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Locating Complex Roots of Quintic Polynomials

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Abstract: *Since there are no general solutions to polynomials of degree higher than four, high school and college students only infrequently investigate quintic polynomials. Additionally, although students commonly investigate real roots of polynomials, only infrequently are complex roots – and, more particularly, the location of complex roots – investigated. This paper considers features of graphs of quintic polynomials and uses analytic constructions to locate the functions' complex roots. Throughout, hyperlinked dynamic applets are provided for the student reader to experientially participate in the paper. This paper is an extension to other investigations regarding locating complex roots (Bauldry, Bossé, & Otey, 2017).*

Keywords: Quartic, Quintic, Polynomials, Complex Roots

Most often, when high school or college students investigate polynomials, they begin with algebraic functions that they are asked to either factor or graph. From the factored form of these functions, they are able to deduce the real and complex roots of the polynomial. These features assist the students in sketching graphs of these polynomials. However, only infrequently are students asked to produce the algebraic factored form of a polynomial based on features of the polynomial's graph. This is particularly daunting when the polynomial is of higher degrees and has some complex roots.

Techniques for locating complex roots of quadratic, cubic, and quartic real monic polynomials using analytic constructions have previously been cataloged. This paper considers how to use analytic constructions and geometric observations to locate the complex roots of real monic quintic polynomials. Analytic constructions are herein defined as geometric constructions based on a limited number of analytic values derived from the graph of the polynomial function.

In order to maximize the reader's experience through this investigation, the contents are separated into two interconnected experiential styles. For the reader who loves to look deeply into mathematical theorems, the theorems that support the findings in this paper are provided. The reader is invited to read and prove these theorems. For the reader who may wish to experiment with these ideas and observe results in a more dynamic manner, hyperlinks to dynamic graphing apps are provided. Manipulating these dynamic graphs allows for the reader to relocate real and complex points on the coordinate plane and observe the effects on the graph. This allows student readers to participate in learning while working through this investigation.

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Some Mathematical Preliminaries

The Fundamental Theorem of Algebra states:

> *Every non-constant single-variable polynomial with complex coefficients has at least one complex root.*

In the context of polynomials with real coefficients, the Fundamental Theorem of Algebra can be restated as:

> *Every polynomial can be factored (over the real numbers) into a product of linear factors and irreducible quadratic factors.*

While "irreducible quadratic factors" are irreducible in the real numbers, they are reducible in the complex numbers. Thus, in respect to polynomials with real coefficients of degree *n*, we know that there exists *n* complex roots, some of which may be real. Therefore, real monic quintic polynomials can be factored as:

$$\underbrace{(x-a)(x-b)(x-c)(x-d)(x-e)}_{\text{distinct or equal real roots}},\ \underbrace{(x-a)(x-b)(x-c)\left((x-d)^2+e^2\right)}_{\substack{\text{three distinct or equal real roots}\\ \text{and one pair of complex roots}}},\ \text{or}$$

$$\underbrace{(x-a)\left((x-b)^2+c^2\right)\left((x-d)^2+e^2\right)}_{\substack{\text{one real roots}\\ \text{and two pairs of distinct or equal complex roots}}}.$$

Quite simply, the real roots and respective odd or even multiplicity of a real polynomial are recognized as the behavior of the *x*-intercepts of the associated graph. Since real roots can be readily located, it is only natural that students inquire regarding the location of these complex roots. Herein, we investigate means of analytically constructing and geometrically approximating the location of complex roots for real monic quintic polynomial functions.

For the remainder of this investigation, the term *quintic* or *quintic polynomial* denotes real monic quintic polynomials (polynomials with real coefficients in which the leading coefficient is 1). While quintics are conventionally graphed on the Cartesian plane (*R*×*R*), non-real complex points belong in the universe of the complex plane (*C*). Therefore, to investigate the location of complex roots of quintics, the complex plane is superimposed on the Cartesian plane ({*R*×*R*, *C*}). (See Figure 1.) This *Superimposed Plane* retains the real-valued *x*-axis of the other planes and has a *y*-axis that is labeled simultaneously with real and imaginary values. The imaginary-valued axis is only used in the context of the complex roots. The Superimposed Plane allows for real zeros of polynomials to be treated precisely the same as on the Cartesian plane.

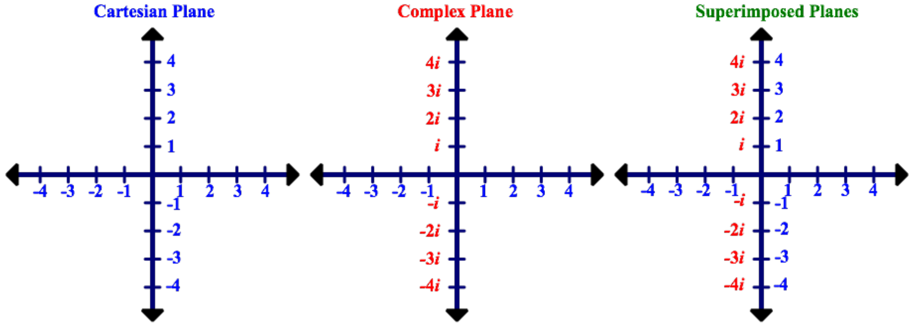

Figure 1. The Development of the Superimposed Plane, {R×R, C}.

In addition to recognizing real roots and their respective odd or even multiplicity on the graph of the quintic, *q*(*x*), to fully appreciate this investigation, readers will benefit from: a recognition of graphical features such as local positive or negative slope of the function, local extrema, and points of inflection; a minimal understanding of the meaning of a function's derivatives; and the ability to sketch the graph's first [$q'(x)$] and second derivative [$q''(x)$] functions. Notably, almost all of these ideas are encountered in high school or early college mathematics classes.

Foundational Theorems

This section provides foundational mathematical theorems for the dynamic experiential discussions in following sections. The reader can interact with the theorems provided in this section by attempting to prove the theorems, exploring the ideas through dynamic applets, and investigating associated theorems in addition to those here. Throughout this paper, we make application of a definition, a number of theorems, and an observation in the analysis and locating of complex roots. These statements are listed below.

Theorem 1. It is well known that a polynomial's roots are continuous functions of its coefficients as seen in Marden (1966): The monic complex polynomial $f(z) = z^n + c_1 z^{n-1} + \cdots + c_n \in \mathbb{C}[z]$ factors as $f(z) = (z - a_1)\cdots(z - a_n)$ where the roots a_k need not be distinct. For every $\varepsilon > 0$, there exists $\delta > 0$ such that every polynomial $g(z) = z^n + d_1 z^{n-1} + \cdots + d_n \in \mathbb{C}[z]$ satisfying $|d_k - c_k| < \delta$ for k=1, …, n can be written $g(z) = (z - b_1)(z - b_2)\cdots(z - b_n)$ with $|b_k - a_k| < \varepsilon$ for k=1, …, n.

More simplistically, this can be stated as: Small changes in the coefficients of a polynomial will only produce small changes in the values of the roots of the polynomial. To observe the meaning of this theorem, use the Marden applet.
(http://appstate.edu/~bossemj/VisualizingRoots/Quintics/Apps/Marden/)

Theorem 2. Extending Theorem 1, let $(h,k) \in \mathbb{R}^2$ with $k \geq 0$. Suppose that the monic quintic real polynomial, $q(x)$, has a positive minimum $q(h) = k$. Then:

$$q(h) = k;\ q'(h) = 0;\ q''(h) > 0;\ \text{and}\ q(h)\cdot q''(h) > 0.$$

Hence, $q(x)$ is given by $q(x) = (x-h)^2 \cdot g(x) + k$ with a monic $g(x) \in P^2$ such that $g(h) \neq 0$, and then

$$q''(h) = 2g(h) > 0\ \text{and}\ q(h)q''(h) = 2kg(h) > 0.$$

By the continuity of the roots with respect to the coefficients $q(x)$ (Theorem 1), the double root at x=h when k=0 becomes a complex conjugate pair of roots $z_\pm$ such that $|\mathrm{Re}(z_\pm) - h|$ is small as k increases; i.e., the conjugate pair $z_\pm$ must be "near" x=h.

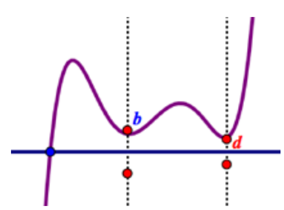

Figure 2

More simplistically, this theorem states that if there is a local extremum and the function is not an x-intercept at that point, then there must be a complex root in the near neighborhood creating that extremum. See Figure 2 and note that two extrema are located at x-values near the x-values of the complex roots. To observe the meaning of this theorem, use the Figure 2 applet.
(http://appstate.edu/~bossemj/VisualizingRoots/Quintics/Apps/Figure2/)

Observation 1. Employing Theorem 2, for the quintic, $q(x)$ with complex roots $(a,\pm bi)$, if b is sufficiently close to zero the graph will possess a local extremum pointing toward the x-axis. However, if b departs sufficiently from 0, the graph will produce a local "flattening". (Numerous examples are provided below.) To observe the meaning of this theorem, use the Figure 3 applet.
http://appstate.edu/~bossemj/VisualizingRoots/Quintics/Apps/Figure3/)

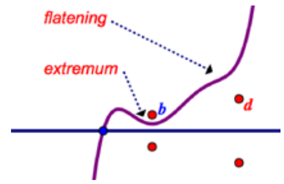

Figure 3

Definition 1. Let Q_3 be the set of monic quintic polynomials having three real roots:
$Q_3 = \{q \in P_5 | q$ is monic, has three real roots r_i, i=1..3, not necessarily distinct and complex roots $z_{\pm} = a \pm bi\}$.

In investigating the mathematics behind the findings of this paper, two new theorems were developed: Theorems 3 and 4. These are provided below.

Theorem 3. Let $q \in Q_3$. Let $s_1 = r_1 + r_2 + r_3$, $s_2 = r_1r_2 + r_1r_3 + r_2r_3$, and $s_3 = r_1r_2r_3$ be the elementary symmetric functions of the three real roots. Then $a = -\frac{1}{2}\left(s_1 + \frac{1}{4!}q^{(4)}(0)\right)$.
The complex roots lie on the intersections of the line $x=a$ and the circle of radius $R = \sqrt{-q(0)/s_3}$ centered at the origin. (See Figure 4 for an example of a quintic with one real root.)

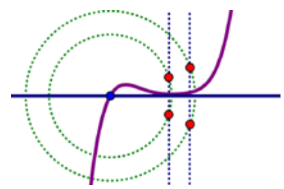
Figure 4

The interpretation of this theorem is that, based only on the sum of the three real roots and the value of the fourth derivative of the quintic at 0, the real component of the complex roots can be determined. Then the location of the imaginary component of the complex roots can be determined using the vertical line through a and a circle centered at the origin with a radius easily calculated using the quintic and the product of the real roots. To observe the meaning of this theorem, use the Figure 4 applet.
(http://appstate.edu/~bossemj/VisualizingRoots/Quintics/Apps/Figure4/)

Theorem 4. Let $q \in Q_3$. Choose a base point $x_0 \neq 0$. Calculate the five values: $q_2 = q(2x_0)$, $q_1 = q(x_0)$, $q_0 = q(0)$, $q_{-1} = q(-x_0)$, and $q_{-2} = q(-2x_0)$. Then

$$q^{(4)}(0) = \frac{q_2 - 4q_1 + 6q_0 - 4q_{-1} + q_{-2}}{x_0^4}.$$

This theorem can be interpreted to say that the value of the fourth derivative of the function at 0 can be calculated analytically sans calculus.

Theorem 5. Let $q \in Q_3$. Choose a base point $x_0 \neq 0$. With s_i and q_j as defined above,

$$a = -\frac{1}{2}\left(s_1 + \frac{q_2 - 4q_1 + 6q_0 - 4q_{-1} + q_{-2}}{4!x_0^4}\right).$$

The complex roots lie on the intersections of the line $x=a$ and the circle of radius $R=\sqrt{-q_0/s_3}$ centered at the origin. If $r=0$ is a root of multiplicity m, then $R=\sqrt{\frac{(-1)^{3-m}}{\prod_{r_i \neq 0} r_i} \cdot \frac{q^{(m)}(0)}{m!}}$.

This theorem combines theorems 3 and 4. It analytically calculates the real part of the complex root without using calculus. It then employs the same elegant calculation and construction to determine the location of the imaginary value of the complex roots. Furthermore, the theorem considers a specific case where there are real roots at 0 of multiplicity m.

Notably, Theorems 3, 4, and 5 are the case $n=3$ of a theorem proven for special polynomials of the form $p(x)=\left((x-a)^2+b^2\right)\cdot\prod(x-r_i)$ for any number $n\geq 3$ of real roots r_i. (See the Appendix.)

Theorem 6. Figure 5 provides an elegant and simple flowchart through which to investigate the graph of a quintic function and determine the number of pairs of complex roots that exists: zero, one, or two. The following notes assist in the interpretation of the flowchart:

- Given that three real roots (r_1, r_2, r_3) are visible, $\prod q'(r_i)=0$ means that $q'(r_1)\cdot q'(r_2)\cdot q'(r_3)=0$. This occurs when there is a critical point (a relative minimum or maximum) at one or more of these real roots.
- When two real roots (r_1, r_2) are visible, $\sum q'(r_i)=0$ means that $q'(r_1)+q'(r_2)=0$. This occurs when there is a critical point at both of these real roots.
- Given that only one real root (r_1) is visible, $q'(r_1)=0$ **and** $q'''(r_i)=0$ means that there is both a critical point at this real root. Notably, $q'''(x)$ can be interpreted as either the function defining the slope of $q''(x)$ or the function defining the rate of change in the concavity of $q(x)$.

These rules can be expressed as tests regarding the even multiplicity of the visible real roots.

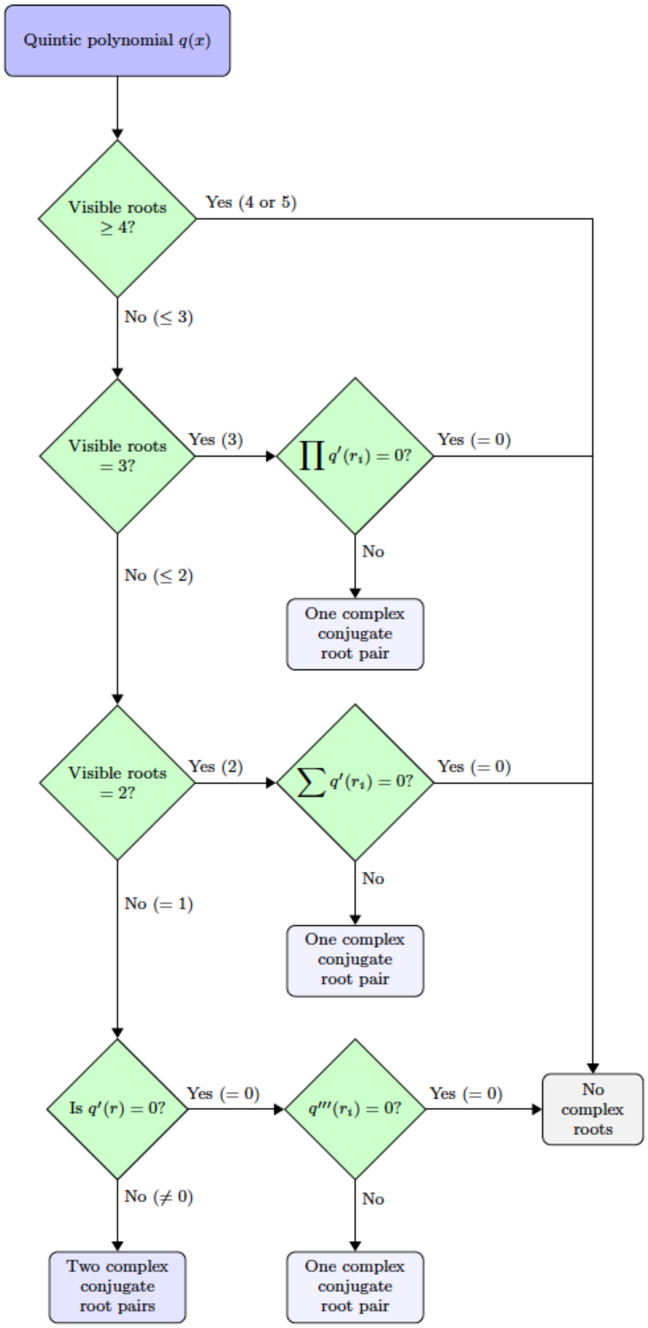

Figure 5

Quintics

In this section, the reader is invited to participate more experientially with the discussion. Dynamic applets are provided throughout for the reader to manipulate the location of complex roots to both confirm the findings reported in the paper and extend these findings based on their observations.

The following investigation of the graphs of quintics is partitioned in a number of ways. First, we introduce characteristics necessary for the behavior of the quintic graph to be interpretable. Second, we segregate our investigation of the location of complex roots of quintics into two cases: one pair of complex roots and two pairs of complex roots.

Interpreting Quintic Graphs for Complex Roots

Throughout this investigation, we consider two behaviors associated with graphs of quintics which shed light on the location of complex roots: critical points (local maximums or minimums) and “flattening”. Since most high school students are familiar with critical points on polynomial functions, where the slope of the graph is zero, little else needs to be discussed in respect to this feature on quintic graphs.

Few high school students have experience with quintics and polynomials of degrees higher than three. Between critical points, the graphs of quintics – even when monic and particularly when the function possesses only real roots – grow vertically at an impressive rate; whether growing positively or negatively away from critical points, the slope of the graph is quite steep. Thus, the steepness of the majority of the graph of the quintic is a readily recognized feature. However, when the function possesses some complex roots, these roots often create a regional flattening near the real value of the complex root. This will be observed in later discussions.

Particular characteristics are necessary to interpret the graph of a quintic and locate complex roots. First, when the real parts of two roots are not equal, if the difference between these real parts is too small, the graph of the quintic may not be distinguishable between two distinct roots and a real root of multiplicity of two, three roots (one real root and one complex pair), or even four roots (two complex pairs). Therefore, in this investigation, when two values are claimed to be distinct, the difference between the values will be sufficient to ensure that the graph behaves as if the values are distinct.

Second, interpreting the graph of a quintic for the location of the complex roots becomes difficult if the imaginary part of the root departs significantly from the *x*-axis. Let us assume that a quintic has complex roots of $a \pm bi$. Without delineating any specific values defining small, medium, and large values for *b*, the behavior of the graph in respect to the magnitude of *b* can be recognized as producing:

$$\begin{cases} b \text{ is small;} & \text{local extrema} \\ b \text{ is medium;} & \text{regional flattening} \\ b \text{ is large;} & \text{indiscernable complex root location} \end{cases}$$

We will begin by considering cases when the imaginary parts of complex roots are sufficiently close to the *x*-axis to observe the local extrema. We will also consider conditions where *b* is of medium magnitude and create regional flattening in the graph. However, we will avoid discussion regarding large values of *b* such that the location of the complex roots cannot be discerned from the graph.

One Pair of Complex Roots

In the case of a quintic with one pair of complex roots, a number of subcases exist. To delineate these cases we consider the factored quintic

$$q(x) = (x-a)(x-b)(x-c)\left((x-d)^2 + e^2\right).$$

Without loss of generality, the cases can now be defined as:

(A) triple real roots: (A1) $a=b=c=\boldsymbol{d}$; (A2) $a=b=c\neq \boldsymbol{d}$

(B) double real roots: (B1) $\boldsymbol{d}<a=b<c$; (B2) $a=b=\boldsymbol{d}<c$; (B3) $a=b<\boldsymbol{d}<c$; (B4) $a=b<c=\boldsymbol{d}$; (B5) $a=b<c<\boldsymbol{d}$

(C) all distinct real roots: (C1) ***d***<*a*<*b*<*c*; (C2) *a*<***d***<*b*<*c*.

Case A1 (*a*=*b*=*c*=*d*). For small and medium values of ***e***, there will be a noticeable flattening of the graph in the region around ***d***. As ***e*** increases, the flattened region of the graph is reduced. Since $q''(d)=0$, the complex root is vertically in line with a point of inflection. (Figure A1. (http://appstate.edu/~bossemj/VisualizingRoots/Quintics/Apps/QGA1/)

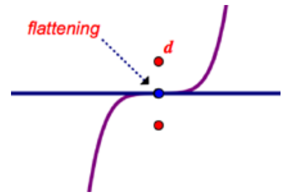

Figure A1

Case A2 (*a*=*b*=*c*≠*d*). For small values of ***e***, the complex roots will be in the neighborhood of the critical point that is a minimum if ***d***<*a* or a maximum if ***d***>*a*. For medium values of ***e***, the graph will demonstrate a flattening near ***d***. If ***d***>*a*, then ***d***>max(x_i) where $q''(x_i)=0$. (Figure A2.) (http://appstate.edu/~bossemj/VisualizingRoots/Quintics/Apps/QGA2/)

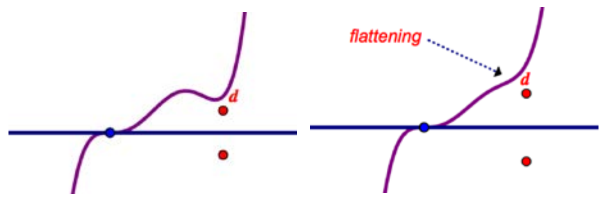

Figure A2

Case B1 (*d*<*a*=*b*<*c*). For small values of ***e***, the complex roots will be in the neighborhood of the local maximum to the left of the double real root. For medium values of ***e***, the graph will demonstrate a flattening on the ***d***-side of *a*. If ***d***<*a*, then ***d***>max(x_i) where $q''(x_i)=0$. (Figure B1.) (http://appstate.edu/~bossemj/VisualizingRoots/Quintics/Apps/QGB1/)

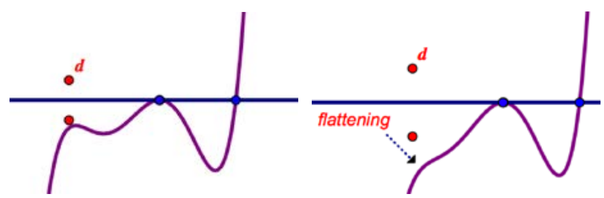

Figure B1

Case B2 (*a*=*b*=*d*<*c*). For small and medium values of ***e***, the complex roots will be in the neighborhood of the flattened double root. (Figure B2.) (http://appstate.edu/~bossemj/VisualizingRoots/Quintics/Apps/QGB2/)

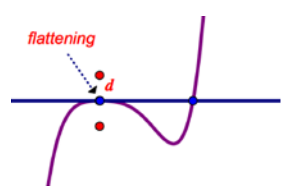

Figure B2

Case B3 (*a*=*b*<*d*<*c*). For small values of ***e***, the complex roots will be in the neighborhood of the local maximum between the real roots. For medium values of ***e***, the graph will demonstrate a flattening in the region of ***d*** between the real roots. In particular cases, there may be $a<x_i<c$ such

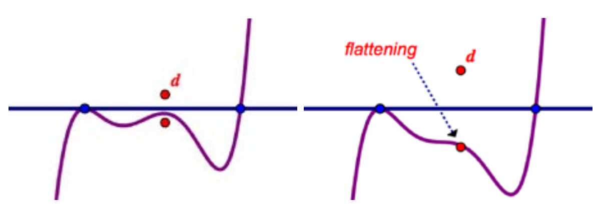

Figure B3

that $q''(x_i)=0$. If x_1 is near *a* or x_2 is near *c*, then $x_1<\mathbf{d}<x_2$. In other words, the complex root is between the leftmost possible and rightmost possible points of inflection between the real roots. Furthermore, there exists at most one $x_i>\mathbf{d}$ such that $q''(x_i)=0$. (Figure B3.)
(http://appstate.edu/~bossemj/VisualizingRoots/Quintics/Apps/QGB3/)

Case B4 (*a*=*b*<*c*=*d*). For small and medium values of ***e***, the complex roots will be in the neighborhood of the flattening of the graph at *c*. Since $q''(d)=0$, ***d*** is vertically aligned with a point of inflection of $q(x)$. (Figure B4.)
(http://appstate.edu/~bossemj/VisualizingRoots/Quintics/Apps/QGB4/)

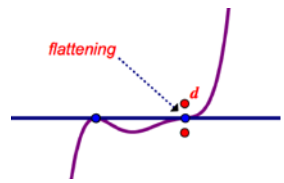

Figure B4

Case B5 (*a*=*b*<*c*<*d*). For small values of ***e***, the complex roots will be in the neighborhood of the local minimum to the right of the real roots. For medium values of ***e***, there is a flattening of the graph to the right of *c*. Notably, ***d***>max(x_i) where $q''(x_i)=0$. (Figure B5.)
(http://appstate.edu/~bossemj/VisualizingRoots/Quintics/Apps/QGB5/)

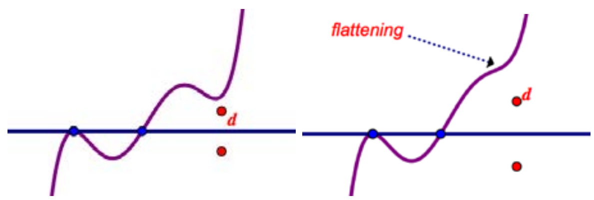

Figure B5

Case C1 (*d*<*a*<*b*<*c*). For small values of ***e***, the complex roots will be in the neighborhood of the local maximum to the left of the real roots. For medium values of ***e***, there is a flattening of the graph to the left of *a*. Notably, ***d***<min(x_i) where $q''(x_i)=0$. (Figure C1.)
(http://appstate.edu/~bossemj/VisualizingRoots/Quintics/Apps/QGC1/)

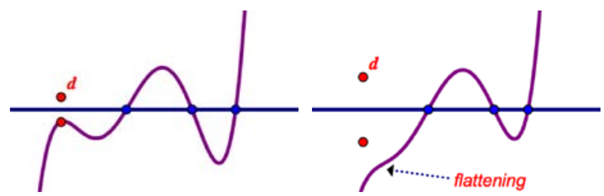

Figure C1

Case C2 (*a*<*d*<*b*<*c*). For small values of ***e***, the complex roots will be in the neighborhood of the local minimum between the real roots. For medium values of ***e***, there is a flattening of the graph between the real roots *a* and *b*. There exists at most one $x_i<\mathbf{d}$ such that $q''(x_i)=0$. (Figure C2.)
(http://appstate.edu/~bossemj/VisualizingRoots/Quintics/Apps/QGC2/)

Figure C2

Two Pairs of Complex Roots

In the case of a quintic with one pair of complex roots, a number of subcases exist. To delineate these cases we will consider the factored quintic

$$q(x)=(x-a)\left((x-b)^2+c^2\right)\left((x-d)^2+e^2\right).$$

Without loss of generality, the cases can now be defined as:

(D) stacked roots: (D1) a=***b***=***d***; (D2) a=***b***≠***d***; (D3) a≠***b***=***d***
(E) all distinct roots: (E1) ***b***<a<***d***; (E2) a<***b***<***d***

Case D1 (*a*=*b*=*d*). For small values of ***c*** and ***e***, Case D1 appears very similar to Case A1, albeit with Case D1 possessing a wider region of flattening in the region around a. For medium values of ***c*** and ***e***, the flattened region of the graph is reduced. Note that $q'(b)=q''(b)=q'(d)=q''(d)=0$. (Figure D1.) (http://appstate.edu/~bossemj/VisualizingRoots/Quintics/Apps/QGD1/)

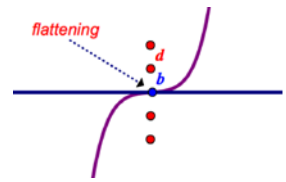

Figure D1

Case D2 (*a*=*b*≠*d*). For small values of ***c*** and ***e***, the complex roots will be in the neighborhood of the local maximum if ***d***<a or the local minimum if ***d***>a. For medium values of ***c*** and ***e***, the graph will demonstrate a flattening in the region of ***d***. Notably, ***b***<min(x_i) and ***d***>max(x_i) where $q''(x_i)=0$. (Figure D2.) (http://appstate.edu/~bossemj/VisualizingRoots/Quintics/Apps/QGD2/)

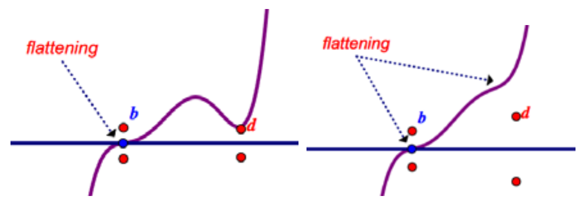

Figure D2

Case D3 (*a*≠*b*=*d*). For small values of ***c*** and ***e***, the complex roots will be in the neighborhood of the flattened local maximum if ***d***<a or the flattened local minimum if ***d***>a. For medium values of ***c*** and ***e***, the graph will demonstrate a flattening in the region of ***d***. Notably, both ***b*** and ***d*** are greater than max(x_i) where $q''(x_i)=0$. (Figure D3.) (http://appstate.edu/~bossemj/VisualizingRoots/Quintics/Apps/QGD3/)

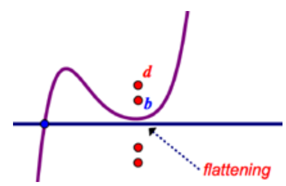

Figure D3

Case E1 (*b*<*a*<*d*). For small values of ***c*** and ***e***, the complex roots associated with ***b*** will be in the neighborhood of the local maximum and the complex roots associated with ***d*** will be in the neighborhood of the local minimum. For medium values of ***c*** and ***e***, the graph will demonstrate flattenings in the regions of ***b*** and ***d***. Notably, ***b***<min(x_i) and ***d***>max(x_i) where $q''(x_i)=0$. (Figure E1.) (http://appstate.edu/~bossemj/VisualizingRoots/Quintics/Apps/QGE1/)

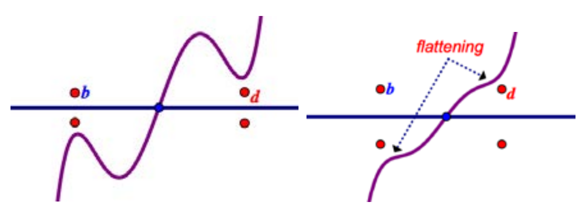

Figure E1

Case E2 (*a*<*b*<*d*). For small values of ***c*** and ***e***, the complex roots associated with ***b*** will be in the neighborhood of a local minimum and the complex roots associated with ***d*** will be in the neighborhood of

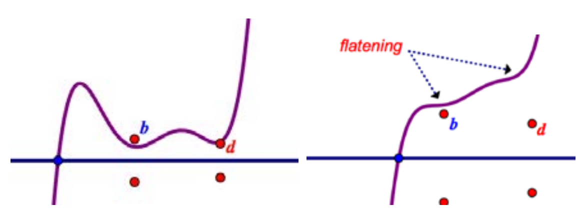

Figure E2

another local minimum. For medium values of ***c*** and ***e***, the graph will demonstrate flattenings in the regions of ***b*** and ***d***. Notably, there exists at most one x_i<***b*** such that $q''(x_i)=0$ and ***d***>max(x_i) where $q''(x_i)=0$. (Figure E2.)
(http://appstate.edu/~bossemj/VisualizingRoots/Quintics/Apps/QGE2/)

Summary & Conclusion

Thus far, one may wonder if this paper's findings are opposite to its intentions. The intention of this investigation is to determine the location of complex roots based on the geometric features of the graph of a real monic quintic polynomial. However, this investigation has considered the features of the graph from known real and complex roots. Experience has shown the writers that, when others are given opportunity to interact with dynamic graphs to investigate these ideas, they very quickly come to be able to closely estimate the real value of complex roots when they are determinable (the imaginary value is not too large). Summarily, in a great many cases of quintics, it is only necessary to investigate whether the graph has local extrema away from real roots or regions of flattening in the graph to approximate the location of the real parts of the complex roots.

Since, unlike quadratics, cubics, and quartics, there is no general solution to a real quintic polynomial, high school and college mathematics students have scant experience with quintic polynomials – and even less with such having complex roots. It is hoped that this investigation shows students that many aspects of quintics are not beyond their understanding and the considering the location of complex roots can be quite intriguing.

Additional extensions to the ideas in this paper can be found in other papers recently written by the authors (Bauldry & Bossé, 2018; Bauldry, Bossé, & Otey, 2017, 2018).

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Appendix

Locating the Complex Roots of Special Monic Polynomials

Definition: Choose *n* real numbers $\{r_i\}_{i=1}^{n}$, not necessarily distinct.

1. Define the *n*+2 degree polynomial *Q* by $q(x)=\left[\prod_{i=1}^{n}(x-r_i)\right]\cdot\left((x-a)^2+b^2\right)$.

2. Fix $x_0\neq 0$, by Q_α denote the values $Q_\alpha = Q(\alpha x_0)$.

3. Define the symmetric difference operator $\Delta_n Q(x) = \frac{1}{x^n}\sum_{k=0}^{n}(-1)^k \binom{n}{k} Q_{\lfloor n/2 \rfloor - k}$.

Lemma. (The (n+1) Derivative of $Q^{(n+1)}(0)$ via Symmetric Differences.) Let $Q(x)$ be a monic n+2 degree polynomial as in Q above. Choose a base point $x_0 \neq 0$. Then, with Q_α and $\Delta_n Q(x)$,

$$Q^{(n+1)}(0) = \Delta_{n+1} Q(x_0) + \cos^2\left(\frac{n\pi}{2}\right) \cdot \frac{(n+2)!}{2} \cdot x_0.$$

Theorem. (The "Circle and line" Construction of Q's Complex Roots). For Q as in the Lemma, the complex roots are located at the intersections of the vertical line

$$x = a = \frac{-1}{2}\left[\frac{Q^{(n+1)}(0)}{(n+1)!} + \sum_{k=1}^{n} r_k\right]$$

and the circle of radius

$$R = \begin{cases} \sqrt{(-1)^n \cdot \dfrac{Q(0)}{\prod_{i=1}^{n} r_i}} & \text{if } \prod_{i=1}^{n} r_i \neq 0 \\ \sqrt{\dfrac{(-1)^{n-m}}{\prod_{i=1}^{n-m} r_i \cdot \dfrac{Q^{(m)}(0)}{m!}}} & \text{if } r_n \text{ is a zero of } Q \text{ of multiplicity } m \end{cases}$$

centered at the origin.