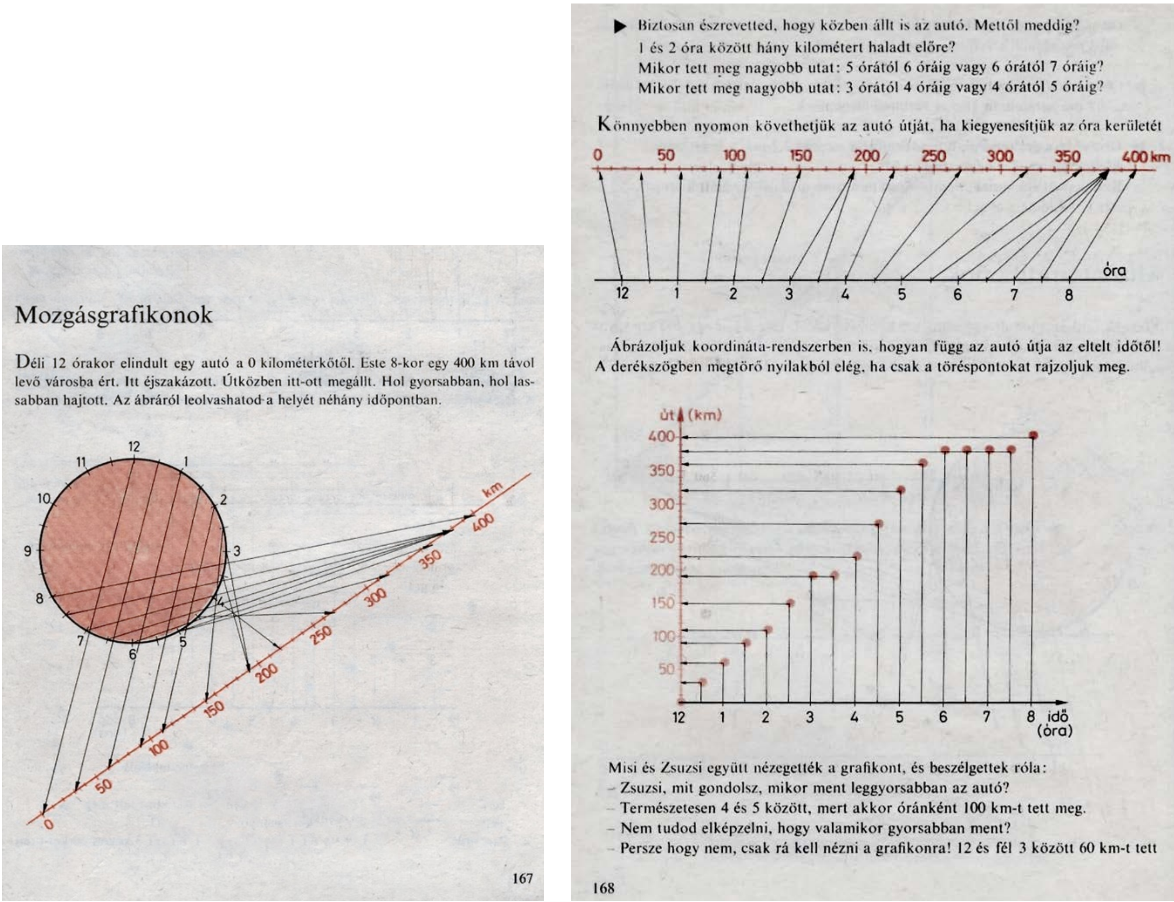


Fig. 2: Examples for teaching mappings, Mathematics 5 (Eglesz et al. 1987, pp. 167,168)

The different graphical presentations for mappings appear in the 8th grade’s textbook as well where the definition for functions are first given (“functions are unique mappings”).

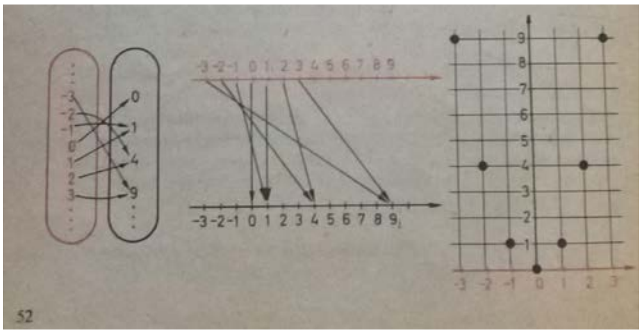
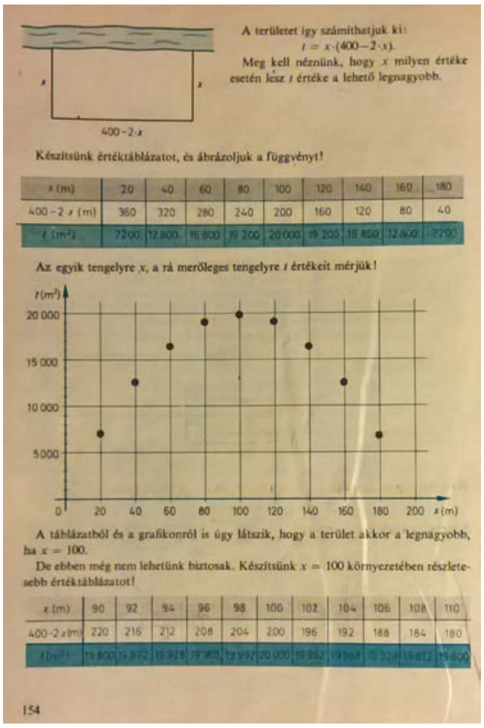


Fig.3: Graphical representations of a function, Mathematics 8 (Imrecze et al. Matematika 8, p. 52)

The word "function" appears first in the 6th grade, in the meaning: "a value is in function with an other value".

Functional relationships are treated in the 6th grade and in the upper grade textbooks not only in different fields (i.e. number theory, geometry) they are used for problem solution as well, like the extremal problem worked out in the textbook. The Fig.4 is an excerpt of the mentioned example-solution for the following problem: *On the riverside we want to rail off a rectangle shaped tract for sports-ground, with 400 m fence. We want to have the largest possible surface. How are the sides of the rectangle in this case?*



A területet így számíthatjuk ki:

$t = x \cdot (400 - 2 \cdot x)$.

Meg kell néznünk, hogy x milyen értéke esetén lesz t értéke a lehető legnagyobb.

Készítsünk értéktáblázatot, és ábrázoljuk a függvényt!

x (m)	20	40	60	80	100	120	140	160	180
$400-2 \cdot x$ (m)	360	320	280	240	200	160	120	80	40
t (m²)	7200	12 800	16 800	19 200	20 000	19 200	16 800	12 800	7200

Az egyik tengelyre x, a rá merőleges tengelyre t értékeit mérjük!

A táblázatból és a grafikonról is úgy látszik, hogy a terület akkor a legnagyobb, ha $x = 100$.

De ebben még nem lehetünk biztosak. Készítsünk $x = 100$ környezetében részletesebb értéktáblázatot!

x (m)	90	92	94	96	98	100	102	104	106	108	110
$400-2 \cdot x$ (m)	220	216	212	208	204	200	196	192	188	184	180
t (m²)	19 800	19 872	19 928	[illegible]	19 992	20 000	19 992	19 968	[illegible]	19 872	19 800

154

Fig. 4: Extremal problem, Mathematics 6

(Eglesz et al., 1983, p. 154)

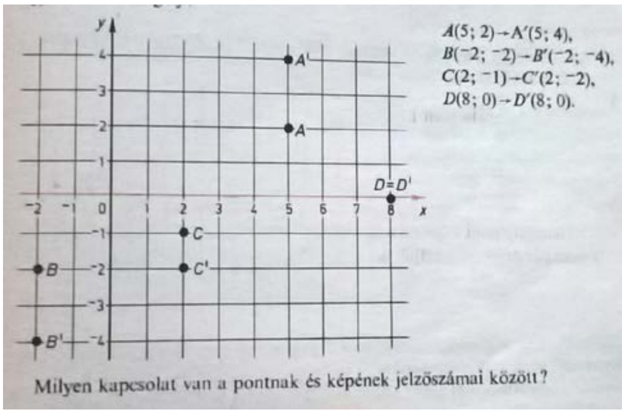


Fig. 5: Geometric transformation, Mathematics 7

(Kovács et al., 1980, p. 147)

According to the first table the conclusion is*: it seems the required side value is x=100, but you can't be sure.* The second table on the Fig.3 give a possible method, how to manage this question at this age - focusing on an interval which contain 100, we get it more and more smaller, this way the previous guess becomes more and more plausible.

The last example (Fig. 5) is from the 7th grade textbook. Geometric transformation is given by coordinates of points. The question is: *What is the relationship between the coordinates of the points A, B, C, D and their images A', B', C', D'?*