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Abstracted Primal-Dual Affine Programming

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Doctoral Dissertation Defense
"Abstracted Primal-Dual Affine Programming"

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The classical study of linear (affine) programs, pioneered by George Dantzig and Albert Tucker, studies both the theory, and methods of solutions for a linear (affine) primal-dual maximization-minimization program, which may be described as follows:

"Given $A \in \mathbb{R}^{m\times n}$, $\vec{b}\in \mathbb{R}^m$, $\vec{c}\in \mathbb{R}^n$, $d \in \mathbb{R}$, find $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} \leq \vec{b}$, and $\vec{x} \geq 0$,
that maximizes the affine functional $f(\vec{x}) \coloneqq \vec{c} \cdot \vec{x} - d$; and find $\vec{y}\in \mathbb{R}^m$ such that $A^{\mathrm{T}}\vec{y} \geq \vec{c}$, and $\vec{y} \geq 0$,
that minimizes the affine functional $g(\vec{y}) \coloneqq \vec{b} \cdot d$."

In this classical setting, there are several canonical results dealing with the primal-dual aspect of affine programming. These include: I: Tucker's Key Equation, II: Weak Duality Theorem, III: Convexity of Solutions, IV: Fundamental Theorem of Linear (Affine) Programming, V: Farkas' Lemma, VI: Complementary Slackness Theorem, VII: Strong Duality Theorem, VIII: Existence-Duality Theorem, IX: Simplex Algorithm.

We note that although the classical setting involves finite dimensional real vector spaces, moreover the classical viewpoint of these problems, the key results, and the solutions are extremely coordinate and basis dependent. However, these problems may be stated in much greater generality. We can define a function-theoretic, rather than coordinate-centric, view of these problem statements. Moreover, we may change the underlying ring, or abstract to a potentially infinite dimensional setting. Integer programming is a well known example of such a generalization. It is natural to ask then, which of the classical facts hold in a general setting, and under what hypothesis would they hold?

We describe the various ways that one may generalize the statement of an affine program. Beginning with the most general case, we prove these facts using as few hypotheses as possible. Given each additional hypothesis, we prove all facts that may be proved in this setting, and provide counterexamples to the remaining facts, until we have successfully established all of our classical results.

Dissertation Committee

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